7-15-1963

Computation of the dipole moment of a coaxial antenna in sea water.

Richard B. Reinman

COAXIAL ANTENNA IN SEA WATER - REINMAN

UNIVERSITY OF NEW MEXICO LIBRARY

Unpublished theses submitted for the Master's and Doctor's degrees and deposited in the University of New Mexico Library are open for inspection, but are to be used only with due regard to the rights of the authors. Bibliographical references may be noted, but passages may be copied only with the permission of the authors, and proper credit must be given in subsequent written or published work. Extensive copying or publication of the thesis in whole or in part requires also the consent of the Dean of the Graduate School of the University of New Mexico.

This thesis by [illegible]
has been used by the following persons, whose signatures attest their acceptance of the above restrictions.

A Library which borrows this thesis for use by its patrons is expected to secure the signature of each user.

NAME AND ADDRESS	DATE

COMPUTATION OF THE DIPOLE MOMENT OF A COAXIAL ANTENNA IN SEA WATER

By

Richard B. Reinman

A Thesis
Submitted in Partial Fulfillment of the
Requirements for the Degree of
Master of Science in Electrical Engineering

The University of New Mexico

1963

This thesis, directed and approved by the candidate's committee, has been accepted by the Graduate Committee of the University of New Mexico in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

Abraham Rosenzweig
Assistant Dean

July 15, 1963
Date

COMPUTATION OF THE DIPOLE MOMENT OF
A COAXIAL ANTENNA IN SEA WATER

BY

Richard B. Reinman

Thesis committee

Richard H. Williams
Chairman

A H Koschmann

Ruben D. Kelly

ACKNOWLEDGMENT

The writer is greatly indebted to Dr. Richard H. Williams who, as thesis adviser, suggested the approach to the solution of this problem.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

COMPUTATION OF THE DIPOLE MOMENT OF A COAXIAL ANTENNA IN SEA WATER

Introduction

As an idealized radiator in its most elementary form, the oscillating electric dipole finds extensive use in wave propagation studies. Basically, a dipole consists of equal and opposite charges separated by an infinitesimal distance. This separation distance in vector form directed from negative to positive charge may be multiplied by charge magnitude to yield the dipole moment. When applied to linear antennas, the charge magnitude varying sinusoidally with time produces an oscillating electric dipole moment.

The prime advantage in the use of the dipole moment lies in its simple relationship with the vector potential, $\vec{A}$, and Hertzian vector, $\vec{\pi}$, from which electric and magnetic fields may be computed.[1] By summing vectorially the contributions of each infinitesimal dipole moment along an antenna, the fields at any point external to the source may be found.

Perhaps the most important application of the dipole moment in antenna theory involves integration of the Poynting vector over a surface to find the power radiated across that surface. In a lossless medium such as air, the power radiated is independent of R, the distance from the dipole to the surface of integration. However, in a dissipative medium

---

[1] J. A. Stratton, Electromagnetic Theory (1st ed.; New York: McGraw-Hill, 1941), pp. 435-36.

such as sea water, Moore points out that because of the complex dielectric constant, a $\frac{1}{R^3}$ term dominates the power expression for small R. In order to maintain finite power as $R \rightarrow 0$, the dipole moment itself must approach zero. Thus, use of the dipole moment in sea water must be confined to distant fields.[2]

The purpose of this paper is to develop a meaningful expression for the dipole moment in a dissipative medium. Since the dipole moment is dependent on the current distribution which in turn is a function of system parameters, one might well start with determination of the input impedance. Two of the most popular methods for determining input impedance to a radiator in an air medium are (1) the induced emf method, and (2) a method whereby the power flow across a spherical surface is calculated as the radius of the sphere becomes infinite. Both methods assume a current distribution; only the first includes reactance as a part of the impedance.

Although the above methods work well in air, their use in a dissipative medium is questionable because of uncertainty of the current distribution. A more valid approach is to treat a coaxial antenna as a transmission line, although the computations involved are more difficult.[3] It is the transmission line approach which will be pursued in the following sections.

---

[2]Richard K. Moore, "The Theory of Radio Communication between Submerged Submarines" (unpublished Ph.D. dissertation, Graduate School, Cornell University, 1951), p. 89.

[3]Ibid. pp. 85-87.

such as sea water, Moore [illegible]

tric constant, $a = \frac{1}{R^2}$ [illegible]

In order to maintain finite power [illegible]

approach zero. Thus, [illegible]

confined to distant fields.[3]

The purpose of this paper is to develop [illegible]

the dipole moment [illegible]

dependent on the current distribution [illegible]

parameters, one might [illegible]

ance. Two of the most [illegible]

to a radiator in [illegible]

method whereby the power [illegible]

the radius of the sphere being [illegible]

distribution; only the [illegible]

Although the above methods [illegible]

sipative medium is [illegible]

distribution. A more [illegible]

transmission line, although [illegible]

It is the transmission line approach which will be [illegible]

sections.

---

[2]Richard K. Moore, [illegible] Submerged Submarines [illegible] Cornell University, [illegible]

[3]Ibid., pp. [illegible]

Coaxial Transmission Line

Consider the infinitely long coaxial transmission line of Figure 1.

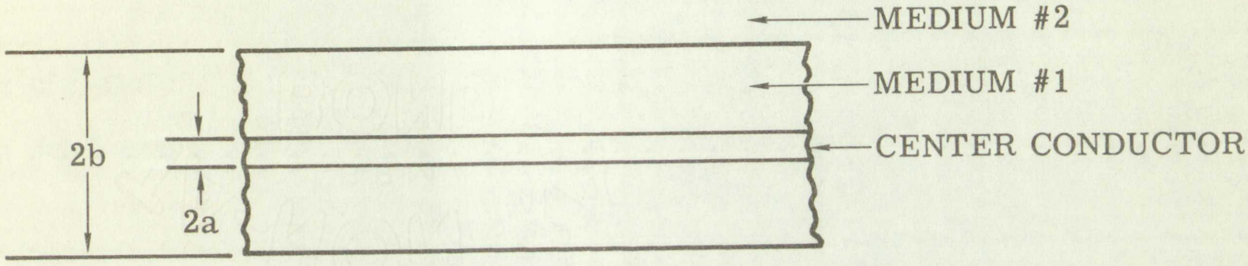

Figure 1. Cross Section of Submerged Coaxial Line

To avoid undue complications, the following assumptions will be made:

1. The center conductor is assumed to have infinite conductivity.

2. Medium No. 1, the dielectric, is assumed to be lossless, and to have the constants μ_o, $\underline{\epsilon_1} = \epsilon_1$(real).

3. Medium No. 2, the outer conductor, is assumed to have the constants σ_2, μ_o, $\underline{\epsilon_2}$,

where

μ = permittivity in henrys/meter

$\underline{\epsilon}$ = complex dielectric constant in farads/meter

σ = conductivity in mhos/meter.

4. The skin depth in the outer conductor is assumed to be much larger than the dielectric radius, b.

That is, $\sqrt{\dfrac{2}{\omega \mu_0 \sigma_2}} \gg b,$

where ω = radians/second

and b = radius in meters.

5. The assumed coordinate system is cylindrical (r, ϕ, z).

6. It is further assumed that only a TM mode exists on the coaxial transmission line, and that the extent of Medium No. 2 is infinite so that all fields are uniform with respect to the ϕ-coordinate direction.

7. Finally, assuming that the fields are time harmonic (the factor $e^{j\omega t}$ being suppressed), the TM fields both in the dielectric and the outer conductor satisfy the relationship:[4]

$$E_z = f(r)e^{-jk_g z} \tag{1}$$

where k_g is the guide propagation constant and where f satisfies Bessel's equation,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(k^2 - k_g^2\right)f = 0.$$

We shall use the notation, $h^2 = k^2 - k_g^2.$ (2)

The transverse fields, E_r and H_ϕ, are:

$$E_r = -\frac{jk_g}{h^2}\frac{\partial E_z}{\partial r} \tag{3}$$

$$H_\phi = \frac{k}{\eta k_g} E_r , \tag{4}$$

[4] W. Panofsky and M. Phillips, Classical Electricity and Magnetism (Reading, Massachusetts: Addison-Wesley, 1955), p. 195.

where

$$\underline{\epsilon} = \epsilon\left(1 - \frac{j\sigma}{\omega\epsilon}\right), \text{ the complex dielectric constant}$$

$$k = \omega\sqrt{\mu_0\,\underline{\epsilon}}\ , \text{ the complex propagation constant}$$

$$\eta = \sqrt{\frac{\mu_0}{\underline{\epsilon}}}\ , \text{ the complex intrinsic impedance.}$$

In the dielectric, there can be standing waves in the radial direction; hence, we let

$$f_1 = A\,J_0(h_1 r) + B\,N_0(h_1 r) \qquad a \leq r \leq b \tag{5}$$

In the outer conductor, the fields must behave like outward traveling waves as $r \to \infty$; hence, we let

$$f_2 = C\,H_0^{(2)}(h_2 r). \tag{6}$$

The boundary conditions to be satisfied are:

1. When $r = a$, then $E_{z1} = 0$.
2. When $r = a$, then $\oint_c \vec{H}_{\phi1} \cdot d\vec{\ell} = I$.
3. When $r = b$, then $E_{z1} = E_{z2}$.
4. When $r = b$, then $H_{\phi1} = H_{\phi2}$.

These four conditions will determine the four unknowns: A, B, C, and k_g.

Thus, when $a \leq r \leq b$,

$$E_{z1} = \frac{h_1}{j\omega\epsilon_1}\,\frac{I}{2\pi a}\left[\frac{N_0(h_1 a)J_0(h_1 r) - J_0(h_1 a)N_0(h_1 r)}{D_1}\right] \tag{7}$$

$$H_{\phi1} = \frac{I}{2\pi a}\left[\frac{N_0(h_1 a)J_1(h_1 r) - J_0(h_1 a)N_1(h_1 r)}{D_1}\right] \tag{8}$$

$$E_{r1} = \frac{k_g}{\omega\epsilon_1}\,H_{\phi1}. \tag{9}$$

When $b \leq r < \infty$,

$$E_{z2} = \left(\frac{h_1}{j\omega\epsilon_1}\right)\left(\frac{I}{2\pi a}\right)\left(\frac{D_2}{D_1}\right)\frac{H_0^{(2)}(h_2 r)}{H_0^{(2)}(h_2 b)} \tag{10}$$

$$H_{\phi 2} = \left(\frac{\epsilon_2}{\epsilon_1}\right)\left(\frac{h_1}{h_2}\right)\left(\frac{I}{2\pi a}\right)\left(\frac{D_2}{D_1}\right)\frac{H_1^{(2)}(h_2 r)}{H_0^{(2)}(h_2 b)} \tag{11}$$

and

$$E_{r2} = \frac{k_g}{\omega\epsilon_2} H_{\phi 2}, \tag{12}$$

where

$$D_1 = J_1(h_1 a)N_0(h_1 a) - J_0(h_1 a)N_1(h_1 a)$$

$$D_2 = J_0(h_1 b)N_0(h_1 a) - J_0(h_1 a)N_0(h_1 b)$$

$$I = I_0 e^{-jk_g z}.$$

Since $H_{\phi 1} = H_{\phi 2}$ at $r = b$,

$$\frac{D_3}{D_2} = \left(\frac{\epsilon_2}{\epsilon_1}\right)\left(\frac{h_1}{h_2}\right)\frac{H_1^{(2)}(h_2 b)}{H_0^{(2)}(h_2 b)}, \tag{13}$$

where

$$D_3 = J_1(h_1 b)N_0(h_1 a) - J_0(h_1 a)N_1(h_1 b).$$

To obtain an approximate expression for k_g, one first assumes the difference between k_g and k_1 is small and then computes a correction term, Δk, such that

$$k_g^2 = k_1^2(1 + \Delta k) \qquad \text{or} \qquad k_g = k_1(1 + \Delta k)^{1/2}$$

$$\mathrm{Im}\left[k_g\right] < 0. \tag{14}$$

The validity of this assumption becomes apparent as the theory develops. If the ratio $\sigma_2/\omega\epsilon_2$ is large (as is normally the case of wave transmission in sea water),

then

$$k_2 \approx \sqrt{-j\omega\mu\sigma_2} \qquad \mathrm{Im}\left[k_2\right] < 0 \tag{15}$$

and

$$k_1 \approx \omega\sqrt{\mu\epsilon_1} \ . \tag{16}$$

It follows that $|k_2|^2 >> k_1^2$

so that

$$h_2 = \sqrt{k_2^2 - k_g^2} \approx k_2. \tag{17}$$

From (2), $k_1^2 - h_1^2 = k_2^2 - h_2^2$. Since $|k_2| >> k_1$, $|h_1|$ is at most the same order of magnitude as $|k_2|$. By the stipulation that the skin depth in the outer conductor is much larger than the dielectric radius, $|k_2 b| << 1$, one deduces that $|h_1 b| << 1$. Taking the equivalent expressions for D_2 and D_3, using small argument approximations for the Bessel functions, and incorporating (13) and (17), one finds that

$$\frac{D_3}{D_2} \approx \frac{-N_1(h_1 b)}{N_0(h_1 a) - N_0(h_1 b)} \approx \frac{-1}{h_1 b \, \ell n\left(\frac{b}{a}\right)} \approx \left(\frac{\epsilon_2}{\epsilon_1}\right)\left(\frac{h_1}{k_2}\right)\frac{H_1^{(2)}(k_2 b)}{H_0^{(2)}(k_2 b)}. \tag{18}$$

Noting that $\frac{k_2}{k_1} = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$, and solving (18) for h_1^2:

$$h_1^2 \approx \frac{-k_1^2}{k_2 b \, \ell n \, \frac{b}{a}} \; \frac{H_0^{(2)}(k_2 b)}{H_1^{(2)}(k_2 b)} . \tag{19}$$

. . . .

Therefore,

$$k_g^2 = k_1^2 - h_1^2 \approx k_1^2\left[1 + \frac{1}{k_2 b\, \ell n\left(\frac{b}{a}\right)}\ \frac{H_0^{(2)}(k_2 b)}{H_1^{(2)}(k_2 b)}\right]. \tag{20}$$

The ratio of the Hankel functions may be evaluated for $|k_2 b| << 1$ as follows:

Using small argument approximations,[5]

$$H_0^{(2)}(k_2 b) \approx 1 - \frac{j2}{\pi}\ \ell n\ \frac{k_2 b}{1.12} = \frac{j2}{\pi}\ \ell n\left(\frac{1.12}{jk_2 b}\right)$$

$$H_1^{(2)}(k_2 b) \approx \frac{j2}{\pi}\left(\frac{1}{k_2 b}\right)$$

$$\frac{H_0^{(2)}(k_2 b)}{H_1^{(2)}(k_2 b)} \approx k_2 b\, \ell n\left(\frac{1.12}{jk_2 b}\right). \tag{21}$$

Combining (14) with (20) and simplifying,

$$k_g = k_1(1 + \Delta k)^{1/2} \approx k_1\left[\frac{\ell n\left(\frac{1.12}{jk_2 a}\right)}{\ell n\left(\frac{b}{a}\right)}\right]^{1/2} \qquad \mathrm{Im}\left[k_g\right] < 0. \tag{22}$$

One can now compute the characteristic impedance, Z_0, of this coaxial transmission line.

The transverse voltage is $V_{ab} = \int_a^b E_{r1}\, dr$.

Using the preceding formulas and the indefinite integral

$\int^x J_1(\sigma)d\sigma = -J_0(x)$, one finds that $V_{ab} = \dfrac{-k_g I D_2}{2\pi a \omega \epsilon_1 h_1 D_1}$.

---

[5] R. F. Harrington, Time-Harmonic Electromagnetic Fields (New York: McGraw-Hill, 1961), p. 203.

Applying small argument approximations,

$$\frac{D_2}{D_1} \approx -h_1 a\, \ln\left(\frac{b}{a}\right)$$

so that the characteristic impedance becomes

$$Z_0 \approx \frac{\eta_1}{2\pi} (1 + \Delta k)^{1/2} \ln\left(\frac{b}{a}\right). \tag{23}$$

Furthermore, the series impedance per unit length, $Z = R + j\omega L$; and the shunt admittance per unit length, $Y = G + j\omega C$, can be computed directly, since, from transmission line theory,

$$R + j\omega L = jk_g Z_0 \approx \frac{j\omega\mu_0}{2\pi} (1 + \Delta k) \ln \frac{b}{a}$$

$$G + j\omega C = \frac{jk_g}{Z_0} \approx \frac{j2\pi\omega\epsilon_1}{\ln\left(\frac{b}{a}\right)} = j\omega C.$$

Coaxial Line Application in Sea Water

The coaxial transmission line finds application as a submerged coaxial antenna in either of two ways:

Case 1 - Open-circuit termination in which the load end of the center conductor is recessed in the end of the dielectric and insulated from the external medium.

Case 2 - Short-circuit termination in which the load end of the center conductor extends beyond the end of the dielectric directly into the external medium.

In both cases, the outer braid of the line is removed so that the medium itself becomes the outer conductor. To analyze these antennas one uses ordinary transmission line theory (see Figure 2).

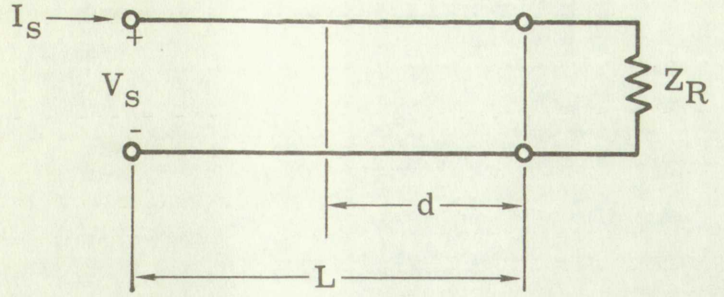

Figure 2. Elementary Transmission Line with Load Z_R

$$\Gamma = \frac{Z_R - Z_0}{Z_R + Z_0} \tag{24}$$

$$V(d) = V_s \left[\frac{e^{jk_g d} + \Gamma e^{-jk_g d}}{e^{jk_g L} + \Gamma e^{-jk_g L}} \right] \tag{25}$$

$$I(d) = I_s \left[\frac{e^{jk_g d} - \Gamma e^{-jk_g d}}{e^{jk_g L} - \Gamma e^{-jk_g L}} \right] \tag{26}$$

$$Z(d) = Z_0 \left[\frac{1 + \Gamma e^{-j2k_g d}}{1 - \Gamma e^{-j2k_g d}} \right], \tag{27}$$

where the subscript s refers to the generator, and Γ is the voltage reflection coefficient.

It has been typical to assume $\Gamma = 1$ for Case 1 and $\Gamma = -1$ for Case 2. Then, for each case, one can compute the driving point impedance and the

current distributions for each antenna. (The current distributions are used in determining the dipole moment.)

Experiment has shown that the assumption of an open-circuit termination in Case 1 is very good, while the assumption of a short-circuit termination in Case 2 is less accurate than desired, especially at the higher frequencies.[6] One purpose of this work is to derive a more accurate expression for the reflection coefficient in Case 2.

Center Conductor in an Infinite Medium

For computation of input impedance, the exposed center conductor will be treated as the load of a coaxial transmission line whose length is finite. First,the load impedance, Z_R, will be expressed as the characteristic impedance of a solid cylinder located in an infinite medium. The analogy between the infinite length of the cylinder implied here and the finite protrusion of the center conductor in practice is proper since wave attenuation in the medium is rapid. Moore suggests a protrusion on the order of one-quarter the wavelength in sea water for the reflected wave from the end of the center conductor to be negligible.[7]

A method for finding the characteristic impedance of a wire in an infinite medium has been described by Stratton.[8] It should be noted that Stratton's expressions were derived for a traveling wave of the type

---

[6] Earl H. Flath, Jr., and Oscar Norgordon, Expressions for Input Impedance and Power Dissipation in Lossy Concentric Lines, (NRL Report R-3436; Washington: Naval Research Laboratory, 1949), p. 6.

[7] 130-133.

[8] 524-533.

$e^{j(hz-\omega t)}$ whereas $e^{j(\omega t - k_g z)}$ is assumed for attenuation and phase of the wave throughout this paper; hence, Stratton's expressions are modified as necessary. Other differences are that: (1) because of the above, the Hankel functions used in this discussion are of the second kind, $H_n^{(2)}(h_2 r)$ rather than of the first kind, and (2) by our notation, Stratton's relationship $\lambda^2 = k^2 - h^2$ becomes $h^2 = k^2 - k_g^2$.

Consider a solid cylinder of conductivity σ, and radius "a" located in an infinite medium of conductivity σ_2 (see Figure 3).

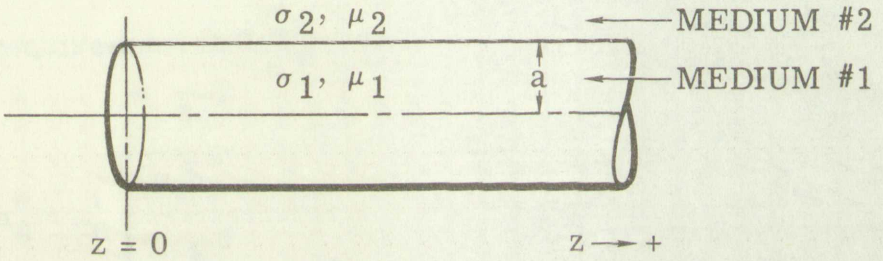

Figure 3. Submerged Solid Cylinder

Extending from z = 0 in the positive z direction, we assume it has infinite length. One might think of this system as an infinitely long coaxial line whose dielectric is the surrounding medium and whose outer conductor is of infinite radius.

In that which follows, superscripts i and e refer to Media No. 1 and No. 2, respectively. Furthermore, in discussing this system, a bar will be placed over certain of the symbols to distinguish them from their counterparts elsewhere in this paper. We shall list (as modified) only those $\vec{E}$ and $\vec{H}$ expressions necessary to the development of this topic.

$e^{j(hz-\omega t)}$ whereas [illegible]

wave throughout this [illegible]

as necessary. One difference [illegible]

Hankel functions [illegible]

rather than of the [illegible]

ship $\lambda^2 = k^2 - h^2$ because [illegible]

Consider a solid [illegible]

in an infinite medium [illegible]

Figure 2. [illegible]

Extending from $z = 0$ [illegible]

length. One might [illegible]

whose dielectric is [illegible]

of infinite radius.

In that which follows [illegible]

and No. 2, respectively. [illegible]

will be placed over certain [illegible]

counterparts elsewhere [illegible]

those E and H [illegible]

$$E_z^i = J_0(\overline{h}_1 r)\, a_0^i\, e^{j(\omega t - \overline{k}_g z)} \tag{28}$$

$$H_\phi^i = \frac{-j\overline{k}_1^2}{\mu_1 \omega \overline{\overline{h}}_1} J_1(\overline{h}_1 r)\, a_0^i\, e^{j(\omega t - \overline{k}_g z)} \tag{29}$$

$$E_r^e = \frac{j\overline{k}_g}{\overline{\overline{h}}_2} H_1^{(2)}(\overline{h}_2 r)\, a_0^e\, e^{j(\omega t - \overline{k}_g z)} \tag{30}$$

$$H_\phi^e = \frac{-j\overline{k}_2^2}{\mu_2 \omega \overline{\overline{h}}_2} H_1^{(2)}(\overline{h}_2 r)\, a_0^e\, e^{j(\omega t - \overline{k}_g z)}\ , \tag{31}$$

where a_0^i, a_0^e are constants.

From the requirement that $H_\phi^i = H_\phi^e$ at $r = a$,

$$a_0^e = a_0^i \left[\frac{\dfrac{\overline{k}_1^2}{\mu_1 \overline{\overline{h}}_1} J_1(\overline{h}_1 a)}{\dfrac{\overline{k}_2^2}{\mu_2 \overline{\overline{h}}_2} H_1^{(2)}(\overline{h}_2 a)} \right]. \tag{32}$$

To find a_0^i we use the relationship that current density $\vec{J} = \sigma \vec{E}$. We know E_z in terms of a_0^i. Hence,

$$I = I_z = \int_0^a \int_0^{2\pi} J_z \, r d\phi dr \tag{33}$$

$$= 2\pi a_0^i \sigma_1 e^{j(\omega t - \overline{k}_g z)} \int_0^a J_0(\overline{h}_1 r) r dr.$$

Since $\int_0^a J_0(\overline{h}_1 r) r dr = \dfrac{a}{\overline{\overline{h}}_1} J_1(\overline{h}_1 a)$,

$$I_z = 2\pi a_0^i \sigma_1 e^{j(\omega t - \overline{k}_g z)} \frac{a}{\overline{\overline{h}}_1} J_1(\overline{h}_1 a).$$

Letting $I = I_0 e^{j(\omega t - \overline{k}_g z)}$ and solving for a_0^i ,

$$a_0^i = \frac{\overline{h}_1 I_0}{2\pi a \sigma_1 J_1(\overline{h}_1 a)} , \qquad (34)$$

leading to the expression via (32),

$$a_0^e = \frac{\mu_2 \overline{k}_1^2 \overline{h}_2 I_0}{\mu_1 \overline{k}_2^2 \, 2\pi a \sigma_1 H_1^{(2)}(\overline{h}_2 a)} . \qquad (35)$$

This value for a_0^e may now be inserted into the expression for E_r^e, (30), which in turn may be integrated across Medium No. 2 to find the voltage. Division of the voltage by current yields the characteristic impedance, Z_R. If $\mu_1 = \mu_2 = \mu_0$,

$$E_r^e = \frac{j\overline{k}_g \overline{k}_1^2 I}{2\pi a \sigma_1 \overline{k}_2^2} \; \frac{H_1^{(2)}(\overline{h}_2 r)}{H_1^{(2)}(\overline{h}_2 a)} ,$$

and

$$Z_R = \frac{V}{I} = \frac{\int_a^\infty E_r^e \, dr}{I} .$$

Since

$$\overline{h}_2 \int_a^\infty H_1^{(2)}(\overline{h}_2 r) dr = H_0^{(2)}(\overline{h}_2 r) \Big|_\infty^a = H_0^{(2)}(\overline{h}_2 a) ,$$

[9]

$$Z_R = \frac{j\overline{k}_g \overline{k}_1^2}{2\pi a \sigma_1 \overline{k}_2^2 \overline{h}_2} \; \frac{H_0^{(2)}(\overline{h}_2 a)}{H_1^{(2)}(\overline{h}_2 a)} . \qquad (36)$$

Equation 36 will be useful only after we express $\overline{k}_g$ in terms of $\overline{k}_2$ and $\overline{h}_2$. To do this, we refer to an equation by Stratton resulting from roots of a previously developed transcendental equation.[10] Modified to our system,

---

[9] (See next page for this footnote)

[10] 527.

[9]One might become concerned when applying formulas in this section to copperweld. For example, the radius of the steel center for 30 percent copperweld is 87 percent that of the total; that for 40 percent copperweld is 80 percent. Since the conductivity of copper is roughly 15 times that of the steel center, formulas for E_z and H_ϕ beginning with (28) would have to be developed for three separate regions. Solutions would become exceedingly difficult. An alternate plan would be to disregard the steel center altogether and treat the wire as a hollow copper tube, since, even at DC, the copper in 30 percent copperweld carries almost 83 percent of the total current.

Due to skin effects, at higher frequencies there is negligible current in the steel. One may determine the skin depth, δ, in copper as $\sqrt{\frac{2}{\omega\mu\sigma}}$. Assuming the steel radius to be 84 percent of the total, only minor error is introduced when the total radius, a, exceeds $3\delta = \frac{1.26 \text{ meters}}{\sqrt{f}}$. For a discussion of currents in copperweld see the article by B. R. Teare, Jr., and E. R. Schatz, "Copper-Covered Steel Wire at Radio Frequencies," Proceedings of the I.R.E. (New York, 1944), pp. 397-403.

Stratton's equation becomes

$$\frac{\overline{k_1}^2 \, J_1(u)}{u J_0(u)} = \frac{\overline{k_2}^2 \, H_1^{(2)}(v)}{v H_0^{(2)}(v)} , \tag{37}$$

where

$$u = \overline{h_1} a$$

$$v = \overline{h_2} a.$$

Note that at practical frequencies along a highly conductive cylinder $|\overline{k_1}^2| \approx |\omega \mu_1 \sigma_1| >> |\overline{k_g}^2|$. Hence $|\overline{h_1}| \approx |\overline{k_1}|$. Now $u = \overline{h_1} a$ which, as the argument of a Bessel function, permits use of the function's asymptotic value. Thus $\frac{J_1(u)}{J_0(u)} = \lim_{u \to \infty} \frac{J_1(u)}{J_0(u)}$.

Using asymptotic values,

$$\lim_{u \to \infty} \frac{J_1(u)}{J_0(u)} = \lim_{u \to \infty} \frac{\cos\left(u - \frac{\pi}{4} - \frac{\pi}{2}\right)}{\cos\left(u - \frac{\pi}{4}\right)} = \lim_{u \to \infty} \frac{\cos\left(\delta - \frac{\pi}{2}\right)}{\cos \delta} ,$$

where

$$\delta = u - \frac{\pi}{4} .$$

$$\frac{\cos\left(\delta - \frac{\pi}{2}\right)}{\cos \delta} = \tan \delta.$$

Since $\delta = \overline{h_1} a - \frac{\pi}{4}$, where $\mathrm{Re}\left[\overline{h_1} a - \frac{\pi}{4}\right] > 0$ and $\mathrm{Im}\left[\overline{h_1} a\right] < 0$, δ takes the form $(\alpha - j\beta)$ $\alpha, \beta > 0$,

and $$\lim_{u \to \infty} \frac{J_1(u)}{J_0(u)} = \lim_{\substack{\alpha \to \infty \\ \beta \to \infty}} \tan(\alpha - j\beta)$$

$$= \lim_{\substack{\alpha \to \infty \\ \beta \to \infty}} -j \left[\frac{1 - e^{-2\beta} \, e^{-2j\alpha}}{1 + e^{-2\beta} \, e^{-2j\alpha}}\right] = -j \approx \frac{J_1(u)}{J_0(u)} . \tag{38}$$

Stratton points out that for a cylinder of infinite conductivity $\overline{k_g} = \overline{k_2}$ whereas for one of high conductivity $\overline{k_g} \approx \overline{k_2}$ and small argument approximations for $H_n^{(2)}(v)$ may be employed.[11] Adapting the results of (21) and replacing the constant (1.12) with $\left(\frac{2}{\gamma}\right)$ where γ is Euler's constant $\approx 1.781\ldots$,

$$\frac{H_1^{(2)}(v)}{H_0^{(2)}(v)} \approx -\frac{1}{v\,\ln\left(\frac{j\gamma v}{2}\right)} . \tag{39}$$

Substituting (38) and (39) into (37),

$$-\frac{j\overline{k_1}^2}{u} \approx -\frac{\overline{k_2}^2}{v^2 \ln\left(\frac{j\gamma v}{2}\right)}$$

or

$$v^2 \ln\left(\frac{j\gamma v}{2}\right) \approx \frac{u\overline{k_2}^2}{j\overline{k_1}^2} \approx -v\,\frac{H_0^{(2)}(v)}{H_1^{(2)}(v)} . \tag{40}$$

But, as noted above, $\overline{k_1} \approx \overline{h_1} = \frac{u}{a}$

so that

$$v^2 \ln\left(\frac{j\gamma v}{2}\right) \approx -\frac{j\overline{k_2}^2 a}{\overline{k_1}} . \tag{41}$$

Equation 41 is transcendental, but note that it is independent of u (and hence, $\overline{h_1}$). Furthermore, it may be solved as follows:

Let

$$x = \left(\frac{j\gamma v}{2}\right)^2 = -\frac{\gamma^2 v^2}{4} . \tag{42}$$

---

[11] 528.

Then

$$\ell n\ x = 2\ \ell n \left(\frac{j\gamma v}{2}\right). \tag{43}$$

Also, let

$$y = x\ \ell n\ x = -\frac{\gamma^2 v^2}{2}\ \ell n \left(\frac{j\gamma v}{2}\right) . \tag{44}$$

From (41) and (44),

$$y = \frac{j\gamma^2 \overline{k}_2^{\,2} a}{2\overline{k}_1} . \tag{45}$$

Simplifying:

$$\frac{\overline{k}_2^{\,2}}{\overline{k}_1} \approx \frac{-j\omega\mu\sigma_2}{\pm(1 - j)\sqrt{\frac{\omega\mu\sigma_1}{2}}} ,$$

where, because $\mathrm{Im}\left[\overline{k}_1\right] < 0$, we choose the + sign for the denominator.

Thus,

$$\frac{\overline{k}_2^{\,2}}{\overline{k}_1} \approx \frac{(1 - j)}{2}\ \sigma_2 \sqrt{\frac{2\,\omega\mu}{\sigma_1}} ,$$

which when substituted into (44) yields

$$y = \frac{(1 + j)}{4}\ \gamma^2 a \sigma_2 \sqrt{\frac{2\,\omega\mu}{\sigma_1}}$$

$$= \frac{\gamma^2 a\,\sigma_2}{2} \sqrt{\frac{\omega\mu}{\sigma_1}}\ \ \underline{/45} . \tag{46}$$

Equation 46 allows solution of y in terms of familiar constants. Evaluation of x is done by a trial-and-error process using successive approximations. Based on the principle that the logarithm of a varying function varies much more slowly than the function itself, one assumes that

$$x_{n+1}\ \ell n\ x_n = y \quad \text{or} \quad x_{n+1} = \frac{y}{\ell n\, x_n} , \tag{47}$$

where the subscript n is the $n^{\underline{th}}$ approximation. The newly computed value, x_{n+1}, replaces x_n in the denominator of (47), and x_{n+2} is computed. These steps are repeated until the difference between x_{n+k} and x_{n+k-1} is negligible. As a starting point, $\ell n\, x_0$ is set at -20. A detailed calculation of x appears in the section, "A Sample Computation." As an aid to calculating Z_R, Figure 4 provides a rapid means for evaluating x over often used values of y.

Determination of x allows calculation of v^2 by (42) and hence $\overline{h_2}$, where the sign of v is taken such that $\mathrm{Im}[v] < 0$. Substituting $\overline{k_2}$ for $\overline{k_g}$ and making use of (40) and (41), the conductor's characteristic impedance of (36) reduces to

$$Z_R \approx \frac{j\mu_0 f}{a\overline{h_2}^2} \sqrt{\frac{\sigma_2}{\sigma_1}} \quad . \tag{48}$$

For purposes of calculation, however, it is more convenient to retain the expression for Z_R in terms of x. Noting that $a\overline{h_2}^2 = \frac{v^2}{a} = -\frac{4x}{a\gamma^2}$, (48) becomes

$$Z_R \approx -\frac{j\mu_0 a\gamma^2 f}{4x\sqrt{\frac{\sigma_1}{\sigma_2}}} \quad . \tag{49}$$

Equation 49 leads to a crude explanation of why an assumed value of -1 for the reflection coefficient becomes less dependable with an increase in frequency. For $\Gamma = -1$, $Z_R = 0$, and in general any increase in $|Z_R|$ results in an increasing deviation from -1 for Γ. Note from Figure 4 that on a log-log plot $|y|$ is almost linear with $|x|$ over the range shown.

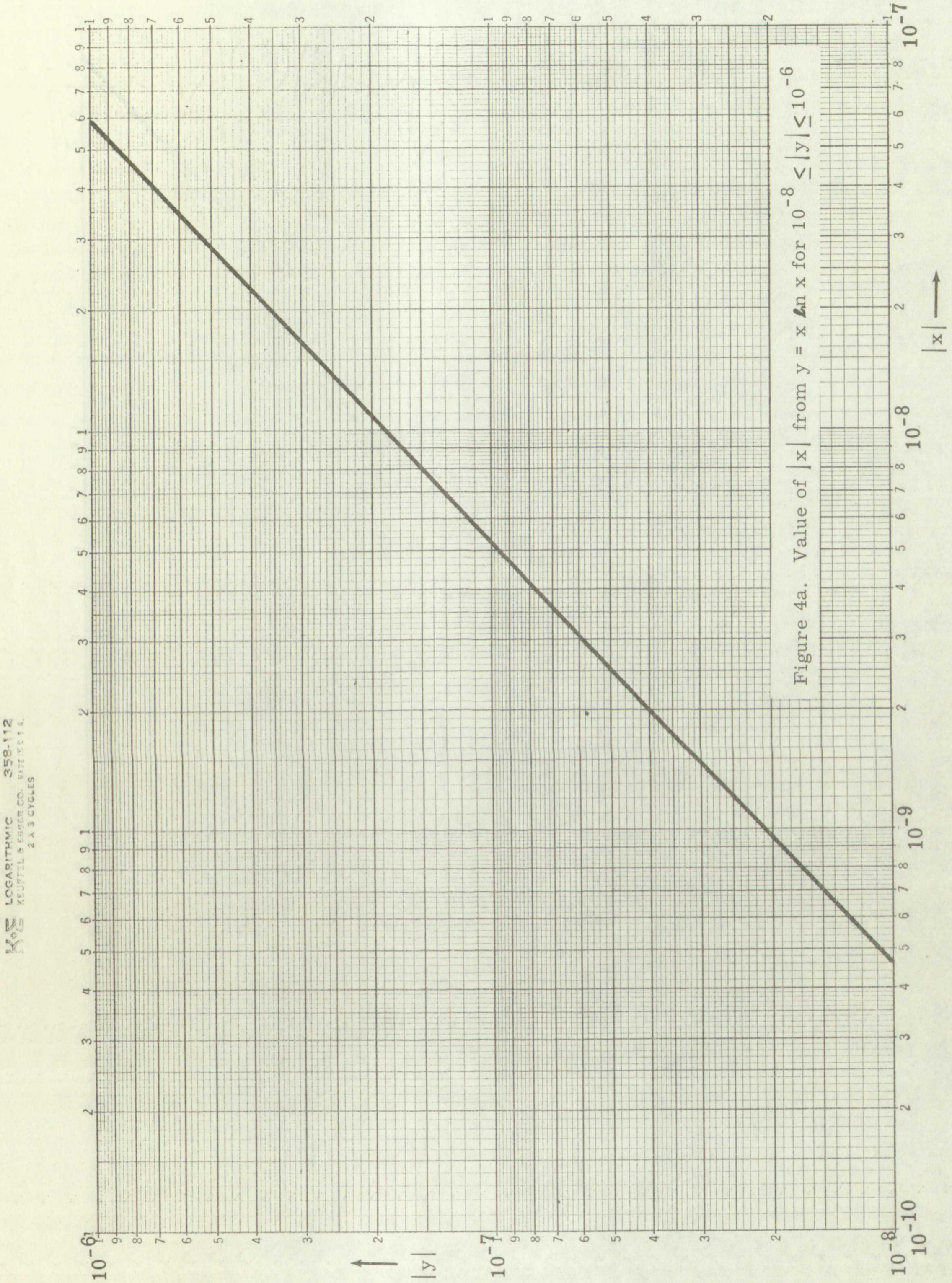

Figure 4a. Value of $|x|$ from $y = x \ln x$ for $10^{-8} \leq |y| \leq 10^{-6}$

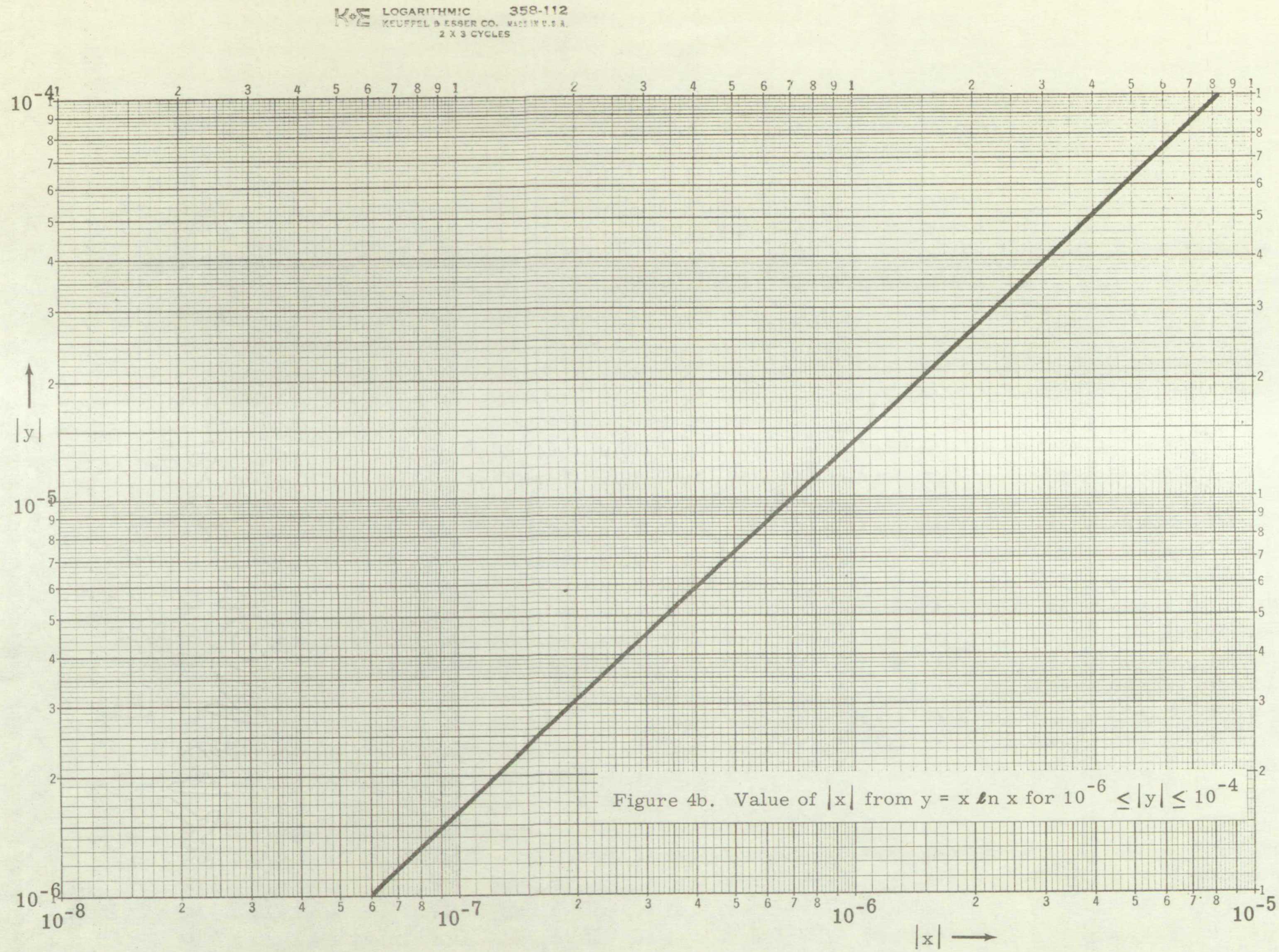

Figure 4b. Value of $|x|$ from $y = x \ln x$ for $10^{-6} \leq |y| \leq 10^{-4}$

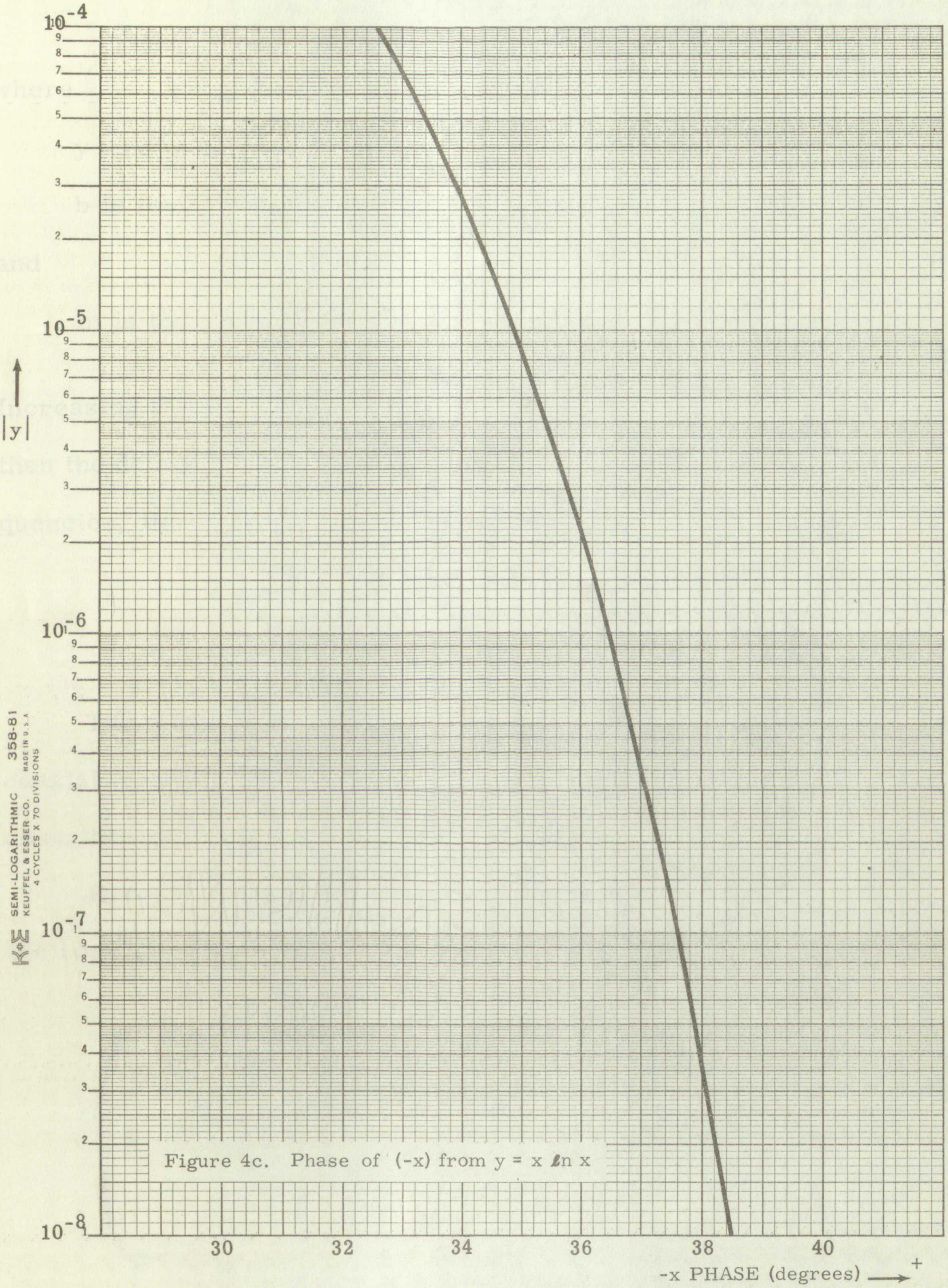

Figure 4c. Phase of (-x) from y = x ℓn x

Therefore

$$|x| \approx \frac{|y| - b}{m} ,$$

where

y (and consequently x) are functions of $\sqrt{f}$

b is the y intercept

and

m is the slope, $\dfrac{\Delta|y|}{\Delta|x|}$.

Increasing the frequency causes the numerator of (49) to increase faster than the denominator, resulting in a larger $|Z_R|$. Thus, at higher frequencies, there is merit in calculating Z_R and computing Γ.

Coaxial Antenna

The coaxial antenna in its most elementary form consists of a coaxial transmission line terminated by a load impedance. Figure 5 describes the antenna with its approximately short-circuited termination.

Except for the finite length of line, all assumptions associated with the coaxial transmission line still apply.

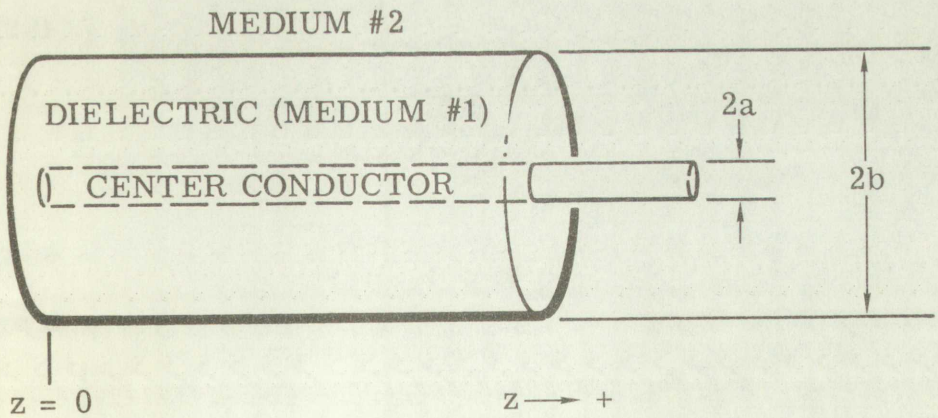

Figure 5. Coaxial Antenna With Short-Circuit Termination

Having computed Z_R (the exposed center conductor acting as the load), one calculates the input impedance as a function of Z_0, Γ, and k_g as noted in (27). From (22),

$$k_g = k_1(1+\Delta k)^{1/2} \approx k_1 \left[\frac{\ln\left(\frac{1.12}{jk_2 a}\right)}{\ln\left(\frac{b}{a}\right)} \right]^{1/2} \qquad \mathrm{Im}\left[k_g\right] < 0$$

The $\ln$ function in the numerator may be calculated or taken directly from Figure 6, once having computed $|k_2 a|$. With constants supplied wherever applicable, the above equation reduces to

$$k_g \approx 2.09 \times 10^{-8} f \left[\frac{\epsilon_r \ln\left(\frac{1.12}{jk_2 a}\right)}{\ln\left(\frac{b}{a}\right)} \right]^{1/2} , \tag{50}$$

where ϵ_r, the relative dielectric constant,

= 2.0 for teflon, 2.26 for polyethylene.

Notice that increasing the frequency reduces $\left[\ln\left(\frac{1.12}{jk_2 a}\right)\right]$. This results in a relative increase in k_g somewhat less than that of f. With an increase in f, the resulting phase angle for k_g increases in magnitude to a more negative value. The significance of these changes appears after replacing η_1 in (23) by $\sqrt{\frac{\mu_0}{\epsilon_1}}$, then simplifying to

$$Z_0 = \frac{k_g}{2\pi\epsilon_1\omega} \ln\left(\frac{b}{a}\right) . \tag{51}$$

As can be seen from (50) and (51), with an increase in f, there is a relative decrease of $|Z_0|$ equal to the relative decrease of

$$\left| \left[\ln\left(\frac{1.12}{jk_2 a}\right) \right]^{1/2} \right| .$$

Having combined [illegible]

load), one calculates the [illegible]

k_g as noted in (27), [illegible]

$$k_g = k_1 ([illegible])$$

The ℓn function [illegible]

from Figure 6, once [illegible]

wherever applicable, [illegible]

$$k_g \approx 2.00 \times 1 [illegible]$$

where ϵ_1, the [illegible]

= 2.0 for [illegible]

Notice that increasing [illegible]

in a relative increase [illegible]

in t, the resulting [illegible]

negative value. The [illegible]

η_1 in (23) by $\sqrt{\frac{\mu_0}{\epsilon_1}}$ [illegible]

$$Z_0 = \frac{[illegible]}{2\pi} [illegible]$$

As can be seen from [illegible]

relative decrease of [illegible]

$$\left| \ell n \left([illegible] \right) \right|$$

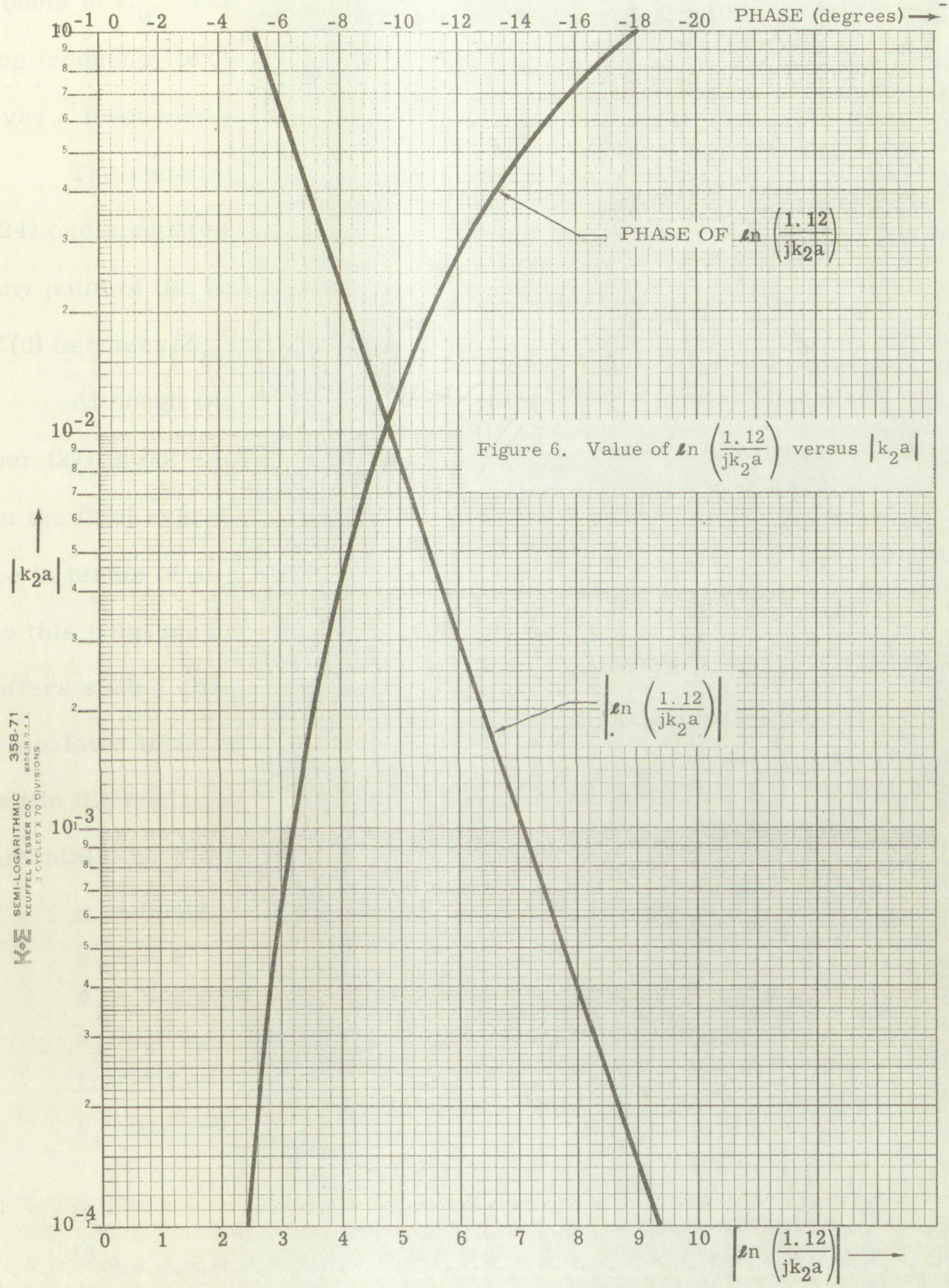

Figure 6. Value of $\ell n \left(\frac{1.12}{jk_2a}\right)$ versus $|k_2a|$

This, coupled with the phase change, ensures a smaller resistive component of Z_0. The fact that both $|Z_0|$ and $Re|Z_0|$ decrease with increasing frequency provides a supplemental check when performing calculations over a frequency range.

The remaining calculations for Z_{in} are straightforward. From (24) one computes Γ which is used in (27) to find the impedance along any point of the line. In particular, if one sets d of (27) equal to L, Z(d) becomes Z_{in} (see Figure 2).

Although (as will be shown) determination of the input impedance per (27) is not essential in calculating the dipole moment, its inclusion in the final expression therein is implicit since this final expression will be in terms of source current. Furthermore, validity of the theory up to this point may be checked against experimental results. Figure 7 offers such a comparison for a particular case. The curves represent calculated input resistances and reactances as functions of frequency while the encircled dots indicate actual values as determined experimentally by Flath and Norgordon.[12] The constants used were as follows:

$a = 0.406 \times 10^{-3}$ meters
$b = 1.47 \times 10^{-3}$ meters
$\sigma_1 = 5.8 \times 10^{7}$ mhos/meter
$\sigma_2 = 50$ mhos/meter
$L = 7$ feet $= 2.13$ meters
ϵ_r (polyethylene) = 2.26.[13]

---

[12] 6.

[13] The cable used was not defined in the data. However, from radii a and b, it is assumed that the cable was either RG-55 or RG-58, both of which have a polyethylene dielectric.

This, coupled with the phase change, ensures a smaller resistive component of Z_0. The fact that both $|Z_0|$ and $Re|Z_0|$ decrease with increasing frequency provides a supplemental check when performing calculations over a frequency range.

The remaining calculations for Z_{in} are straightforward. From (24) one computes Γ which is used in (27) to find the impedance along any point of the line. In particular, if one sets d of (27) equal to L, Z(d) becomes Z_{in} (see Figure 2).

Although (as will be shown) determination of the input impedance per (27) is not essential in calculating the dipole moment, its inclusion in the final expression therein is implicit since this final expression will be in terms of source current. Furthermore, validity of the theory up to this point may be checked against experimental results. Figure 7 offers such a comparison for a particular case. The curves represent calculated input resistances and reactances as functions of frequency while the encircled dots indicate actual values as determined experimentally by Flath and Norgordon.[12] The constants used were as follows:

$a = 0.406 \times 10^{-3}$ meters
$b = 1.47 \times 10^{-3}$ meters
$\sigma_1 = 5.8 \times 10^{7}$ mhos/meter
$\sigma_2 = 50$ mhos/meter
L = 7 feet = 2.13 meters
ϵ_r (polyethylene) = 2.26.[13]

---

[12] 6.

[13] The cable used was not defined in the data. However, from radii a and b, it is assumed that the cable was either RG-55 or RG-58, both of which have a polyethylene dielectric.

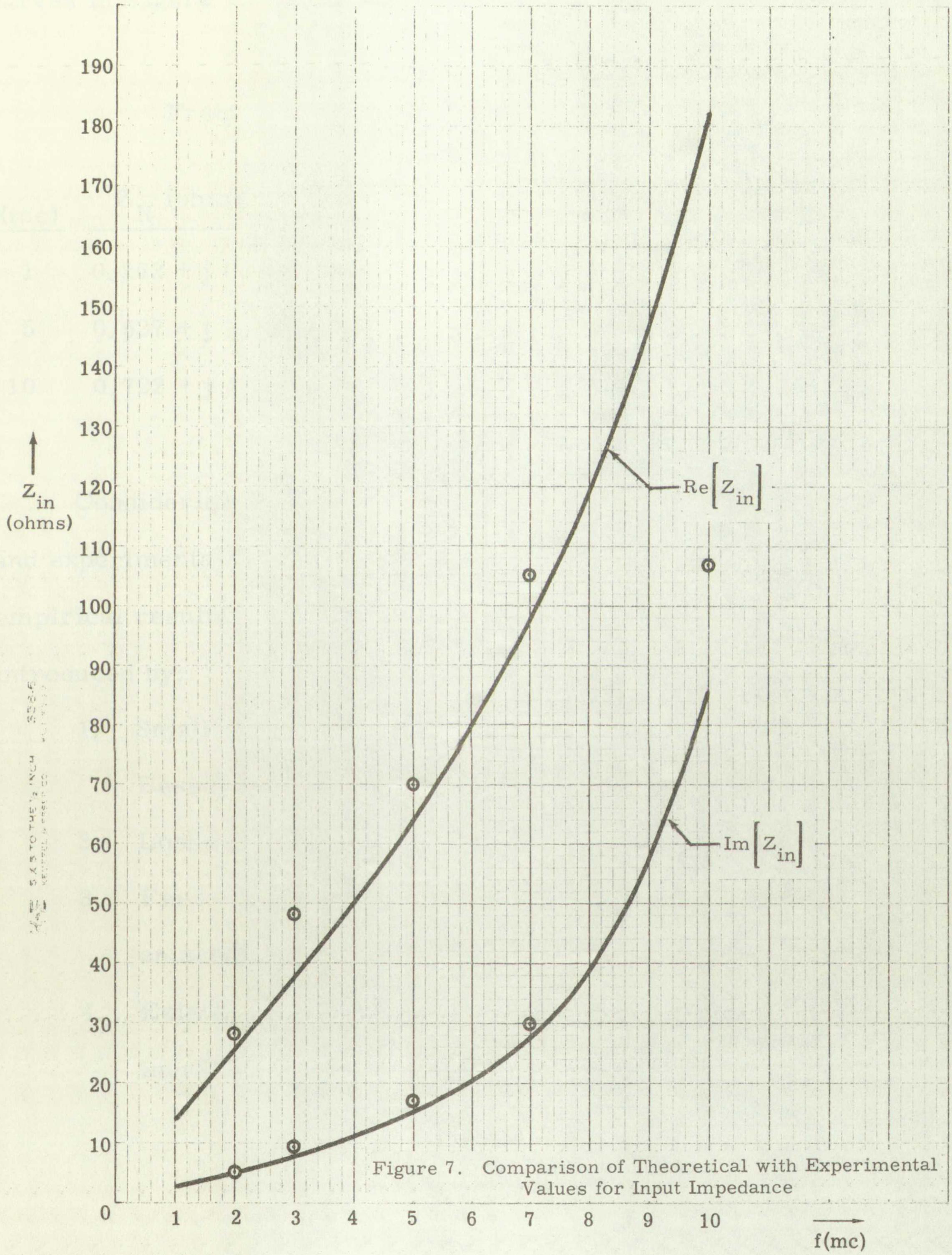

Figure 7. Comparison of Theoretical with Experimental Values for Input Impedance

Table I shows calculated values for various parameters leading to curves in Figure 7.

TABLE I

Frequency Effects on Parameters Leading to Computation of Input Impedance

f(mc)	Z_R (ohms)	Z_0 (ohms)	k_g	Γ
1	0.262 + j 0.378	102.4 - j 8.06	0.062 - j 0.005	$-0.994 \angle -0.44$
5	0.527 + j 0.781	92.5 - j 8.66	0.284 - j 0.027	$-0.988 \angle -1.00$
10	0.722 + j 1.081	88.7 - j 9.09	0.543 - j 0.056	$-0.987 \angle -1.45$

Considering inherent errors introduced by slide-rule calculations and experimental measurements, the theory exhibits good correlation with empirical results. Additional reasons for noted deviations are errors introduced by:

1. Small argument and asymptotic approximations for Bessel functions
2. Losses in the dielectric
3. Finite conductance of the center conductor (contrary to an assumption when computing Z_0)
4. Existence of higher-order modes. This factor becomes increasingly important at high frequencies.

The Dipole Moment

The relationship between dipole moment and current begins with the Equation of Continuity:

$$(\nabla \cdot \vec{J}) + \frac{\partial \rho}{\partial t} = 0, \tag{52}$$

where $\vec{J}$ is current density expressed in amp/meter^2 and ρ is charge density expressed in coul/meter^3. Both $\vec{J}$ and ρ may be expressed in terms of a single vector $\vec{P}$ such that (52) is satisfied:

$$\vec{J} = \frac{\partial \vec{P}}{\partial t} \tag{53}$$

$$\rho = - \nabla \cdot \vec{P}. \tag{54}$$

From a physical standpoint, $\vec{P}$ represents the dipole moment per unit volume of free-charge distribution. If its magnitude is sinusoidal with time, polarization may be written as

$$\vec{P} = \mathrm{Re}\left[\vec{P}_0(x)e^{j\omega t}\right], \tag{55}$$

where x is a space coordinate.

Differentiating (55) with respect to time,

$$\vec{P}_0\,(x) = \frac{\vec{J}_0}{j\omega} \quad \text{where} \quad \vec{J} = \mathrm{Re}\left[\vec{J}_0\, e^{j\omega t}\right].$$

Since $\vec{P}_0$ (x) is the dipole moment per unit volume, the total dipole moment $\vec{p}$ is

$$\vec{p} = \int_V \vec{P}_0\,(x)\,dv = \frac{1}{j\omega}\int_L \int_A \vec{J}_0 \; dA\; dz \text{ coulomb-meters}$$

Let $\int_A \vec{J}_0 \, dA = I(z)\,\hat{z}$

Then $\vec{p} = \frac{1}{j\omega}\int_L I(z)dz\;\hat{z},$ (56)

where

dA is an elemental crossectional area

L is the length of the line

$\hat{z}$ is a unit vector in the positive z direction (from load toward generator).

Letting $z = L - d$, (56) in terms of generator current I_s and through use of (26) becomes

$$\vec{p} = \frac{I_s}{j\omega} \int_0^L \left[\frac{e^{jk_g(L-z)} - \Gamma e^{-jk_g(L-z)}}{e^{jk_gL} - \Gamma e^{-jk_gL}} \right] \hat{z}$$

$$= \frac{I_s}{j\omega} \left[\frac{e^{jk_gL} + \Gamma e^{-jk_gL} - (1 + \Gamma)}{e^{jk_gL} - \Gamma e^{-jk_gL}} \right] \hat{z} \qquad (57)$$

Note that if $|1 + \Gamma| << \left| e^{jk_gL} + \Gamma e^{-jk_gL} \right|$

then $$\vec{p} \approx \frac{I_s}{\omega k_g} \left(\frac{Z_{in}}{Z_0} \right) \hat{z} \qquad (58)$$

A Sample Computation

In this section a logical sequence of operations will be offered for the application of preceding formulas. For steps leading to computation of Z_0, evaluation of constants in the formulas will later reduce computing time. The following constants may be replaced by their numerical values:

$$\mu_0 = 4\pi \times 10^{-7}$$

$$\epsilon_0 = \frac{10^{-9}}{36\pi}$$

ϵ_r (sea water) = 81.

Table II gives the values of simplified parameters when proper substitutions are made.

TABLE II

Formulas With Physical Constants Replaced by Their Numerical Values

Original equation	Equivalent equation
(15)	$k_2 = \sqrt{-j\omega\mu\sigma_2} \approx 2.81 \times 10^{-3} \sqrt{\sigma_2 f}\ \underline{/-45}$
(46)	$y = \dfrac{\gamma^2 a \sigma_2}{2} \sqrt{\dfrac{\omega\mu}{\sigma_1}}\ \underline{/45} = 4.46 \times 10^{-3}\ a\sigma_2 \sqrt{\dfrac{f}{\sigma_1}}\ \underline{/45}$
	$= 5.85 \times 10^{-7}\ a\sigma_2 \sqrt{f}\ \underline{/45}$ for copper center conductor
(49)	$Z_R = -\dfrac{j\mu_0\ a\gamma^2 f}{4x\sqrt{\dfrac{\sigma_1}{\sigma_2}}} = -\dfrac{j\,10^{-6}\ af}{x\sqrt{\dfrac{\sigma_1}{\sigma_2}}}$
	$= -\dfrac{j\,1.31 \times 10^{-10}\ af\sqrt{\sigma_2}}{x}$ for copper center conductor
(22)	$k_1 = \omega\sqrt{\mu\epsilon_1} = 2.09 \times 10^{-8}\ f\sqrt{\epsilon_r}$
(23)	$Z_0 = \dfrac{\eta_1}{2\pi}\left(1 + \Delta k\right)^{1/2} \ln\left(\dfrac{b}{a}\right) = 60 \left\{\dfrac{\left[\ln\left(\dfrac{1.12}{jk_2 a}\right)\right]\left[\ln\left(\dfrac{b}{a}\right)\right]}{\epsilon_r}\right\}^{1/2}$

For the sample computation, consider the example cited earlier. Suppose it is desired to evaluate $\vec{p}$ at a frequency of 7 megacycles. Thus, we have

$a = 0.406 \times 10^{-3}$ meters

$b = 1.47 \times 10^{-3}$ meters

$\sigma_1 = \sigma_{copper} = 5.8 \times 10^{7}$ mhos/meter

$\sigma_2 = 50$ mhos/meter

$L = 2.13$ meters

$\epsilon_r = 2.26$

$f = 7$ mc.

The first two steps are performed to see how well the system under consideration fits the theory. Recall that small argument approximations are the basis for many of the equations derived from Bessel functions. Generally, the condition that $|k_2 b| << 1$ is the most stringent for these, and this may be precluded by too high a frequency. Also, if the frequency is too high for the condition $\sigma_2/\omega\epsilon_2 >> 1$ to be met, the real part of the complex dielectric $\overline{\epsilon_2}$ becomes important, and one can no longer assume that $k_2 = \sqrt{\omega\mu\sigma_2}\ \underline{/-45}$. If one takes the ratio $\omega\epsilon_2/\sigma_2$ and compares it to $|k_2 b|$, it turns out that $|k_2 b| > \omega\epsilon_2/\sigma_2$ for $6.25 \times 10^{5}\, b\sigma_2^{3/2} f^{1/2} > 1$. Fortunately, this is true for all practical cases; hence, it is necessary to show only that $|k_2 b| << 1$.

The second step is performed only if the center conductor is copper-weld. It was stated that negligible error is introduced so long as the radius of the center conductor exceeds three times the skin depth. Thus,

with a copperweld center conductor there are two frequency considerations which oppose each other.

Note that even though one of the above conditions is not met, useful information may still be obtained. However, the validity of the theory is sacrificed as the deviation from these conditions broadens.

The first two steps are as follows:

1. Evaluate $|k_2 b| = 2.81 \times 10^{-3}\, b \sqrt{\sigma_2 f}$

$$= 2.81 \times 10^{-3}\,(1.47 \times 10^{-3})\ \sqrt{50(7 \times 10^{6})}$$

$$= 7.73 \times 10^{-2} << 1.$$

2. If the center conductor is copperweld, compare three times the skin depth $= \dfrac{1.26 \text{ meters}}{\sqrt{f}}$ with radius "a".

$$\frac{1.26}{\sqrt{f}} = \frac{1.26}{\sqrt{7 \times 10^{6}}} = 4.76 \times 10^{-4} \text{ meters.}$$

Since $a = 4.06 \times 10^{-4}$ meters, this requirement is not quite met. However, the multiplier three times the skin depth is somewhat arbitrary, so for this case where radius "a" is almost 2.6 times the skin depth, only minor error will be introduced. It should be recognized, however, that this error increases with a lowering of frequency unless "a" is increased.

The remaining steps deal with actual calculations toward $\vec{p}$, having determined that the system considered here is quite suited to the theory presented earlier.

3. From (46) calculate $y = 5.85 \times 10^{-7}(a\sigma_2\sqrt{f}) \angle 45$

$$= 5.85 \times 10^{-7}(0.406 \times 10^{-3})(50)\sqrt{7 \times 10^{6}} \angle 45$$

$$= 3.14 \times 10^{-5} \angle 45 .$$

4. Determine x by either (a) or (b):

(a) From Figures 4(b) and (c) for $|y| = 3.14 \times 10^{-5}$,

$x \approx -2.4 \times 10^{-6} \angle 33.9$.

(b) $y = x \ln x$

Assume $\ln x_0 = -20$.

$$x_1 = \frac{y}{\ln x_0} = \frac{3.14 \times 10^{-5} \angle 45}{-20}$$

$$= -1.57 \times 10^{-6} \angle 45$$

$$= 1.57 \times 10^{-6} \angle -135$$

$$\ln x_1 = -(13.38 + j\,2.36) = -13.58 \angle 10.0$$

$$x_2 = \frac{3.14 \times 10^{-5} \angle 45}{-13.58 \angle 10.0} = 2.31 \times 10^{-6} \angle -145$$

$$\ln x_2 = -(12.98 + j\,2.53) = -13.22 \angle 11.0$$

$$x_3 = \frac{3.14 \times 10^{-5} \angle 45}{-13.22 \angle 11.0} = 2.37 \times 10^{-6} \angle -146$$

$$\ln x_3 = -(12.95 + j\,2.55) = -13.20 \angle 11.1$$

$$x_4 = \frac{3.14 \times 10^{-5} \angle 45}{-13.20 \angle 11.1} = 2.38 \times 10^{-6} \angle -146.1$$

$$= -2.38 \times 10^{-6} \angle 33.9$$

which is very close to x_3.

5. From (49) determine

$$Z_R = \frac{-1.31 \times 10^{-10}(0.406 \times 10^{-3})(7 \times 10^{6})\sqrt{50}\ \underline{/90}}{\left(-2.38 \times 10^{-6}\ \underline{/33.9}\right)}$$

$$= 1.105\ \underline{/56.1} = 0.616 + j\,0.918$$

6. Find $k_1 = 2.09 \times 10^{-8}\ f\sqrt{\epsilon_r} = 2.09 \times 10^{-8}\left(7 \times 10^{6}\right)\sqrt{2.26} = 0.22.$

7. From Step 1 evaluate $k_2a = \left(|k_2b|\ \underline{/-45}\right)\left(\frac{a}{b}\right)$

$$= \left(7.73 \times 10^{-2}\ \underline{/-45}\right)\left(\frac{0.406}{1.47}\right) = 2.14 \times 10^{-2}\ \underline{/-45}\ .$$

8. Evaluate $\ell n\left(\frac{1.12}{jk_2a}\right)$ by either (a) or (b):

(a) From Figure 6, $\ell n\left(\frac{1.12}{jk_2a}\right) = 4.05\ \underline{/-11.2}\ .$

(b) $\ell n\left(\frac{1.12}{jk_2a}\right) = \ell n\left(\frac{1.12\ \underline{/-45}}{0.0214}\right) = 3.96 - j\,0.785 = 4.04\ \underline{/-11.2}\ .$

9. Find $\ell n\left(\frac{b}{a}\right) = \ell n\left(\frac{1.47}{0.406}\right) = 1.288.$

10. From (22) evaluate $k_g = k_1\left[\frac{\ell n\left(\frac{1.12}{jk_2a}\right)}{\ell n\left(\frac{b}{a}\right)}\right]^{1/2}$

$$= 0.22\left[\frac{4.04\ \underline{/-11.2}}{1.288}\right]^{1/2} = 0.390\ \underline{/-5.6} = 0.388 - j\,0.038.$$

11. Solve (23) for $Z_0 = 60 \left\{ \frac{\left[\ell n \left(\frac{1.12}{jk_2 a}\right)\right] \left[\ell n \left(\frac{b}{a}\right)\right]}{\epsilon_r} \right\}^{1/2}$

$$= 60 \left[\frac{\left(4.04 \angle -11.2\right)\left(1.288\right)}{2.26} \right]^{1/2} = 91.1 \angle -5.6 = 90.5 - j\,8.9.$$

12. From (24) find $\Gamma = - \frac{(Z_0 - Z_R)}{Z_0 + Z_R}$

$$= - \frac{(90.5 - j\,8.9 - 0.616 - j\,0.918)}{90.5 - j\,8.9 + 0.616 + j\,0.918}$$

$$= -0.988 \angle -1.2 = 0.988 \angle 178.8 .$$

13. Evaluate $\Gamma e^{-j2k_g L}$.

$$e^{-j2k_g L} = e^{-j2(2.13)(0.388 - j0.038)} = 0.8505 \angle -94.9$$

$$\Gamma e^{-j2k_g L} = \left(0.988 \angle 178.8\right)\left(0.8505 \angle -94.9\right)$$

$$= 0.840 \angle 83.9$$

$$= 0.089 + j\,0.853.$$

14. Evaluate $\left[\frac{1 + e^{-j2k_g L}}{1 - e^{-j2k_g L}}\right]$. This may be done with a Smith

Chart or by calculation:

$$\left[\frac{1 + e^{-j2k_g L}}{1 - e^{-j2k_g L}}\right] = \frac{1.089 + j\,0.835}{0.911 - j\,0.835} = 1.111 \angle 80.0 .$$

15. From (27) find $Z_{in} = Z_0 \left[\dfrac{1 + e^{-j2k_g L}}{1 - e^{-j2k_g L}} \right]$

$$= \left(1.111 \angle 80.0\right)\left(91.1 \angle -5.6\right) = 101.3 \angle 74.4$$

$$= 27.3 + j\,97.7.$$

16. Compute $\vec{p}$: (a) from (57), then (b) from (58).

a) $$\vec{p} = \frac{I_s}{\omega k_g} \left[\frac{e^{jk_g L} + \Gamma e^{-jk_g L} - (1 + \Gamma)}{e^{jk_g L} - \Gamma e^{-jk_g L}} \right] \hat{z}$$

$$e^{jk_g L} = 1.0845 \angle 47.3$$

$$\Gamma e^{-jk_g L} = -0.911 \angle -48.5$$

$$\vec{p} = \frac{I_s}{2\pi\left(7 \times 10^6\right)\left(0.390 \angle -5.6\right)} \left[\frac{1.507 \angle 85.46}{1.345 \angle 3.04} \right] \hat{z}$$

$$= 6.54 \times 10^{-8} \angle 88.0\, I_s\, \hat{z}.$$

b) $$\vec{p} \approx \frac{I_s}{\omega k_g} \left(\frac{Z_{in}}{Z_0} \right) \hat{z} = \frac{I_s}{\omega k_g} \left[\frac{1 + e^{-j2k_g L}}{1 - e^{-j2k_g L}} \right] \hat{z}$$

$$= \frac{\left(1.111 \angle 80.0\right) I_s}{2\pi\left(7 \times 10^6\right)\left(0.390 \angle -5.6\right)} \hat{z} = 6.49 \times 10^{-8} \angle 85.6\, I_s\, \hat{z}.$$

Conclusions

By performing the steps described in the previous section, one can compute the dipole moment of a short-circuit termination coaxial antenna in sea water for most practical situations. Correlation of calculated input impedance with experimental data is very good at the lower frequencies where small argument approximations of Bessel functions are most valid and where higher order modes are negligible. In other works which analyze the input impedance from a loaded transmission line viewpoint, the line is restricted to one of short electrical length over which the current distribution is assumed uniform. In such cases, the input impedance expression is in the form of an infinite series for which higher ordered terms may be neglected.

In this paper no restriction is placed on line length. Therefore, a uniform current distribution along a line cannot be assumed constant without further investigation.

At higher frequencies, the effects of an imperfect short-circuit termination become noticeable, and the voltage reflection coefficient cannot be assumed as -1 without sacrificing accuracy. On the other hand, at lower frequencies the value is so close to -1 that the assumption of this value introduces negligible error while simplifying the calculations. Just where one makes this assumption depends on its actual value and the accuracy of impedance and dipole moment determinations desired.