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Conversion of Electromagnetic Waves into Magnetohydrodynamic Waves at the Interface of a Dielectric and Incomprehensible Fluid

Ralph L. Marsten

CONVERSION OF ELECTROMAGNETIC WAVES - MARSTON

CONVERSION OF ELECTROMAGNETIC WAVES INTO
MAGNETOHYDRODYNAMIC WAVES AT THE INTERFACE OF A
DIELECTRIC AND AN INCOMPRESSIBLE FLUID

By
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This thesis, directed and approved by the candidate's committee, has been accepted by the Graduate Committee of the University of New Mexico in partial fulfillment of the requirements for the degree of

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THE PROBLEM

Consider space to be separated into two semi-infinite regions by a rigid boundary. On one side lies medium A -- a perfect, rigid, dielectric, with electric permittivity ϵ and magnetic permeability μ. On the other side is medium B; a perfect fluid, perfectly conducting, also with magnetic permeability μ. Throughout all space exists a uniform static magnetic field $\mathbf{H}_0$.

Now such a fluid as medium B is capable of supporting a magnetohydrodynamic (MHD) wave; and of course a perfect dielectric will support an electromagnetic (EM) wave. Suppose then, that a uniform plane EM wave in medium A is incident at arbitrary angle on the boundary. Will the EM wave generate an MHD wave at the boundary? If so, to what degree? In what manner will the interaction between the EM wave and the MHD fluid be dependent on the characteristics of the media and on the angle of incidence of the EM wave?

THE WAVE EQUATIONS

The essential characteristics of the media and fields involved may be listed as follows: (see Appendix for nomenclature)

Medium A	Medium B
μ	μ
ϵ	$\nu = 0$
$\sigma = 0$	$\sigma = \infty$
$H(\mathbf{r}, t)$	$\rho =$ constant
$\mathbf{E}(\mathbf{r}, t)$	$p(\mathbf{r}, t)$
	$\mathbf{H}(\mathbf{r}, t)$
	$\mathbf{q}(\mathbf{r}, t)$
	$\mathbf{E}(\mathbf{r}, t)$

The EM field will set the fluid in motion at the boundary. This motion in turn will set up a magnetic field in the fluid. The velocity of the fluid $\mathbf{q}$ and the magnetic field intensity H_B are related by the magnetohydrodynamic equations,

$$(1) \quad \frac{\partial H_B}{\partial t} = \nabla \times (\mathbf{q} \times H_B), \quad (2) \quad \frac{d\mathbf{q}}{dt} = -\frac{\nabla p}{\rho} - \frac{\mu}{\rho}(\nabla \times H_B) \times H_B .$$

Since the fluid is perfect and the conductivity is infinite, we also have

$$(3) \quad E_B = -\mu \mathbf{q} \times H_B , \quad (4) \quad \nabla \cdot \mathbf{q} = 0 .$$

In medium A, Maxwell's equations become

(5) $\nabla \times E_A = -\mu \dfrac{\partial H_A}{\partial t}$, (6) $\nabla \times H_A = \epsilon \dfrac{\partial E_A}{\partial t}$,

(7) $\nabla \cdot E_A = 0$.

In both media $\nabla \cdot H = 0$.

To linearize equations (1) and (2), we suppose that the time-varying fields are sufficiently less than the static field H_0 that we can neglect products and powers of q and h .[1] Writing

$$H_B = H_0 + h_B \quad \text{and} \quad H_A = H_0 + h_A \ ,$$

we obtain for medium B

(8) $\dfrac{\partial h_B}{\partial t} = (H_0 \cdot \nabla)\, q$,

(9) $\dfrac{\partial q}{\partial t} = -\dfrac{1}{\rho} \nabla (P + \mu H_0 \cdot h_B) + \dfrac{\mu}{\rho} (H_0 \cdot \nabla)\, h_B$;

and for medium A

(10) $\nabla \times E_A = -\mu \dfrac{\partial h_A}{\partial t}$, (11) $\nabla \times h_A = \epsilon \dfrac{\partial E_A}{\partial t}$.

To simplify writing, let

(12) $\varphi = \dfrac{1}{\rho} \left[P + \dfrac{1}{2} (H_0 + h_B)^2 \right]$;

then (9) becomes

(13) $$\frac{\partial q}{\partial t} = -\nabla\varphi + \frac{\mu}{\rho}(H_0\cdot\nabla)h_s \ .$$

On taking the divergence of this equation, it is seen that

(14) $$\nabla^2\varphi = 0 \ .$$

As a solution to Laplace's equation, φ must of course be everywhere finite. If there were no boundary, this would require that φ be a constant.(2) However, the presence of the boundary relieves this restriction. Instead, we must have φ vanish far from the boundary, but $\nabla\varphi\neq 0$ in general. The significance of this is that h_s does not satisfy a wave equation of the type

$$\frac{\partial^2 a}{\partial t^2} = \frac{\mu}{\rho}(H_0\cdot\nabla)^2 a$$

except in special cases.(3) The essence of the development from here through the next section is due to Roberts.(3)

To circumvent this difficulty, let us define a vector a by the equations

(15) $$h_s = (H_0\cdot\nabla)a,$$ (16) $$q = \frac{\partial a}{\partial t} \ .$$

Obviously equations (4) and (8) are satisfied by a . We may specify a by solenoidal and irrotational components

(17) $$a = b + \nabla\psi,$$

where $\mathbf{b}$ is defined by

(18) $\nabla \cdot \mathbf{b} = 0$, (19) $\dfrac{\partial^2 \mathbf{b}}{\partial t^2} = V^2 \dfrac{\partial^2 \mathbf{b}}{\partial s^2}$,

where s is measured along $\mathbf{H}_o$, and

(20) $V^2 = \frac{\mu}{\rho} H_o^2$ is the Alfvén velocity.

With these definitions, we must have

(21) $\nabla^2 \psi = 0$.

Equations (19) and (21) constitute wave equations for the vector $\mathbf{a}$. However it remains to show that $\mathbf{a}$, as defined by equations (15), (16), (17), satisfies equation (9).

For medium A we have the usual electromagnetic wave equation

(22) $\dfrac{\partial^2 \mathbf{h}_A}{\partial t^2} = c^2 \nabla^2 \mathbf{h}_A$, where $c^2 = \dfrac{1}{\mu \epsilon}$.

SOLUTIONS TO THE WAVE EQUATIONS

As solutions to (19) and (21) we may take

$$\text{(23)} \qquad \boldsymbol{b} = \mathbf{B}\, e^{i\omega\left(t - \frac{\hat{n}_2 \cdot \boldsymbol{r}}{N}\right)},$$

$$\text{(24)} \qquad \psi = \Psi\, e^{i\omega\left(t - \frac{\boldsymbol{m} \cdot \boldsymbol{r}}{N}\right)},$$

where $\hat{n}_2$ is a unit vector in the direction of wave propagation, $\boldsymbol{m}$ is a constant complex vector of zero length, and where $N = V\left|\hat{\lambda}_5 \cdot \hat{n}_2\right|$ is the component of wave velocity in the direction of H_o, $\hat{\lambda}_5$ being a unit vector in that direction. That $\boldsymbol{m}$ must have zero magnitude may be seen by substituting ψ in (21). Combining (23) and (24) according to (17),

$$\text{(25)} \qquad \boldsymbol{a} = \mathbf{B}e^{i\omega\left(t - \frac{n_2 \cdot \boldsymbol{r}}{N}\right)} - i\frac{\omega}{N}\Psi\, \boldsymbol{m}\, e^{i\omega\left(t - \frac{\boldsymbol{m} \cdot \boldsymbol{r}}{N}\right)}.$$

On substitution of (17), using (15) and (16), equation (13) becomes

$$\frac{\partial^2 \boldsymbol{b}}{\partial t^2} + \frac{\partial^2}{\partial t^2}(\nabla \psi) = V^2 \frac{\partial^2 \boldsymbol{b}}{\partial 5^2} + V^2 \frac{\partial^2}{\partial 5^2}(\nabla \psi) - \nabla \varphi .$$

Using (19) leaves $\quad \nabla\left(\frac{\partial^2 \psi}{\partial t^2} - V^2 \frac{\partial^2 \psi}{\partial 5^2} + \varphi\right) = 0$

or,

$$\text{(26)} \qquad \frac{\partial^2 \psi}{\partial t^2} = V^2 \frac{\partial^2 \psi}{\partial 5^2} - \varphi + \varphi_o ,$$

where φ_0 is an arbitrary constant. As a solution to (14), φ may be written

$$(27) \qquad \varphi = \varphi_0 + \Phi e^{i\omega\left(t - \frac{\mathbf{m}\cdot\mathbf{r}}{N}\right)} ,$$

where φ_0 , being arbitrary, is the same as in (26). Substituting (24) and (27) in (26) reveals

$$(28) \qquad \Phi = \omega^2 \left[1 - \left(\frac{\mathbf{m}\cdot\hat{\lambda}_5}{\hat{n}_2\cdot\hat{\lambda}_5}\right)^2\right]\Psi$$

for consistancy.

For medium A we take as the solution of equation (22)

$$(29) \qquad \mathbf{h}_A = A_0\, e^{i\omega\left(t - \frac{\hat{n}_0\cdot\mathbf{r}}{c}\right)} + A_1\, e^{i\omega\left(t - \frac{\hat{n}_1\cdot\mathbf{r}}{c}\right)} ,$$

where $\hat{n}_0$ and $\hat{n}_1$ are unit vectors in the directions of propagation of the incident and reflected waves respectively.

THE FIELD QUANTITIES

Equations (25) and (29) are the fundamental solutions for their respective media; from them may be obtained all the field quantities of interest.

To obtain the magnetic field in medium B we use (15); obtaining

$$(30)\quad \mathbf{h}_B = -i\frac{\omega}{N}(\mathbf{H}_o\cdot\hat{n}_2)\,\mathbf{B}\,e^{i\omega\left(t-\frac{\hat{n}_2\cdot\mathbf{r}}{N}\right)} - \frac{\omega^2}{N^2}\mathbb{V}\,\mathbf{m}(\mathbf{H}_o\cdot\mathbf{m})\,e^{i\omega\left(t-\frac{\mathbf{m}\cdot\mathbf{r}}{N}\right)}.$$

Similarly for the velocity field; from (16),

$$(31)\quad \mathbf{q} = i\omega\,\mathbf{B}\,e^{i\omega\left(t-\frac{\hat{n}_2\cdot\mathbf{r}}{N}\right)} + \frac{\omega^2}{N}\mathbb{V}\,\mathbf{m}\,e^{i\omega\left(t-\frac{\mathbf{m}\cdot\mathbf{r}}{N}\right)}.$$

The electric field in medium B may be obtained from equation (3), which under the perturbation assumption becomes $\mathbf{E}_B = \mu\,\mathbf{H}_o\times\mathbf{q}$. We have

$$(32)\quad \mathbf{E}_B = i\mu\omega(\mathbf{H}_o\times\mathbf{B})\,e^{i\omega\left(t-\frac{\hat{n}_2\cdot\mathbf{r}}{N}\right)} + \frac{\mu\omega^2}{N}\mathbb{V}(\mathbf{H}_o\times\mathbf{m})\,e^{i\omega\left(t-\frac{\mathbf{m}\cdot\mathbf{r}}{N}\right)}.$$

To find the electric field in medium A one uses equation (11), assuming the usual time dependence for $\mathbf{E}_A$. First we

write $$\nabla \times (\mathbf{A}_0 \mathscr{H}_0 + \mathbf{A}_1 \mathscr{H}_1) = i \epsilon \omega \mathbf{E}_A ,$$

where

$$\mathscr{H}_0 = e^{i\omega(t - \frac{\hat{n}_0 \cdot \mathbf{r}}{c})}, \quad \mathscr{H}_1 = e^{i\omega(t - \frac{\hat{n}_1 \cdot \mathbf{r}}{c})} .$$

Expanding the curl, $-\mathbf{A}_0 \times \nabla \mathscr{H}_0 - \mathbf{A}_1 \times \nabla \mathscr{H}_1 = i \epsilon \omega \mathbf{E}_A$.

Since $\nabla \mathscr{H} = -i \frac{\omega}{c} \hat{n} \mathscr{H}$, we have

$$\mathbf{A}_0 \times \hat{n}_0 \mathscr{H}_0 + \mathbf{A}_1 \times \hat{n}_1 \mathscr{H}_1 = c \epsilon \mathbf{E}_A ,$$

or

$$\mathbf{E}_A = -\eta (\hat{n}_0 \times \mathbf{h}_0 + \hat{n}_1 \times \mathbf{h}_1), \text{ where } \eta = \sqrt{\frac{\mu}{\epsilon}} , \tag{33}$$

and where $\mathbf{h}_0$ and $\mathbf{h}_1$ are the magnetic fields of the incident and reflected waves respectively.

Summarizing, we have

Medium A

$$\mathbf{h}_A = \mathbf{A}_0 e^{i\omega(t - \frac{\hat{n}_0 \cdot \mathbf{r}}{c})} + \mathbf{A}_1 e^{i\omega(t - \frac{\hat{n}_1 \cdot \mathbf{r}}{c})} \tag{29}$$

$$\mathbf{E}_A = -\eta \hat{n}_0 \times \mathbf{A}_0 e^{i\omega(t - \frac{\hat{n}_0 \cdot \mathbf{r}}{c})} - \eta \hat{n}_1 \times \mathbf{A}_1 e^{i\omega(t - \frac{\hat{n}_1 \cdot \mathbf{r}}{c})} \tag{34}$$

Medium B

$$\mathbf{h}_B = \mathbf{A}_2 e^{i\omega(t - \frac{\hat{n}_2 \cdot \mathbf{r}}{N})} + \boldsymbol{\Omega} e^{i\omega(t - \frac{\mathbf{m} \cdot \mathbf{r}}{N})} , \tag{35}$$

where

$$(36)\quad A_2 = -i\frac{\omega}{N}(\mathbf{H}_0\cdot\hat{n}_2)\mathbf{B}, \quad \boldsymbol{\Omega} = -\frac{\omega^2}{N^2}(\mathbf{H}_0\cdot\mathbf{m})\,\Psi\,\mathbf{m}$$

$$(31)\quad \mathbf{q} = i\omega\mathbf{B}\,e^{i\omega\left(t-\frac{\hat{n}_2\cdot\mathbf{r}}{N}\right)} + \frac{\omega^2}{N}\Psi\,\mathbf{m}\,e^{i\omega\left(t-\frac{\mathbf{m}\cdot\mathbf{r}}{N}\right)}$$

$$(37)\quad \mathbf{E}_B = \mathbf{D}\,e^{i\omega\left(t-\frac{\hat{n}_2\cdot\mathbf{r}}{N}\right)} + \boldsymbol{\Pi}\,e^{i\omega\left(t-\frac{\mathbf{m}\cdot\mathbf{r}}{N}\right)},$$

where

$$(38)\quad \mathbf{D} = -\mu N\frac{\mathbf{H}_0\times\mathbf{A}_2}{\mathbf{H}_0\cdot\hat{n}_2}, \quad \boldsymbol{\Pi} = \frac{\mu\omega^2}{N}\Psi\,\mathbf{H}_0\times\mathbf{m}.$$

THE LAWS OF REFLECTION AND REFRACTION

The coefficients in equations (29) and (35) are constant vector amplitudes, to be evaluated at the boundary. Since these must bear some certain relationship to one another independent of time and position, the exponents must be equal at any given time and place on the boundary. Thus

$$\frac{\hat{n}_0\cdot\mathbf{r}_s}{c} = \frac{\hat{n}_1\cdot\mathbf{r}_s}{c} = \frac{\hat{n}_2\cdot\mathbf{r}_s}{N} = \frac{\mathbf{m}\cdot\mathbf{r}_s}{N},$$

where $\mathbf{r}_s$ is a point on the boundary surface. Defining a unit vector normal to the boundary by $\hat{\lambda}_z\cdot\mathbf{r}_s = 0$, one can easily show that $\mathbf{r}_s = -\hat{\lambda}_z\times(\hat{\lambda}_z\times\mathbf{r}_s)$. Hence, substituting and manipulating the triple scalar products

$$\hat{\lambda}_z\times\mathbf{r}_s\cdot\frac{\hat{n}_0\times\hat{\lambda}_z}{C} = \hat{\lambda}_z\times\mathbf{r}_s\cdot\frac{\hat{n}_1\times\hat{\lambda}_z}{C} = \hat{\lambda}_z\times\mathbf{r}_s\cdot\frac{\hat{n}_2\times\hat{\lambda}_z}{N} = \hat{\lambda}_z\times\mathbf{r}_s\cdot\frac{\mathbf{m}\times\hat{\lambda}_z}{N},$$

or

$$(39) \qquad \frac{\hat{\lambda}_z \times \hat{n}_0}{C} = \frac{\hat{\lambda}_z \times \hat{n}_1}{C} = \frac{\hat{\lambda}_z \times \hat{n}_2}{N} = \frac{\hat{\lambda}_z \times m}{N}.$$

Since each of the cross-products in (39) has the same direction, they must all be normal to the same plane. This plane, containing $\hat{n}_0$, $\hat{n}_1$, $\hat{n}_2$, $\hat{\lambda}_z$, may be called the plane of incidence, in analogy to optics. The first two equalities yield the laws of reflection and refraction respectively. The subscripts 0, 1, 2 refer to the incident, reflected, refracted waves, in that order.

Since the normal to the boundary, $\hat{\lambda}_z$, lies in the plane of incidence, this plane is perpendicular to the boundary. Then the line of intersection of the plane of incidence with the boundary, and the normal to the plane of incidence, together with $\hat{\lambda}_z$, form a convenient mutually perpendicular triad, as shown in Figure 1.

In terms of this coordinate system,

$$(40) \qquad \hat{n}_0 = -\hat{\lambda}_x \sin\theta_0 - \hat{\lambda}_z \cos\theta_0$$

$$(41) \qquad \hat{n}_1 = -\hat{\lambda}_x \sin\theta_1 + \hat{\lambda}_z \cos\theta_1$$

$$(42) \qquad \hat{n}_2 = -\hat{\lambda}_x \sin\theta_2 - \hat{\lambda}_z \cos\theta_2$$

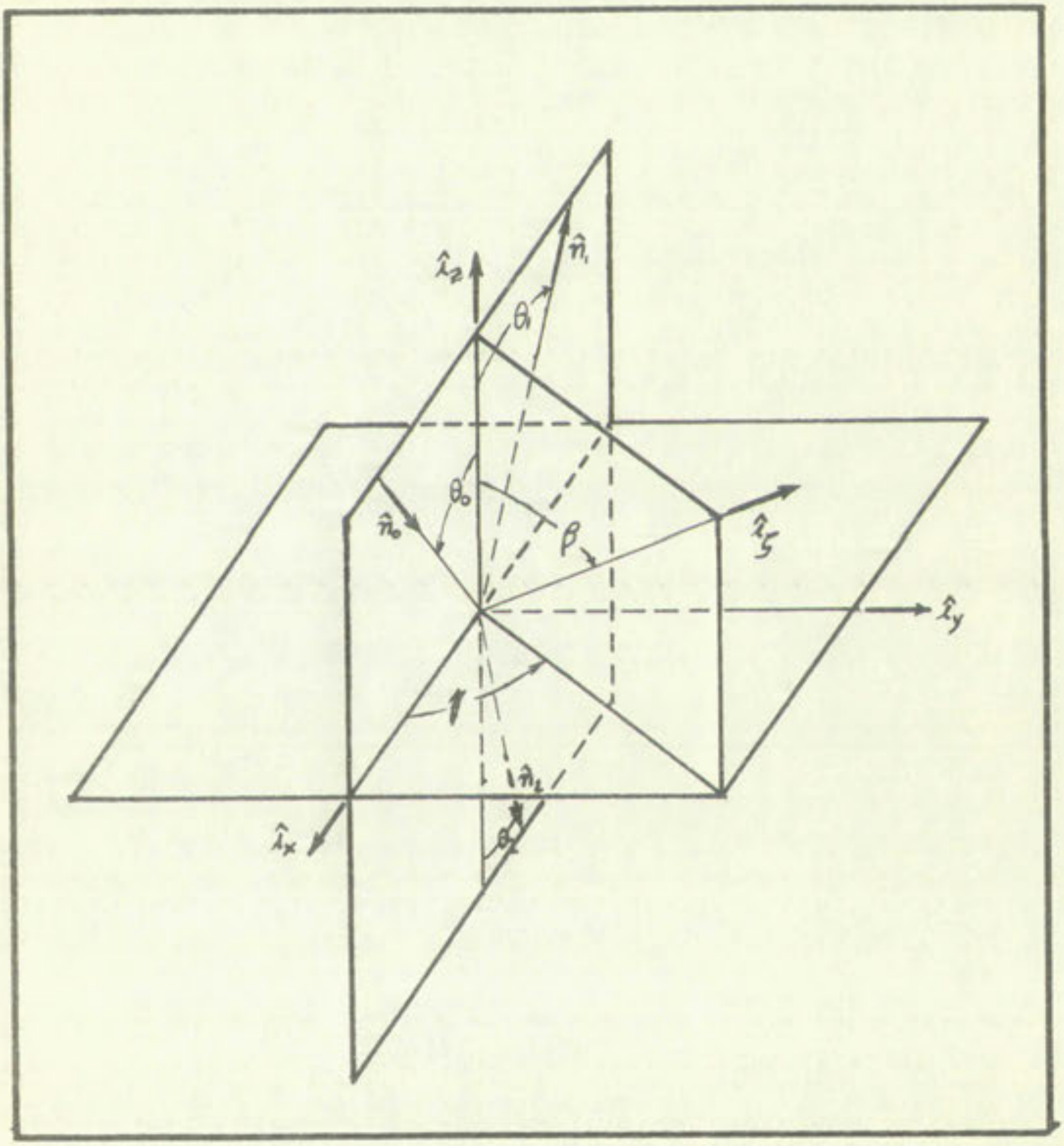
$$(43) \qquad \hat{\lambda}_\zeta = \hat{\lambda}_x \sin\beta\cos\gamma + \hat{\lambda}_y \sin\beta\sin\gamma + \hat{\lambda}_z \cos\beta .$$

With these relations,

$$(44) \qquad N = V\,(\sin\beta\cos\gamma\sin\theta_2 + \cos\beta\cos\theta_2) = -V\,(\hat{n}_2 \cdot \hat{\lambda}_\zeta),$$

FIGURE 1

Orthogonal Coordinate System



The x-y plane is the boundary.

The x-z plane is the plane of incidence.

The angles β and γ fix the direction of $\hat{\imath}_\xi$.

for β and γ in the first quadrants. If β or γ were in any other quadrant, the absolute value brackets would have to be retained, because the refracted wave must go in the -Z direction as does the incident wave.

The first equality of (39) yields

(45) $\theta_1 = \theta_0$, the law of reflection.

The second equality yields

$$- \hat{\lambda}_y \sin \theta_0 = -\frac{c}{V} \frac{\hat{\lambda}_y \sin \theta_2}{\sin \beta \cos \gamma \sin \theta_2 + \cos \beta \cos \theta_2},$$

or

$$(46) \quad \tan \theta_2 = \frac{\frac{V}{c} \sin \theta_0 \cos \beta}{1 - \frac{V}{c} \sin \beta \cos \gamma \sin \theta_0},$$

the law of refraction.

Now this development has been non-relativistic; hence we must consider $V^2 \ll c^2$. With this in mind, (46) gives evidence that θ_2 is quite small, regardless of the orientation of $\mathbf{H}_0$ or the angle of incidence.

EVALUATION OF m

ψ as given by (24) must vanish at distances sufficiently far from the boundary. Hence m must be complex in order to represent ψ as a damped wave. In addition, to satisfy Laplace's equation (21), the magnitude of m must be zero. A third condition is obtained from the third equality of (39). We will show that these conditions, together with a fourth to be introduced presently, are sufficient to determine m uniquely.

Let us write $m = m_r \pm i\, m_i$. Then if we put

$$(m_r \pm i\, m_i) \cdot (m_r \pm i\, m_i) = 0,$$

the first two conditions will be satisfied. Since

$$m_r \cdot m_r - m_i \cdot m_i \pm 2i\, m_r \cdot m_i = 0$$

, we have

(a) $|m_r| = |m_i|$, and (b) $m_r \cdot m_i = 0$

The third condition requires $\hat{\lambda}_z \times m = \hat{\lambda}_z \times \hat{n}_2$, or

$$\hat{\lambda}_z \times (m_r \pm i\, m_i) = \hat{\lambda}_z \times (-\hat{\lambda}_x \sin\theta_2 - \hat{\lambda}_z \cos\theta_2)$$

from (43). This breaks down to

(c) $\hat{\lambda}_z \times m_r = -\hat{\lambda}_y \sin\theta_2$, and (d) $\hat{\lambda}_z \times m_i = 0$.

The y-component of (c) is

(e) $\hat{\lambda}_x \cdot m_r = -\sin\theta_2$.

(e) gives the x-component of m_r. From (c), m_r

cannot have a y-component. From (d), $\mathbf{m}_i$ can only have a z-component. It follows from (b) that $\mathbf{m}_r$ cannot have a z-component. Hence

(f) $\mathbf{m}_r = -\hat{\lambda}_x \sin\theta_2$.

From (a) and (d) then

(g) $\mathbf{m}_i = \hat{\lambda}_z \sin\theta_2$,

and $\mathbf{m} = -\hat{\lambda}_x \sin\theta_2 \pm i\,\hat{\lambda}_z \sin\theta_2$.

The fourth condition is that the sign of the imaginary part be chosen so that ψ is damped as it travels in the -z direction. We consider the exponent of ψ, writing

$$\mathrm{Re}\; i\omega\left(t - \frac{\mathbf{m}\cdot\mathbf{r}}{N}\right) < 0 .$$

Realizing that N > 0, letting $\mathbf{r}$ become $-\hat{\lambda}_z |z|$, we have

$$-i\,(\pm i\,\hat{\lambda}_z \sin\theta_2)(-\hat{\lambda}_z |z|) < 0,$$

resulting in

(47) $\mathbf{m} = -\hat{\lambda}_x \sin\theta_2 + i\,\hat{\lambda}_z \sin\theta_2$.

It is easily shown that this is equivalent to

(48) $\mathbf{m} = -\hat{\lambda}_z \times (\hat{\lambda}_z \times \hat{n}_2) - i\,\hat{\lambda}_x \times (\hat{\lambda}_z \times \hat{n}_2)$.

BOUNDARY CONDITIONS

After Roberts (3) we take the following:

$$(49) \qquad \hat{\lambda}_z \times \Delta \mathbf{h} = 0, \qquad \Delta \mathbf{h} = \mathbf{h}_A - \mathbf{h}_B$$

$$(50) \qquad \hat{\lambda}_z \times \Delta \mathbf{E} = 0, \qquad \Delta \mathbf{E} = \mathbf{E}_A - \mathbf{E}_B$$

$$(51) \qquad \hat{\lambda}_z \cdot \mathbf{q} = 0.$$

AMPLITUDES OF THE FIELDS

When the expressions for the fields, equations (29) through (34), are substituted in the boundary conditions, the exponentials cancel. Sufficient equations for determining the coefficients A_1, B, Ψ are available. Using (44) on (30) and substituting, we obtain

$$(52) \qquad \hat{\lambda}_x \cdot \mathbf{A}_0 + \hat{\lambda}_x \cdot \mathbf{A}_1 = i \frac{\omega H_0}{V} \hat{\lambda}_x \cdot \mathbf{B} - \frac{\omega^2 H_0}{N^2} (\mathbf{m} \cdot \hat{\lambda}_\zeta) \Psi (\hat{\lambda}_x \cdot \mathbf{m})$$

$$(53) \qquad \hat{\lambda}_y \cdot \mathbf{A}_0 + \hat{\lambda}_y \cdot \mathbf{A}_1 = i \frac{\omega H_0}{V} \hat{\lambda}_y \cdot \mathbf{B}$$

$$(54) \qquad -\eta \hat{\lambda}_x \cdot \hat{n}_0 \times \mathbf{A}_0 - \eta \hat{\lambda}_x \cdot \hat{n}_1 \times \mathbf{A}_1 = i \mu \omega \hat{\lambda}_x \cdot \mathbf{H}_0 \times \mathbf{B} + \frac{\mu \omega^2}{N} \Psi \hat{\lambda}_x \cdot \mathbf{H}_0 \times \mathbf{m}$$

$$(55) \qquad -\eta \hat{\lambda}_y \cdot \hat{n}_0 \times \mathbf{A}_0 - \eta \hat{\lambda}_y \cdot \hat{n}_1 \times \mathbf{A}_1 = i \mu \omega \hat{\lambda}_y \cdot \mathbf{H}_0 \times \mathbf{B} + \frac{\mu \omega^2}{N} \Psi \hat{\lambda}_y \cdot \mathbf{H}_0 \times \mathbf{m}$$

$$(56) \qquad i \omega \hat{\lambda}_z \cdot \mathbf{B} + \frac{\omega^2}{N} \Psi \hat{\lambda}_z \cdot \mathbf{m} = 0.$$

The divergence conditions show that the polarization vectors $\mathbf{A}_0$, $\mathbf{A}_1$, $\mathbf{B}$, are normal respectively to $\hat{n}_0$, $\hat{n}_1$, $\hat{n}_2$. Therefore the planes of polarization of the incident, reflected, and refracted waves make angles θ_0, θ_1, θ_2 respectively with the boundary, as shown in Figures 2, 3, 4. Referred to the plane of incidence, the polarization angles are respectively α_0, α_1, α_2. From the figures

$$(57) \quad \mathbf{A}_0 = A_0\left(\hat{\lambda}_x \cos\alpha_0 \cos\theta_0 + \hat{\lambda}_y \sin\alpha_0 - \hat{\lambda}_z \cos\alpha_0 \sin\theta_0\right)$$

$$(58) \quad \mathbf{A}_1 = A_1\left(\hat{\lambda}_x \cos\alpha_1 \cos\theta_1 + \hat{\lambda}_y \sin\alpha_1 + \hat{\lambda}_z \cos\alpha_1 \sin\theta_1\right)$$

$$(59) \quad \mathbf{B} = B\left(\hat{\lambda}_x \cos\alpha_2 \cos\theta_2 + \hat{\lambda}_y \sin\alpha_2 - \hat{\lambda}_z \cos\alpha_2 \sin\theta_2\right).$$

Note that if the polarization of the reflected wave is reversed in direction, so that $\pi < \alpha_1 < \frac{3\pi}{2}$, equation (58) still properly represents $\mathbf{A}_1$.

Using equations (40) through (43 and (57), (58), (59) to expand (52) through (56) we obtain

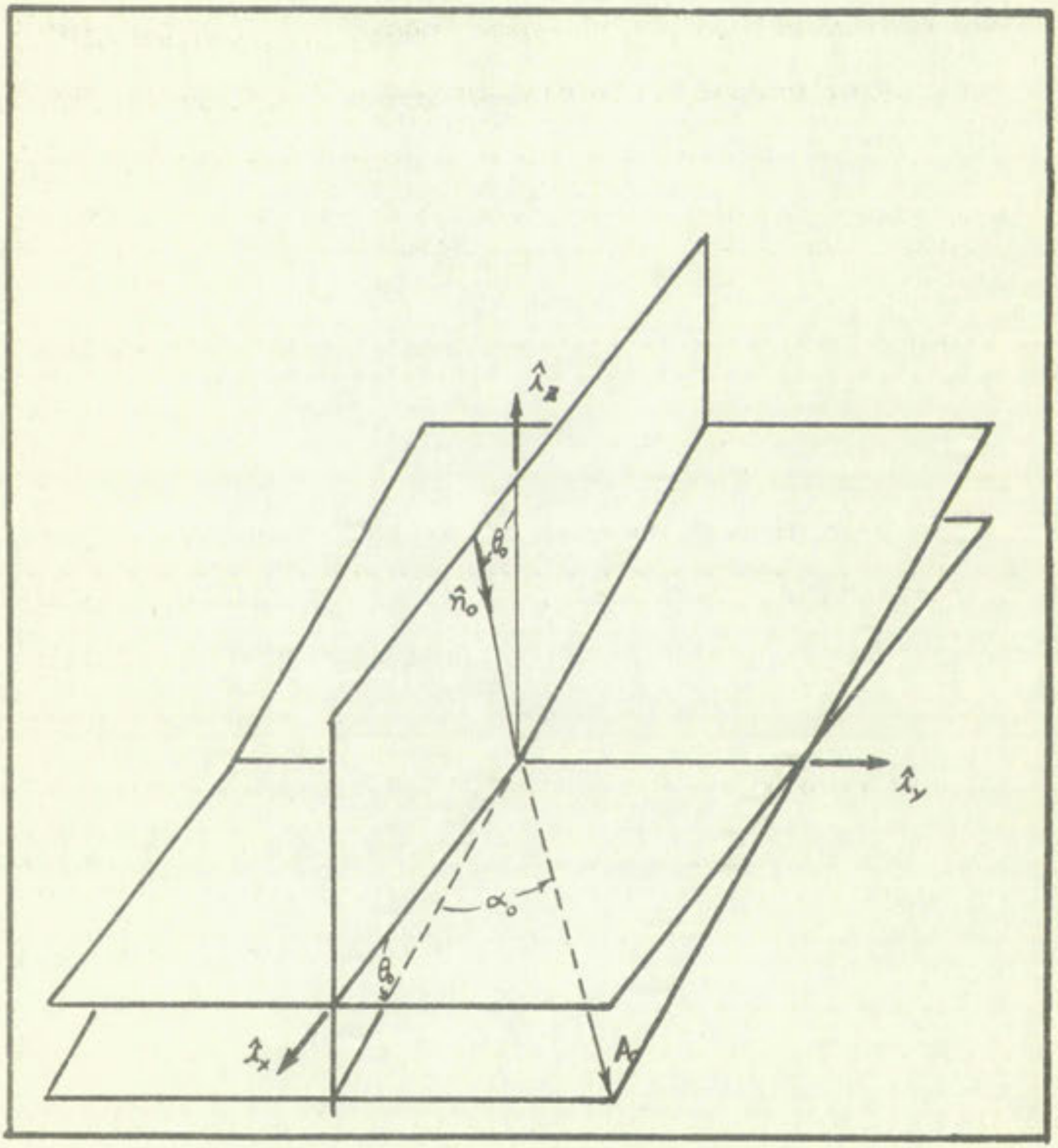
$$(60) \quad A_1 \sin\alpha_1 - i\frac{\omega H_0}{V} B \sin\alpha_2 = -A_0 \sin\alpha_0$$

$$(61) \quad A_1 \cos\alpha_1 \cos\theta_1 - i\frac{\omega H_0}{V}\cos\theta_2 \, B\cos\alpha_2 + \frac{\omega^2 H_0}{N^2}\Psi \sin^2\theta_2 (\sin\beta\cos\gamma - i\cos\beta) = -A_0 \cos\alpha_0 \cos\theta_0$$

$$(62) \quad \eta A_1 \cos\alpha_1 + i\frac{\mu\omega H_0 N}{V} B\cos\alpha_2 - i\frac{\mu\omega^2 H_0}{N}\Psi \sin\theta_2 (\sin\beta\cos\gamma - i\cos\beta) = \eta A_0 \cos\alpha_0$$

FIGURE 2

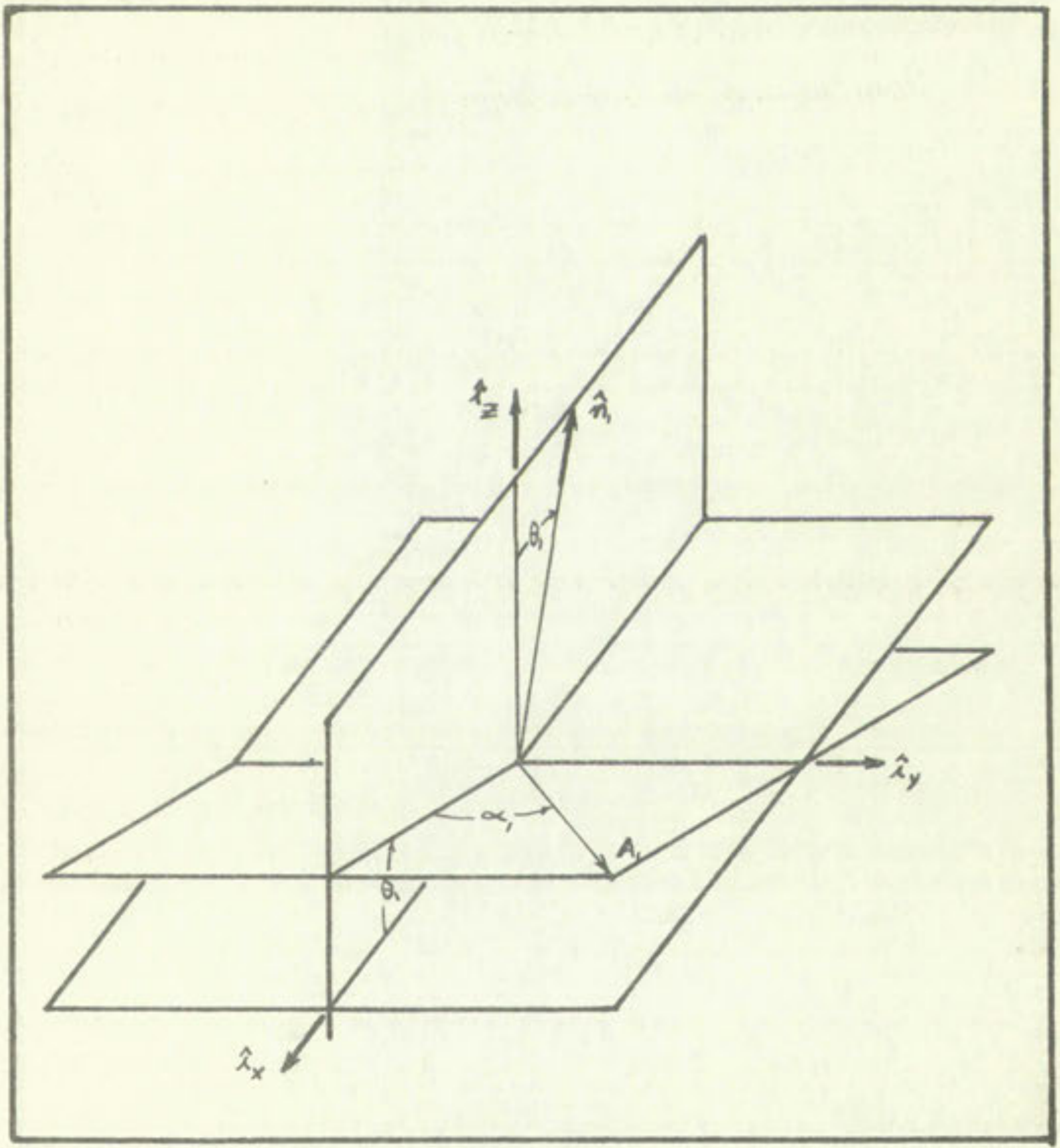
Polarization of Incident Wave



The plane containing $\hat{\lambda}_y$ and A_0, normal to $\hat{n}_0$, is the plane of the incident wave front.

FIGURE 3

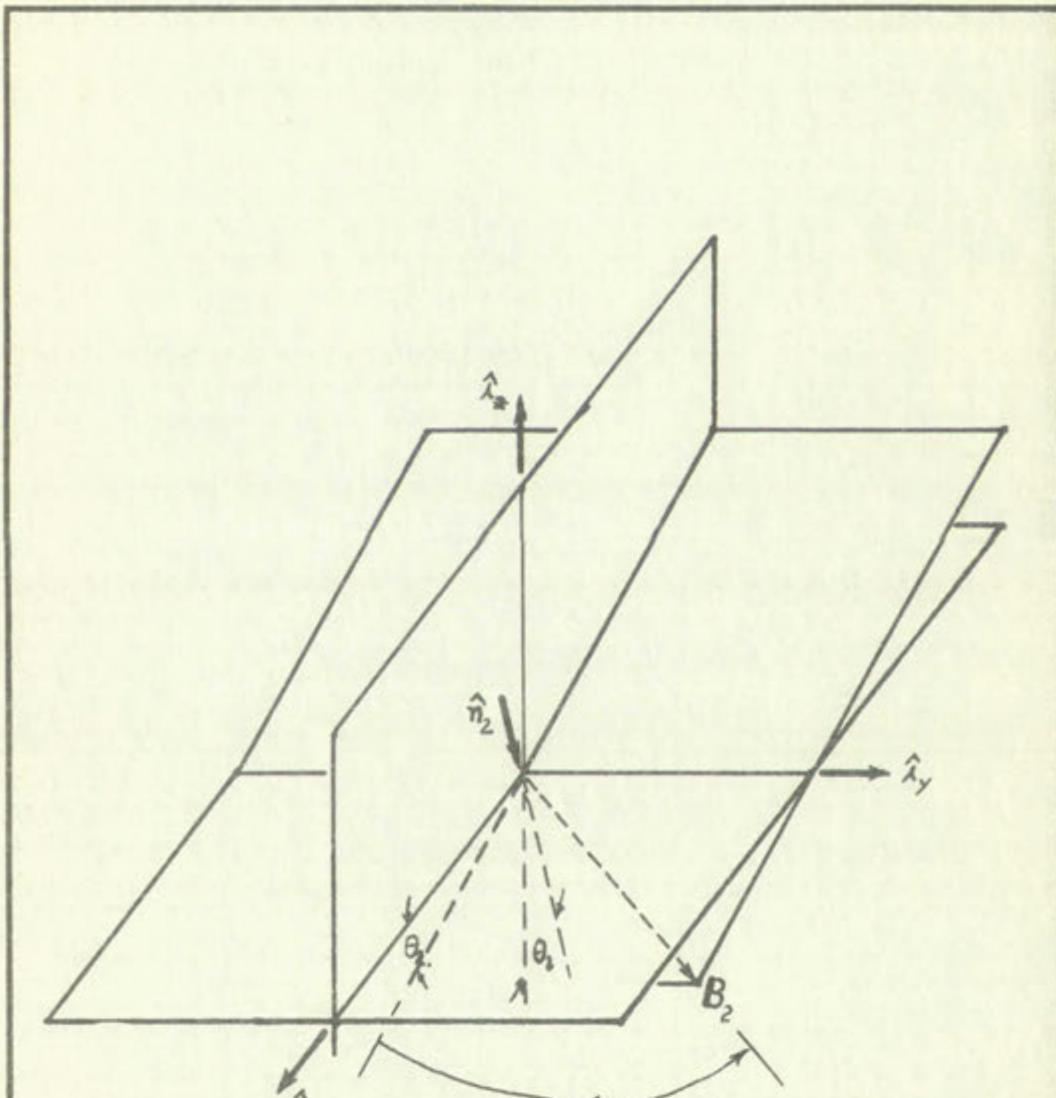
Polarization of Reflected Wave



The plane containing $\hat{\lambda}_y$ and A_1, normal to $\hat{n}_1$, is the plane of the reflected wave front.

FIGURE 4

Polarization of Refracted Wave



The plane containing $\hat{\lambda}_y$ and B, normal to $\hat{n}_z$, is the plane of the refracted wave front.

$$(63)\quad \eta A_1 \sin\alpha_1 \cos\theta_1 + i\mu\omega H_0 \cos\beta\, B \sin\alpha_2$$

$$+ i\mu\omega H_0 \sin\beta\ \sin\gamma\ \sin\theta_2\ B\cos\alpha_2$$

$$- i\frac{\mu\omega^2 H_0}{N}\Psi \sin\beta\ \sin\gamma\ \sin\theta_2 = \eta A_0 \sin\alpha_0 \cos\theta_0$$

$$(64)\quad B\cos\alpha_2 - \frac{\omega}{N}\Psi = 0.$$

Since $V^2 << c^2$, $\sin^2\theta_2 << 1$. With this simplification, the determinant of equations (60) through (64) is easier to solve for the polarization components $A_1 \sin\alpha_1$, $A_1\cos\alpha_1$, $B\sin\alpha_2$, $B\cos\alpha_2$, and Ψ . Making use of the relation $\eta = \mu c$, substituting for the angles θ_1 and θ_2 from (45) and (46), we may write the solutions in the following form, where

$$D = \frac{\eta^2\omega^3 H_0^2}{V^3 \cos\beta}\left(\frac{V}{c}\cos\beta + \cos\theta_0\right)\left[1 - \frac{V}{c}k_3 \sin\theta_0 + \frac{V}{c}\cos\beta\cos\theta_0\right]$$

is the denominator determinant.

$$D\, A_1 \sin\alpha_1 = -\frac{\eta^2\omega^3 H_0^2}{V^3\cos\beta}\left(\frac{V}{c}\cos\beta - \cos\theta_0\right)\left[1 - \frac{V}{c}k_3 \sin\theta_0 + \frac{V}{c}\cos\beta\cos\theta_0\right] A_0 \sin\alpha_0$$

$$D\, A_1 \cos\alpha_1 = \frac{\eta^2\omega^3 H_0^2}{V^3\cos\beta}\left(\frac{V}{c}\cos\beta + \cos\theta_0\right)\left[1 - \frac{V}{c}k_3 \sin\theta_0 - \frac{V}{c}\cos\beta\cos\theta_0\right] A_0 \cos\alpha_0$$

$$D\, B\sin\alpha_2 = 2i\frac{\eta^2\omega^2 H_0}{V^2\cos\beta}\left[1 - \frac{V}{c}k_3 \sin\theta_0 + \frac{V}{c}\cos\beta\cos\theta_0\right] A_0 \sin\alpha_0 \cos\theta_0$$

$$D\, B\cos\alpha_2 = 2i\frac{\eta^2\omega^2 H_0}{V^2\cos\beta}\left(\frac{V}{c}\cos\beta + \cos\theta_0\right)\left[1 - 2k_3\frac{V}{c}\sin\theta_0\right]^{1/2} A_0 \cos\alpha_0 \cos\theta_0$$

$$D\,\Psi = -2i\frac{\eta^2\omega H_0}{V}\left(\frac{V}{c}\cos\beta + \cos\theta_0\right) A_0 \cos\alpha_0 \cos\theta_0 ,$$

where $k_3 = \sin\beta \cos\gamma$.

Substituting for B from equation (36), one obtains by eliminating D

$$(65) \quad A_1 \sin\alpha_1 = - A_0 \sin\alpha_0 \frac{\frac{V}{c}\cos\beta - \cos\theta_0}{\frac{V}{c}\cos\beta + \cos\theta_0}$$

$$(66) \quad A_1 \cos\alpha_1 = A_0 \cos\alpha_0 \frac{1 - \frac{V}{c}k_3 \sin\theta_0 - \frac{V}{c}\cos\beta\cos\theta_0}{1 - \frac{V}{c}k_3 \sin\theta_0 + \frac{V}{c}\cos\beta\cos\theta_0}$$

$$(67) \quad A_2 \sin\alpha_2 = A_0 \sin\alpha_0 \frac{2\cos\theta_0}{\frac{V}{c}\cos\beta + \cos\theta_0}$$

$$(68) \quad A_2 \cos\alpha_2 = A_0 \cos\alpha_0 \frac{2\cos\theta_0 \left[1 - 2\frac{V}{c}k_3 \sin\theta_0\right]^{1/2}}{1 - \frac{V}{c}k_3 \sin\theta_0 + \frac{V}{c}\cos\beta\cos\theta_0}$$

$$(69) \quad \Psi = - A_0 \cos\alpha_0 \frac{2i \frac{V^2}{\omega^2 H_0}\cos\beta\cos\theta_0}{1 - \frac{V}{c}k_3 \sin\theta_0 + \frac{V}{c}\cos\beta\cos\theta_0}$$

It is possible to simplify these expressions slightly by specifying the direction of $\mathbf{H}_0$. If we let $\beta = 0$, the polarization angles are given by

$$\tan\alpha_1 = \tan\alpha_0 \frac{\cos\theta_0 - \frac{V}{c}\sin^2\theta_0}{\cos\theta_0 + \frac{V}{c}\sin^2\theta_0}$$

$$\tan\alpha_2 = \tan\alpha_0 \frac{1 + \frac{V}{c}\cos\theta_0}{\frac{V}{c} + \cos\theta_0} .$$

Thus the polarization angles are not simply related to that of the incident wave. Only if we consider normal incidence do we

get something simple;

$\tan \alpha_2 = \tan \alpha_1 = \tan \alpha_0$

regardless of the orientation of the static magnetic field.

The coefficient Ω may be obtained from equations (36) and (69);

$$(70) \quad \Omega = -A_0 \cos\alpha_0 \frac{V}{C} \frac{\sin 2\theta_0 \left[1 - 2\frac{V}{C} k_3 \sin\theta_0\right]^{1/2}}{1 - \frac{V}{C} k_3 \sin\theta_0 + \frac{V}{C}\cos\beta\cos\theta_0} (\cos\beta + i \sin\beta\cos t).$$

Note that $\Omega = 0$ for normal incidence.

Perhaps more significant than the amplitude Ω is the skin depth of the ψ disturbance. This depends on the real part of the exponent of the ψ function.

$$\frac{\omega}{N} \hat{\lambda}_x \times (\hat{\lambda}_z \times \hat{n}_2) \cdot \hat{\lambda}_z \, |z| \, .$$

It is given by $d = -\frac{N}{\omega} \left[\hat{\lambda}_z \cdot \hat{\lambda}_x \times (\hat{\lambda}_z \times \hat{n}_2)\right]^{-1}.$

Manipulating the vectors results in

$$d = \frac{N}{\omega \sin\theta_2} \, .$$

Substituting from (44) and (46) yields

$$(71) \quad d = \frac{c}{\omega} \csc\theta_0 \, .$$

Note that the skin depth is independent of the magnitude and direction of H_0, and becomes infinite as $\theta_0 \to 0$. Thus, for oblique incidence, Ω may be large but quickly damped; while

get something similar;

$\tan \alpha_2 = \tan \alpha_1 = \tan \alpha_0$

regardless of the orientation of the earth's magnetic field.

The coefficient Ω may be obtained from equations (68) and (69);

$$(70) \quad \Omega = -A_0 \cos\alpha_0 \frac{\nu}{c} \frac{\sin 2\theta_0 \left[1 - \ldots \right]}{1 - \ldots} \ldots$$

Note that $\Omega = 0$ for normal incidence.

Perhaps more significant than the expression Ω is the skin depth of the ψ disturbance. This depends on the real part of the exponent of the ψ function.

It is given by $d = \frac{N}{\omega} \left[\ldots \right]$.

Manipulating any results in

$$d = \frac{N}{\omega \sin \ldots}$$

Substituting from (...) and (...) yields

$$(71) \quad d = \frac{c}{\omega} \cos\theta_0 .$$

Note that the skin depth is independent of the magnitude and direction of H_0, and becomes infinite only ... Thus, for oblique incidence, Ω may be large ...

for nearly normal incidence, Ω is small but slowly damped.

ENERGY IN THE WAVES

Consider a fixed volume V enclosed by a surface S in an electromagnetic medium. If the volume contains a source of EM energy W, then

$$(72) \quad \frac{\partial W}{\partial t} = \int_S \mathbf{E} \times \mathbf{H} \cdot \hat{s}\, d\sigma$$

is the rate at which energy leaves the volume ($\hat{s}$ being the outward normal to the surface)..

We seek an expression for the energy flow per unit area at a point. Transforming (72) to a volume integral, it is evident that only that part of the integrand whose divergence is NOT zero will contribute to the energy flow. Hence one cannot in general simply use the integrand of (72) to represent the energy flow at a point. In addition, if complex notation is used for field quantities, we must consider only the real parts to obtain the actual energy flow. Finally, since we want the flow under equilibrium conditions, we must obtain the time average of (72).

In medium A the total magnetic field is $\mathbf{H}_A = \mathbf{H}_0 + \mathbf{h}_A$,

where $\mathbf{H}_o$ is the static field, and $\mathbf{h}_A$ is the wave field. Substituting in (72), and taking the real part,

$$\mathcal{R}e\,\frac{\partial W}{\partial t} = \int_S \mathcal{R}e\,\mathbf{E}_A \times \mathcal{R}e\,\mathbf{h}_A \cdot \hat{s}\,d\sigma + \int_S \mathcal{R}e\,\mathbf{E}_A \times \mathcal{R}e\,\mathbf{H}_o \cdot \hat{s}\,d\sigma\,.$$

The time average of this may be written

$$(73)\quad \overline{\mathcal{R}e\,\frac{\partial W}{\partial t}} = \frac{1}{\tau}\int_0^{\tau}\int_S \mathcal{R}e\,\mathbf{E}_A \times \mathcal{R}e\,\mathbf{h}_A \cdot \hat{s}\,d\sigma\,dt$$

$$+ \int_S \frac{1}{\tau}\int_0^{\tau} \mathcal{R}e\,\mathbf{E}_A \times \mathcal{R}e\,\mathbf{H}_o\,dt \cdot \hat{s}\,d\sigma\,,$$

where τ is the period of the wave. Since the surface is fixed in space we can interchange order of integration. Now $\mathbf{H}_o$ is real and constant, so the second integral is

$$-\int_S \frac{1}{\tau}\,\mathbf{H}_o \times \int_0^{\tau} \mathcal{R}e\,\mathbf{E}_A\,dt \cdot \hat{s}\,d\sigma\,.$$

It is easily shown that the time integral vanishes; i.e. $\overline{\mathcal{R}e\,\mathbf{E}_A} = 0$. Therefore the static magnetic field does not contribute to the energy flow in medium A. Switching the order of integration in the first integral, we arrive at

$$\overline{\mathcal{R}e\,\frac{\partial W}{\partial t}} = \int_S \overline{\mathcal{R}e\,\mathbf{E}_A \times \mathcal{R}e\,\mathbf{h}_A} \cdot \hat{s}\,d\sigma = \int_S \frac{1}{2}\,\mathcal{R}e\,\mathbf{E}_A \times \mathbf{h}_A^* \cdot \hat{s}\,d\sigma\,,$$

a well known expression. If we transform this surface integral, we see that no part of the integrand is solenoidal; hence the integrand may be used to represent the energy flow per unit area at a point. Treating the incident and reflected waves separately, we have [4]

$$(74)\quad \overline{P_0} = \tfrac{1}{2}\,\mathcal{R}e\,\mathbf{E}_0\times\mathbf{h}_0^*, \qquad (75)\quad \overline{P_1} = \tfrac{1}{2}\,\mathcal{R}e\,\mathbf{E}_1\times\mathbf{h}_1^* .$$

For a fixed volume of a magnetohydrodynamic medium, enclosed by a surface S, one can write the energy flow as[5]

$$(76)\quad \frac{\partial W}{\partial t} = \int_S (\mathbf{E}\times\mathbf{H} + \tfrac{1}{2}\rho q^2 \mathbf{q} + p\mathbf{q})\cdot\hat{s}\,d\sigma .$$

The term $\mathbf{E}\times\mathbf{H}$ represents the energy flowing with a MHD wave, the second term indicates a convection of kinetic energy, and $p\mathbf{q}$ denotes a flow of mechanical energy by means of momentum transfer. As for medium A, we seek the time average of that real part of the integrand whose divergence does not vanish. Again one can show that the static field $\mathbf{H}_0$ makes no average contribution to the energy flow.

It may be shown that, for our particular case, the second and third terms of the integrand of (76) make no real contribution to the energy flow across the boundary, on the average.

First let us define

$$(77)\quad \delta = \frac{\hat{n}_2\cdot\mathbf{r}}{N}, \qquad \epsilon = \frac{\mathbf{m}\cdot\mathbf{r}}{N} .$$

Then we can write the fluid velocity (from (31), using (36)) as

$$(78)\quad \mathbf{q} = \frac{V}{H_0}\mathbf{A}_2\, e^{i\omega(t-\delta)} + \mathbf{Q}\, e^{i\omega(t-\epsilon)} ,$$

where $\mathbf{Q} = \mathbf{Q}_r + i\,\mathbf{Q}_i = \frac{\omega^2}{N}\Psi\,\mathbf{m}$.

We may express Ψ in terms of A; from (64), using (36),

$$(79)\quad \Psi = i\frac{N^2}{\omega^2}\frac{A_2\cos\alpha_2}{\mathbf{H}_0\cdot\hat{n}_2} = -i\frac{NV}{\omega^2 H_0}A_2\cos\alpha_2 .$$

Substituting for m and Ψ, we obtain from (78)

$$\mathcal{R}e\, q = \frac{V}{H_0}\left\{\mathbf{A}_2\cos\omega(t-\delta) - \hat{\imath}_z\cdot\mathbf{A}_2\left[\hat{\imath}_z\cos\omega(t-\epsilon) - \hat{\imath}_x\sin\omega(t-\epsilon)\right]\right\}$$

Notice that $\hat{\imath}_z\cdot\mathbf{A}_2$ is of order $\sin\theta_2 \propto \frac{V}{c}$ (from (59));
neglecting terms with higher powers of $\hat{\imath}_z\cdot\mathbf{A}_2$,

$$(\mathcal{R}e\, q)^2\,\mathcal{R}e\, q = \frac{V^3}{H_0^3}\Big\{\mathbf{A}_2 A_2^2\cos^3\omega(t-\delta) + 2\mathbf{A}_2(\hat{\imath}_z\cdot\mathbf{A}_2)\hat{\imath}_x\cdot\mathbf{A}_2 \times$$

$$\cos^2\omega(t-\delta)\cos\omega(t-\epsilon) - A_2^2\,\hat{\imath}_z\cdot\mathbf{A}_2\cos^2\omega(t-\delta)\left[\hat{\imath}_z\cos\omega(t-\epsilon) - \hat{\imath}_x\sin\omega(t-\epsilon)\right]\Big\}$$

Now energy flow across the boundary depends on the z-component of this expression. It is apparent that

$$\tfrac{1}{2}\rho(\mathcal{R}e\, q)^2\,\mathcal{R}e\, q\cdot\hat{\imath}_z = 0 .$$

That is, the convection term does not contribute to the energy flow across the boundary.

To evaluate the third term of the integrand of (76), one must determine the pressure. The following relations, repeated for convenience, will be of use.

$$(12)\quad \varphi = \frac{1}{\rho}\left[p + \frac{1}{2}\left(\mathbf{H}_0 + \mathbf{h}_8\right)^2\right], \text{ or } p = \rho\varphi - \frac{1}{2}\left(H_0^2 + 2\mathbf{H}_0\cdot\mathbf{h}_2\right),$$

$$(27)\quad \varphi = \varphi_0 + \Phi e^{i\omega(t-\epsilon)},$$

(28) $$\Phi = \omega^2\left[1-\left(\frac{\mathfrak{m}\cdot\hat{\lambda}_\zeta}{\hat{n}_2\cdot\hat{\lambda}_\zeta}\right)^2\right]\Psi = \omega^2\left[1-\frac{V^2}{N^2}(\mathfrak{m}\cdot\hat{\lambda}_\zeta)^2\right]\Psi.$$

Since φ_0 is a constant for all time it can be evaluated for static conditions; from (12),

(80) $$\varphi_0 = \frac{1}{\rho}\left(p_0 + \frac{1}{2}H_0^2\right),$$

where p_0 is the hydrostatic pressure of the fluid. To evaluate Φ, note that $(\mathfrak{m}\cdot\hat{\lambda}_\zeta)^2 \propto \sin^2\theta_2$ is vanishingly small; hence

(81) $$\Phi = \omega^2\Psi = -i\frac{NV}{H_0}A_2\cos\alpha_2 = i\Phi_i .$$

Substituting in (27) we discover

$$\mathcal{R}e\,\varphi = \frac{1}{\rho}\left(p_0 + \frac{1}{2}H_0^2\right) - \Phi_i \sin\omega(t-\epsilon).$$

With this result (12) yields

(82) $$\mathcal{R}e\,p = p_0 - \rho\Phi_i\sin\omega(t-\epsilon) - \mathbf{H}_0\cdot\mathbf{A}_2\cos\omega(t-\delta).$$

From (78) we obtain

$$\mathcal{R}e\,\mathbf{q} = \frac{V}{H_0}\mathbf{A}_2\cos\omega(t-\delta) + \mathbf{Q}_r\cos\omega(t-\epsilon) - \mathbf{Q}_i\sin\omega(t-\epsilon).$$

One can show that the time average of the product $\mathcal{R}e\,p\,\mathcal{R}e\,\mathbf{q}$, evaluated at the boundary, is

$$\overline{\mathcal{R}e\,p\,\mathcal{R}e\,\mathbf{q}}\Big)_{z=0} = -\frac{1}{2}\left[(\mathbf{H}_0\cdot\mathbf{A}_2)\mathbf{Q}_r + \frac{V}{H_0}(\mathbf{H}_0\cdot\mathbf{A}_2)\mathbf{A}_2 + \rho\Phi_i\mathbf{Q}_i\right].$$

Substituting for $\mathfrak{m}$ and Ψ in (78) gives

$$\mathbf{Q}_r = -\frac{V}{H_0}(\hat{\lambda}_z\cdot\mathbf{A}_2)\hat{\lambda}_z, \quad \mathbf{Q}_i = \frac{V}{H_0}(\hat{\lambda}_z\cdot\mathbf{A}_2)\hat{\lambda}_x .$$

Thus $\hat{\lambda}_z \cdot \overline{\mathcal{R}e\, p\, \mathcal{R}e\, q}\Big)_{z=0} = 0$;

there is no real energy transferred across the boundary by momentum, on the average.

The only term left to the integrand of (76) is $\mathbf{E}_2 \times \mathbf{h}_2$. It is easily seen that no part of div ($\mathbf{E}_2 \times \mathbf{h}_2$) vanishes, so we can safely take

$$(83) \quad \overline{\mathbf{P}_2} = \tfrac{1}{2}\, \mathcal{R}e\, \mathbf{E}_2 \times \mathbf{h}_2^{*} \quad .$$

REFLECTION AND TRANSMISSION COEFFICIENTS

We define[4]

$$(84) \quad \Gamma \equiv \frac{\hat{\lambda}_z \cdot \overline{\mathbf{P}_1}}{\hat{\lambda}_z \cdot \overline{\mathbf{P}_0}} \quad \text{, reflection coefficient,}$$

$$(85) \quad \mathrm{T} \equiv \frac{\hat{\lambda}_z \cdot \overline{\mathbf{P}_2}}{\hat{\lambda}_z \cdot \overline{\mathbf{P}_0}} \quad \text{, transmission coefficient,}$$

where $\overline{\mathbf{P}_0}$, $\overline{\mathbf{P}_1}$, $\overline{\mathbf{P}_2}$ are "complex" Poynting vectors, defined by equations (74), (75), (83).

Γ and T are determined by the fields (35) through (38). In order to find the real parts of the fields, it will be convenient to separate Ω and Π into real and imaginary parts. Using equations (47), (59), (79) we may write

Thus $\quad \xi_{21} = \xi_0 \, p \sqrt{\frac{\mu_2}{\varepsilon_2}} = 0$

there is no real energy transferred across the boundary by momentum, on the average.

The only term left is the integral of $\Re$ in $E_2 \times h_2$. It is easily seen that no part of $\Re\,(E_1 \times h_2)$ vanishes, so we can safely take

$$\overline{P_2} = \frac{1}{2}\Re\, E_2 \times h_2^* \tag{83}$$

REFLECTION AND TRANSMISSION COEFFICIENTS

We define(13)

$$\Gamma \equiv \frac{\overline{P_2}}{\overline{P_0}} \text{, reflection coefficient,} \tag{84}$$

$$T \equiv \frac{\overline{P_3}}{\overline{P_0}} \text{, transmission coefficient,} \tag{85}$$

where $\overline{P_0}$, $\overline{P_1}$, $\overline{P}$ are normalized Poynting vectors, defined by equations (81), (72), (83).

Γ and T are determined by the fields (38) to (39). In order that the real parts of the fields, to correspond to physical Γ or T are taken imaginary parts. Using equations (81), (82), (83) we can write

$\boldsymbol{\Omega} = \boldsymbol{\Omega}_r + i\boldsymbol{\Omega}_i$, where

(86) $\boldsymbol{\Omega}_r = -\frac{V}{N}\sin\theta_2(\hat{\lambda}_z\cdot\mathbf{A}_2)\left[\hat{\lambda}_x(\hat{\lambda}_z\cdot\hat{\lambda}_\zeta)+\hat{\lambda}_z(\hat{\lambda}_x\cdot\hat{\lambda}_\zeta)\right]$

$\boldsymbol{\Omega}_i = -\frac{V}{N}\sin\theta_2(\hat{\lambda}_z\cdot\mathbf{A}_2)\left[\hat{\lambda}_x(\hat{\lambda}_x\cdot\hat{\lambda}_\zeta)-\hat{\lambda}_z(\hat{\lambda}_z\cdot\hat{\lambda}_\zeta)\right]$,

and $\boldsymbol{\Pi} = \boldsymbol{\Pi}_r + i\boldsymbol{\Pi}_i$

, where

(87) $\boldsymbol{\Pi}_r = -\mu V(\hat{\lambda}_z\cdot\mathbf{A}_2)\,\hat{\lambda}_\zeta\times\hat{\lambda}_z$

$\boldsymbol{\Pi}_i = -\mu V(\hat{\lambda}_z\cdot\mathbf{A}_2)\,\hat{\lambda}_\zeta\times\hat{\lambda}_x$.

We may rewrite (35) and (37) as

$$\mathbf{h}_2 = \mathbf{A}_2 e^{i\omega(t-\delta)} + (\boldsymbol{\Omega}_r + i\boldsymbol{\Omega}_i)\, e^{i\omega(t-\epsilon)},$$

$$\mathbf{E}_2 = \mathbf{D}\, e^{i\omega(t-\delta)} + (\boldsymbol{\Pi}_r + i\boldsymbol{\Pi}_i)\, e^{i\omega(t-\epsilon)}.$$

On forming the cross-product $\mathbf{E}_2\times\mathbf{h}_2^*$, we obtain

$$\mathbf{E}_2\times\mathbf{h}_2^* = \mathbf{D}\times\mathbf{A}_2 + (\boldsymbol{\Pi}_r + i\boldsymbol{\Pi}_i)\times\mathbf{A}_2\, e^{i\omega(\delta-\epsilon)}$$
$$+ \mathbf{D}\times(\boldsymbol{\Omega}_r - i\boldsymbol{\Omega}_i)\, e^{-i\omega(\delta-\epsilon)} + (\boldsymbol{\Pi}_r + i\boldsymbol{\Pi}_i)\times(\boldsymbol{\Omega}_r - i\boldsymbol{\Omega}_i).$$

From (77),

$$\delta - \epsilon = \frac{\mathbf{r}}{N}\cdot(-\cos\theta_2 - i\sin\theta_2)\hat{\lambda}_z = -\frac{z}{N}(\cos\theta_2 + i\sin\theta_2)$$

But $\mathbf{E}_2\times\mathbf{h}_2^*$ is to be evaluated at the boundary ($z = 0$), so

$$e^{i\omega(\delta-\epsilon)} = e^{-i\omega(\delta-\epsilon)} = 1 .$$

Expanding the cross-product and selecting the real part, we get

$$\mathcal{Re}\,\mathbf{E}_2 \times \mathbf{h}_2^*\Big)_{z=0} = (\mathbf{D} + \mathbf{\Pi}_r) \times (\mathbf{A}_2 + \mathbf{\Omega}_r) + \mathbf{\Pi}_i \times \mathbf{\Omega}_i \ .$$

Now it is easily seen that $\mathbf{\Omega}$ is vanishingly small (of order $\sin^2\theta_2$); hence

$$\text{(88)} \quad \mathcal{Re}\,\mathbf{E}_2 \times \mathbf{h}_2^*\Big)_{z=0} = (\mathbf{D} + \mathbf{\Pi}_r) \times \mathbf{A}_2$$

Using (44) on (38) we obtain a simpler expression

$$\mathbf{D} = \mu V\, \hat{\lambda}_\zeta \times \mathbf{A}_2 \quad ;$$

hence

$$(\mathbf{D} + \mathbf{\Pi}_r) \times \mathbf{A}_2 = \mu V (\hat{\lambda}_\zeta \times \mathbf{A}_2) \times \mathbf{A}_2 - \mu V (\hat{\lambda}_z \cdot \mathbf{A}_2)(\hat{\lambda}_\zeta \times \hat{\lambda}_z) \times \mathbf{A}_2$$

$$= \mu V \Big[\mathbf{A}_2 (\hat{\lambda}_\zeta \cdot \mathbf{A}_2) - \hat{\lambda}_\zeta A_2^2 - \hat{\lambda}_z (\hat{\lambda}_z \cdot \mathbf{A}_2)(\hat{\lambda}_\zeta \cdot \mathbf{A}_2) + \hat{\lambda}_\zeta (\hat{\lambda}_z \cdot \mathbf{A}_2)^2 \Big]$$

The last term in brackets is vanishingly small. The component of (88) normal to the boundary is simply

$$\text{(89)} \quad \hat{\lambda}_z \cdot \overline{\mathbf{P}_2} = -\mu V A_2^2 \cos\beta \ .$$

$\overline{\mathbf{P}_0}$ and $\overline{\mathbf{P}_1}$ (equations (74) and (75)) are more easily obtained. From (29) and (34),

$$\text{(90)} \quad \begin{aligned} \mathbf{E}_0 &= -\eta\, \hat{n}_0 \times \mathbf{A}_0\, e^{i\omega\left(t - \frac{\hat{n}_0 \cdot \mathbf{r}}{c}\right)} \\ \mathbf{h}_0^* &= \mathbf{A}_0\, e^{-i\omega\left(t - \frac{\hat{n}_0 \cdot \mathbf{r}}{c}\right)} \end{aligned}$$

$$\text{(91)} \quad \begin{aligned} \mathbf{E}_1 &= -\eta\, \hat{n}_1 \times \mathbf{A}_1\, e^{i\omega\left(t - \frac{\hat{n}_1 \cdot \mathbf{r}}{c}\right)} \\ \mathbf{h}_1^* &= \mathbf{A}_1\, e^{-i\omega\left(t - \frac{\hat{n}_1 \cdot \mathbf{r}}{c}\right)} \ . \end{aligned}$$

Substituting (90) and (91) in equations (74) and (75) yields

$$\overline{\mathbf{P}_0} = -\eta\,(\hat{n}_0 \times \mathbf{A}_0)\times\mathbf{A}_0 = \eta\left[\hat{n}_0 A_0^2 - \mathbf{A}_0(\hat{n}_0\cdot\mathbf{A}_0)\right]$$

$$\overline{\mathbf{P}_1} = -\eta\,(\hat{n}_1 \times \mathbf{A}_1)\times\mathbf{A}_1 = \eta\left[\hat{n}_1 A_1^2 - \mathbf{A}_1(\hat{n}_1\cdot\mathbf{A}_1)\right] .$$

Since $\mathbf{A}_0$ and $\mathbf{A}_1$ are perpendicular to $\hat{n}_0$ and $\hat{n}_1$ respectively,

(92) $$\hat{\imath}_z \cdot \overline{\mathbf{P}_0} = -\eta \cos\theta_0 \, A_0^2 ,$$

(93) $$\hat{\imath}_z \cdot \overline{\mathbf{P}_1} = \eta \cos\theta_1 \, A_1^2 .$$

Using the relation $\eta = \mu c$, (89) may be put

(94) $$\hat{\imath}_z \cdot \overline{\mathbf{P}_2} = -\eta \frac{V}{c} \cos\beta \, A_2^2 .$$

With these expressions, the coefficients become

(95) $$\Gamma = -\frac{A_1^2}{A_0^2} \quad \text{, reflection coefficient,}$$

(96) $$T = \frac{V}{c}\frac{\cos\beta}{\cos\theta_0}\frac{A_2^2}{A_0^2}\text{, transmission coefficient.}$$

It remains to evaluate A_1 and A_2 by means of equations (65) through (68). We obtain

(97) $$\frac{A_1^2}{A_0^2} = \sin^2\alpha_0 \left(\frac{\frac{V}{c}\cos\beta - \cos\theta_0}{\frac{V}{c}\cos\beta + \cos\theta_0}\right)^2 + \cos^2\alpha_0\left[\frac{1 - \frac{V}{c}(k_3 \sin\theta_0 + \cos\beta\cos\theta_0)}{1 - \frac{V}{c}(k_3\sin\theta_0 - \cos\beta\cos\theta_0)}\right]^2 ,$$

$$(98) \qquad \frac{A_2^2}{A_o^2} = 4 \cos^2\theta_o \left\{ \frac{\sin^2\alpha_o}{(\frac{V}{c}\cos\beta + \cos\theta_o)^2} + \frac{\cos^2\alpha_o\,(1 - 2\frac{V}{c}k_3 \sin\theta_o)}{\left[1 - \frac{V}{c}(k_3 \sin\theta_o - \cos\beta\cos\theta_o)\right]^2} \right\}.$$

Equations (95) through (98) show that the transfer of energy from as EM wave to an MHD fluid is a function of orientation of the static magnetic field, polarization of the incident wave, and the angle of incidence. The exact functional dependences are obscured by the complexity of equations (97) and (98). Therefore it is instructive to consider special cases.

The dependence on β is clearly shown for the case of normal incidence; for $\theta_o = 0$, equations (97) and (98) reduce to

$$\frac{A_1^2}{A_o^2} = \frac{1}{1 + 4\frac{V}{c}\cos\beta}\,, \qquad \frac{A_2^2}{A_o^2} = \frac{4(1 + 2\frac{V}{c}\cos\beta)}{1 + 4\frac{V}{c}\cos\beta}\,.$$

wherein the coefficients become

$$(99) \quad \Gamma = -\frac{1}{1 + 4\frac{V}{c}\cos\beta}\,,$$

$$(100) \quad T = \frac{4\frac{V}{c}\cos\beta}{1 + 4\frac{V}{c}\cos\beta}\,.$$

The (-) sign of course signifies reflection. Note that $|T| + |\Gamma| = 1$, as it must for conservation of energy.

FUNCTIONAL DEPENDENCE OF Γ AND T

To correctly interpret the curves of Figures 5, 6, 7, it is important to keep in mind that Γ and T are defined in of the <u>normal</u> components of the Poynting vectors. While the normal component of $\overline{P_0}$ decreases smoothly as θ_0 increases, the percentage of $\overline{P_2}$ normal to the boundary remains essentially constant because θ_2 is always small. Since $\theta_1 = \theta_0$, a change in Γ indicates a corresponding change in the total reflected energy, but there is a tendency for T to increase with increasing θ_0 simply because the normal component of $\overline{P_0}$ decreases.

However this does not at all explain the curves of Figures 5, 6, 7. The behavior for the case $\alpha_0 = \frac{\pi}{2}$ is most surprising. Inspection of Figure 2 reveals that the polarization of the h_0 wave is entirely parallel to the boundary for this case, for all θ_0. Apparently as θ_0 increases the transmission of the parallel component of polarization increases, reflection decreasing correspondingly. At a certain angle ($\theta_0 \simeq 85°$), the entire incident wave is transmitted. From Equations 46, 97, it may be seen that at this point $\frac{\pi}{2} - \theta_0 \simeq \theta_2$.

Γ and T behave in the same way regardless of β, although obviously their numerical values are a function of β.

[illegible]

[illegible] throughout, the curves of Figure [illegible], it is important to keep in mind that Γ and T are measures of the normal components of the [illegible]. [illegible] normal component of $\overline{E_2}$ [illegible] percentage of $\overline{E_2}$ normal to the [illegible] constant because θ_t is always small. Since [illegible], a change in Γ indicates a corresponding change in [illegible] energy, but there is [illegible] increasing θ_o simply because the [illegible] of $\overline{E_2}$ [illegible].

[illegible] Figures 5, 6, 7. [illegible] behavior [illegible] pretation. Inspection of Figure [illegible] the h_o wave is [illegible] parallel to the [illegible] for all θ_o. [illegible] as θ_o [illegible] the parallel component of [illegible] creating [illegible]. At [illegible] entire incident wave is transmitted [illegible] may be seen that [illegible] $\frac{\pi}{2}$ [illegible].

Γ and T behave in [illegible] although obviously [illegible].

FIGURE 5

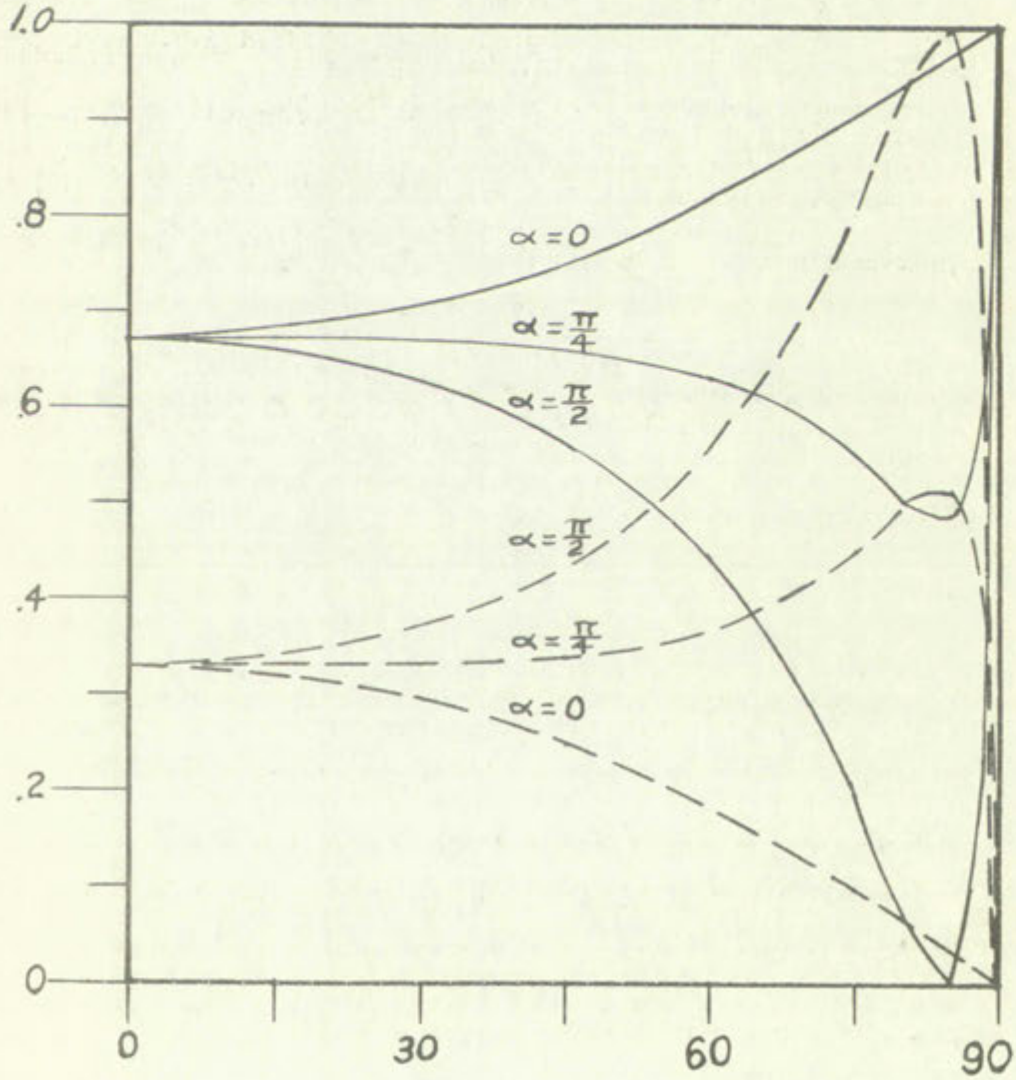
Dependence of Γ and T on Angle of Incidence,

With Polarization Angle of Incident Wave as Parameter,

For the Case $\beta = 0$.

Γ solid curves

T dashed curves



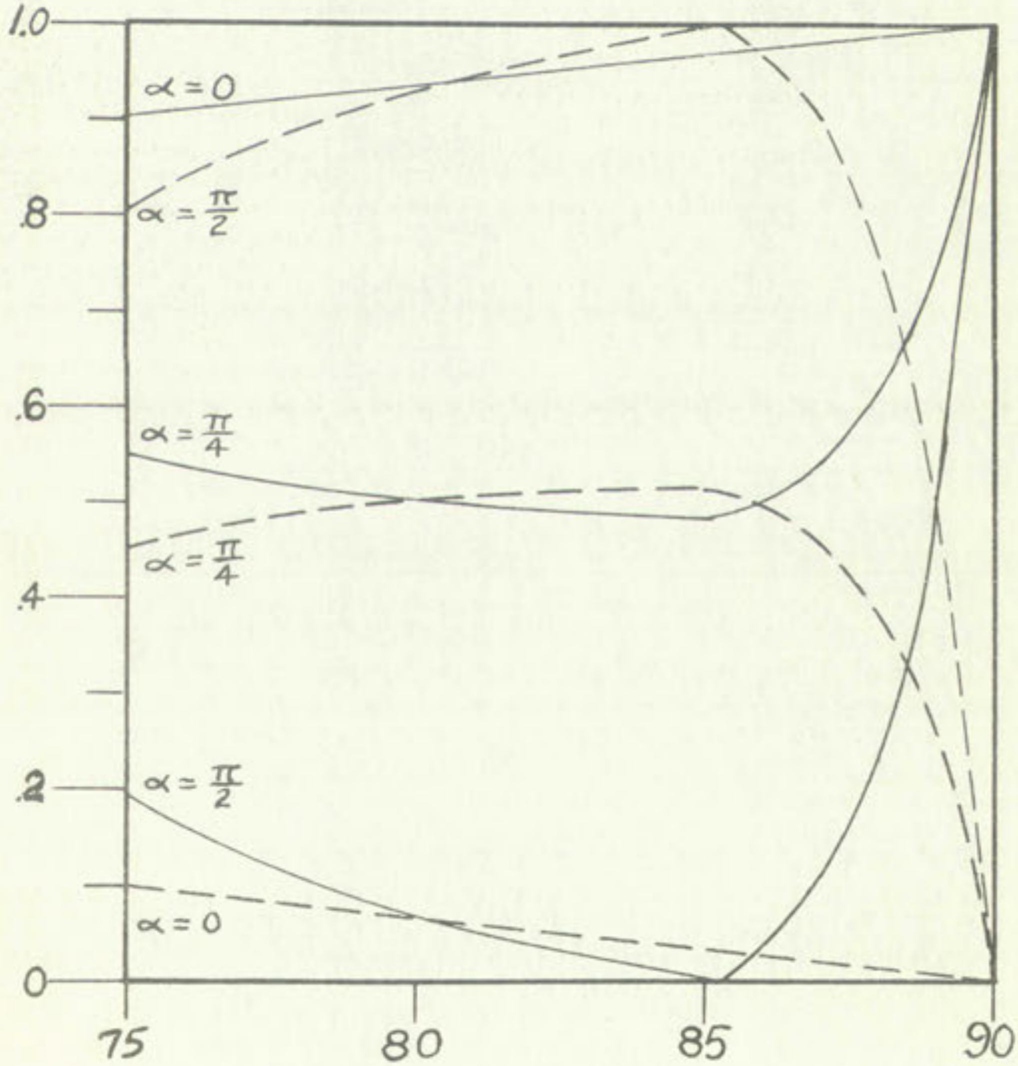
Angle of Incidence (θ_0) (degrees)

FIGURE 6

Dependence of Γ and T on Angle of Incidence, With Polarization Angle of Incident Wave as Parameter, For the Case $\beta = 0$ (Part of Figure 5 Expanded).

Γ solid curves

T dashed curves



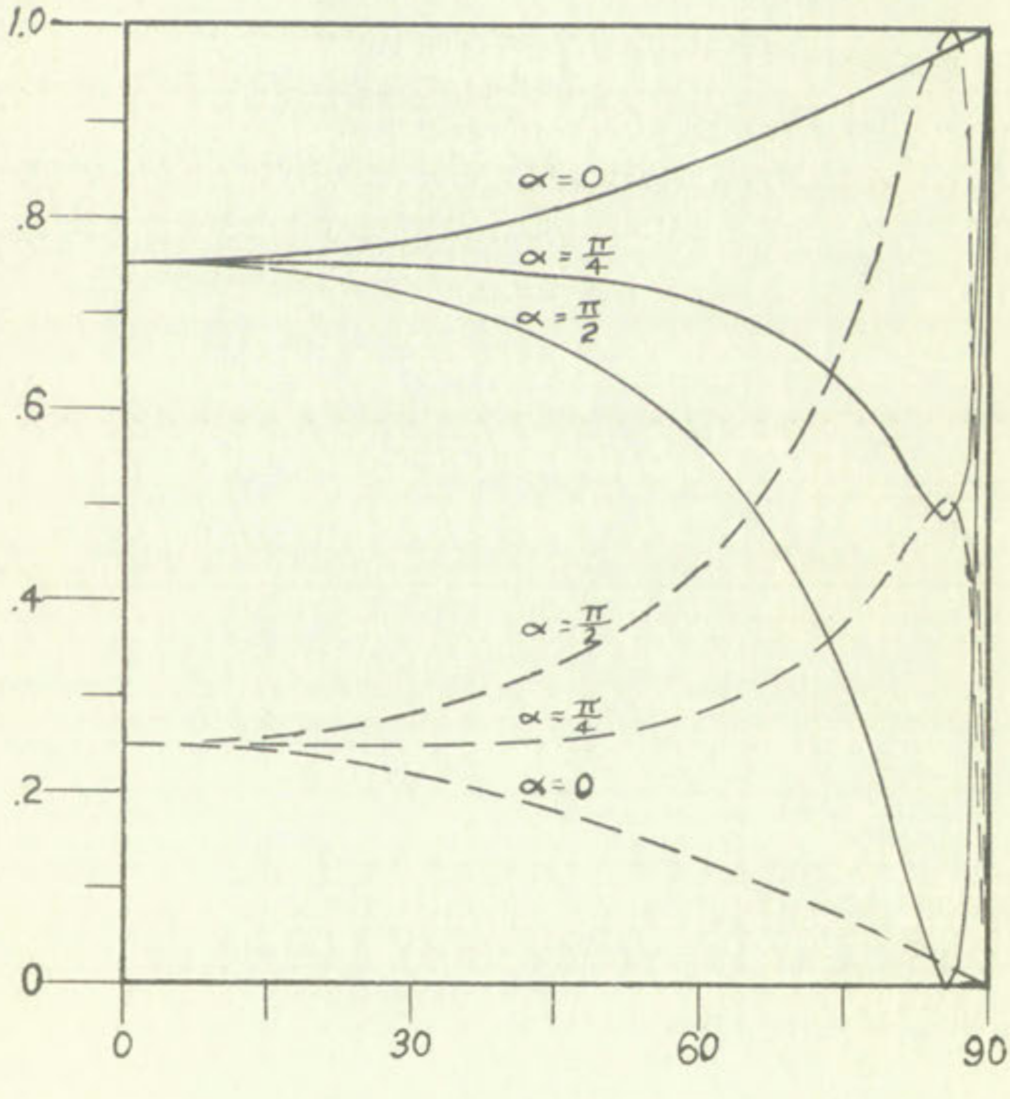
Angle of Incidence (θ_o) (degrees)

FIGURE 7

Dependence of Γ and T on Angle of Incidence,
With Polarization Angle of Incident Wave as Parameter,
For the Case $\beta = \frac{\pi}{4} = \gamma$.

Γ solid curves

T dashed curves



Angle of Incidence (θ_o) (degrees)

In addition to the data for the cases $\beta = 0$ and $\beta = \frac{\pi}{4} = \gamma$ (Figures 5, 6, 7), data was obtained for $\beta = \frac{\pi}{4}$ with $\gamma = 0, \frac{\pi}{2}$; since the curves hardly differed numerically from those of Figure 7, they are not shown. It seems that variation with azimuthal orientation of the static magnetic field is strictly second order compared to variation with the polar orientation.

For the case $\alpha_0 = 0$ the behavior is regular. Although the tangential component of polarization tends to produce an increase of transmission as θ_0 increases, the tangential component itself decreases so fast that the tendency is overcome. For $\alpha_0 = \frac{\pi}{4}$, the behavior is intermediate.

Worthy of mention also is the behavior of Γ and $\top$ with V/c. The value chosen for Figures 5, 6, 7 is V/c = .1 (small enough to avoid the need of a relativistic analysis, yet large enough to provide considerable transmission). Inspection of Equations 96, 98 reveals that in general $\top \rightarrow 0$ as $V/c \rightarrow 0$. However, provided V/c is not vanishingly small, the anomaly still exists for the case $\alpha_0 = \frac{\pi}{2}$. It appears that, as V/c decreases, the angular width of the transmission region decreases sharply, and the region moves toward $\theta_0 = \frac{\pi}{2}$. With V/c vanishingly small the "spike" is at $\theta_0 = \frac{\pi}{2}$, where of course there can be neither reflection nor transmission because the $\mathbf{h}_0$ wave is not actually incident on the boundary.

In addition to the [illegible] $\alpha_0 = 0$ and [illegible] (Figures 5, 6, 7), [illegible] since the curves hardly [illegible] are 7, they are not [illegible] orientation of the [illegible] compared to variation with the [illegible].

For the case α_0 = [illegible] the tangential component [illegible] trends to [illegible] crease of [illegible] θ_0 [illegible] itself [illegible] $\alpha_0 = \frac{\pi}{4}$, the behavior is [illegible].

[illegible] with γ/a. The [illegible] (small enough to avoid the [illegible] large enough to [illegible] of Equations [illegible] However, provided γ/a is not [illegible] exists for the case α = [illegible] creases, the angular [illegible] sharply, and the [illegible] ingly small [illegible] be neither [illegible] not actually [illegible] on the [illegible].

ACKNOWLEDGMENT

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APPENDIX

DESIGNATION OF SYMBOLS

$\mathcal{a}$	vector field defined by Equations 15, 16
$\mathbf{A}$	vector amplitude of magnetic field
A	magnitude of $\mathbf{A}$
$\mathbf{b}$	solenoidal part of $\mathcal{a}$
$\mathbf{B}$	vector amplitude of $\mathbf{b}$
B	magnitude of $\mathbf{B}$
c	speed of light in medium A
d	skin depth of the ψ disturbance
$\mathbf{D}$	vector amplitude of electric field in medium B
d	total differential operator
e	base of natural logarithms
$\mathbf{E}$	electric field intensity
EM	electromagnetic
$\mathbf{h}$	time dependent magnetic field
$\mathbf{H}$	total magnetic field
$\mathcal{H}$	exponential of magnetic waves
i	$\sqrt{-1}$
$\hat{i}$	unit vector
$\mathcal{I}m$	imaginary part
k_3	$\sin \beta \cos \gamma$
$\mathbf{m}$	propagation vector of ψ disturbance
MHD	magnetohydrodynamic
$\hat{n}$	unit vector in direction of wave propagation

N	component of Alfvén velocity in direction of propagation
p	pressure of fluid
$\overline{P}$	Poynting vector
$\mathbf{q}$	fluid velocity
$\mathbf{Q}$	vector amplitude of damped part of $\mathbf{q}$ (Equation 78)
$\mathbf{r}$	position vector (with respect to a point on the boundary)
$\mathcal{R}e$	real part
$\hat{s}$	unit vector normal to arbitrary fixed surface
S	arbitrary fixed surface
t	time
$\mathcal{T}$	transmission coefficient
V	magnitude of Alfvén velocity of MHD wave
V	volume enclosed by S
W	energy
x	coordinate axis
y	coordinate axis
z	coordinate axis
α	polarization angle (Figures 2, 3, 4)
β	polar angle of orientation of $\mathbf{H}_o$
γ	azimuth angle of orientation of $\mathbf{H}_o$
Γ	reflection coefficient
δ	$\hat{n}_2 \cdot \mathbf{r}/N$
ϵ	$\mathbf{m} \cdot \mathbf{r}/N$
ϵ	electric permittivity of medium A
ζ	distance measured along $\mathbf{H}_o$

η	$\sqrt{\mu/\epsilon}$, characteristic impedance of medium A
ν	viscosity of fluid
μ	magnetic permeability
π	3.1416
$\mathbf{\Pi}$	vector amplitude of damped part of $\mathbf{E}$ (Equation 38)
ρ	density of fluid
σ	conductivity of fluid
τ	period of wave
φ	disturbance created by boundary (Equation 12)
φ_o	constant of integration
Φ	amplitude of φ-function
ψ	potential function leading to part of $\mathbf{a}$ (Equation 17)
Ψ	amplitude of ψ-function
ω	angular frequency of wave
Ω	vector amplitude of damped part of $\mathbf{h}_B$ (Equation 36)
∂	partial differential operator
∇	vector differential operator

Subscripts

A	pertaining to medium A
B	pertaining to medium B
i	imaginary part
r	real part
s	situated on the boundary

x	in the direction of the +x-axis
y	in the direction of the +y-axis
z	in the direction of the +z axis
o	original static condition before incident wave arrives
0	pertaining to the incident wave
1	pertaining to the reflected wave
2	pertaining to the refracted or transmitted wave

Miscellaneous

^	indicates a unit vector; other vectors are indicated by making dark the left side of a symbol.
*	complex conjugate
——	time-averaged quantity

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