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Conversion of Electromagnetic Waves into Magnetohydrodynamic Waves at the Interface of a Dielectric and Incompressible Fluid

Ralph L. Marsten

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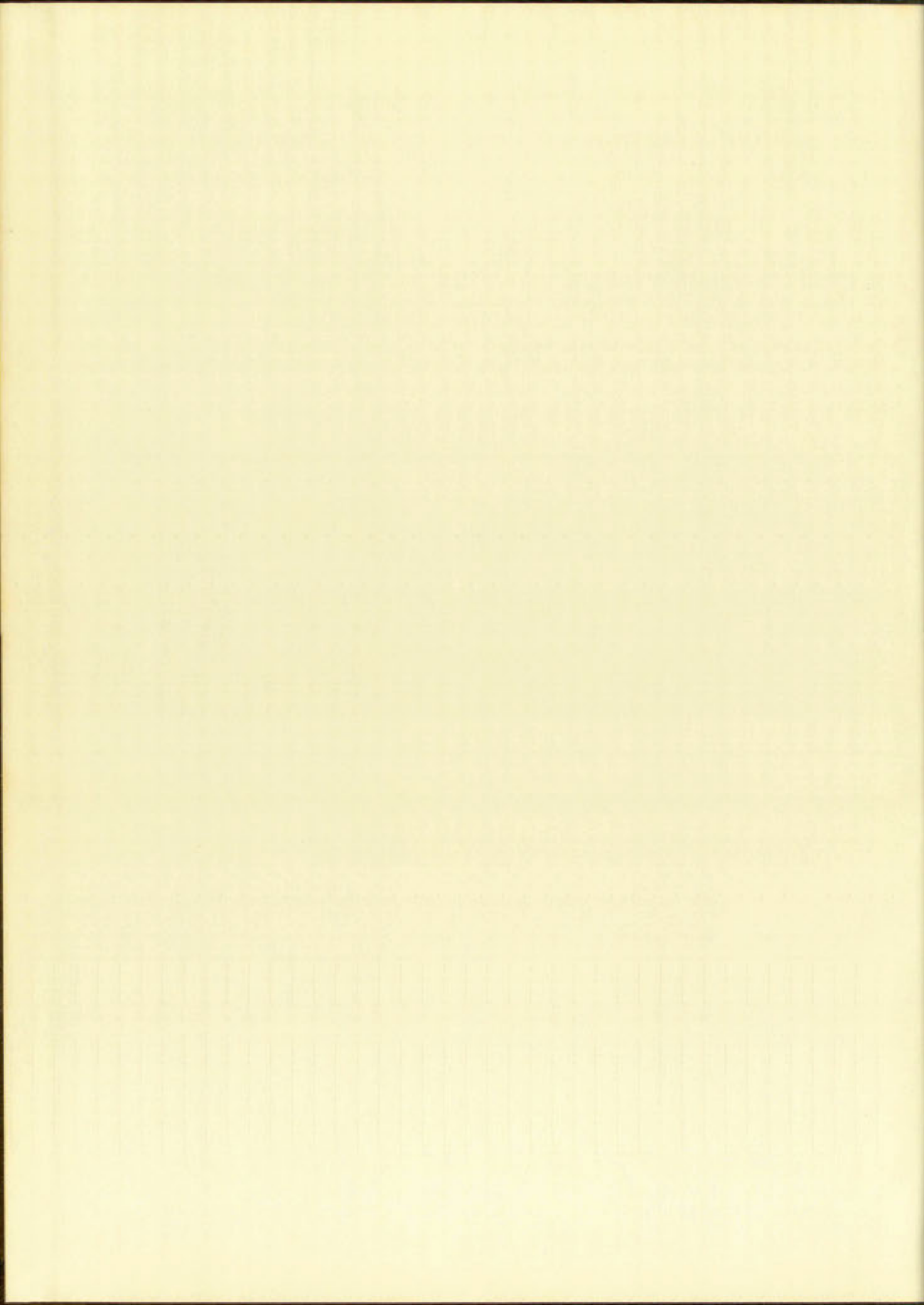
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CONVERSION OF ELECTROMAGNETIC WAVES INTO
MAGNETOHYDRODYNAMIC WAVES AT THE INTERFACE OF A
DIELECTRIC AND AN INCOMPRESSIBLE FLUID

By

Ralph L. Marston

A Thesis

Submitted in Partial Fulfillment of the
Requirements for the Degree of
Master of Science in Physics

The University of New Mexico

1959

WILFRED HALLS
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The University of the South
SOUTH AFRICA

Faculty of Science

Department of Chemistry

Submitted in partial fulfillment of

the requirements for the degree of

Bachelor of Science

This thesis, directed and approved by the candidate's committee, has been accepted by the Graduate Committee of the University of New Mexico in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

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DATE June 9, 1959

Thesis committee

Jack Katzenstein
CHAIRMAN
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The first thing I saw when I stepped out of the car was a man in a suit and tie, who had been waiting for me. He was a man of about 40, with dark hair and a friendly smile. He introduced himself as Mr. Jones, and said that he was the manager of the hotel. He told me that the room was ready for me, and that he would show me to it. I thanked him and followed him to the room. The room was on the second floor, and it was a nice, comfortable room. I was very tired, so I went to bed. I slept very well, and when I woke up in the morning, I felt refreshed. I had a good breakfast, and then I went out for a walk. The hotel was in a nice location, and the view was beautiful. I was very happy with my stay, and I would recommend it to anyone who is looking for a nice, comfortable place to stay.

ALBERT J. JONES

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The Problem

The New Situation

Solutions to the New Situation

The Field Committee

The Law of Reflection and Refraction

Reflection of Light

Refraction of Light

Optics of the Field

Energy in the Waves

Reflection and Transmission of Waves

Practical Applications of Optics

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THE PROBLEM

Consider space to be separated into two semi-infinite regions by a rigid boundary. On one side lies medium A -- a perfect, rigid, dielectric, with electric permittivity ϵ and magnetic permeability μ . On the other side is medium B; a perfect fluid, perfectly conducting, also with magnetic permeability μ . Throughout all space exists a uniform static magnetic field H_0 .

Now such a fluid as medium B is capable of supporting a magnetohydrodynamic (MHD) wave; and of course a perfect dielectric will support an electromagnetic (EM) wave. Suppose then, that a uniform plane EM wave in medium A is incident at arbitrary angle on the boundary. Will the EM wave generate an MHD wave at the boundary? If so, to what degree? In what manner will the interaction between the EM wave and the MHD fluid be dependent on the characteristics of the media and on the angle of incidence of the EM wave?

THE WAVE EQUATIONS

The essential characteristics of the media and fields involved may be listed as follows: (see Appendix for nomenclature)

Medium A

 μ ϵ $\sigma = 0$ $H(r, t)$ $E(r, t)$

Medium B

 μ $v = 0$ $\sigma = \infty$ $\rho = \text{constant}$ $p(r, t)$ $H(r, t)$ $g(r, t)$ $E(r, t)$

The EM field will set the fluid in motion at the boundary. This motion in turn will set up a magnetic field in the fluid. The velocity of the fluid g and the magnetic field intensity H_B are related by the magnetohydrodynamic equations,

$$(1) \quad \frac{\partial H_B}{\partial t} = \nabla \times (g \times H_B), \quad (2) \quad \frac{dg}{dt} = -\frac{\nabla p}{\rho} - \frac{\mu}{\rho} (\nabla \times H_B) \times H_B.$$

Since the fluid is perfect and the conductivity is infinite, we also have

$$(3) \quad E_B = -\mu g \times H_B, \quad (4) \quad \nabla \cdot g = 0.$$

The essential characteristics of the wave equation involved may be listed as follows: (see Figure 1 for notation)

Medium 1	Medium 2
μ	μ
ϵ	ϵ
$\sigma = 0$	$\sigma = 0$
$\rho = \text{constant}$	$\rho = \text{constant}$
$E(r, t)$	$E(r, t)$
$H(r, t)$	$H(r, t)$
$\Phi(r, t)$	$\Phi(r, t)$
$\Psi(r, t)$	$\Psi(r, t)$

The EM field and the fluid motion at the boundary. The motion in turn will set up a magnetic field in the fluid. The velocity of the fluid \mathbf{v} and the magnetic field \mathbf{H} are related by the magnetohydrodynamic equation,

$$\frac{\partial \mathbf{H}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{H}), \quad (1)$$

where the fluid is perfect and the conductivity is infinite, so also have

$$\mathbf{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad (2)$$

In medium A, Maxwell's equations become

$$(5) \quad \nabla \times E_A = -\mu \frac{\partial H_A}{\partial t}, \quad (6) \quad \nabla \times H_A = \epsilon \frac{\partial E_A}{\partial t},$$

$$(7) \quad \nabla \cdot E_A = 0.$$

In both media $\nabla \cdot H = 0.$

To linearize equations (1) and (2), we suppose that the time-varying fields are sufficiently less than the static field H_0 that we can neglect products and powers of \mathcal{G} and h .⁽¹⁾

Writing

$$H_B = H_0 + h_B \quad \text{and} \quad H_A = H_0 + h_A,$$

we obtain for medium B

$$(8) \quad \frac{\partial h_B}{\partial t} = (H_0 \cdot \nabla) \mathcal{G},$$

$$(9) \quad \frac{\partial \mathcal{G}}{\partial t} = -\frac{1}{\rho} \nabla (p + \mu H_0 \cdot h_B) + \frac{\mu}{\rho} (H_0 \cdot \nabla) h_B;$$

and for medium A

$$(10) \quad \nabla \times E_A = -\mu \frac{\partial h_A}{\partial t}, \quad (11) \quad \nabla \times h_A = \epsilon \frac{\partial E_A}{\partial t}.$$

To simplify writing, let

$$(12) \quad \varphi = \frac{1}{\rho} \left[p + \frac{1}{2} (H_0 + h_B)^2 \right];$$

In region 1, Maxwell's equations reduce

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \quad \nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} \quad (2)$$

$$\nabla \cdot \mathbf{E} = 0 \quad (3)$$

$$\nabla \cdot \mathbf{H} = 0 \quad (4)$$

To eliminate eqs. (3) and (4), we use the vector identity

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \nabla \times (\nabla \times \mathbf{A})$$

where \mathbf{A} is any vector. Then we can replace eqs. (3) and (4) by

$$\nabla^2 \mathbf{E} = \mu \frac{\partial^2 \mathbf{H}}{\partial t^2} \quad \text{and} \quad \nabla^2 \mathbf{H} = \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

we obtain for region 1

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\mu} \nabla^2 \mathbf{E} \quad (5)$$

$$\frac{\partial^2 \mathbf{H}}{\partial t^2} = \frac{1}{\epsilon} \nabla^2 \mathbf{H} \quad (6)$$

and for region 2

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \quad \nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} \quad (7)$$

To simplify writing, let

$$\mathbf{Q} = \frac{1}{2} [\mathbf{E} + \mathbf{H}] \quad (8)$$

then (9) becomes

$$(13) \quad \frac{\partial \mathcal{G}}{\partial t} = -\nabla \phi + \frac{\mu}{\rho} (H_0 \cdot \nabla) h_B.$$

On taking the divergence of this equation, it is seen that

$$(14) \quad \nabla^2 \phi = 0.$$

As a solution to Laplace's equation, ϕ must of course be everywhere finite. If there were no boundary, this would require that ϕ be a constant.⁽²⁾ However, the presence of the boundary relieves this restriction. Instead, we must have ϕ vanish far from the boundary, but $\nabla \phi \neq 0$ in general. The significance of this is that h_B does not satisfy a wave equation of the type

$$\frac{\partial^2 a}{\partial t^2} = \frac{\mu}{\rho} (H_0 \cdot \nabla)^2 a$$

except in special cases.⁽³⁾ The essence of the development from here through the next section is due to Roberts.⁽³⁾

To circumvent this difficulty, let us define a vector a by the equations

$$(15) \quad h_B = (H_0 \cdot \nabla) a, \quad (16) \quad \mathcal{G} = \frac{\partial a}{\partial t}.$$

Obviously equations (4) and (8) are satisfied by a . We may specify a by solenoidal and irrotational components

$$(17) \quad a = b + \nabla \psi,$$

then (9) becomes

$$(13) \quad \frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi + \frac{\partial}{\partial t} (H \cdot \nabla) \psi = 0$$

On taking the divergence of this equation, it is seen that

$$(14) \quad \nabla^2 \psi = 0$$

As a solution to Laplace's equation, ψ must be constant

at every point inside. It thus follows that ψ must be

constant throughout the region, and hence ψ must be

constant on the boundary. Hence, we may take $\psi = 0$

without loss of generality, and $\nabla^2 \psi = 0$ becomes

the wave equation for ψ , and ψ must be constant on the

boundary.

$$\frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi = 0$$

except in special cases. (15) The solution of this equation

is given by the next section in the case of a sphere.

To determine the solution, we assume a form

for the solution

$$(16) \quad \psi = (H \cdot \nabla) \psi + \frac{\partial^2 \psi}{\partial t^2}$$

Obviously equations (15) and (16) are satisfied if

equation (16) is satisfied and ψ is constant on the

$$(17) \quad \psi = \psi + \nabla^2 \psi$$

where \mathbf{b} is defined by

$$(18) \quad \nabla \cdot \mathbf{b} = 0, \quad (19) \quad \frac{\partial^2 \mathbf{b}}{\partial t^2} = V^2 \frac{\partial^2 \mathbf{b}}{\partial \zeta^2},$$

where ζ is measured along \mathbf{H}_0 , and

$$(20) \quad V^2 = \frac{\mu}{\rho} H_0^2 \text{ is the Alfvén velocity.}$$

With these definitions, we must have

$$(21) \quad \nabla^2 \psi = 0.$$

Equations (19) and (21) constitute wave equations for the vector \mathbf{a} . However it remains to show that \mathbf{a} , as defined by equations (15), (16), (17), satisfies equation (9).

For medium A we have the usual electromagnetic wave equation

$$(22) \quad \frac{\partial^2 \mathbf{h}_A}{\partial t^2} = c^2 \nabla^2 \mathbf{h}_A, \text{ where } c^2 = \frac{1}{\mu \epsilon}.$$

where \mathbf{b} is defined by

$$\nabla \cdot \mathbf{b} = 0. \quad (18)$$

$$\frac{\partial \mathbf{b}}{\partial t} + \nabla \times \mathbf{a} = 0. \quad (19)$$

where \mathbf{a} is a vector along \mathbf{H}_0 , and

$$\mathbf{a} = \frac{1}{H_0} \nabla \times \mathbf{H}_0. \quad (20)$$

With these definitions, we can write

$$\nabla^2 \psi = 0. \quad (21)$$

Equations (17) and (21) constitute a boundary value problem for

the vector \mathbf{Q} . However, it remains to show that \mathbf{Q} is a vector.

By operations (15), (16), (17), and (21), we obtain

For definiteness we take the curl of (21) and obtain

Equation

$$\frac{1}{H_0} \nabla^2 \nabla \times \mathbf{H}_0 = \frac{1}{H_0} \nabla^2 \nabla \times \mathbf{H}_0. \quad (22)$$

SOLUTIONS TO THE WAVE EQUATIONS

As solutions to (19) and (21) we may take

$$(23) \quad \mathbf{b} = B e^{i\omega(t - \frac{\hat{n}_2 \cdot \mathbf{r}}{N})},$$

$$(24) \quad \psi = \Psi e^{i\omega(t - \frac{\mathbf{m} \cdot \mathbf{r}}{N})},$$

where \hat{n}_2 is a unit vector in the direction of wave propagation, \mathbf{m} is a constant complex vector of zero length, and where $N = v |\hat{\lambda}_2 \cdot \hat{n}_2|$ is the component of wave velocity in the direction of H_0 , $\hat{\lambda}_2$ being a unit vector in that direction. That \mathbf{m} must have zero magnitude may be seen by substituting ψ in (21). Combining (23) and (24) according to (17),

$$(25) \quad \mathbf{a} = B e^{i\omega(t - \frac{\hat{n}_2 \cdot \mathbf{r}}{N})} - i \frac{\omega}{N} \Psi \mathbf{m} e^{i\omega(t - \frac{\mathbf{m} \cdot \mathbf{r}}{N})}.$$

On substitution of (17), using (15) and (16), equation (13) becomes

$$\frac{\partial^2 \mathbf{b}}{\partial t^2} + \frac{\partial^2}{\partial t^2} (\nabla \psi) = v^2 \frac{\partial^2 \mathbf{b}}{\partial s^2} + v^2 \frac{\partial^2}{\partial s^2} (\nabla \psi) - \nabla \varphi.$$

Using (19) leaves $\nabla \left(\frac{\partial^2 \psi}{\partial t^2} - v^2 \frac{\partial^2 \psi}{\partial s^2} + \varphi \right) = 0$

or,

$$(26) \quad \frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial s^2} - \varphi + \varphi_0,$$

As solutions to (25) and (26) we suppose

$$\phi = B e^{i\omega(t - \frac{\hat{r}_1 \cdot \mathbf{r}}{N})} \quad (27)$$

$$\psi = \bar{B} e^{i\omega(t - \frac{\hat{r}_2 \cdot \mathbf{r}}{N})} \quad (28)$$

where \hat{r}_1 is a unit vector in the direction of wave propagation,

we have a constant complex vector of unit length, and where

$N = \sqrt{\frac{1}{2}(\hat{r}_1^2 + \hat{r}_2^2)}$ is the component of wave velocity in the direction

of \hat{r}_1 , $\frac{1}{2}$ being a unit vector in the direction. Then we

must have some signposts and be seen by the relation $\hat{r}_1 \cdot \hat{r}_2 = 0$.

Combining (27) and (28) according to (25)

$$\alpha = B e^{i\omega(t - \frac{\hat{r}_1 \cdot \mathbf{r}}{N})} - i \frac{\omega}{N} \bar{B} e^{i\omega(t - \frac{\hat{r}_2 \cdot \mathbf{r}}{N})} \quad (29)$$

on substitution of (29) into (26), we obtain

(30) becomes

$$\frac{\partial^2 \alpha}{\partial t^2} - \nabla^2 \alpha = \frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi = 0$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \nabla^2 \alpha = \frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi = 0 \quad (31)$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \nabla^2 \alpha = \frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi = 0 \quad (32)$$

where φ_0 is an arbitrary constant. As a solution to (14), φ may be written

$$(27) \quad \varphi = \varphi_0 + \Phi e^{i\omega(t - \frac{\mathbf{m} \cdot \mathbf{r}}{N})},$$

where φ_0 , being arbitrary, is the same as in (26). Substituting (24) and (27) in (26) reveals

$$(28) \quad \Phi = \omega^2 \left[1 - \left(\frac{\mathbf{m} \cdot \hat{\mathbf{r}}}{\hat{n}_2 \cdot \hat{\mathbf{r}}} \right)^2 \right] \Psi$$

for consistency.

For medium A we take as the solution of equation (22)

$$(29) \quad h_A = A_0 e^{i\omega(t - \frac{\hat{n}_0 \cdot \mathbf{r}}{c})} + A_1 e^{i\omega(t - \frac{\hat{n}_1 \cdot \mathbf{r}}{c})},$$

where \hat{n}_0 and \hat{n}_1 are unit vectors in the directions of propagation of the incident and reflected waves respectively.

where Φ_0 is an arbitrary constant, Φ_0 is a function of ω and ω may be written

$$(27) \quad \Phi = \Phi_0 + \Phi_1 \omega + \Phi_2 \omega^2 + \dots$$

where Φ_0 , being arbitrary, is the same as in (26). Substituting (27) in (26) we obtain

$$(28) \quad \Phi = \Phi_0 + \Phi_1 \omega + \Phi_2 \omega^2 + \dots$$

For convenience,

for medium 1 we take as the incident wave

$$(29) \quad A_1 = A_0 \left(\frac{1}{2} + \frac{1}{2} \frac{v_1}{v_2} \right) e^{i(k_1 x - \omega t)}$$

where \hat{n}_1 and \hat{n}_2 are unit vectors in the direction of propagation of the incident and reflected waves respectively.

THE FIELD QUANTITIES

Equations (25) and (29) are the fundamental solutions for their respective media; from them may be obtained all the field quantities of interest.

To obtain the magnetic field in medium B we use (15); obtaining

$$(30) \quad \mathbf{h}_B = -i \frac{\omega}{N} (\mathbf{H}_0 \cdot \hat{\mathbf{n}}_2) \mathbf{B} e^{i\omega(t - \frac{\hat{\mathbf{n}}_2 \cdot \mathbf{r}}{N})} - \frac{\omega^2}{N^2} \Psi \mathbf{m} (\mathbf{H}_0 \cdot \mathbf{m}) e^{i\omega(t - \frac{\mathbf{m} \cdot \mathbf{r}}{N})}.$$

Similarly for the velocity field; from (16),

$$(31) \quad \mathbf{q} = i\omega \mathbf{B} e^{i\omega(t - \frac{\hat{\mathbf{n}}_2 \cdot \mathbf{r}}{N})} + \frac{\omega^2}{N} \Psi \mathbf{m} e^{i\omega(t - \frac{\mathbf{m} \cdot \mathbf{r}}{N})}.$$

The electric field in medium B may be obtained from equation (3), which under the perturbation assumption becomes $\mathbf{E}_B = \mu \mathbf{H}_0 \times \mathbf{q}$.

We have

$$(32) \quad \mathbf{E}_B = i\mu\omega (\mathbf{H}_0 \times \mathbf{B}) e^{i\omega(t - \frac{\hat{\mathbf{n}}_2 \cdot \mathbf{r}}{N})} + \frac{\mu\omega^2}{N} \Psi (\mathbf{H}_0 \times \mathbf{m}) e^{i\omega(t - \frac{\mathbf{m} \cdot \mathbf{r}}{N})}.$$

To find the electric field in medium A one uses equation (11), assuming the usual time dependence for \mathbf{E}_A . First we

Equations (17) and (18) are the generalized equations

for their respective fields. From these we obtain (19) and

field quantities of interest.

To obtain the average field intensity we assume (20)

obtaining

$$(20) \quad E = -\lambda \frac{W}{H} (H - h) E \quad \text{and} \quad \frac{H - h}{H} = \frac{h}{H}$$

$$= \frac{1}{2} \sqrt{1 - \left(\frac{h}{H} \right)^2}$$

Similarly for the electric field, from (18),

$$(21) \quad \phi = \lambda W B \quad \text{and} \quad \frac{H - h}{H} = \frac{h}{H}$$

The electric field is uniform in region 2 and is directed from region 1 to

region 2. The potential difference between regions 1 and 2 is

We have

$$(22) \quad E = -\lambda W (H - h) \quad \text{and} \quad \frac{H - h}{H} = \frac{h}{H}$$

TO FIND THE ELECTRIC FIELD IN REGION 1, WE ASSUME

$$\frac{H - h}{H} = \frac{h}{H}$$

To find the electric field in region 1, we assume

from (11), assuming the field lines are parallel to the z -axis we

write

$$\nabla \times (\mathbf{A}_0 \mathcal{H}_0 + \mathbf{A}_1 \mathcal{H}_1) = i\epsilon\omega \mathbf{E}_A,$$

where

$$\mathcal{H}_0 = e^{i\omega(t - \frac{\hat{n}_0 \cdot \mathbf{r}}{c})}, \quad \mathcal{H}_1 = e^{i\omega(t - \frac{\hat{n}_1 \cdot \mathbf{r}}{c})}.$$

Expanding the curl, $-\mathbf{A}_0 \times \nabla \mathcal{H}_0 - \mathbf{A}_1 \times \nabla \mathcal{H}_1 = i\epsilon\omega \mathbf{E}_A$.

Since $\nabla \mathcal{H} = -i\frac{\omega}{c} \hat{n} \mathcal{H}$, we have

$$\mathbf{A}_0 \times \hat{n}_0 \mathcal{H}_0 + \mathbf{A}_1 \times \hat{n}_1 \mathcal{H}_1 = c\epsilon \mathbf{E}_A,$$

or

$$(33) \quad \mathbf{E}_A = -\eta (\hat{n}_0 \times \mathbf{h}_0 + \hat{n}_1 \times \mathbf{h}_1), \text{ where } \eta = \sqrt{\frac{\mu}{\epsilon}},$$

and where \mathbf{h}_0 and \mathbf{h}_1 are the magnetic fields of the incident and reflected waves respectively.

Summarizing, we have

Medium A

$$(29) \quad \mathbf{h}_A = \mathbf{A}_0 e^{i\omega(t - \frac{\hat{n}_0 \cdot \mathbf{r}}{c})} + \mathbf{A}_1 e^{i\omega(t - \frac{\hat{n}_1 \cdot \mathbf{r}}{c})}$$

$$(34) \quad \mathbf{E}_A = -\eta \hat{n}_0 \times \mathbf{A}_0 e^{i\omega(t - \frac{\hat{n}_0 \cdot \mathbf{r}}{c})} - \eta \hat{n}_1 \times \mathbf{A}_1 e^{i\omega(t - \frac{\hat{n}_1 \cdot \mathbf{r}}{c})}$$

Medium B

$$(35) \quad \mathbf{h}_B = \mathbf{A}_2 e^{i\omega(t - \frac{\hat{n}_2 \cdot \mathbf{r}}{N})} + \mathbf{\Omega} e^{i\omega(t - \frac{\mathbf{m} \cdot \mathbf{r}}{N})},$$

write
where

$$\nabla \times (A_0 \hat{r}) = A_0 \nabla \times \hat{r}$$

$$\hat{r} = e^{-i\omega t - i\frac{\hat{r} \cdot \nabla}{c}} \hat{r} = e^{-i\omega t - i\frac{\hat{r} \cdot \nabla}{c}} \hat{r}$$

$$\text{Separating the ansatz, } -A_0 \nabla \times \hat{r} = A_0 \nabla \times \hat{r}$$

$$\text{Since } \nabla \times \hat{r} = -\frac{\omega}{c} \hat{r} \times \hat{r} = 0$$

$$A_0 \nabla \times \hat{r} = A_0 \nabla \times \hat{r}$$

or

$$(32) \quad E_A = -\gamma(\hat{r} \times \hat{r} + \gamma \hat{r} \times \hat{r})$$

and where \hat{r} and \hat{r} are the direction of the incident and reflected waves respectively.

Consequently, we have

where

$$(33) \quad \hat{r} = A_0 e^{-i\omega t - i\frac{\hat{r} \cdot \nabla}{c}} \hat{r}$$

$$(34) \quad E_A = -\gamma \hat{r} \times A_0 e^{-i\omega t - i\frac{\hat{r} \cdot \nabla}{c}} \hat{r}$$

where

$$(35) \quad \hat{r} = A_0 e^{-i\omega t - i\frac{\hat{r} \cdot \nabla}{c}} \hat{r}$$

where

$$(36) \quad A_2 = -i \frac{\omega}{N} (H_0 \cdot \hat{n}_2) B, \quad \Omega = -\frac{\omega^2}{N^2} (H_0 \cdot m) \mathcal{V} m$$

$$(31) \quad \mathcal{G} = i\omega B e^{i\omega(t - \frac{\hat{n}_2 \cdot \mathbf{r}}{N})} + \frac{\omega^2}{N} \mathcal{V} m e^{i\omega(t - \frac{m \cdot \mathbf{r}}{N})}$$

$$(37) \quad E_B = D e^{i\omega(t - \frac{\hat{n}_2 \cdot \mathbf{r}}{N})} + \Pi e^{i\omega(t - \frac{m \cdot \mathbf{r}}{N})},$$

where

$$(38) \quad D = -\mu N \frac{H_0 \times A_2}{H_0 \cdot \hat{n}_2}, \quad \Pi = \frac{\mu \omega^2}{N} \mathcal{V} H_0 \times m.$$

THE LAWS OF REFLECTION AND REFRACTION

The coefficients in equations (29) and (35) are constant vector amplitudes, to be evaluated at the boundary. Since these must bear some certain relationship to one another independent of time and position, the exponents must be equal at any given time and place on the boundary. Thus

$$\frac{\hat{n}_0 \cdot \mathbf{r}_s}{c} = \frac{\hat{n}_1 \cdot \mathbf{r}_s}{c} = \frac{\hat{n}_2 \cdot \mathbf{r}_s}{N} = \frac{m \cdot \mathbf{r}_s}{N},$$

where \mathbf{r}_s is a point on the boundary surface. Defining a unit vector normal to the boundary by $\hat{\lambda}_z \cdot \mathbf{r}_s = 0$, one can easily show that $\mathbf{r}_s = -\hat{\lambda}_z \times (\hat{\lambda}_z \times \mathbf{r}_s)$. Hence, substituting and manipulating the triple scalar products

$$\hat{\lambda}_z \times \mathbf{r}_s \cdot \frac{\hat{n}_0 \times \hat{\lambda}_z}{c} = \hat{\lambda}_z \times \mathbf{r}_s \cdot \frac{\hat{n}_1 \times \hat{\lambda}_z}{c} = \hat{\lambda}_z \times \mathbf{r}_s \cdot \frac{\hat{n}_2 \times \hat{\lambda}_z}{N} = \hat{\lambda}_z \times \mathbf{r}_s \cdot \frac{m \times \hat{\lambda}_z}{N},$$

or

$$(39) \quad \frac{\hat{\lambda}_z \times \hat{n}_0}{c} = \frac{\hat{\lambda}_z \times \hat{n}_1}{c} = \frac{\hat{\lambda}_z \times \hat{n}_2}{N} = \frac{\hat{\lambda}_z \times \mathbf{m}}{N}.$$

Since each of the cross-products in (39) has the same direction, they must all be normal to the same plane. This plane, containing $\hat{n}_0, \hat{n}_1, \hat{n}_2, \hat{\lambda}_z$, may be called the plane of incidence, in analogy to optics. The first two equalities yield the laws of reflection and refraction respectively. The subscripts 0, 1, 2 refer to the incident, reflected, refracted waves, in that order.

Since the normal to the boundary, $\hat{\lambda}_z$, lies in the plane of incidence, this plane is perpendicular to the boundary. Then the line of intersection of the plane of incidence with the boundary, and the normal to the plane of incidence, together with $\hat{\lambda}_z$, form a convenient mutually perpendicular triad, as shown in Figure 1.

In terms of this coordinate system,

$$(40) \quad \hat{n}_0 = -\hat{\lambda}_x \sin \theta_0 - \hat{\lambda}_z \cos \theta_0$$

$$(41) \quad \hat{n}_1 = -\hat{\lambda}_x \sin \theta_1 + \hat{\lambda}_z \cos \theta_1$$

$$(42) \quad \hat{n}_2 = -\hat{\lambda}_x \sin \theta_2 - \hat{\lambda}_z \cos \theta_2$$

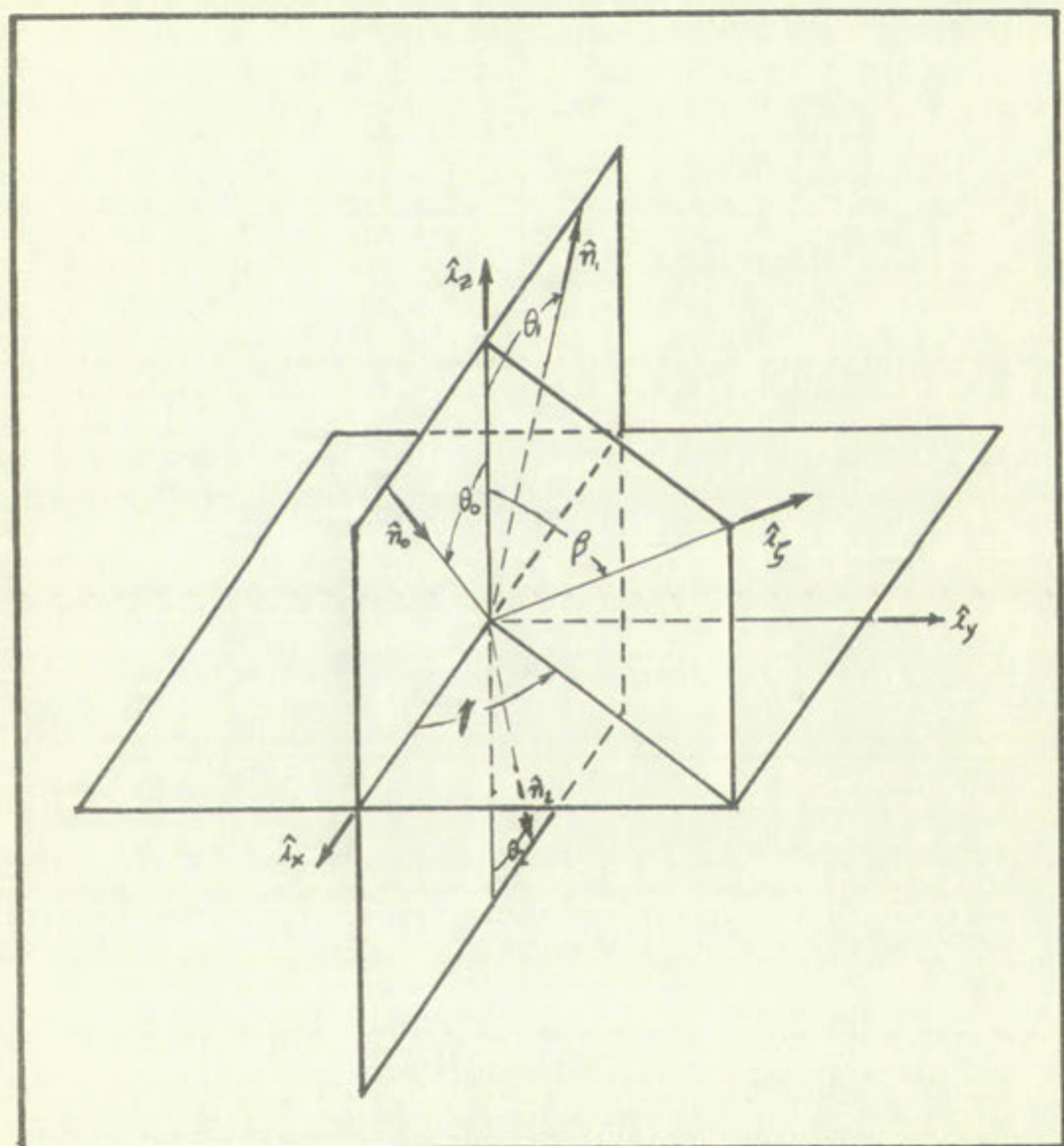
$$(43) \quad \hat{\lambda}_z = \hat{\lambda}_x \sin \beta \cos \tau + \hat{\lambda}_y \sin \beta \sin \tau + \hat{\lambda}_z \cos \beta.$$

With these relations,

$$(44) \quad N = V (\sin \beta \cos \tau \sin \theta_2 + \cos \beta \cos \theta_2) = -V (\hat{n}_2 \cdot \hat{\lambda}_z),$$

FIGURE 1

Orthogonal Coordinate System



The x-y plane is the boundary.

The x-z plane is the plane of incidence.

The angles β and γ fix the direction of \hat{n}_1 .

for β and τ in the first quadrants. If β or τ were in any other quadrant, the absolute value brackets would have to be retained, because the refracted wave must go in the $-Z$ direction as does the incident wave.

The first equality of (39) yields

$$(45) \quad \theta_1 = \theta_0, \text{ the law of reflection.}$$

The second equality yields

$$-\hat{\lambda}_y \sin \theta_0 = -\frac{c}{V} \frac{\hat{\lambda}_y \sin \theta_2}{\sin \beta \cos \tau \sin \theta_2 + \cos \beta \cos \theta_2},$$

or

$$(46) \quad \tan \theta_2 = \frac{\frac{V}{c} \sin \theta_0 \cos \beta}{1 - \frac{V}{c} \sin \beta \cos \tau \sin \theta_0},$$

the law of refraction.

Now this development has been non-relativistic; hence we must consider $V^2 \ll c^2$. With this in mind, (46) gives evidence that θ_2 is quite small, regardless of the orientation of H_0 or the angle of incidence.

for θ and γ in the first quadrant, θ is the angle between the incident ray and the normal, and γ is the angle between the reflected ray and the normal. The angle of incidence is θ and the angle of reflection is γ . The angle of refraction is θ' and the angle of total internal reflection is γ' .

The first equality is the law of reflection.

$$(1) \quad \theta = \gamma, \quad \text{the law of reflection.}$$

The second equality is the law of refraction.

$$- \lambda_1 \sin \theta = - \lambda_2 \sin \theta' \quad \text{or} \quad \lambda_1 \sin \theta = \lambda_2 \sin \theta'$$

or

$$(2) \quad \sin \theta = \frac{\lambda_2}{\lambda_1} \sin \theta' = \frac{v_2}{v_1} \sin \theta'$$

the law of refraction.

Now this theorem can be proved by using the principle of least time. We must consider θ and θ' as variables. The angle of incidence θ is given, and the angle of refraction θ' is given. The angle of incidence θ is given, and the angle of refraction θ' is given. The angle of incidence θ is given, and the angle of refraction θ' is given.

EVALUATION OF m

ψ as given by (24) must vanish at distances sufficiently far from the boundary. Hence m must be complex in order to represent ψ as a damped wave. In addition, to satisfy Laplace's equation (21), the magnitude of m must be zero. A third condition is obtained from the third equality of (39). We will show that these conditions, together with a fourth to be introduced presently, are sufficient to determine m uniquely.

Let us write $m = m_r + i m_i$. Then if we put

$$(m_r + i m_i) \cdot (m_r + i m_i) = 0,$$

the first two conditions will be satisfied. Since

$$m_r \cdot m_r - m_i \cdot m_i + 2 i m_r \cdot m_i = 0, \text{ we have}$$

$$(a) \quad |m_r| = |m_i|, \text{ and (b) } m_r \cdot m_i = 0$$

The third condition requires $\hat{\lambda}_z \times m = \hat{\lambda}_z \times \hat{n}_2$, or

$$\hat{\lambda}_z \times (m_r + i m_i) = \hat{\lambda}_z \times (-\hat{\lambda}_x \sin \theta_2 - \hat{\lambda}_z \cos \theta_2)$$

from (43). This breaks down to

$$(c) \quad \hat{\lambda}_z \times m_r = -\hat{\lambda}_y \sin \theta_2, \text{ and (d) } \hat{\lambda}_z \times m_i = 0.$$

The y-component of (c) is

$$(e) \quad \hat{\lambda}_x \cdot m_r = -\sin \theta_2.$$

(e) gives the x-component of m_r . From (c), m_r

cannot have a y-component. From (d), m_i can only have a z-component. It follows from (b) that m_r cannot have a z-component.

Hence

$$(f) \quad m_r = -\hat{\lambda}_x \sin \theta_2 .$$

From (a) and (d) then

$$(g) \quad m_i = \hat{\lambda}_z \sin \theta_2 ,$$

$$\text{and } m = -\hat{\lambda}_x \sin \theta_2 + i \hat{\lambda}_z \sin \theta_2 .$$

The fourth condition is that the sign of the imaginary part be chosen so that ψ is damped as it travels in the $-z$ direction. We consider the exponent of ψ , writing

$$\operatorname{Re} i\omega(t - \frac{m \cdot r}{N}) < 0 .$$

Realizing that $N > 0$, letting r become $-\hat{\lambda}_z |z|$, we have

$$-i(+i \hat{\lambda}_z \sin \theta_2)(-\hat{\lambda}_z |z|) < 0 ,$$

resulting in

$$(47) \quad m = -\hat{\lambda}_x \sin \theta_2 + i \hat{\lambda}_z \sin \theta_2 .$$

It is easily shown that this is equivalent to

$$(48) \quad m = -\hat{\lambda}_z \times (\hat{\lambda}_z \times \hat{n}_2) - i \hat{\lambda}_x \times (\hat{\lambda}_z \times \hat{n}_2) .$$

cannot have a γ -component. From (7), (8) and (9) it follows

that, it follows from (5) that γ must have a γ -component.

From

$$(1) \quad \gamma_1 = -\hat{\lambda}_1 \sin \theta_1$$

from (a) and (b) from

$$(2) \quad \gamma_2 = \hat{\lambda}_2 \sin \theta_2$$

$$\text{and } \gamma_3 = -\hat{\lambda}_3 \sin \theta_3 = \hat{\lambda}_2 \sin \theta_2$$

The fourth condition is that the right-hand side of (10)

must be chosen so that γ is a vector in the plane of the

direction. We consider the case of γ in the

$$\text{plane } \lambda(\omega) \left(\frac{\pi}{2} - \theta \right) < 0$$

Realizing that $\theta > 0$ is chosen to be $\theta = \theta_1$, we get

$$-\lambda(\omega) \hat{\lambda}_2 \sin \theta_2 - \hat{\lambda}_1 \sin \theta_1 < 0$$

resulting in

$$(11) \quad \gamma_1 = -\hat{\lambda}_1 \sin \theta_1 + \hat{\lambda}_2 \sin \theta_2$$

It is easily seen that this is equivalent to

$$(12) \quad \gamma_1 = -\hat{\lambda}_2 \sin \theta_2 + \hat{\lambda}_1 \sin \theta_1$$

BOUNDARY CONDITIONS

After Roberts (3) we take the following:

$$(49) \quad \hat{\lambda}_z \times \Delta h = 0, \quad \Delta h = h_A - h_B$$

$$(50) \quad \hat{\lambda}_z \times \Delta E = 0, \quad \Delta E = E_A - E_B$$

$$(51) \quad \hat{\lambda}_z \cdot \mathcal{G} = 0.$$

AMPLITUDES OF THE FIELDS

When the expressions for the fields, equations (29) through (34), are substituted in the boundary conditions, the exponentials cancel. Sufficient equations for determining the coefficients A_i , B , Ψ are available. Using (44) on (30) and substituting, we obtain

$$(52) \quad \hat{\lambda}_x \cdot A_0 + \hat{\lambda}_x \cdot A_1 = i \frac{\omega H_0}{V} \hat{\lambda}_x \cdot B - \frac{\omega^2 H_0}{N^2} (m \cdot \hat{\lambda}_z) \Psi (\hat{\lambda}_x \cdot m)$$

$$(53) \quad \hat{\lambda}_y \cdot A_0 + \hat{\lambda}_y \cdot A_1 = i \frac{\omega H_0}{V} \hat{\lambda}_y \cdot B$$

$$(54) \quad -\eta \hat{\lambda}_x \cdot \hat{n}_0 \times A_0 - \eta \hat{\lambda}_x \cdot \hat{n}_1 \times A_1 = i \mu \omega \hat{\lambda}_x \cdot H_0 \times B + \frac{\mu \omega^2}{N} \Psi \hat{\lambda}_x \cdot H_0 \times m$$

$$(55) \quad -\eta \hat{\lambda}_y \cdot \hat{n}_0 \times A_0 - \eta \hat{\lambda}_y \cdot \hat{n}_1 \times A_1 = i \mu \omega \hat{\lambda}_y \cdot H_0 \times B + \frac{\mu \omega^2}{N} \Psi \hat{\lambda}_y \cdot H_0 \times m$$

$$(56) \quad i \omega \hat{\lambda}_z \cdot B + \frac{\omega^2}{N} \Psi \hat{\lambda}_z \cdot m = 0.$$

After solving (2) for \hat{A} and \hat{B}

(19)

$$\hat{A} \times \Delta A = 0$$

(20)

$$\hat{B} \times \Delta B = 0$$

(21)

$$\hat{A} \cdot \hat{B} = 0$$

METHODS OF THE PAPER

When the expressions for the fields, \hat{A} and \hat{B} , are substituted in the equations of motion, the exponential terms, $\exp(i\omega t)$, cancel out. The remaining equations are then solved for the coefficients \hat{A} , \hat{B} , and \hat{C} . The results are then substituted in the expressions for the fields, \hat{A} and \hat{B} , and the final results are obtained.

(22)

$$\hat{A} \times \Delta A = \hat{B} \times \Delta B = \hat{C} \times \Delta C = 0$$

(23)

$$\hat{A} \cdot \hat{B} = \hat{A} \cdot \hat{C} = \hat{B} \cdot \hat{C} = 0$$

(24)

$$\hat{A} \times \Delta A = \hat{B} \times \Delta B = \hat{C} \times \Delta C = 0$$

(25)

$$\hat{A} \cdot \hat{B} = \hat{A} \cdot \hat{C} = \hat{B} \cdot \hat{C} = 0$$

(26)

$$\hat{A} \times \Delta A = \hat{B} \times \Delta B = \hat{C} \times \Delta C = 0$$

The divergence conditions show that the polarization vectors A_0 , A_1 , B , are normal respectively to \hat{n}_0 , \hat{n}_1 , \hat{n}_2 . Therefore the planes of polarization of the incident, reflected, and refracted waves make angles θ_0 , θ_1 , θ_2 respectively with the boundary, as shown in Figures 2, 3, 4. Referred to the plane of incidence, the polarization angles are respectively α_0 , α_1 , α_2 . From the figures

$$(57) \quad A_0 = A_0 (\hat{x} \cos \alpha_0 \cos \theta_0 + \hat{y} \sin \alpha_0 - \hat{z} \cos \alpha_0 \sin \theta_0)$$

$$(58) \quad A_1 = A_1 (\hat{x} \cos \alpha_1 \cos \theta_1 + \hat{y} \sin \alpha_1 + \hat{z} \cos \alpha_1 \sin \theta_1)$$

$$(59) \quad B = B (\hat{x} \cos \alpha_2 \cos \theta_2 + \hat{y} \sin \alpha_2 - \hat{z} \cos \alpha_2 \sin \theta_2).$$

Note that if the polarization of the reflected wave is reversed in direction, so that $\pi < \alpha_1 < \frac{3\pi}{2}$, equation (58) still properly represents A_1 .

Using equations (40) through (43) and (57), (58), (59) to expand (52) through (56) we obtain

$$(60) \quad A_1 \sin \alpha_1 - i \frac{\omega H_0}{V} B \sin \alpha_2 = -A_0 \sin \alpha_0$$

$$(61) \quad A_1 \cos \alpha_1 \cos \theta_1 - i \frac{\omega H_0}{V} \cos \theta_2 B \cos \alpha_2 + \frac{\omega^2 H_0}{N^2} \mathcal{V} \sin^2 \theta_2 (\sin \beta \cos \tau - i \cos \beta) = -A_0 \cos \alpha_0 \cos \theta_0$$

$$(62) \quad \eta A_1 \cos \alpha_1 + i \frac{\mu \omega H_0 N}{V} B \cos \alpha_2 - i \frac{\mu \omega^2 H_0}{N} \mathcal{V} \sin \theta_2 (\sin \beta \cos \tau - i \cos \beta) = \eta A_0 \cos \alpha_0$$

The divergence condition shows that the polarization vectors A_0, A_1, B , are linearly independent. Therefore the planes of polarization of the incident, reflected, and refracted waves must differ. It is with this boundary, as shown in Figure 1, that the plane of incidence, the polarization of the incident wave, α_0 , and the plane of reflection, α_1 , are the same.

$$\begin{aligned} (27) \quad A_0 &= A_0 (\hat{x} \cos \alpha_0 \cos \theta_0 + \hat{y} \sin \alpha_0 \cos \theta_0 + \hat{z} \sin \theta_0) \\ (28) \quad A_1 &= A_1 (\hat{x} \cos \alpha_1 \cos \theta_1 + \hat{y} \sin \alpha_1 \cos \theta_1 + \hat{z} \sin \theta_1) \\ (29) \quad B &= B (\hat{x} \cos \alpha_2 \cos \theta_2 + \hat{y} \sin \alpha_2 \cos \theta_2 + \hat{z} \sin \theta_2) \end{aligned}$$

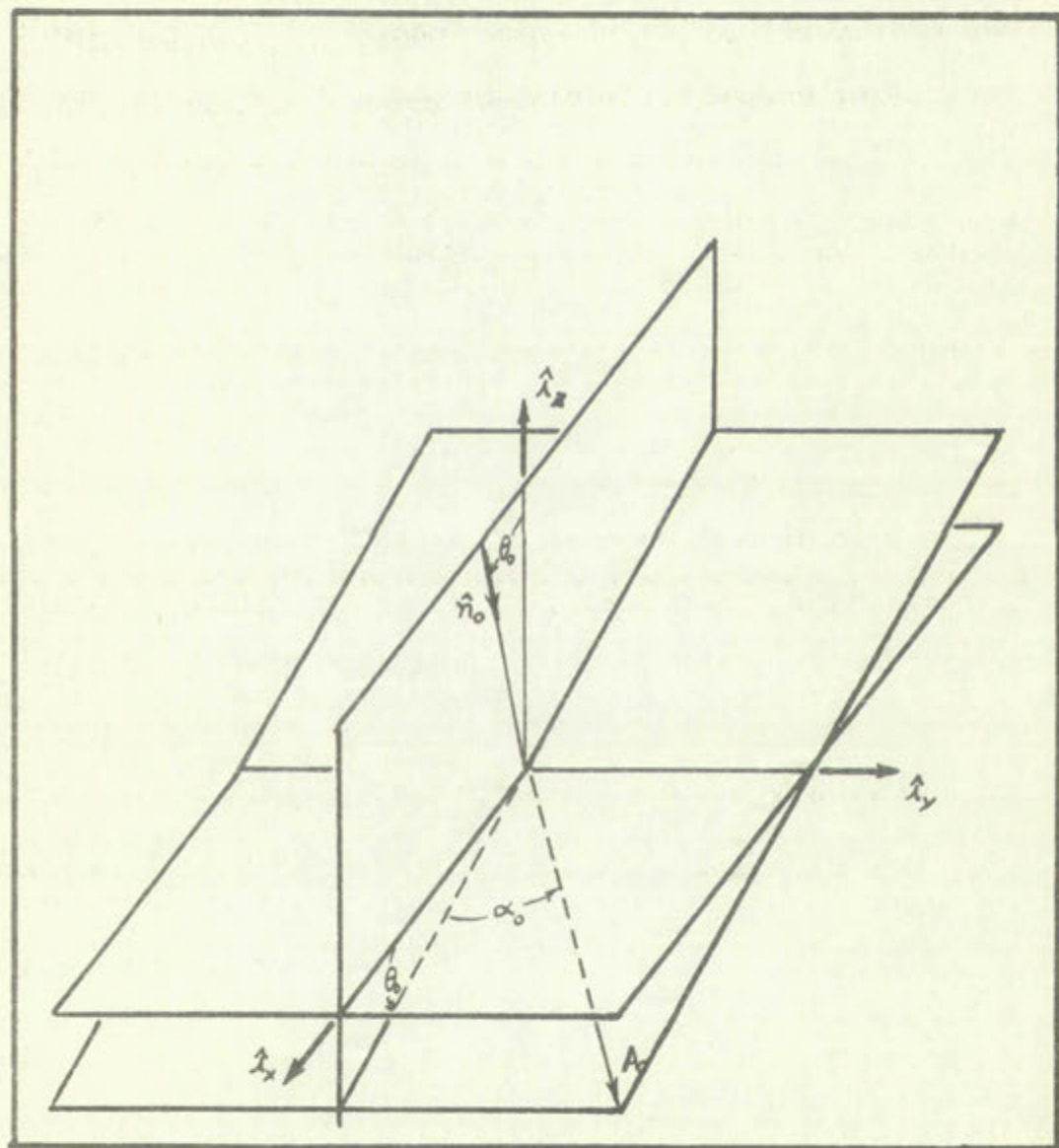
Note that in the polarization of the reflected wave, α_1 , is in the plane of incidence, so that $\pi < \alpha_1 < \frac{3\pi}{2}$, whereas α_0 is in the plane of reflection, α_0 .

Using equation (10) through (13) and (27) through (29) to expand (25) through (28) we obtain

$$\begin{aligned} (30) \quad A_0 \sin \alpha_0 &= \frac{1}{\sqrt{1 - \epsilon_2}} B \sin \alpha_2 \cos \theta_2 \\ (31) \quad A_1 \cos \alpha_1 &= \frac{1}{\sqrt{1 - \epsilon_2}} B \cos \alpha_2 \cos \theta_2 \\ (32) \quad A_0 \sin \alpha_0 &= \frac{1}{\sqrt{1 - \epsilon_2}} B \sin \alpha_2 \cos \theta_2 + \frac{1}{\sqrt{1 - \epsilon_2}} B \sin \alpha_2 \sin \theta_2 \end{aligned}$$

FIGURE 2

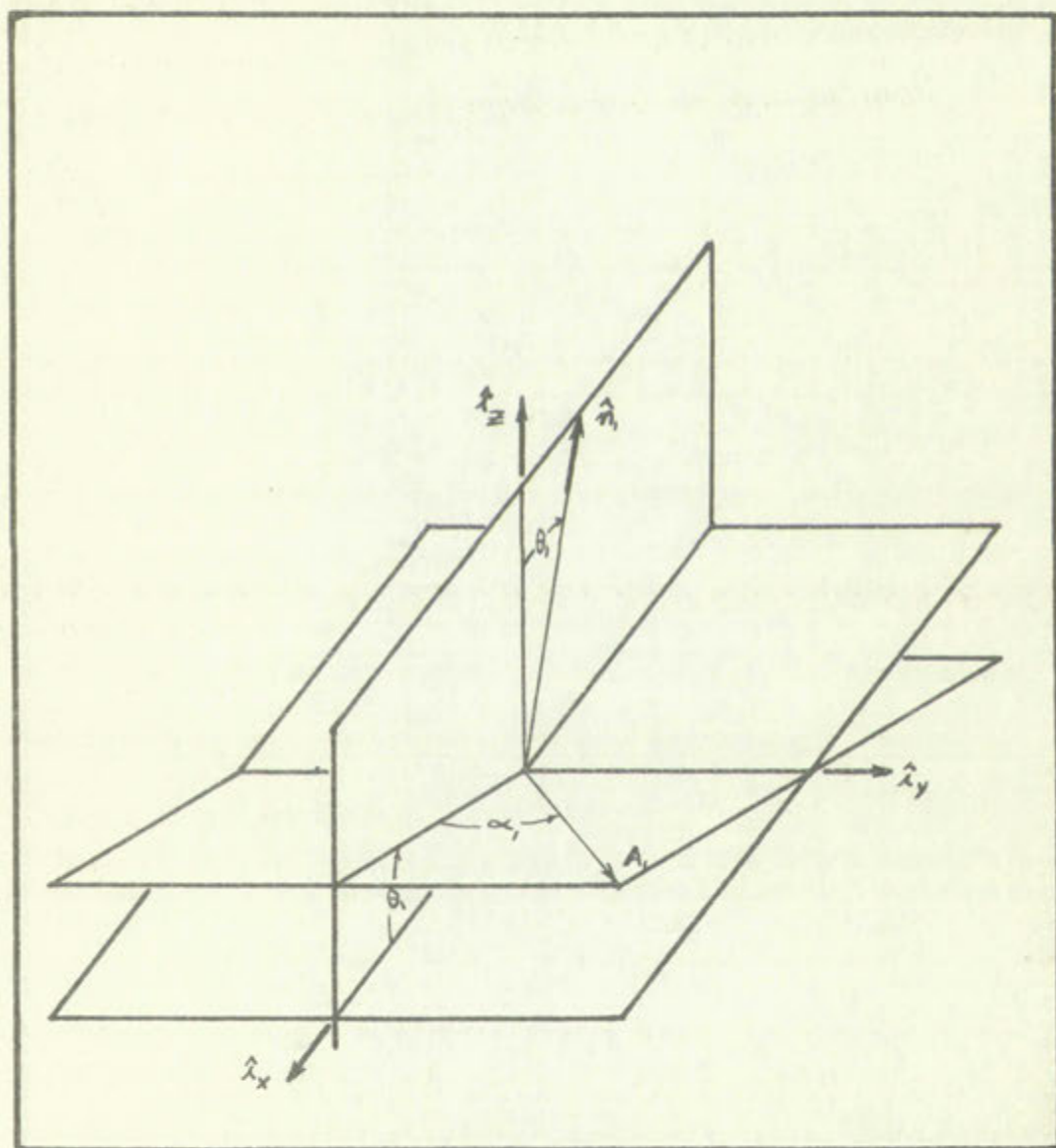
Polarization of Incident Wave



The plane containing \hat{x}_y and A_0 , normal to \hat{n}_0 , is the plane of the incident wave front.

FIGURE 3

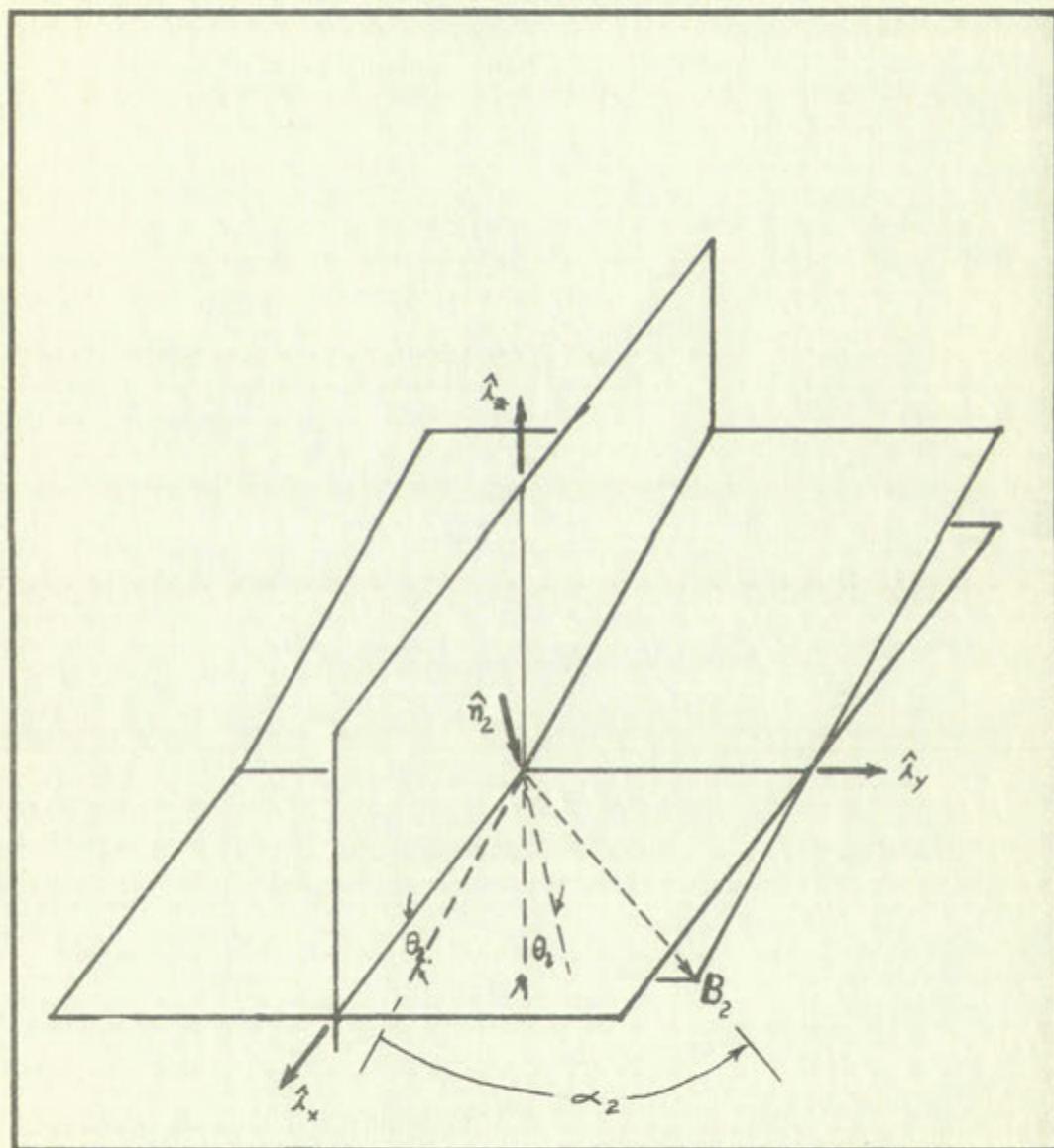
Polarization of Reflected Wave



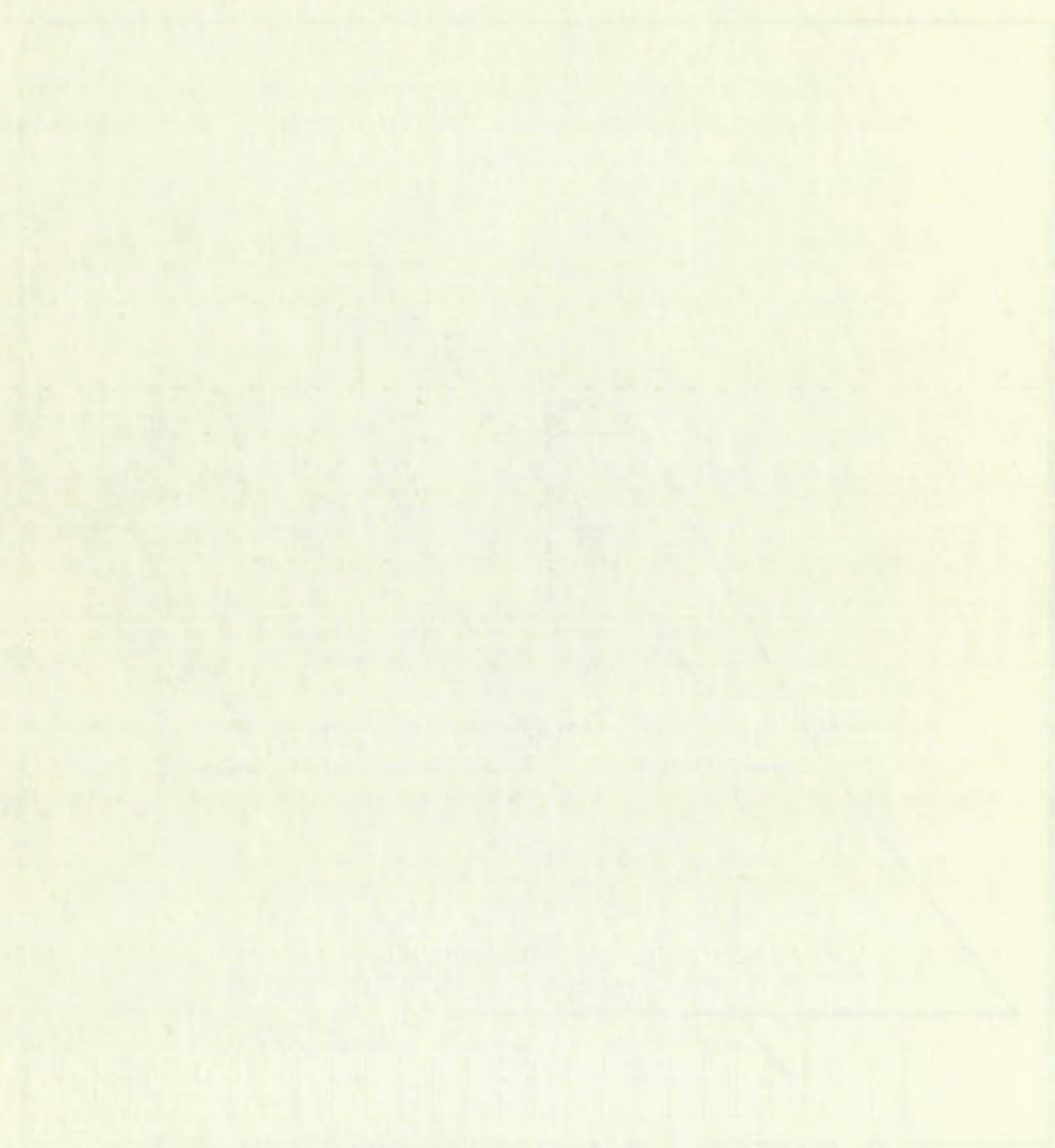
The plane containing \hat{y} and A_i , normal to \hat{n}_i , is the plane of the reflected wave front.

FIGURE 4

Polarization of Refracted Wave



The plane containing $\hat{\lambda}_y$ and \mathbf{B}_2 , normal to \hat{n}_2 , is the plane of the refracted wave front.



$$\begin{aligned}
 (63) \quad & \eta A_1 \sin \alpha_1 \cos \theta_1 + i \mu \omega H_0 \cos \beta B \sin \alpha_2 \\
 & + i \mu \omega H_0 \sin \beta \sin \gamma \sin \theta_2 B \cos \alpha_2 \\
 & - i \frac{\mu \omega^2 H_0}{N} \Psi \sin \beta \sin \gamma \sin \theta_2 = \eta A_0 \sin \alpha_0 \cos \theta_0
 \end{aligned}$$

$$(64) \quad B \cos \alpha_2 - \frac{\omega}{N} \Psi = 0.$$

Since $V^2 \ll c^2$, $\sin^2 \theta_2 \ll 1$. With this simplification, the determinant of equations (60) through (64) is easier to solve for the polarization components $A_1 \sin \alpha_1$, $A_1 \cos \alpha_1$, $B \sin \alpha_2$, $B \cos \alpha_2$, and Ψ . Making use of the relation $\eta = \mu c$, substituting for the angles θ_1 and θ_2 from (45) and (46), we may write the solutions in the following form, where

$$D = \frac{\eta^2 \omega^3 H_0^2}{V^3 \cos \beta} \left(\frac{V}{c} \cos \beta + \cos \theta_0 \right) \left[1 - \frac{V}{c} k_3 \sin \theta_0 + \frac{V}{c} \cos \beta \cos \theta_0 \right]$$

is the denominator determinant.

$$D A_1 \sin \alpha_1 = - \frac{\eta^2 \omega^3 H_0^2}{V^3 \cos \beta} \left(\frac{V}{c} \cos \beta - \cos \theta_0 \right) \left[1 - \frac{V}{c} k_3 \sin \theta_0 + \frac{V}{c} \cos \beta \cos \theta_0 \right] A_0 \sin \alpha_0$$

$$D A_1 \cos \alpha_1 = \frac{\eta^2 \omega^3 H_0^2}{V^3 \cos \beta} \left(\frac{V}{c} \cos \beta + \cos \theta_0 \right) \left[1 - \frac{V}{c} k_3 \sin \theta_0 - \frac{V}{c} \cos \beta \cos \theta_0 \right] A_0 \cos \alpha_0$$

$$D B \sin \alpha_2 = 2i \frac{\eta^2 \omega^2 H_0}{V^2 \cos \beta} \left[1 - \frac{V}{c} k_3 \sin \theta_0 + \frac{V}{c} \cos \beta \cos \theta_0 \right] A_0 \sin \alpha_0 \cos \theta_0$$

$$D B \cos \alpha_2 = 2i \frac{\eta^2 \omega^2 H_0}{V^2 \cos \beta} \left(\frac{V}{c} \cos \beta + \cos \theta_0 \right) \left[1 - 2k_3 \frac{V}{c} \sin \theta_0 \right]^{\frac{1}{2}} A_0 \cos \alpha_0 \cos \theta_0$$

$$D \Psi = -2i \frac{\eta^2 \omega H_0}{V} \left(\frac{V}{c} \cos \beta + \cos \theta_0 \right) A_0 \cos \alpha_0 \cos \theta_0,$$

$$(2) \quad \nabla A = 0 \quad \text{where } \nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

$$\nabla H = 0 \quad \text{where } H = \frac{1}{2} \omega^2 x^2 + \frac{1}{2} \omega^2 y^2$$

$$\nabla H = 0 \quad \text{where } H = \frac{1}{2} \omega^2 x^2 + \frac{1}{2} \omega^2 y^2$$

$$(3) \quad B = 0 \quad \text{where } B = \frac{1}{2} \omega^2 x^2 + \frac{1}{2} \omega^2 y^2$$

$$\text{Since } \nabla^2 < 0, \text{ the point } (0,0) \text{ is a local maximum.}$$

From the above, it is clear that the point (0,0) is a local maximum.

Since for the above, the point (0,0) is a local maximum.

B is the point (0,0) and the point (0,0) is a local maximum.

$\nabla^2 H < 0$, indicating for the point (0,0) that it is a local maximum.

we may write the above as the following:

$$D = \frac{1}{2} \omega^2 x^2 + \frac{1}{2} \omega^2 y^2 \quad \text{where } \omega = \frac{1}{2} \sqrt{\frac{1}{m}}$$

is the same as the above.

$$D A = \frac{1}{2} \omega^2 x^2 + \frac{1}{2} \omega^2 y^2 \quad \text{where } \omega = \frac{1}{2} \sqrt{\frac{1}{m}}$$

$$D A = \frac{1}{2} \omega^2 x^2 + \frac{1}{2} \omega^2 y^2 \quad \text{where } \omega = \frac{1}{2} \sqrt{\frac{1}{m}}$$

$$D B = \frac{1}{2} \omega^2 x^2 + \frac{1}{2} \omega^2 y^2 \quad \text{where } \omega = \frac{1}{2} \sqrt{\frac{1}{m}}$$

$$D B = \frac{1}{2} \omega^2 x^2 + \frac{1}{2} \omega^2 y^2 \quad \text{where } \omega = \frac{1}{2} \sqrt{\frac{1}{m}}$$

$$D \psi = \frac{1}{2} \omega^2 x^2 + \frac{1}{2} \omega^2 y^2 \quad \text{where } \omega = \frac{1}{2} \sqrt{\frac{1}{m}}$$

where $k_3 = \sin \beta \cos \gamma$.

Substituting for B from equation (36), one obtains by eliminating D

$$(65) \quad A_1 \sin \alpha_1 = -A_0 \sin \alpha_0 \frac{\frac{V}{C} \cos \beta - \cos \theta_0}{\frac{V}{C} \cos \beta + \cos \theta_0}$$

$$(66) \quad A_1 \cos \alpha_1 = A_0 \cos \alpha_0 \frac{1 - \frac{V}{C} k_3 \sin \theta_0 - \frac{V}{C} \cos \beta \cos \theta_0}{1 - \frac{V}{C} k_3 \sin \theta_0 + \frac{V}{C} \cos \beta \cos \theta_0}$$

$$(67) \quad A_2 \sin \alpha_2 = A_0 \sin \alpha_0 \frac{2 \cos \theta_0}{\frac{V}{C} \cos \beta + \cos \theta_0}$$

$$(68) \quad A_2 \cos \alpha_2 = A_0 \cos \alpha_0 \frac{2 \cos \theta_0 \left[1 - 2 \frac{V}{C} k_3 \sin \theta_0 \right]^{1/2}}{1 - \frac{V}{C} k_3 \sin \theta_0 + \frac{V}{C} \cos \beta \cos \theta_0}$$

$$(69) \quad \Psi = -A_0 \cos \alpha_0 \frac{2i \frac{V^2}{\omega^2 H_0} \cos \beta \cos \theta_0}{1 - \frac{V}{C} k_3 \sin \theta_0 + \frac{V}{C} \cos \beta \cos \theta_0}$$

It is possible to simplify these expressions slightly by specifying the direction of H_0 . If we let $\beta = 0$, the polarization angles are given by

$$\tan \alpha_1 = \tan \alpha_0 \frac{\cos \theta_0 - \frac{V}{C} \sin^2 \theta_0}{\cos \theta_0 + \frac{V}{C} \sin^2 \theta_0}$$

$$\tan \alpha_2 = \tan \alpha_0 \frac{1 + \frac{V}{C} \cos \theta_0}{\frac{V}{C} + \cos \theta_0}$$

Thus the polarization angles are not simply related to that of the incident wave. Only if we consider normal incidence do we

$$\text{where } \beta = \sin \delta \cos \lambda$$

Substituting for β from equation (20), and

eliminating λ

$$\frac{A_1 \sin \alpha_1 - \frac{V}{C} \cos \delta}{A_2 \sin \alpha_2 - \frac{V}{C} \cos \delta} = \frac{A_1 \sin \alpha_1}{A_2 \sin \alpha_2} \quad (21)$$

$$\frac{A_1 \cos \alpha_1 - \frac{V}{C} \sin \delta}{A_2 \cos \alpha_2 - \frac{V}{C} \sin \delta} = \frac{A_1 \cos \alpha_1}{A_2 \cos \alpha_2} \quad (22)$$

$$\frac{A_1 \sin \alpha_1}{A_2 \sin \alpha_2} = \frac{A_1 \cos \alpha_1}{A_2 \cos \alpha_2} \quad (23)$$

$$\frac{A_1 \sin \alpha_1}{A_2 \sin \alpha_2} = \frac{A_1 \cos \alpha_1}{A_2 \cos \alpha_2} \quad (24)$$

$$\frac{A_1 \sin \alpha_1}{A_2 \sin \alpha_2} = \frac{A_1 \cos \alpha_1}{A_2 \cos \alpha_2} \quad (25)$$

It is possible to simplify these equations by

by specifying the direction of H_0 . If we let $\delta = 0$, the

incident angles are given by

$$\begin{aligned} \tan \alpha_1 &= \frac{\cos \theta_0 - \frac{V}{C} \sin \theta_0}{\cos \theta_0 + \frac{V}{C} \sin \theta_0} \\ \tan \alpha_2 &= \frac{1 + \frac{V}{C} \sin \theta_0}{\frac{V}{C} + \cos \theta_0} \end{aligned}$$

Thus the polarization angles are not strictly related to θ_0 .

the incident wave. Only if we consider waves incident at $\theta_0 = 0$

get something simple;

$$\tan \alpha_2 = \tan \alpha_1 = \tan \alpha_0$$

regardless of the orientation of the static magnetic field.

The coefficient Ω may be obtained from equations (36) and (69);

$$(70) \quad \Omega = -A_0 \cos \alpha_0 \frac{V}{c} \frac{\sin 2\theta_0 \left[1 - 2 \frac{V}{c} k_3 \sin \theta_0 \right]^{1/2}}{1 - \frac{V}{c} k_3 \sin \theta_0 + \frac{V}{c} \cos \beta \cos \theta_0} (\cos \beta + i \sin \beta \cos t).$$

Note that $\Omega = 0$ for normal incidence.

Perhaps more significant than the amplitude Ω is the skin depth of the ψ disturbance. This depends on the real part of the exponent of the ψ function.

$$\frac{\omega}{N} \hat{\lambda}_x \times (\hat{\lambda}_z \times \hat{n}_2) \cdot \hat{\lambda}_z |\neq|.$$

$$\text{It is given by } d = -\frac{N}{\omega} \left[\hat{\lambda}_z \cdot \hat{\lambda}_x \times (\hat{\lambda}_z \times \hat{n}_2) \right]^{-1}.$$

Manipulating the vectors results in

$$d = \frac{N}{\omega \sin \theta_2}.$$

Substituting from (44) and (46) yields

$$(71) \quad d = \frac{c}{\omega} \csc \theta_0.$$

Note that the skin depth is independent of the magnitude and direction of H_0 , and becomes infinite as $\theta_0 \rightarrow 0$. Thus, for oblique incidence, Ω may be large but quickly damped; while

and assuming

$$\tan \alpha_2 = \tan \alpha_1$$

regardless of the orientation of the unit vectors \mathbf{e}_1 and \mathbf{e}_2 .

The coefficient Ω can be obtained from equation (10)

and (9):

$$(10) \quad \Omega = -A_{\text{osc}} \frac{\frac{1}{2} \sin \alpha_1 \left(1 - \frac{1}{2} \sin \alpha_1 \right)}{\left(1 - \frac{1}{2} \sin \alpha_1 \right) \frac{1}{2} \sin \alpha_1}$$

Note that $\Omega = 0$ for $\alpha_1 = 0$ and $\alpha_1 = \pi$.

Perhaps the simplest form of the expression for Ω is

when α_1 is the angle of the \mathbf{e}_1 vector. It is assumed that \mathbf{e}_2 is

of the order of the \mathbf{e}_1 vector.

$$\frac{1}{2} \sin \alpha_1 \left(1 - \frac{1}{2} \sin \alpha_1 \right)$$

$$\text{It is given by } \frac{1}{2} \left[1 - \frac{1}{2} \sin \alpha_1 \right]$$

Manipulating and using the identity

$$b = \frac{1}{2} \sin \alpha_1$$

Substituting from (11) and (12) into (10)

$$(11) \quad b = \frac{1}{2} \sin \alpha_1$$

Note that the value of b is independent of the orientation of the

direction of \mathbf{e}_1 and \mathbf{e}_2 and hence is a scalar.

Substituting Ω and b into equation (9) yields

for nearly normal incidence, Ω is small but slowly damped.

ENERGY IN THE WAVES

Consider a fixed volume V enclosed by a surface S in an electromagnetic medium. If the volume contains a source of EM energy W , then

$$(72) \quad \frac{\partial W}{\partial t} = \int_S \mathbf{E} \times \mathbf{H} \cdot \hat{\mathbf{s}} d\sigma$$

is the rate at which energy leaves the volume ($\hat{\mathbf{s}}$ being the outward normal to the surface)..

We seek an expression for the energy flow per unit area at a point. Transforming (72) to a volume integral, it is evident that only that part of the integrand whose divergence is NOT zero will contribute to the energy flow. Hence one cannot in general simply use the integrand of (72) to represent the energy flow at a point. In addition, if complex notation is used for field quantities, we must consider only the real parts to obtain the actual energy flow. Finally, since we want the flow under equilibrium conditions, we must obtain the time average of (72).

In medium A the total magnetic field is $H_A = H_0 + h_A$,

for nearly normal incidence, $\Omega \approx \omega$, the energy density

$$W = \frac{1}{2} \epsilon_0 E^2$$

Consider a thin volume V within a medium in an electromagnetic field. The volume contains a charge Q . In energy W , then

$$(12) \quad \frac{\partial W}{\partial t} = \int_V E \cdot \frac{\partial D}{\partial t} dV$$

is the rate at which energy leaves the volume V (plus an outward normal to the surface).

We seek an expression for the energy W in terms of the rate of change of the field. Transformation (11) to a frame moving with velocity v relative to the medium shows that the energy density W is not zero but is proportional to v^2 . This is the energy in general simply due to the motion of the medium. The energy flux S is defined in analogy with the energy flux used for electromagnetic waves. We have written S in terms of E and D to obtain the energy flux. Finally, there is a term due to the rate of change of the field, $\partial D / \partial t$, in the energy flux S .

In medium at rest the energy density is $W = \frac{1}{2} \epsilon_0 E^2$.

where H_0 is the static field, and h_A is the wave field.

Substituting in (72), and taking the real part,

$$\operatorname{Re} \frac{\partial W}{\partial t} = \int_S \operatorname{Re} E_A \times \operatorname{Re} h_A \cdot \hat{s} d\sigma + \int_S \operatorname{Re} E_A \times \operatorname{Re} H_0 \cdot \hat{s} d\sigma.$$

The time average of this may be written

$$(73) \quad \overline{\operatorname{Re} \frac{\partial W}{\partial t}} = \frac{1}{\tau} \int_0^\tau \int_S \operatorname{Re} E_A \times \operatorname{Re} h_A \cdot \hat{s} d\sigma dt \\ + \int_S \frac{1}{\tau} \int_0^\tau \operatorname{Re} E_A \times \operatorname{Re} H_0 \cdot \hat{s} dt d\sigma,$$

where τ is the period of the wave. Since the surface is fixed in space we can interchange order of integration. Now H_0 is real and constant, so the second integral is

$$- \int_S \frac{1}{\tau} H_0 \times \int_0^\tau \operatorname{Re} E_A dt \cdot \hat{s} d\sigma.$$

It is easily shown that the time integral vanishes; i.e. $\overline{\operatorname{Re} E_A} = 0$.

Therefore the static magnetic field does not contribute to the energy flow in medium A. Switching the order of integration in the first integral, we arrive at

$$\overline{\operatorname{Re} \frac{\partial W}{\partial t}} = \int_S \overline{\operatorname{Re} E_A \times \operatorname{Re} h_A} \cdot \hat{s} d\sigma = \int_S \frac{1}{2} \operatorname{Re} E_A \times h_A^* \cdot \hat{s} d\sigma,$$

a well known expression. If we transform this surface integral, we see that no part of the integrand is solenoidal; hence the integrand may be used to represent the energy flow per unit area at a point. Treating the incident and reflected waves separately, we have (4)

where H_0 is the static field, and H_1 is the dynamic field.

Substituting in (12), and taking the average

$$\overline{K \frac{\partial W}{\partial t}} = \int_0^{\infty} K E_0 \times K E_1 \sin \omega t + \int_0^{\infty} K E_1 \times K E_0 \sin \omega t$$

The time average of each term is obtained

$$(12) \quad \overline{K \frac{\partial W}{\partial t}} = \frac{1}{T} \int_0^T K E_0 \times K E_1 \sin \omega t + \int_0^T K E_1 \times K E_0 \sin \omega t$$

$$\int_0^T K E_0 \times K E_1 \sin \omega t + \int_0^T K E_1 \times K E_0 \sin \omega t$$

where T is the period of the wave. The first term is

equal to zero because the average of $\sin \omega t$ is zero.

H_0 is very small compared to the static field H_0 .

$$-\int_0^T H_1 \times H_0 \sin \omega t + \int_0^T H_0 \times H_1 \sin \omega t$$

It is easily seen that the first term is zero, and the second

term is equal to $H_0 H_1 \sin \omega t$ and the average of this is zero.

Therefore the average of the first term is zero, and the average of the

second term is $H_0 H_1 \sin \omega t$, and the average of this is zero.

$$\overline{K \frac{\partial W}{\partial t}} = \int_0^{\infty} K E_0 \times K E_1 \sin \omega t + \int_0^{\infty} K E_1 \times K E_0 \sin \omega t$$

a well known approximation. It is not necessary to know the exact

value of H_0 or H_1 to obtain the average of the first term, but the

value of H_0 is needed to obtain the average of the second term.

Therefore the dynamic field H_1 is very small compared to the static

$$(74) \quad \overline{P}_0 = \frac{1}{2} R_0 E_0 \times h_0^*, \quad (75) \quad \overline{P}_1 = \frac{1}{2} R_0 E_1 \times h_1^*.$$

For a fixed volume of a magnetohydrodynamic medium, enclosed by a surface S , one can write the energy flow as⁽⁵⁾

$$(76) \quad \frac{\partial W}{\partial t} = \int_S (\mathbf{E} \times \mathbf{H} + \frac{1}{2} \rho \mathbf{q}^2 \mathbf{q} + p \mathbf{q}) \cdot \hat{\mathbf{s}} d\sigma.$$

The term $\mathbf{E} \times \mathbf{H}$ represents the energy flowing with a MHD wave, the second term indicates a convection of kinetic energy, and $p \mathbf{q}$ denotes a flow of mechanical energy by means of momentum transfer. As for medium A, we seek the time average of that real part of the integrand whose divergence does not vanish. Again one can show that the static field \mathbf{H}_0 makes no average contribution to the energy flow.

It may be shown that, for our particular case, the second and third terms of the integrand of (76) make no real contribution to the energy flow across the boundary, on the average.

First let us define

$$(77) \quad \delta = \frac{\hat{n}_2 \cdot \mathbf{r}}{N}, \quad \epsilon = \frac{\mathbf{m} \cdot \mathbf{r}}{N}.$$

Then we can write the fluid velocity (from (31), using (36)) as

$$(78) \quad \mathbf{q} = \frac{V}{H_0} A_2 e^{i\omega(t-\delta)} + \mathbf{Q} e^{i\omega(t-\epsilon)},$$

$$\text{where } \mathbf{Q} = \mathbf{Q}_r + i \mathbf{Q}_i = \frac{\omega^2}{N} \Psi \mathbf{m}.$$

We may express Ψ in terms of A ; from (64), using (36),

$$(79) \quad \Psi = i \frac{N^2}{\omega^2} \frac{A_2 \cos \alpha_2}{H_0 \cdot \hat{n}_2} = -i \frac{NV}{\omega^2 H_0} A_2 \cos \alpha_2.$$

Substituting for m and Ψ , we obtain from (78)

$$\text{Re } \mathcal{Q} = \frac{V}{H_0} \left\{ A_2 \cos \omega(t-\delta) - \hat{\lambda}_z \cdot A_2 [\hat{\lambda}_z \cos \omega(t-\epsilon) - \hat{\lambda}_x \sin \omega(t-\epsilon)] \right\}$$

Notice that $\hat{\lambda}_z \cdot A_2$ is of order $\sin \theta_2 \propto \frac{V}{C}$ (from (59));

neglecting terms with higher powers of $\hat{\lambda}_z \cdot A_2$,

$$\begin{aligned} (\text{Re } \mathcal{Q})^2 \text{Re } \mathcal{Q} = \frac{V^3}{H_0^3} \left\{ A_2^2 \cos^3 \omega(t-\delta) + 2 A_2 (\hat{\lambda}_z \cdot A_2) \hat{\lambda}_x \cdot A_2 \times \right. \\ \left. \cos^2 \omega(t-\delta) \cos \omega(t-\epsilon) - A_2^2 \hat{\lambda}_z \cdot A_2 \cos^2 \omega(t-\delta) [\hat{\lambda}_z \cos \omega(t-\epsilon) - \hat{\lambda}_x \sin \omega(t-\epsilon)] \right\} \end{aligned}$$

Now energy flow across the boundary depends on the z -component of this expression. It is apparent that

$$\frac{1}{2} \rho (\text{Re } \mathcal{Q})^2 \text{Re } \mathcal{Q} \cdot \hat{\lambda}_z = 0.$$

That is, the convection term does not contribute to the energy flow across the boundary.

To evaluate the third term of the integrand of (76), one must determine the pressure. The following relations, repeated for convenience, will be of use.

$$(12) \quad \varphi = \frac{1}{\rho} \left[P + \frac{1}{2} (H_0 + h_2)^2 \right], \text{ or } p = \rho \varphi - \frac{1}{2} (H_0^2 + 2 H_0 \cdot h_2),$$

$$(27) \quad \varphi = \varphi_0 + \Phi e^{i\omega(t-\epsilon)},$$

We may express \bar{V} in terms of A_1 and A_2 as follows:

$$(17) \quad \bar{V} = \frac{1}{H} \left[\frac{A_1^2 \cos^2 \delta}{2} + \frac{A_2^2 \cos^2 \delta}{2} + \frac{A_1 A_2 \cos \delta}{2} \right]$$

Substituting for \bar{V} in eq. (16), we obtain:

$$k\phi = \frac{V}{H} \left[A_1^2 \cos^2 \delta + A_2^2 \cos^2 \delta + 2 A_1 A_2 \cos \delta \right]$$

Notice that $\frac{1}{2} A_1 A_2$ is $V \cos \delta$ and $\delta = \frac{1}{2} \pi - \alpha$.

neglecting terms with $\cos^2 \delta$ and $\cos \delta$:

$$(k\phi)^2 = \frac{V^2}{H^2} \left[A_1^2 \cos^2 \delta + A_2^2 \cos^2 \delta + 2 A_1 A_2 \cos \delta \right]$$

$$\cos^2 \delta (1 - \delta) \cos^2 \delta - A_1^2 \cos^2 \delta - A_2^2 \cos^2 \delta - 2 A_1 A_2 \cos \delta$$

Now using the values of A_1 and A_2 in the above eq.

of this expression, it is obtained that:

$$\frac{1}{2} \rho (k\phi)^2 = 0$$

That is, the above eq. has been satisfied for all values of δ .

That means the system is in equilibrium.

To evaluate the value of ϕ in the case of $\delta = 0$,

one may calculate the value of ϕ for $\delta = 0$ and $\alpha = 0$.

For $\delta = 0$, the value of ϕ is given by:

$$(18) \quad \phi = \frac{1}{2} \left[\frac{1}{2} (H - 1) + \frac{1}{2} (H + 1) \right]$$

$$(19) \quad \phi = \frac{1}{2} (H - 1)$$

$$(28) \quad \Phi = \omega^2 \left[1 - \left(\frac{\mathbf{m} \cdot \hat{\lambda}_z}{\hat{n}_z \cdot \hat{\lambda}_z} \right)^2 \right] \Psi = \omega^2 \left[1 - \frac{V^2}{N^2} (\mathbf{m} \cdot \hat{\lambda}_z)^2 \right] \Psi.$$

Since Φ_0 is a constant for all time it can be evaluated for static conditions; from (12),

$$(80) \quad \Phi_0 = \frac{1}{\rho} \left(p_0 + \frac{1}{2} H_0^2 \right),$$

where p_0 is the hydrostatic pressure of the fluid. To evaluate Φ , note that $(\mathbf{m} \cdot \hat{\lambda}_z)^2 \propto \sin^2 \theta_2$ is vanishingly small; hence

$$(81) \quad \Phi = \omega^2 \Psi = -i \frac{NV}{H_0} A_2 \cos \alpha_2 = i \Phi_i.$$

Substituting in (27) we discover

$$\text{Re } \Phi = \frac{1}{\rho} \left(p_0 + \frac{1}{2} H_0^2 \right) - \Phi_i \sin \omega(t - \epsilon).$$

With this result (12) yields

$$(82) \quad \text{Re } p = p_0 - \rho \Phi_i \sin \omega(t - \epsilon) - H_0 \cdot \mathbf{A}_2 \cos \omega(t - \delta).$$

From (78) we obtain

$$\text{Re } q = \frac{V}{H_0} \mathbf{A}_2 \cos \omega(t - \delta) + Q_r \cos \omega(t - \epsilon) - Q_i \sin \omega(t - \epsilon).$$

One can show that the time average of the product $\text{Re } p \text{ Re } q$, evaluated at the boundary, is

$$\overline{\text{Re } p \text{ Re } q}_{z=0} = -\frac{1}{2} \left[(H_0 \cdot \mathbf{A}_2) Q_r + \frac{V}{H_0} (H_0 \cdot \mathbf{A}_2) \mathbf{A}_2 \cdot \rho \Phi_i \mathbf{Q}_i \right].$$

Substituting for \mathbf{m} and Ψ in (78) gives

$$Q_r = -\frac{V}{H_0} (\hat{\lambda}_z \cdot \mathbf{A}_2) \hat{\lambda}_z, \quad Q_i = \frac{V}{H_0} (\hat{\lambda}_z \cdot \mathbf{A}_2) \hat{\lambda}_x.$$

$$\left[\frac{1}{\omega} \frac{d}{dt} \left(\frac{m}{\omega} \right) - \frac{1}{\omega} \right] \left[\frac{1}{\omega} \frac{d}{dt} \left(\frac{m}{\omega} \right) - \frac{1}{\omega} \right] \Phi = 0 \quad (25)$$

Since Φ is a function of t and ω only, we have

$$\Phi = \frac{1}{\omega} \left(\frac{m}{\omega} \right) \quad (26)$$

where $\frac{1}{\omega}$ is the inverse of the frequency ω .
 Φ , now being a function of t and ω only, we have

$$\Phi = \frac{1}{\omega} \left(\frac{m}{\omega} \right) \quad (27)$$

Substituting in (27) we have

$$\Phi = \frac{1}{\omega} \left(\frac{m}{\omega} \right) \quad (28)$$

where $\frac{1}{\omega}$ is the inverse of the frequency ω .

$$\Phi = \frac{1}{\omega} \left(\frac{m}{\omega} \right) \quad (29)$$

Thus (29) is satisfied.

$$\Phi = \frac{1}{\omega} \left(\frac{m}{\omega} \right) \quad (30)$$

It now remains to show that the above is the only solution.

$$\Phi = \frac{1}{\omega} \left(\frac{m}{\omega} \right) \quad (31)$$

Substituting in (31) we have

$$\Phi = \frac{1}{\omega} \left(\frac{m}{\omega} \right) \quad (32)$$

Thus $\hat{\lambda}_z \cdot \overline{\mathcal{R} \text{Re} \mathbf{q}} \Big|_{z=0} = 0$;

there is no real energy transferred across the boundary by momentum, on the average.

The only term left to the integrand of (76) is $\mathbf{E}_2 \times \mathbf{h}_2$. It is easily seen that no part of $\text{div} (\mathbf{E}_2 \times \mathbf{h}_2)$ vanishes, so we can safely take

$$(83) \quad \overline{\mathbf{P}}_2 = \frac{1}{2} \mathcal{R} \mathbf{E}_2 \times \mathbf{h}_2^* .$$

REFLECTION AND TRANSMISSION COEFFICIENTS

We define⁽¹⁴⁾

$$(84) \quad \Gamma \equiv \frac{\hat{\lambda}_z \cdot \overline{\mathbf{P}}_1}{\hat{\lambda}_z \cdot \overline{\mathbf{P}}_0} , \text{ reflection coefficient,}$$

$$(85) \quad T \equiv \frac{\hat{\lambda}_z \cdot \overline{\mathbf{P}}_2}{\hat{\lambda}_z \cdot \overline{\mathbf{P}}_0} , \text{ transmission coefficient,}$$

where $\overline{\mathbf{P}}_0$, $\overline{\mathbf{P}}_1$, $\overline{\mathbf{P}}_2$ are "complex" Poynting vectors, defined by equations (74), (75), (83).

Γ and T are determined by the fields (35) through (38). In order to find the real parts of the fields, it will be convenient to separate Ω and Π into real and imaginary parts. Using equations (47), (59), (79) we may write

$$\Omega = \Omega_r + i\Omega_i, \text{ where}$$

$$(86) \quad \Omega_r = -\frac{V}{N} \sin \theta_2 (\hat{\lambda}_z \cdot \mathbf{A}_2) \left[\hat{\lambda}_x (\hat{\lambda}_z \cdot \hat{\lambda}_5) + \hat{\lambda}_z (\hat{\lambda}_x \cdot \hat{\lambda}_5) \right] \\ \Omega_i = -\frac{V}{N} \sin \theta_2 (\hat{\lambda}_z \cdot \mathbf{A}_2) \left[\hat{\lambda}_x (\hat{\lambda}_x \cdot \hat{\lambda}_5) - \hat{\lambda}_z (\hat{\lambda}_z \cdot \hat{\lambda}_5) \right],$$

$$\text{and } \Pi = \Pi_r + i\Pi_i, \text{ where}$$

$$(87) \quad \Pi_r = -\mu V (\hat{\lambda}_z \cdot \mathbf{A}_2) \hat{\lambda}_5 \times \hat{\lambda}_z \\ \Pi_i = -\mu V (\hat{\lambda}_z \cdot \mathbf{A}_2) \hat{\lambda}_5 \times \hat{\lambda}_x.$$

We may rewrite (35) and (37) as

$$\mathbf{h}_2 = \mathbf{A}_2 e^{i\omega(t-\delta)} + (\Omega_r + i\Omega_i) e^{i\omega(t-\epsilon)}, \\ \mathbf{E}_2 = \mathbf{D} e^{i\omega(t-\delta)} + (\Pi_r + i\Pi_i) e^{i\omega(t-\epsilon)}.$$

On forming the cross-product $\mathbf{E}_2 \times \mathbf{h}_2^*$, we obtain

$$\mathbf{E}_2 \times \mathbf{h}_2^* = \mathbf{D} \times \mathbf{A}_2 + (\Pi_r + i\Pi_i) \times \mathbf{A}_2 e^{i\omega(\delta-\epsilon)} \\ + \mathbf{D} \times (\Omega_r - i\Omega_i) e^{-i\omega(\delta-\epsilon)} + (\Pi_r + i\Pi_i) \times (\Omega_r - i\Omega_i).$$

From (77),

$$\delta - \epsilon = \frac{\mathbf{r}}{N} \cdot (-\cos \theta_2 - i \sin \theta_2) \hat{\lambda}_z = -\frac{z}{N} (\cos \theta_2 + i \sin \theta_2)$$

But $\mathbf{E}_2 \times \mathbf{h}_2^*$ is to be evaluated at the boundary ($z = 0$), so

$$e^{i\omega(\delta-\epsilon)} = e^{-i\omega(\delta-\epsilon)} = 1.$$

$$U_1 = -\frac{V}{N} \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \left[\frac{1}{2} (\sigma_{\mathbf{k}}^x + i \sigma_{\mathbf{k}}^y) \right] \quad (66)$$

$$U_2 = -\frac{V}{N} \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \left[\frac{1}{2} (\sigma_{\mathbf{k}}^x - i \sigma_{\mathbf{k}}^y) \right]$$

$$\text{and } \Pi = \Pi_1 + \Pi_2$$

$$(67) \quad \Pi_1 = -\frac{N}{2} \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \left[\frac{1}{2} (\sigma_{\mathbf{k}}^x + i \sigma_{\mathbf{k}}^y) \right]$$

$$\Pi_2 = -\frac{N}{2} \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \left[\frac{1}{2} (\sigma_{\mathbf{k}}^x - i \sigma_{\mathbf{k}}^y) \right]$$

We now rewrite (67) and (68) as

$$H = A_1 e^{i\omega(t-\mathbf{r})} + (H_1 + H_2) e^{-i\omega(t-\mathbf{r})}$$

$$E_2 = D e^{i\omega(t-\mathbf{r})} + (H_1 + H_2) e^{-i\omega(t-\mathbf{r})}$$

On taking the time-derivative of (67), we obtain

$$E_2 \times \mathbf{h}_2^* = D \times \mathbf{A}_1^* + (H_1 - H_2) \times \mathbf{A}_1^*$$

$$D \times (\mathbf{u}_1 - \mathbf{u}_2) e^{-i\omega(t-\mathbf{r})} = (H_1 - H_2) \times \mathbf{A}_1^* e^{-i\omega(t-\mathbf{r})}$$

or

$$E_2 \times \mathbf{h}_2^* = \frac{E}{N} \left[(\mathbf{u}_1 - \mathbf{u}_2) \times \mathbf{A}_1^* \right] e^{-i\omega(t-\mathbf{r})}$$

and $E_2 \times \mathbf{h}_2^*$ is an expression of the form $\mathbf{E} \times \mathbf{h}^*$

$$E \times \mathbf{h}^* = \frac{E}{N} \left[(\mathbf{u}_1 - \mathbf{u}_2) \times \mathbf{A}_1^* \right] e^{-i\omega(t-\mathbf{r})}$$

Expanding the cross-product and selecting the real part, we get

$$\operatorname{Re} \left(\mathbf{E}_2 \times \mathbf{h}_2^* \right)_{z=0} = (\mathbf{D} + \mathbf{\Pi}_r) \times (\mathbf{A}_2 + \mathbf{\Omega}_r) + \mathbf{\Pi}_i \times \mathbf{\Omega}_i.$$

Now it is easily seen that $\mathbf{\Omega}$ is vanishingly small (of order $\sin^2 \theta_2$); hence

$$(88) \quad \operatorname{Re} \left(\mathbf{E}_2 \times \mathbf{h}_2^* \right)_{z=0} = (\mathbf{D} + \mathbf{\Pi}_r) \times \mathbf{A}_2$$

Using (44) on (38) we obtain a simpler expression

$$\mathbf{D} = \mu V \hat{\lambda}_s \times \mathbf{A}_2 ;$$

hence

$$\begin{aligned} (\mathbf{D} + \mathbf{\Pi}_r) \times \mathbf{A}_2 &= \mu V (\hat{\lambda}_s \times \mathbf{A}_2) \times \mathbf{A}_2 - \mu V (\hat{\lambda}_z \cdot \mathbf{A}_2) (\hat{\lambda}_s \times \hat{\lambda}_z) \times \mathbf{A}_2 \\ &= \mu V \left[\mathbf{A}_2 (\hat{\lambda}_s \cdot \mathbf{A}_2) - \hat{\lambda}_s A_2^2 - \hat{\lambda}_z (\hat{\lambda}_z \cdot \mathbf{A}_2) (\hat{\lambda}_s \cdot \mathbf{A}_2) + \hat{\lambda}_s (\hat{\lambda}_z \cdot \mathbf{A}_2)^2 \right] \end{aligned}$$

The last term in brackets is vanishingly small. The component of (88) normal to the boundary is simply

$$(89) \quad \hat{\lambda}_z \cdot \overline{\mathbf{P}}_2 = -\mu V A_2^2 \cos \beta.$$

$\overline{\mathbf{P}}_0$ and $\overline{\mathbf{P}}_1$ (equations (74) and (75)) are more easily obtained. From (29) and (34),

$$(90) \quad \begin{aligned} \mathbf{E}_0 &= -\eta \hat{n}_0 \times \mathbf{A}_0 e^{i\omega(t - \frac{\hat{n}_0 \cdot \mathbf{r}}{c})} \\ \mathbf{h}_0^* &= \mathbf{A}_0 e^{-i\omega(t - \frac{\hat{n}_0 \cdot \mathbf{r}}{c})} \end{aligned}$$

$$(91) \quad \begin{aligned} \mathbf{E}_1 &= -\eta \hat{n}_1 \times \mathbf{A}_1 e^{i\omega(t - \frac{\hat{n}_1 \cdot \mathbf{r}}{c})} \\ \mathbf{h}_1^* &= \mathbf{A}_1 e^{-i\omega(t - \frac{\hat{n}_1 \cdot \mathbf{r}}{c})} . \end{aligned}$$

expanding the cross-product and using the fact that

$$A^2 E_2 \times E_2 = (E_2 \cdot E_2) A = 0$$

we find a result from the fact that E_2 is a vector and A is a scalar

and $E_2^2 = 0$; hence

$$(68) \quad A^2 E_2 \times E_2 = (E_2 \cdot E_2) A = 0$$

Using (61) in (68) we obtain a similar result

$$D = \mu \nu A^2 \times A^2$$

hence

$$(D + I) \times A = \mu \nu A^2 \times A^2 \times A = \mu \nu A^2 \times A^2 \times A$$

$$= \mu \nu A^2 (A^2 \times A) = \mu \nu A^2 (A \times A) = 0$$

The last two results are in accordance with the hypothesis

of (65) namely that the product of two vectors

$$(69) \quad A^2 \times A^2 = -\mu \nu A^2 \times A^2$$

\bar{P} and \bar{P} (equation (65) and (67) respectively)

obtained from (6) and (7)

$$(70) \quad E_2 = \frac{1}{\sqrt{2}} (A_1 + A_2)$$

$$H_2 = A_1 \times A_2 = \frac{1}{\sqrt{2}} (A_1 \times A_2)$$

$$E_1 = \frac{1}{\sqrt{2}} (A_1 - A_2)$$

$$(71) \quad H_1 = A_2 \times A_1 = \frac{1}{\sqrt{2}} (A_2 \times A_1)$$

Substituting (90) and (91) in equations (74) and (75) yields

$$\begin{aligned}\overline{P}_0 &= -\eta (\hat{n}_0 \times \mathbf{A}_0) \times \mathbf{A}_0 = \eta [\hat{n}_0 A_0^2 - \mathbf{A}_0 (\hat{n}_0 \cdot \mathbf{A}_0)] \\ \overline{P}_1 &= -\eta (\hat{n}_1 \times \mathbf{A}_1) \times \mathbf{A}_1 = \eta [\hat{n}_1 A_1^2 - \mathbf{A}_1 (\hat{n}_1 \cdot \mathbf{A}_1)] .\end{aligned}$$

Since \mathbf{A}_0 and \mathbf{A}_1 are perpendicular to \hat{n}_0 and \hat{n}_1 respectively,

$$(92) \quad \hat{\lambda}_z \cdot \overline{P}_0 = -\eta \cos \theta_0 A_0^2 ,$$

$$(93) \quad \hat{\lambda}_z \cdot \overline{P}_1 = \eta \cos \theta_1 A_1^2 .$$

Using the relation $\eta = \mu c$, (89) may be put

$$(94) \quad \hat{\lambda}_z \cdot \overline{P}_2 = -\eta \frac{V}{c} \cos \beta A_2^2 .$$

With these expressions, the coefficients become

$$(95) \quad \Gamma = - \frac{A_1^2}{A_0^2} , \text{ reflection coefficient,}$$

$$(96) \quad T = \frac{V}{c} \frac{\cos \beta}{\cos \theta_0} \frac{A_2^2}{A_0^2} , \text{ transmission coefficient.}$$

It remains to evaluate A_1 and A_2 by means of equations (65) through (68). We obtain

$$(97) \quad \frac{A_1^2}{A_0^2} = \sin^2 \alpha_0 \left(\frac{\frac{V}{c} \cos \beta - \cos \theta_0}{\frac{V}{c} \cos \beta + \cos \theta_0} \right)^2 + \cos^2 \alpha_0 \left[\frac{1 - \frac{V}{c} (k_3 \sin \theta_0 + \cos \beta \cos \theta_0)}{1 - \frac{V}{c} (k_3 \sin \theta_0 - \cos \beta \cos \theta_0)} \right]^2 ,$$

$$\begin{aligned} \bar{P} - \gamma(A \cdot A) &= \gamma(A \cdot A) \cdot \bar{P} \\ \bar{P} - \gamma(A \cdot A) &= \gamma(A \cdot A) \cdot \bar{P} \end{aligned}$$

Given A and \bar{A} are the two sides of a right triangle.

$$\bar{P} - \gamma(A \cdot A) = \gamma(A \cdot A) \cdot \bar{P} \quad (32)$$

$$\bar{P} - \gamma(A \cdot A) = \gamma(A \cdot A) \cdot \bar{P} \quad (33)$$

Using the identity $\gamma(A \cdot A) = \frac{1}{2}(A \cdot A)$

$$\bar{P} - \frac{1}{2}(A \cdot A) = \frac{1}{2}(A \cdot A) \cdot \bar{P} \quad (34)$$

Using the identity $\gamma(A \cdot A) = \frac{1}{2}(A \cdot A)$

$$\bar{P} - \frac{1}{2}(A \cdot A) = \frac{1}{2}(A \cdot A) \cdot \bar{P} \quad (35)$$

$$\bar{P} - \frac{1}{2}(A \cdot A) = \frac{1}{2}(A \cdot A) \cdot \bar{P} \quad (36)$$

Using the identity $\gamma(A \cdot A) = \frac{1}{2}(A \cdot A)$

(37) through (39) are the same.

$$\bar{P} - \frac{1}{2}(A \cdot A) = \frac{1}{2}(A \cdot A) \cdot \bar{P} \quad (37)$$

$$\bar{P} - \frac{1}{2}(A \cdot A) = \frac{1}{2}(A \cdot A) \cdot \bar{P}$$

$$(98) \quad \frac{A_2^2}{A_0^2} = 4 \cos^2 \theta_0 \left\{ \frac{\sin^2 \alpha_0}{\left(\frac{V}{c} \cos \beta + \cos \theta_0 \right)^2} + \frac{\cos^2 \alpha_0 \left(1 - 2 \frac{V}{c} k_3 \sin \theta_0 \right)}{\left[1 - \frac{V}{c} \left(k_3 \sin \theta_0 - \cos \beta \cos \theta_0 \right) \right]^2} \right\}.$$

Equations (95) through (98) show that the transfer of energy from an EM wave to an MHD fluid is a function of orientation of the static magnetic field, polarization of the incident wave, and the angle of incidence. The exact functional dependences are obscured by the complexity of equations (97) and (98). Therefore it is instructive to consider special cases.

The dependence on β is clearly shown for the case of normal incidence; for $\theta_0 = 0$, equations (97) and (98) reduce to

$$\frac{A_1^2}{A_0^2} = \frac{1}{1 + 4 \frac{V}{c} \cos \beta}, \quad \frac{A_2^2}{A_0^2} = \frac{4(1 + 2 \frac{V}{c} \cos \beta)}{1 + 4 \frac{V}{c} \cos \beta}.$$

wherein the coefficients become

$$(99) \quad \Gamma = - \frac{1}{1 + 4 \frac{V}{c} \cos \beta},$$

$$(100) \quad T = \frac{4 \frac{V}{c} \cos \beta}{1 + 4 \frac{V}{c} \cos \beta}.$$

The (-) sign of course signifies reflection. Note that

$|T| + |\Gamma| = 1$, as it must for conservation of energy.

$$\frac{A_2^2}{A_1^2} = \frac{1}{1 + \frac{1}{2} \frac{A_2^2}{A_1^2}} \quad (98)$$

$$\frac{A_2^2}{A_1^2} = \frac{1}{1 + \frac{1}{2} \frac{A_2^2}{A_1^2}}$$

From (97) and (98) we have the relation
 energy loss in the shock is a function of the
 density of the gas, the angle of the shock, and the
 ratio of the specific heats, γ . The angle of the shock
 is determined by the upstream Mach number, M_1 , and
 the ratio of the specific heats, γ . Therefore, the
 energy loss is a function of the upstream Mach number
 and the ratio of the specific heats, γ .

REFERENCES

The following references are given for the purpose of
 providing information on the subject of shock waves.

$$\frac{A_2^2}{A_1^2} = \frac{1}{1 + \frac{1}{2} \frac{A_2^2}{A_1^2}} \quad (99)$$

where the coefficient is given by

$$\frac{A_2^2}{A_1^2} = \frac{1}{1 + \frac{1}{2} \frac{A_2^2}{A_1^2}} \quad (100)$$

$$\frac{A_2^2}{A_1^2} = \frac{1}{1 + \frac{1}{2} \frac{A_2^2}{A_1^2}}$$

The sign of the energy loss is positive when the
 upstream Mach number is greater than one, and negative
 when it is less than one.

FUNCTIONAL DEPENDENCE OF Γ AND T

To correctly interpret the curves of Figures 5, 6, 7, it is important to keep in mind that Γ and T are defined in terms of the normal components of the Poynting vectors. While the normal component of \overline{P}_0 decreases smoothly as θ_0 increases, the percentage of \overline{P}_2 normal to the boundary remains essentially constant because θ_2 is always small. Since $\theta_1 = \theta_0$, a change in Γ indicates a corresponding change in the total reflected energy, but there is a tendency for T to increase with increasing θ_0 simply because the normal component of \overline{P}_0 decreases.

However this does not at all explain the curves of Figures 5, 6, 7. The behavior for the case $\alpha_0 = \frac{\pi}{2}$ is most surprising. Inspection of Figure 2 reveals that the polarization of the h_0 wave is entirely parallel to the boundary for this case, for all θ_0 . Apparently as θ_0 increases the transmission of the parallel component of polarization increases, reflection decreasing correspondingly. At a certain angle ($\theta_0 \simeq 85^\circ$), the entire incident wave is transmitted. From Equations 46, 97, it may be seen that at this point $\frac{\pi}{2} - \theta_0 \simeq \theta_2$.

Γ and T behave in the same way regardless of β , although obviously their numerical values are a function of β .

THE T-TEST

The T-test is a statistical test used to determine if there is a significant difference between the means of two groups. It is based on the assumption that the data is normally distributed. The test statistic is calculated as follows:

$$T = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{s^2}{n_1} + \frac{s^2}{n_2}}}$$

where \bar{X}_1 and \bar{X}_2 are the sample means, s^2 is the pooled variance, and n_1 and n_2 are the sample sizes. The test is then compared to a critical value from the T-distribution table to determine if the difference is statistically significant.

There are two main types of T-tests: the one-tailed T-test and the two-tailed T-test. The one-tailed T-test is used when the researcher has a specific hypothesis about the direction of the difference between the means. The two-tailed T-test is used when the researcher is simply interested in whether there is a difference between the means, regardless of the direction. The choice of which test to use depends on the research question and the hypotheses being tested.

It is important to note that the T-test is only valid if the data is normally distributed and the variances of the two groups are equal. If these assumptions are violated, the results of the T-test may be unreliable.

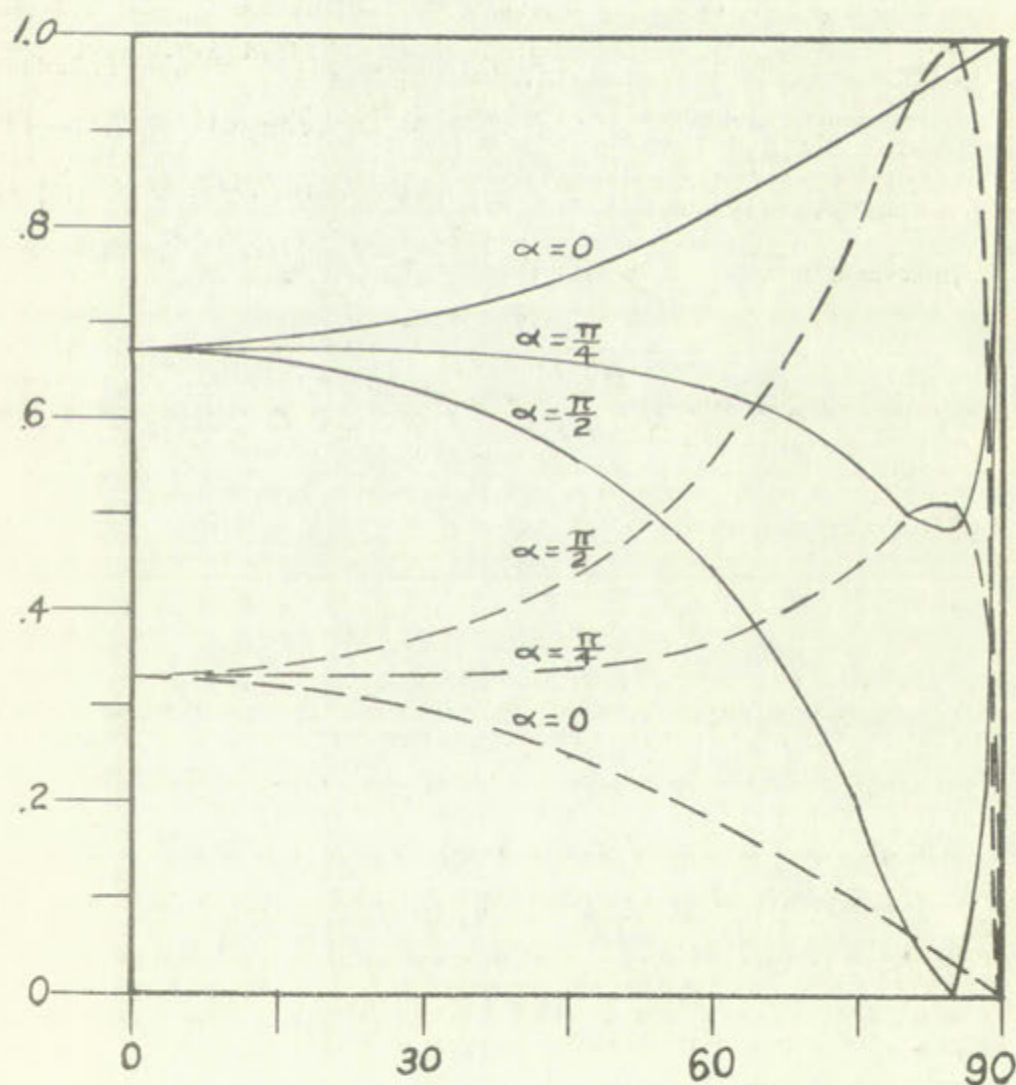
FIGURE 5

Dependence of Γ and T on Angle of Incidence,
With Polarization Angle of Incident Wave as Parameter,

For the Case $\beta = 0$.

Γ solid curves

T dashed curves



Angle of Incidence (θ_0) (degrees)

PROBLEM 1

Consider a system of two particles, each of mass m , moving in a one-dimensional potential $V(x)$. The particles are initially at rest at positions x_1 and x_2 . The potential is defined by the following graph.

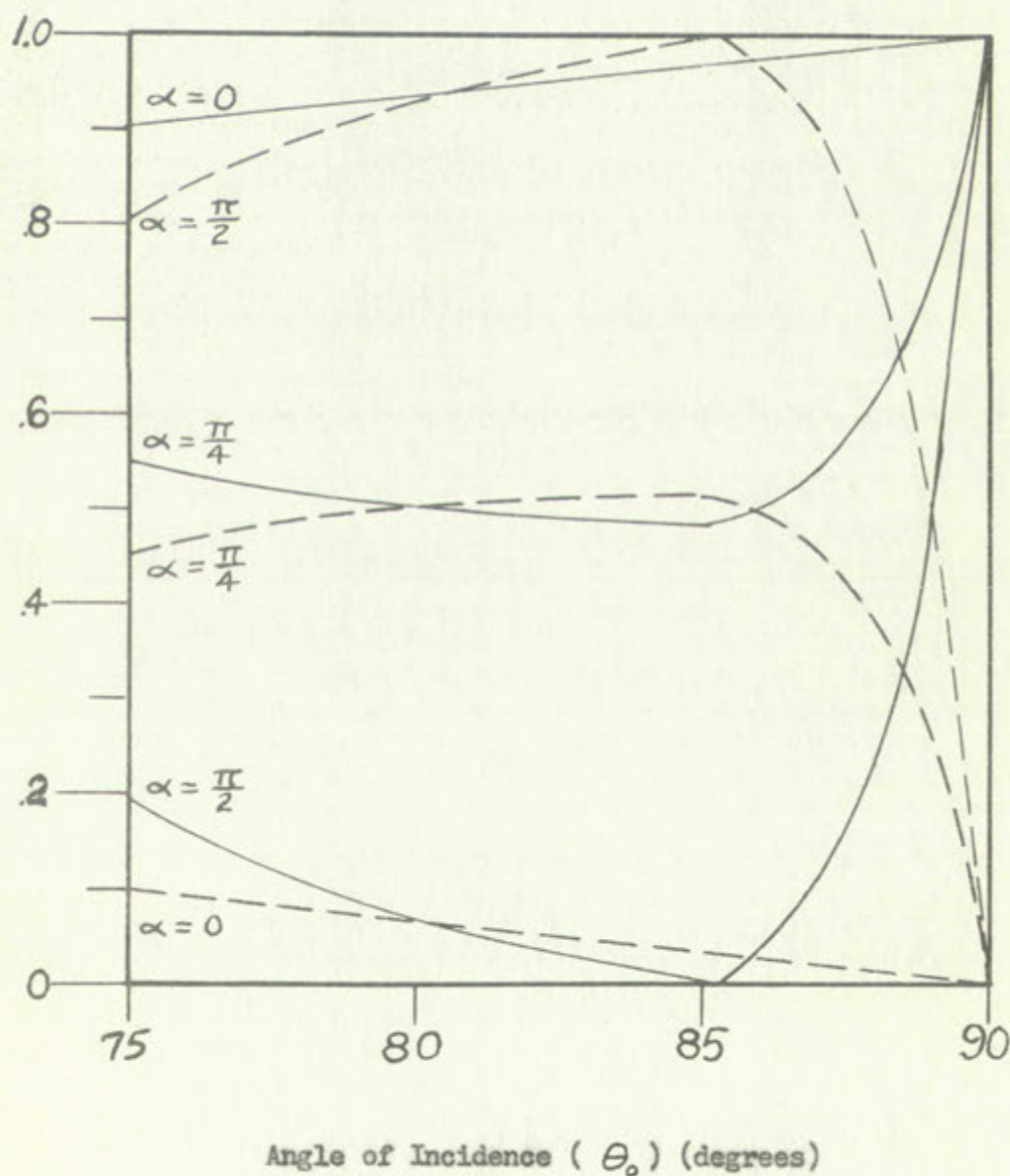


Find the energy levels of the system.

FIGURE 6

Dependence of Γ and T on Angle of Incidence,
 With Polarization Angle of Incident Wave as Parameter,
 For the Case $\beta = 0$ (Part of Figure 5 Expanded).

Γ solid curves
 T dashed curves



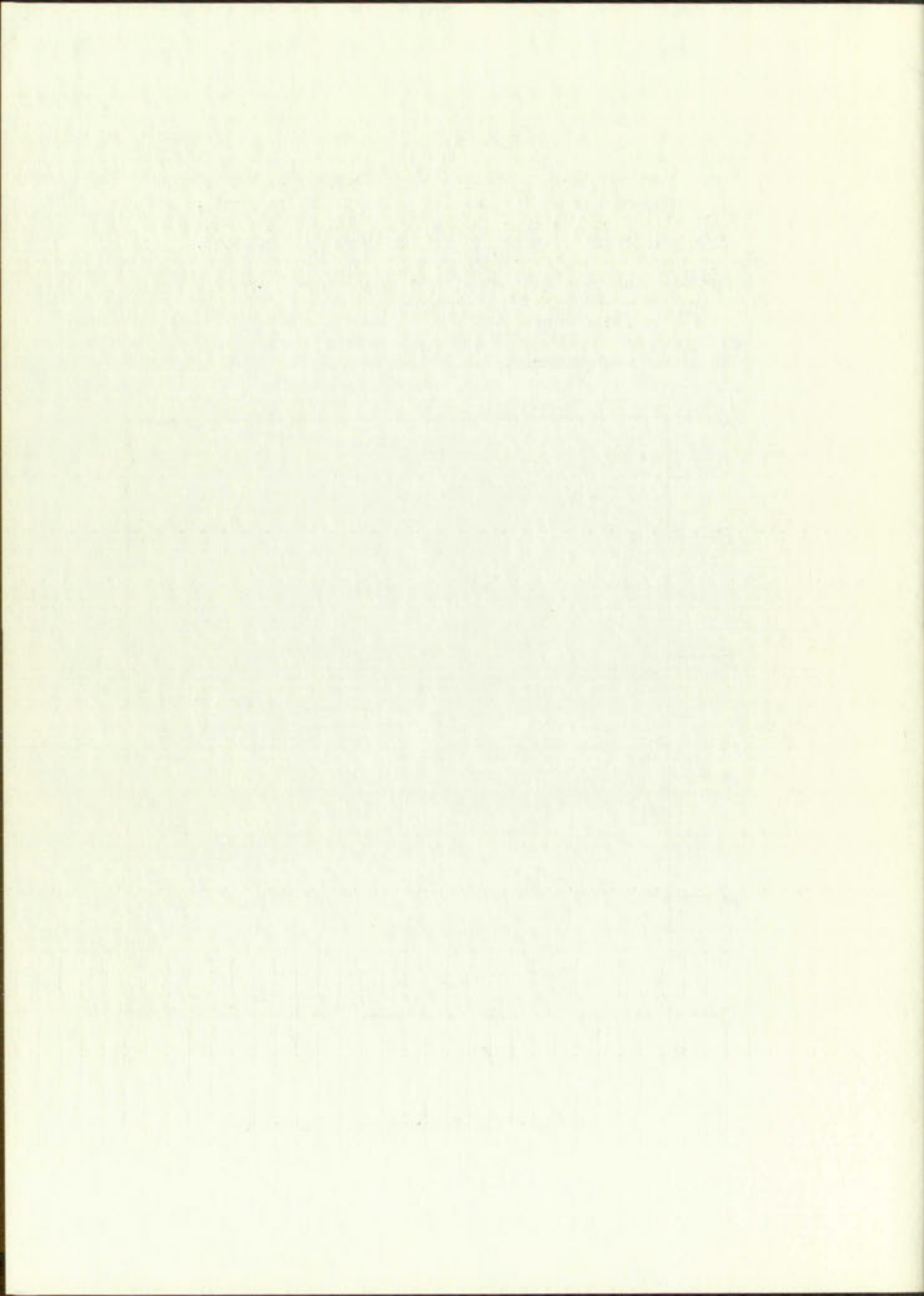


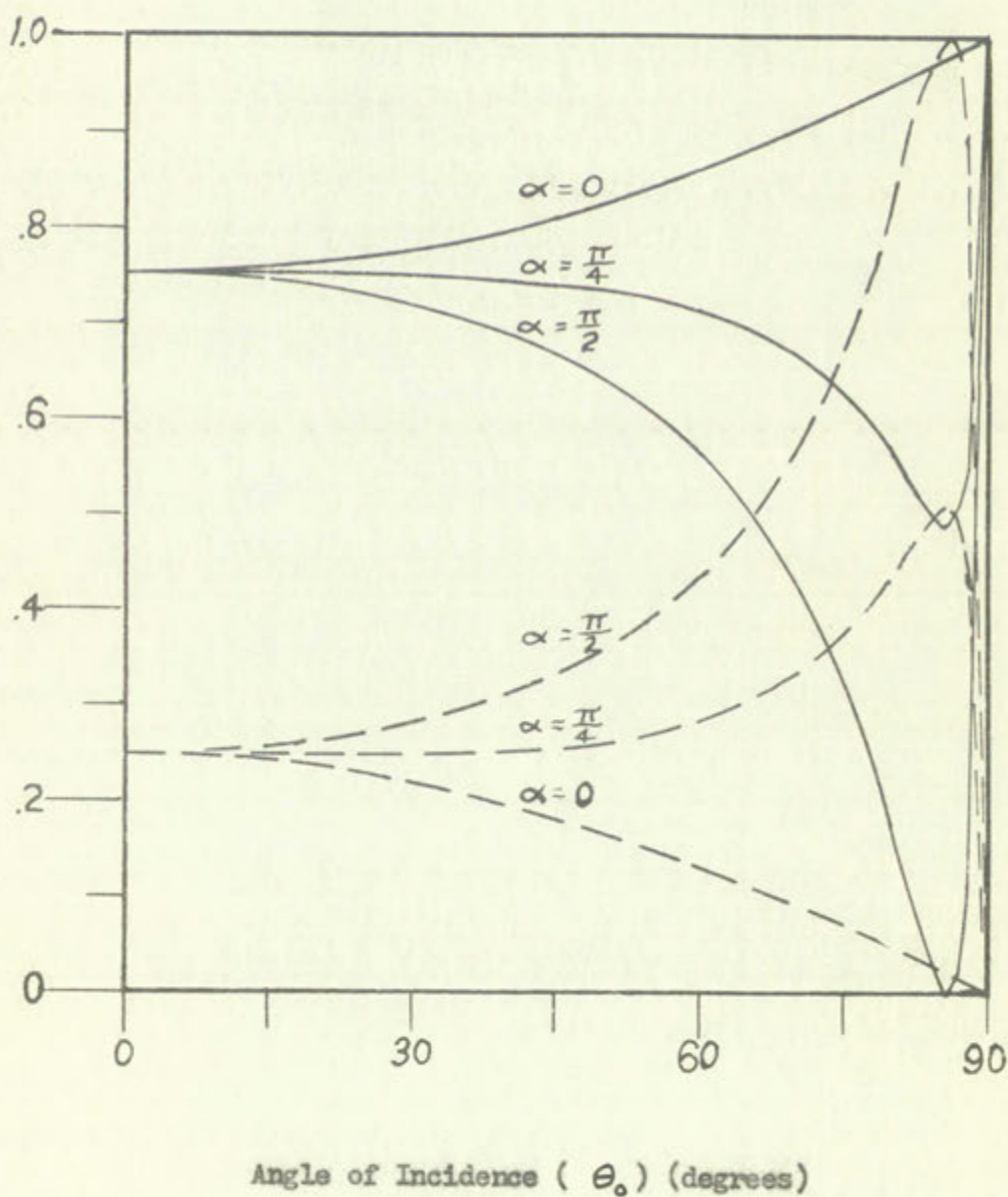
FIGURE 7

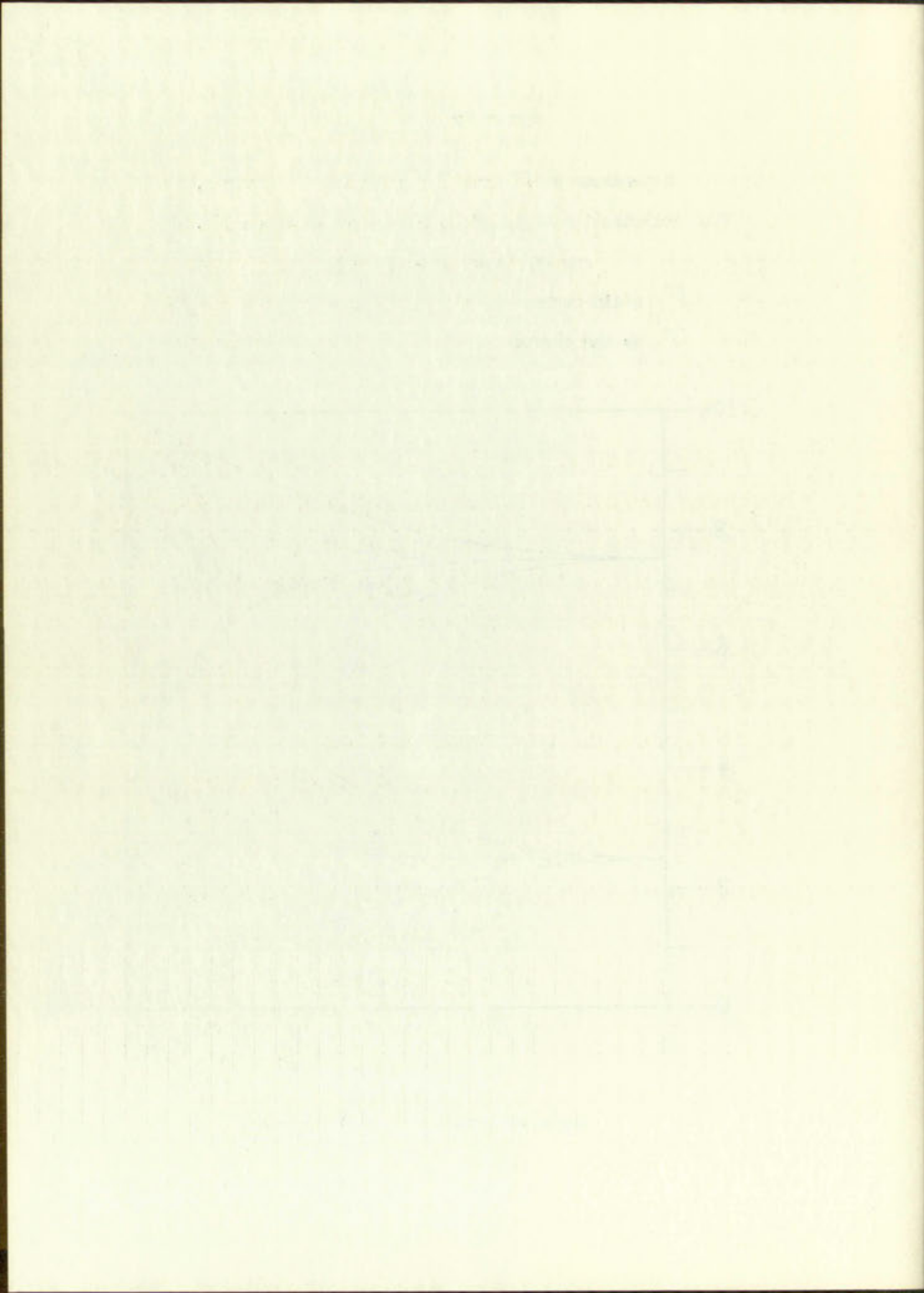
Dependence of Γ and T on Angle of Incidence,
With Polarization Angle of Incident Wave as Parameter,

For the Case $\beta = \frac{\pi}{4} = \gamma$.

Γ solid curves

T dashed curves





In addition to the data for the cases $\beta = 0$ and $\beta = \frac{\pi}{4} = \tau$ (Figures 5, 6, 7), data was obtained for $\beta = \frac{\pi}{4}$ with $\tau = 0, \frac{\pi}{2}$; since the curves hardly differed numerically from those of Figure 7, they are not shown. It seems that variation with azimuthal orientation of the static magnetic field is strictly second order compared to variation with the polar orientation.

For the case $\alpha_0 = 0$ the behavior is regular. Although the tangential component of polarization tends to produce an increase of transmission as θ_0 increases, the tangential component itself decreases so fast that the tendency is overcome. For $\alpha_0 = \frac{\pi}{4}$, the behavior is intermediate.

Worthy of mention also is the behavior of Γ and \mathcal{T} with V/c . The value chosen for Figures 5, 6, 7 is $V/c = .1$ (small enough to avoid the need of a relativistic analysis, yet large enough to provide considerable transmission). Inspection of Equations 96, 98 reveals that in general $\mathcal{T} \rightarrow 0$ as $V/c \rightarrow 0$. However, provided V/c is not vanishingly small, the anomaly still exists for the case $\alpha_0 = \frac{\pi}{2}$. It appears that, as V/c decreases, the angular width of the transmission region decreases sharply, and the region moves toward $\theta_0 = \frac{\pi}{2}$. With V/c vanishingly small the "spike" is at $\theta_0 = \frac{\pi}{2}$, where of course there can be neither reflection nor transmission because the h_0 wave is not actually incident on the boundary.

ACKNOWLEDGMENT

The author is greatly indebted to Professor Donald E. Skabelund for suggesting this problem, and for helpful guidance in carrying on the investigation.

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The author is indebted to the following persons:

1. Dr. J. H. ...
2. Dr. ...

REFERENCES

1. J. H. ...
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5. ...

APPENDIX

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DESIGNATION OF SYMBOLS

\mathbf{a}	vector field defined by Equations 15, 16
\mathbf{A}	vector amplitude of magnetic field
A	magnitude of \mathbf{A}
\mathbf{b}	solenoidal part of \mathbf{a}
\mathbf{B}	vector amplitude of \mathbf{b}
B	magnitude of \mathbf{B}
c	speed of light in medium A
d	skin depth of the ψ disturbance
\mathbf{D}	vector amplitude of electric field in medium B
d	total differential operator
e	base of natural logarithms
E	electric field intensity
EM	electromagnetic
\mathbf{h}	time dependent magnetic field
\mathbf{H}	total magnetic field
\mathcal{H}	exponential of magnetic waves
i	$\sqrt{-1}$
$\hat{\mathbf{i}}$	unit vector
Im	imaginary part
k_3	$\sin \beta \cos \gamma$
\mathbf{m}	propagation vector of ψ disturbance
MHD	magnetohydrodynamic
$\hat{\mathbf{n}}$	unit vector in direction of wave propagation

CONTENTS

vector field defined by $\mathbf{F}(x, y, z)$	2
vector magnitude of vector field	A
direction of a	A
scalar field of \mathbf{F}	B
vector magnitude of \mathbf{F}	B
direction of \mathbf{F}	B
stream of fluid in vector \mathbf{F}	C
line integral of the \mathbf{F} -field	D
vector magnitude of electric field in the field	E
total differential vector	F
base of vector field	G
electric field intensity	H
electrostatic	I
the magnetic vector field	J
total magnetic field	K
potential of electric field	L
$\nabla \cdot \mathbf{F}$	M
unit vector	N
length of \mathbf{F}	O
the \mathbf{F} and \mathbf{G}	P
properties of vector of \mathbf{F} and \mathbf{G}	Q
vector field	R
unit vector in direction of \mathbf{F} and \mathbf{G}	S

N	component of Alfvén velocity in direction of propagation
p	pressure of fluid
\vec{P}	Poynting vector
\mathcal{Q}	fluid velocity
Q	vector amplitude of damped part of \mathcal{Q} (Equation 78)
\mathbf{r}	position vector (with respect to a point on the boundary)
Re	real part
\hat{s}	unit vector normal to arbitrary fixed surface
S	arbitrary fixed surface
t	time
T	transmission coefficient
V	magnitude of Alfvén velocity of MHD wave
V	volume enclosed by S
W	energy
x	coordinate axis
y	coordinate axis
z	coordinate axis
α	polarization angle (Figures 2, 3, 4)
β	polar angle of orientation of H_0
γ	azimuth angle of orientation of H_0
Γ	reflection coefficient
δ	$\hat{n}_2 \cdot \mathbf{r}/N$
ϵ	$\mathbf{m} \cdot \mathbf{r}/N$
ϵ	electric permittivity of medium A
S	distance measured along H_0

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vector axis of induction of \vec{E}	30
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$\vec{E} \cdot \vec{v}$	33
electric permeability of medium	34
distance measured along \vec{E}	35

η	$\sqrt{\mu\epsilon}$, characteristic impedance of medium A
ν	viscosity of fluid
μ	magnetic permeability
π	3.1416
Π	vector amplitude of damped part of E (Equation 38)
ρ	density of fluid
σ	conductivity of fluid
τ	period of wave
ϕ	disturbance created by boundary (Equation 12)
ϕ_0	constant of integration
Φ	amplitude of ϕ -function
ψ	potential function leading to part of a (Equation 17)
Ψ	amplitude of ψ -function
ω	angular frequency of wave
Ω	vector amplitude of damped part of h_θ (Equation 36)
∂	partial differential operator
∇	vector differential operator

Subscripts

A	pertaining to medium A
B	pertaining to medium B
i	imaginary part
r	real part
s	situated on the boundary

x	in the direction of the +x-axis
y	in the direction of the +y-axis
z	in the direction of the +z axis
0	original static condition before incident wave arrives
0	pertaining to the incident wave
1	pertaining to the reflected wave
2	pertaining to the refracted or transmitted wave

Miscellaneous

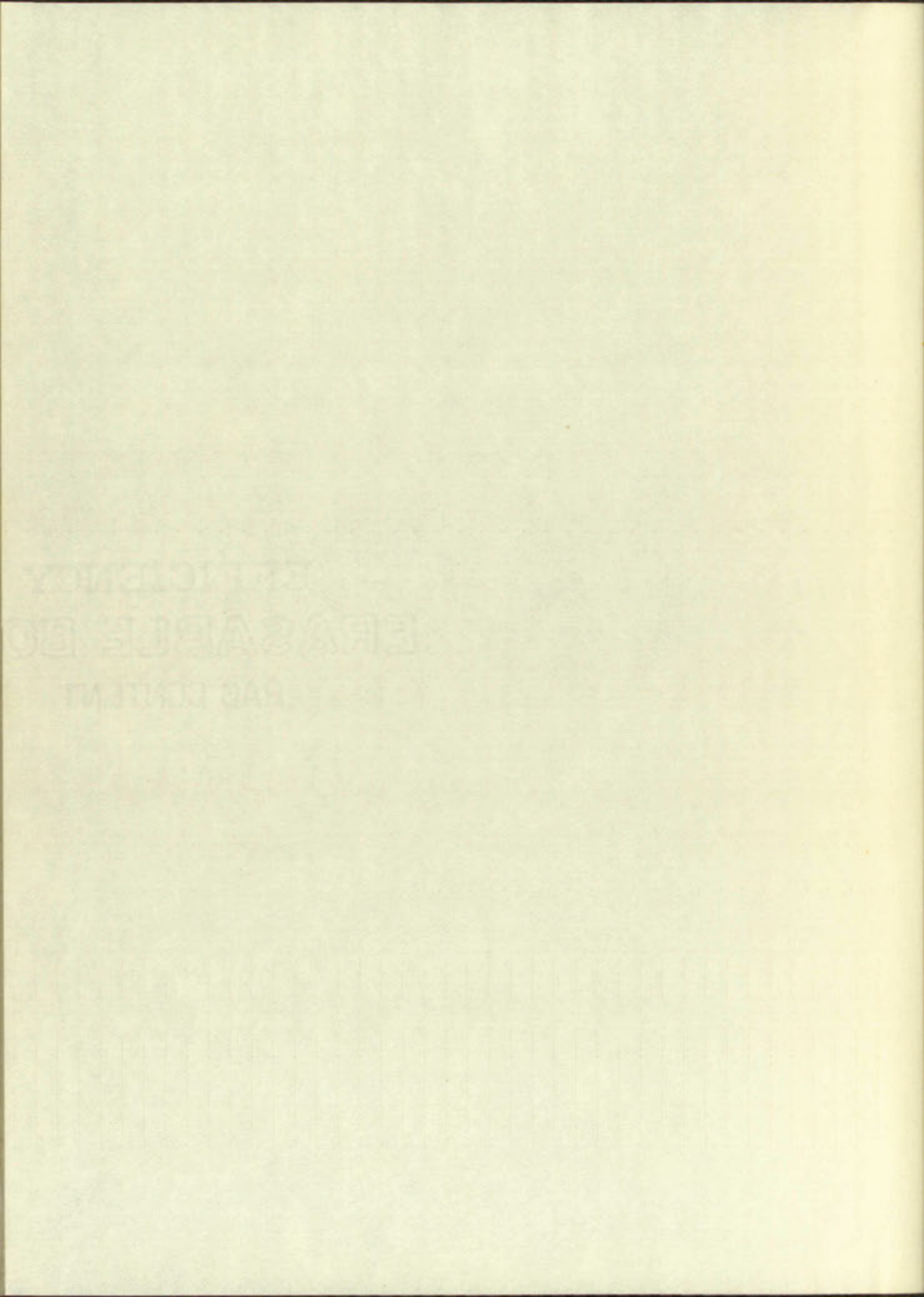
^	indicates a unit vector; other vectors are indicated by making dark the left side of a symbol.
*	complex conjugate
—	time-averaged quantity

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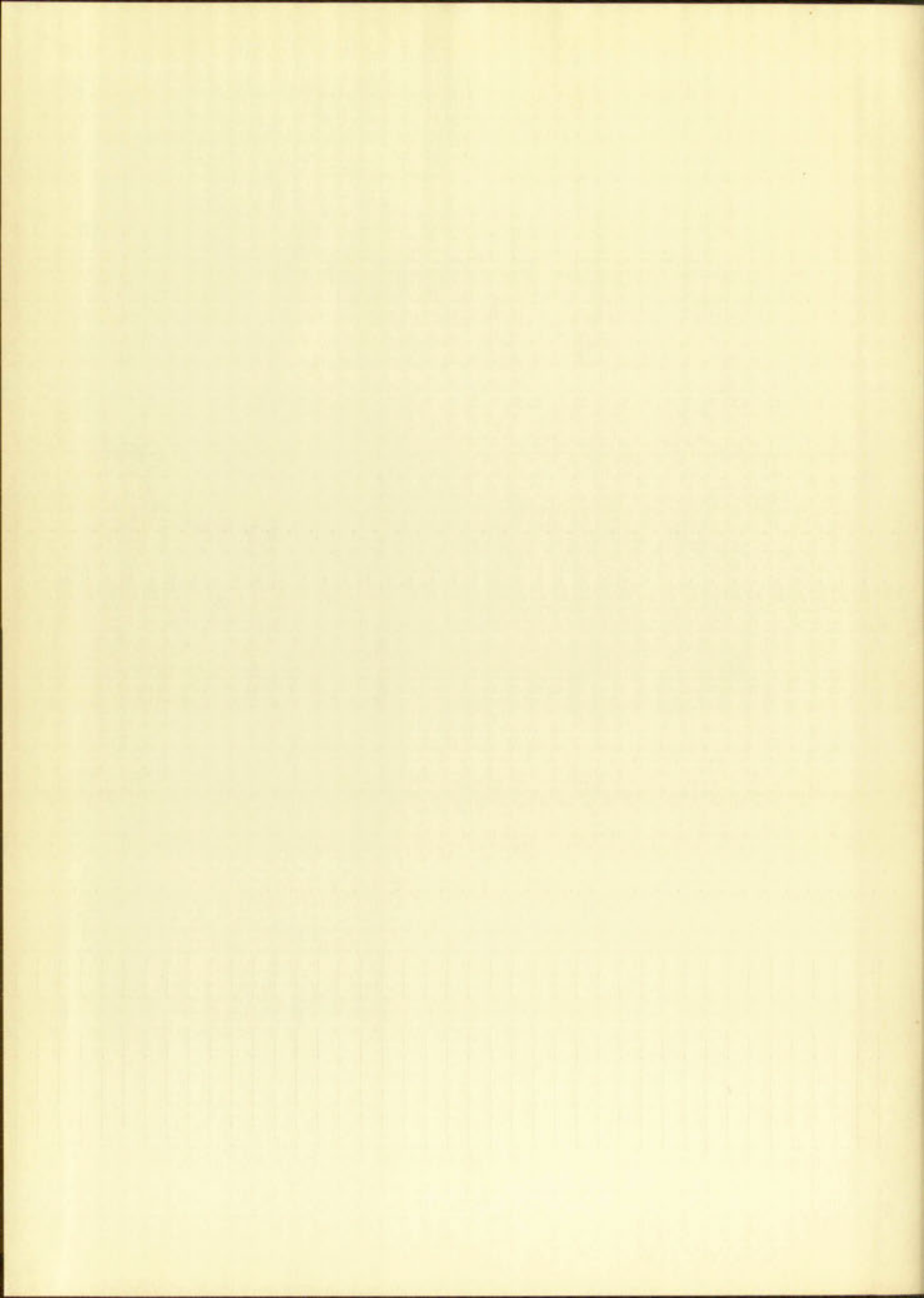
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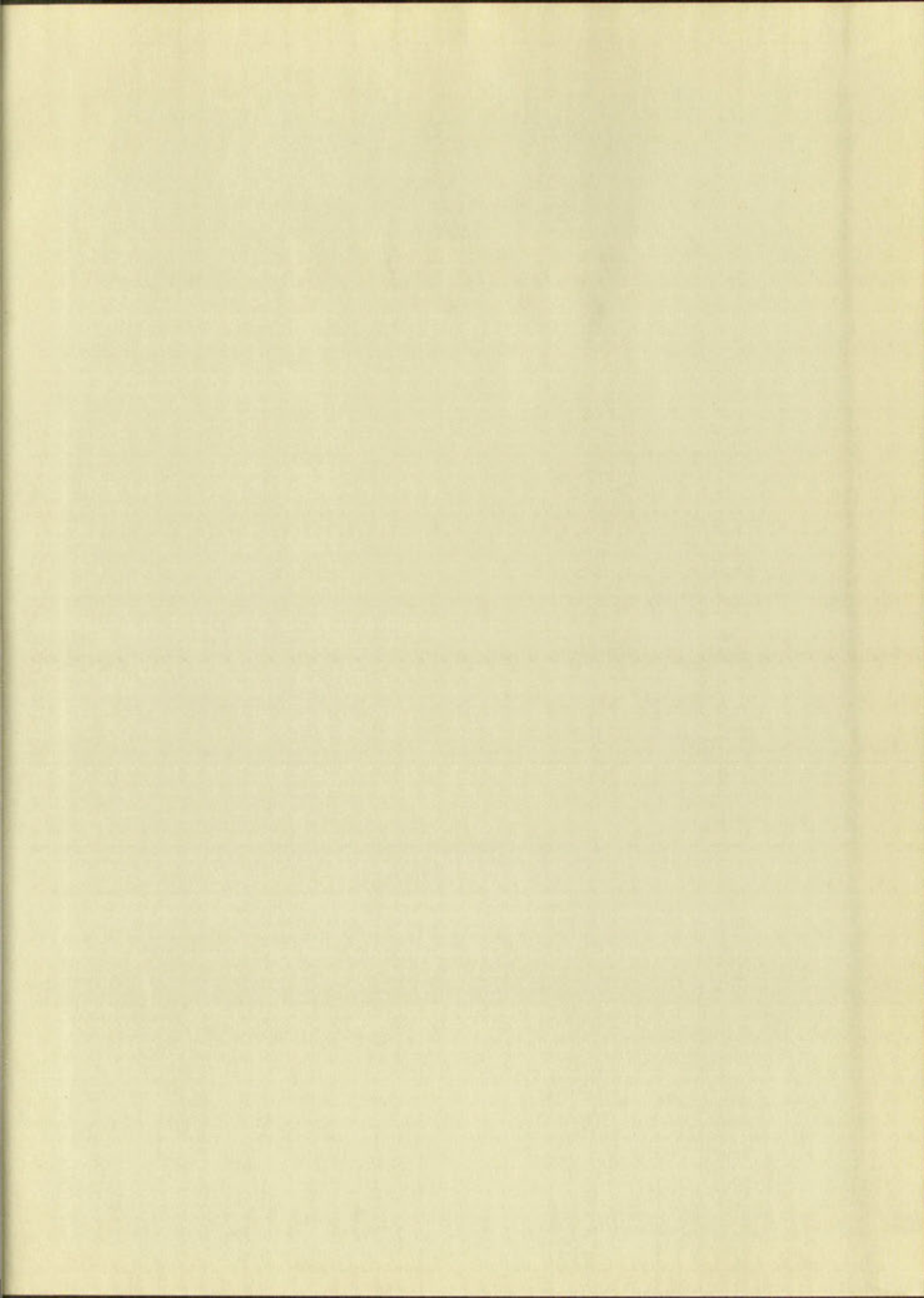
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