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Cerenkov Radiation from an Electron Traveling in a Circle Through a Dielectric Medium

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CERENKOV RADIATION FROM AN ELECTRON TRAVELING IN A CIRCLE THROUGH A DIELECTRIC MEDIUM

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John Joseph Newman

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ABSTRACT

The theoretical problem of determining the Cerenkov radiation from an electron traveling in a circle through a medium has been advanced and the fields and resultant radiation from such a geometry have been determined. Various direct methods of solution were originally attempted and were determined to be impossible with available techniques. These direct methods included setting up the wave equation for the electron's motion in various coordinate systems such as cylindrical, spherical, and toroidal coordinates. A solution was finally achieved by a transformation of the fields as determined by Frank and Tamm in their original article on Cerenkov radiation. This transformation resulted in a solution as a summation of eigenfrequencies each separated by the frequency of the electron's periodic motion. The total radiation for one period of the electron motion then takes the form:

$$W = \frac{e^2}{c^2}\,\frac{8\pi^3 a}{T^2}\sum_m m\left(1 - \frac{1}{\beta^2 n^2}\right)$$

In the limit, as the period of the electron's motion becomes infinite, this expression approaches the classical expression for the radiation as given by Frank and Tamm. As a direct result of the method of solution, the field expressions for the circular geometry are obtained in a form which is immediately applicable to calculations of radiated power flow in directions of interest.

Two hardware configurations are discussed which might be used as vehicles for checking the accuracy of the theory. It is hoped that one of these configurations might lead to a device which would yield a useful amount of power in the submillimeter region of the microwave spectrum.

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CERENKOV RADIATION FROM AN ELECTRON TRAVELING IN A CIRCLE THROUGH A DIELECTRIC MEDIUM

Statement of the Problem

At the present time, considerable interest exists in obtaining sources of electromagnetic energy in the infrared and visible regions of the electromagnetic spectrum. In line with this interest, an investigation of Cerenkov radiation was proposed as a means for obtaining radiation of very short wavelength.

This investigation was begun with a survey of the literature to determine what theoretical work had been done and what methods were available to obtain usable power from the Cerenkov effect. The survey indicated that very few useful configurations had been investigated which would be capable of use as a power source. Most of the articles were concerned with the Cerenkov radiation obtained from particles (charged and neutral) traveling in straight line paths or with the interaction of particles with abrupt boundaries.

During this survey, it was noticed that no work had been done on the problem of a particle traveling in a circular geometry. Since a circular geometry yields the capability of obtaining a long path length in a small space, it was felt that Cerenkov radiation obtained in such a geometry would be of large interest in the construction of a usable microwave power source.

It was decided that, before a proposal for a Cerenkov power source could be made, an investigation of the radiation from a particle in a circular geometry would be necessary. Since it appeared to be an excellent thesis problem, it was undertaken as an interesting theoretical problem and as a vehicle for obtaining a better understanding of the Cerenkov effect.

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A straightforward solution of the wave equation for a particle traveling in a circle seemed to be the most obvious method of approach. After attacking the problem in several coordinate systems, it became obvious that solution in this manner would be extremely difficult, if not impossible, at this time. The problem was set up in cylindrical, spherical, and toroidal coordinates and each of these systems was individually investigated. Toroidal coordinates seemed to show the most promise but there appears to be no analytical solution of the wave equation in these coordinates.

When these attempts at direct solution failed, the Frank and Tamm linear particle solution was examined to see if it could be transformed into a useful solution to the problem. This has proved to be not only feasible but practical and has resulted in a technique which should also be applicable to other similar problems. The method entails a coordinate transformation of the cylindrical coordinates of Frank and Tamm, with a particle traveling along the z axis, into a set of cylindrical coordinates with a particle traveling in the ϕ direction at a distance a from the center of the coordinate system. Before a discussion of this transformation is undertaken, however, a general discussion of Cerenkov radiation and its history will be given.

History

Cerenkov radiation was observed as early as 1910 when Mme. Curie noticed a pale blue glow coming from bottles containing concentrated radium solutions. At that time she was preoccupied with the problems of radioactivity and made no investigations into the cause of the glow. The Cerenkov effect was observed by many investigators working in the field of radioactivity, long before the effect was studied and explained.

Mallet made the first deliberate attempt to study the phenomenon, and discussed his work in three papers in 1926, 1928 and 1929. He discovered several important facts about the radiation – that the light emitted from a large variety of materials, when placed near a radioactive source, always had the same bluish-white quality, and that the spectrum of the

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radiation was continuous rather than discrete or banded. In spite of these discoveries, the work of Mallet largely has been forgotten. This is probably because he did not continue his investigations and because he offered no explanation for the effect.

In 1934, Cerenkov began an intense investigation of the phenomenon which he discovered while studying fluorescence. He continued his investigations until 1938 and obtained remarkable agreement with the theory of Frank and Tamm which was advanced in 1937. Cerenkov's experiments stand as excellent examples of simplicity and good practice.

In 1940, Ginsburg advanced a quantum theory of the phenomenon. His paper stands as one of the best quantum treatments of the subject. There were no contributions to the theory of Cerenkov radiation during the war, but, from 1947 on, considerable work has been done. The photomultiplier was developed by Curran and Baker in 1944 and has found large application in the investigation of Cerenkov radiation.

Cerenkov radiation counters have become important in the field of high energy physics for the investigation of high speed particles. They have been used with the high energy particle machines and for the detection of cosmic rays.

The various theoretical studies in Cerenkov radiation have included radiation in anisotropic media, radiation in ferromagnetics, and radiation from electric and magnetic dipoles. In 1947, Ginsburg suggested Cerenkov radiation as a possible source for microwaves. Many papers on Cerenkov radiation contain schemes for its practical application. Most of these are discussed at length in Jelley's book, <u>Cerenkov Radiation and Its Applications.</u>[1]

Qualitative Description of the Cerenkov Effect

An electron may be considered to be moving with a relatively slow velocity through a dielectric medium along a path from A to B and it may be assumed that it encounters no collisions with the atoms of the material. Then, in the neighborhood of the particle, there will be a polarization of

the atoms of the material (the electrons of the atoms in the dielectric will suffer a displacement from their equilibrium position due to the charge of the electron). Because of the symmetry of the polarization around the electron (P), there will be no resultant field at large distances from the track of the electron (see Figure 1a).

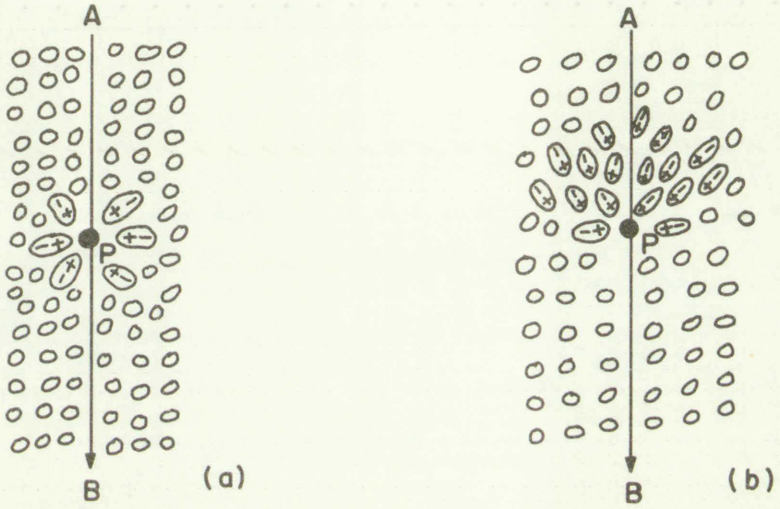

Figure 1. The Polarization Set Up in a Dielectric by the Passage of a Charged Particle (a) At Low Velocity; (b) At High Velocity (Jelley)[1]

Now if the electron is moving with a velocity comparable with or greater than the speed of light in the medium, the polarization field is no longer symmetrical. Since the electron is traveling at a velocity greater than the signal velocity in the medium, the electron's field can only affect the atoms behind it. In this situation a quite noticeable dipole field is set up which is measurable even at long distances. Then each atom along the track will have a momentary dipole field set up in it and will radiate after the electron is past.

The elementary properties of the radiation may be studied with a Huygens construction of the wavelets generated at various points along the path. Such a construction is shown in Figure 2. Here the electron is traveling from A to B with a constant velocity $v = \beta c$, while the light is traveling from A to C. For coherent radiation the wavelets from P_1, P_2, P_3, etc.,

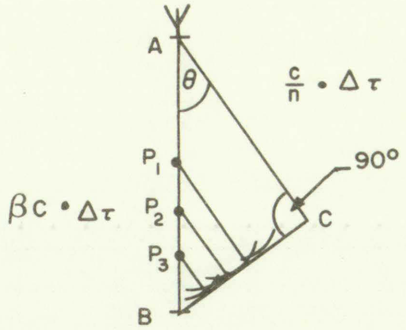

Figure 2. Huygens Construction to Illustrate Coherence (Jelley)[1]

must add in phase along some line CB. Since the velocity of light in the medium is $\frac{c}{n}$ where n is the index of refraction of the medium, the distance AB will be greater than the distance AC ($V > \frac{c}{n}$). Then if $\Delta\tau$ is the time required for the electron to travel from A to B, AB = $\beta c \cdot \Delta\tau$ and AC = $\Delta\tau \cdot (\frac{c}{n})$ which yields the result:

$$\cos \theta = \frac{1}{\beta n} \tag{1}$$

which is the Cerenkov relation.

The Cerenkov relation indicates that for a medium with a refractive index n there is a minimum velocity, $c\beta_{min} = \frac{c}{n}$, below which there is no radiation. At a lower velocity the radiation is along the direction of the particle. For an ultrarelativistic particle ($\beta = 1$) there is a maximum angle of radiation, $\theta_{max} = \cos^{-1}(\frac{1}{n})$. In addition, it may be seen from the Cerenkov relation that most of the radiation occurs in the visible and near-visible regions of the electromagnetic spectrum. In these regions $n > 1$. Radiation cannot occur in the X-ray region of the spectrum, since $n < 1$ and (1) cannot be satisfied.

The above discussion treats the radiation from the electron in only one plane. In three-dimensional space, the radiation would be emitted in a cone whose semivertical angle is the angle θ from the axis. The radiation has a very sharp maximum in the θ direction and approximates a δ-function distribution.

Figure 8. Huygens Construction to Illustrate Cerenkov Radiation

wave and is perpendicular to line AC. Since the velocity of light in the medium is $\frac{c}{n}$ where n is the index of refraction of the medium, the distance AB will be greater than the distance AC $(V > \frac{c}{n})$. Then if Δt is the time required for the electron to travel from A to B, $AB = \beta c \Delta t$ and $AC = \Delta t \left(\frac{c}{n}\right)$ which yields the result:

$$\cos\theta = \frac{1}{\beta n} \qquad (9)$$

which is the Cerenkov relation.

The Cerenkov relation indicates that for a medium with a refractive index n there is a threshold velocity, $\beta_{min} = \frac{1}{n}$, below which there is no radiation. At a lower velocity the radiation is along the direction of the particle. For an ultrarelativistic particle $(\beta \approx 1)$ there is a maximum angle of radiation, $\theta_{max} = \cos^{-1}(1/n)$. In addition, θ may be [illegible] dispersive relation that most of the radiation occurs in the visible and near visible regions of the electromagnetic spectrum. In these regions $n > 1$. Radiation cannot occur in the X-ray region of the spectrum, since $n < 1$ and (9) cannot be satisfied.

The above construction covers the radiation from one direction in only one plane. In three dimensional space, the radiation would be emitted in a cone whose semivertical angle is the angle θ from the axis. The radiation has a very sharp maximum in the θ direction and approximately a [illegible] function distribution.

There are several analogs of the Cerenkov effect in everyday life. Among these are the bow wave created by a ship exceeding the speed of the surface waves on water, and the shock wave created by an airplane exceeding the speed of sound.

Solution to the Problem

As stated, the solution to the problem of Cerenkov radiation from an electron traveling in a circle will be obtained by a transformation of the field expressions obtained by Frank and Tamm. Since the results of Frank and Tamm will form an integral portion of the solution, the basic assumptions and conditions will have to be essentially the same as those used by Frank and Tamm. They are (as stated by Jelley):[1]

1. The medium is considered as a continuum, so that microscopic structure is ignored; the dielectric constant is then the only parameter used to describe the behavior of the medium.
2. Dispersion is ignored, at least to a first approximation.
3. Radiation reaction is neglected.
4. The medium is assumed to be a perfect isotropic dielectric, so that the conductivity is zero, the magnetic permeability $\mu = 1$, and there is no absorption of radiation.
5. The electron is assumed to move at constant [tangential]* velocity; i. e., slowing down as a result of ionization, and the multiple Coulomb scattering are ignored.
6. The medium is unbounded and the track length infinite.

To reduce the difficulties of solving the problem, an additional assumption will be made. It will be assumed that the effects of acceleration

*The word [tangential] is added here to make the assumption more closely fit the problem at hand.

on the electron can be neglected. Although this seems at first glance to be a rather risky assumption, it must be remembered that radiation due to acceleration is a second-order effect while the above conditions limit the solution obtained by Frank and Tamm to a first-order approximation. Since the basis for the solution of this problem is only good to first order, there seems to be no justification for considering a second-order effect.

Frank and Tamm's solution is obtained in cylindrical coordinates (see Appendix) with the electron traveling the z direction. The solution for an electron traveling in a circle can most easily be represented in cylindrical coordinates with the electron traveling in the ϕ direction. To employ the Frank and Tamm field representation, a transformation is required which will have the effect of displacing the path of the electron by a distance a and bending the electron's path (displaced z axis) into a circle in the ϕ' direction. This process may best be illustrated by Figure 3.

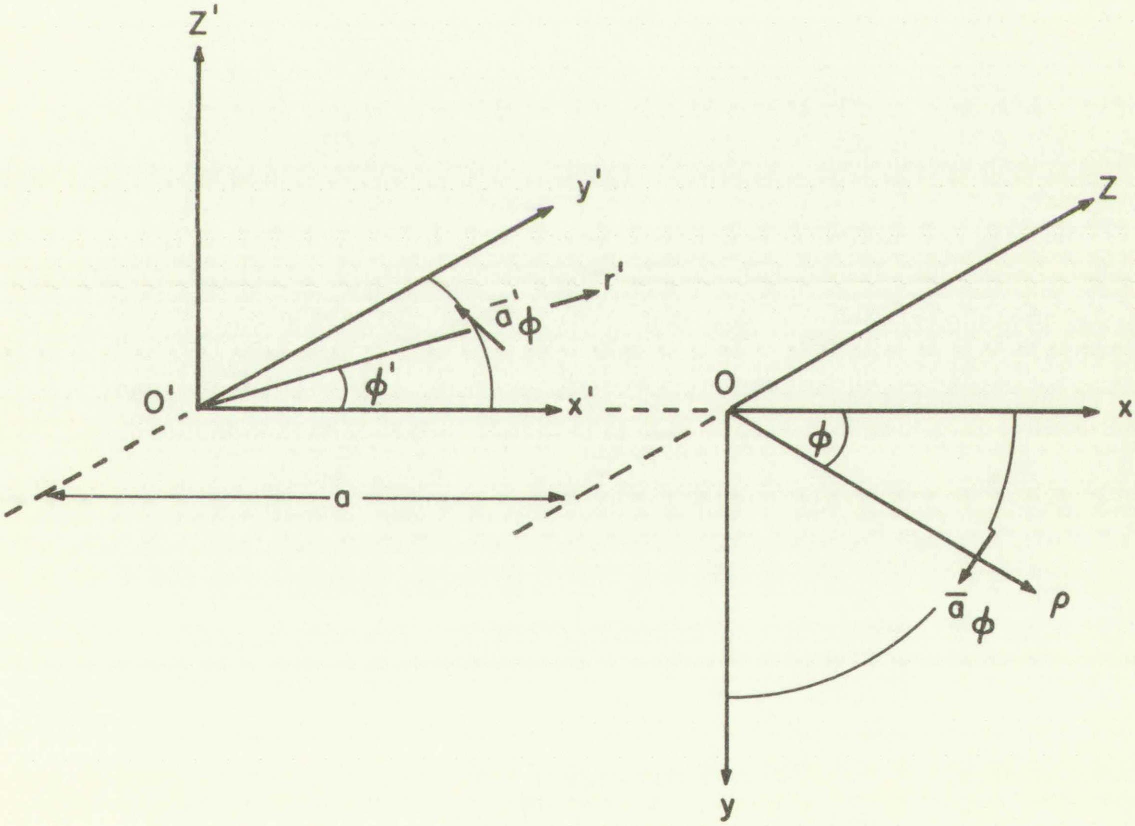

Figure 3. The Frank and Tamm Coordinate System (Unprimed) and the New Coordinate System (Primed)

The unprimed coordinates are those of Frank and Tamm, and the primed coordinates are the new set. Then, from the vector relations between the unit vectors, the following relations may be written for the unprimed set:

$$\bar{a}_\rho = \bar{a}_x \cos\phi + \bar{a}_y \sin\phi \tag{2}$$

$$\bar{a}_\phi = -\bar{a}_x \sin\phi + \bar{a}_y \cos\phi\,. \tag{3}$$

Upon examination of Figure 4, it will be seen that the following relations also hold:

$$\cos\phi = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x' - a}{\sqrt{(x' - a)^2 + z'^2}} \tag{4}$$

$$\sin\phi = \frac{y}{\sqrt{x^2 + y^2}} = \frac{-z'}{\sqrt{(x' - a)^2 + z'^2}} \tag{5}$$

$$\bar{a}_x = \bar{a}'_x \tag{6}$$

$$\bar{a}_y = -\,\bar{a}'_z \tag{7}$$

Then, upon substituting (4), (5), (6), and (7) into (2) and (3), the following relations are obtained:

$$\bar{a}_\rho = \bar{a}'_x \frac{x' - a}{\sqrt{(x' - a)^2 + z'^2}} + \bar{a}'_z \frac{z'}{\sqrt{(x' - a)^2 + z'^2}} \tag{8}$$

$$\bar{a}_\phi = \bar{a}'_x \frac{z'}{\sqrt{(x' - a)^2 + z'^2}} - \bar{a}'_z \frac{x' - a}{\sqrt{(x' - a)^2 + z'^2}} \tag{9}$$

Now there are two problems remaining to the transformation: (1) how to relate the z axis to the ϕ' axis, and (2) how to relate the x' in the above equations to r' of the new cylindrical coordinate system.

The first of these problems may be solved in one of two ways. The easiest and least rigorous solution is to imagine the z axis simply curved or bent around into the ϕ' direction. This results in the relation:

$$\bar{a}_z = \bar{a}'_\phi \tag{10}$$

between the two coordinates.

The more rigorous solution to the problem may be outlined as follows with the help of Figure 4.

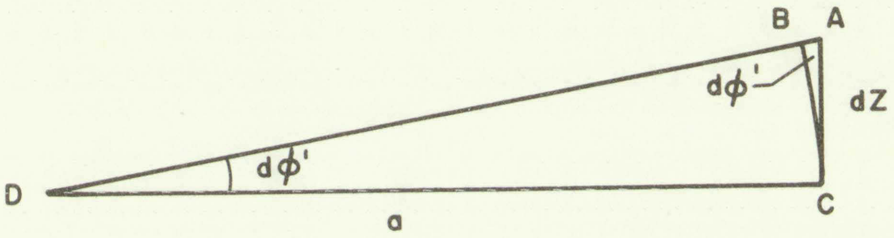

Figure 4. Illustration of Relation Between z and ϕ'

From Figure 4:

$$\overline{AC} = \overline{BC} + \overline{BA} \tag{11}$$

$$\overline{AC} = \bar{a}_z dz \tag{12}$$

$$\left|\overline{AB}\right| = \left|\sin (d\phi') \, dz\right| \doteq dz d\phi' \tag{13}$$

$$\left|\cos (d\phi') dz\right| = \left|\overline{BC}\right| \doteq dz \quad . \tag{14}$$

Substituting (12), (13), and (14) into (11), the relation:

$$\bar{a}_z dz = \bar{a}'_\phi dz + \bar{a}'_r d\phi' dz \tag{15}$$

is obtained. Then, dividing through by dz:

$$\bar{a}_z = \bar{a}'_\phi + \bar{a}'_r \, d\phi' \quad . \tag{16}$$

The first order correction [illegible] is not of [illegible]. The second and least [illegible] relation is to integrate the [illegible] of those [illegible] the r-direction. This results in the relation

$$\xi_k = x_j \qquad (16)$$

between the two coordinates.

The more exact relation of the problem may be [illegible] with the help of Figure 4.

Figure 4. Illustration of Relation Between ξ and x

From Figure 4:

$$AC + BC = AB \qquad (17)$$

$$AC = [illegible] \qquad (18)$$

$$|AB| = [illegible] \qquad (19)$$

$$[illegible] = |BC| [illegible] \qquad (20)$$

Substituting (18), (19), and (20) into (17), the relation

$$[illegible] \qquad (21)$$

is obtained. Then, [illegible] by [illegible]

$$[illegible]$$

Since $\bar{a}'_r$ is multiplied by an infinitesimal, it may be neglected with respect to $\bar{a}'_\phi$ and the final result is:

$$\bar{a}_z = \bar{a}'_\phi \tag{10}$$

which is in complete agreement with the intuitive result.

The answer to the problem of relating x' to r' is not nearly so straightforward. Perhaps the simplest solution is to let $r' = |x'|$, which in the present case amounts to replacing x' by r'. In the primed coordinates, x' would, to be technical, contain direction information; however, any information as to direction which x' contains is superfluous since ϕ' controls the direction in the $r' - \phi'$ plane.

After making these substitutions into equations (8) and (9), the complete transformation is obtained as:

$$\bar{a}_\rho = \bar{a}'_r \frac{r' - a}{\sqrt{(r' - a)^2 + z'^2}} + \bar{a}'_z \frac{z'}{\sqrt{(r' - a)^2 + z'^2}} \tag{17}$$

$$\bar{a}_\phi = \bar{a}'_r \frac{z'}{\sqrt{(r' - a)^2 + z'^2}} - \bar{a}'_z \frac{r' - a}{\sqrt{(r' - a)^2 + z'^2}} \tag{18}$$

$$\bar{a}_z = \bar{a}'_\phi . \tag{19}$$

Now consider the field expressions as obtained by Frank and Tamm* (see Appendix for derivation):

$$E_\rho = \frac{e}{c^2} \int_{\omega > o} \sqrt{\frac{2}{\pi S \rho}} \frac{\sqrt{\beta^2 n^2 - 1}}{\beta^2 n^2} \cos[A] \omega d\omega \tag{20}$$

*The signs on the field expressions in the reprint of Frank and Tamm's original article are incorrect. The signs should be as represented here. This is a self-cancelling error and does not affect the conclusions of Frank and Tamm.

Since E_y is multiplied by an infinitesimal ϵ, it may be neglected with respect to E_x and the final result is:

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After making these substitutions into equations (6) and (8), the near-plane field expression is obtained as:

$$\frac{2}{\sqrt{(r'-a)^2 + z^2}} \qquad (17)$$

$$\frac{\ldots}{\sqrt{(r'-a)^2 + z^2}} \qquad (18)$$

$$(19)$$

After obtaining the field expressions as obtained by Frank and Tamm (see Appendix for derivation):

$$E_r = \frac{\ldots}{\ldots} \int \sqrt{\ldots} \cos(k|z|) d\omega, \qquad (20)$$

The near-plane field expressions in the papers of Frank and Tamm's … should be as represented here. There is a self-consistency error and does not affect the conclusions of Frank and Tamm.

$$E_z = -\frac{e}{c^2}\int_{\omega>o} \sqrt{\frac{2}{\pi S\rho}}\left(1 - \frac{1}{\beta^2 n^2}\right) \cos [A] \omega d\omega \tag{21}$$

$$H_\phi = \frac{e}{c}\int_{\omega>o} \sqrt{\frac{2S}{\pi\rho}} \cos [A] d\omega \tag{22}$$

$$A = \left[\omega\left(t - \frac{z}{v}\right) - S\rho + \frac{\pi}{4}\right]$$

where S is defined by $S^2 = (\omega^2/v^2)(\beta^2 n^2 - 1)$. These relations are the expressions for the fields in a set of cylindrical coordinates which view the electron as traveling in the positive z direction, along the axis, with a velocity $v > \frac{c}{n}$. This treatment will not consider any results at a velocity $v < \frac{c}{n}$ since in the case of a linearly moving electron the fields vanish, and in the case of an electron moving in a circle the fields are only those caused by radial acceleration of the electron and are assumed negligible. It must also be remembered that in these solutions velocity or tangential velocity is considered to be constant.

The field expressions of Frank and Tamm can be transformed into the fields as seen by the moving electron by the application of the following transformation:

$$\overline{E}' = \overline{E} + \frac{\overline{V}}{c} \times \overline{B} \tag{23}$$

$$\overline{B}' = \overline{B} - \frac{\overline{V}}{c} \times \overline{E} \tag{24}$$

where the prime indicates the field after transformation. It will be noticed that these are the nonrelativistic field transformations. Since the Frank and Tamm treatment is nonrelativistic, the assumption must be made that $\beta \ll 1$, which is the condition for application of the nonrelativistic transformations. It will be seen, however, at the conclusion of the following two transformations, that there is an error of $(1 - \beta^2)$ introduced and that this error could be removed by including the relativistic correction in the transformations.

[illegible] (21)

[illegible] (22)

where k is defined by [illegible]. These relations are the expressions for the fields in terms of cylindrical coordinates when the electrons are moving in the positive z direction, along the axis, with a velocity [illegible]. This treatment will not consider any results at a velocity [illegible] since in the case of a linearly moving electron the fields [illegible], and in the case of an electron moving in a circle the fields are only those caused by radial acceleration of the electrons and are assumed negligible. It must also be remembered that in these solutions velocity or tangential velocity is considered to be constant.

The field components of Frank and Tamm can be transformed into the fields as seen by the moving electron by the application of the following transformation:

$$E' = E + \frac{V}{c} \times H \qquad (23)$$

$$H' = H - \frac{V}{c} \times E \qquad (24)$$

where the prime indicates the field after transformation. It will be noticed that these are the non-relativistic field transformations. Since the [illegible] and [illegible] is [illegible], the assumption must be made that [illegible] which is the condition for application of the non-relativistic transformations. It will be shown, however, in the application of the following two transformations, that there is an error of [illegible] and that this error could be removed by including the relativistic correction in the transformations.

The individual components of the transformed field may be written immediately from equations (23) and (24). Since $\mu = 1$, $\overline{B} = \overline{H}$ and the components of the transformation become:

$$E'_\rho = E_\rho - \frac{v}{c} H \tag{25}$$

$$E'_z = E_z \tag{26}$$

$$H'_\phi = H_\phi - \frac{v}{c} E_\rho \tag{27}$$

Applying (25), (26), and (27) to (20), (21), and (22) yields:

$$E'_\rho = \frac{e}{c} \int_{\omega > o} \sqrt{\frac{2}{\pi\rho}} \left\{ \frac{\omega\sqrt{\beta^2 n^2 - 1}}{c\beta^2 n^2 \sqrt{S}} - \frac{v}{c}\sqrt{S} \right\} \cos\,[\mathbf{B}]\,d\omega \tag{28}$$

$$E'_z = -\frac{e}{c^2} \int_{\omega > o} \sqrt{\frac{2}{\pi S\rho}} \left(1 - \frac{1}{\beta^2 n^2}\right) \omega \cos[\,B]\,d\omega \tag{29}$$

$$H'_\phi = \frac{e}{c} \int_{\omega > o} \sqrt{\frac{2}{\pi\rho}} \left\{ \sqrt{S} - \frac{v\omega\sqrt{\beta^2 n^2 - 1}}{c^2\beta^2 n^2 \sqrt{S}} \right\} \cos\,[\,B]\,d\omega \tag{30}$$

$$B = \left[\omega\,(t - t_o) - S\rho + \frac{\pi}{4}\right]$$

where $t_o = \frac{z}{v}$ with $z = vt_o$. Here the use of a dummy variable, t_o, is merely an artifice for keeping track of the time dependence.

The coordinate transformation ((17), (18), and (19)) may now be applied to the static fields. It must be realized, that after the electron path is translated to the right, the fields will appear as though they were generated by either a medium moving in the minus z direction past the electron or by a medium moving in the minus ϕ' direction past the electron. To anticipate the latter case, it is expedient to replace the Fourier integral representation of the fields with a Fourier series representation. This step is required because of the periodicity of the motion of an electron in a circle. This requires replacing ω with $\frac{2m\pi}{T}$, $d\omega$ with $\frac{2\pi}{T}$, and the integration

with a summation. Here T is the time required for the electron to make one trip around the circle and m is an integer (m = 1, 2, 3, . . .). Then the field representation of an electron traveling in a circle, as seen from the electron, can be written as:

$$\overline{a}'_\rho E'_\rho = \left\{ \overline{a}''_r \frac{(r' - a)}{\sqrt{(r' - a)^2 + z'^2}} + \overline{a}''_z \frac{z'}{\sqrt{(r' - a)^2 + z'^2}} \right\} \frac{2e\pi}{cT} \sum_m \sqrt{\frac{2}{\pi}} \times$$

$$\left\{ \frac{2m\pi\sqrt{\beta^2 n^2 - 1}}{Tc\ \beta^2 n^2 \sqrt{S}} - \frac{v}{c}\sqrt{S} \right\} \frac{\cos [C]}{\left[(r' - a)^2 + z'^2\right]^{1/4}}$$

$$\overline{a}'_z E'_z = - \overline{a}''_\phi \frac{4e\pi^2}{c^2 T^2} \sum_m \sqrt{\frac{2}{\pi S}} \frac{m\left(1 - \frac{1}{\beta^2 n^2}\right)}{\left[(r' - a)^2 + z'^2\right]^{1/4}} \cos [C]$$

$$\overline{a}'_\phi H_\phi = \left\{ \overline{a}''_r \frac{z'}{\sqrt{(r' - a)^2 + z'^2}} - \overline{a}''_z \frac{(r' - a)}{\sqrt{(r' - a)^2 + z'^2}} \right\} \frac{2e\pi}{cT} \sum_m \sqrt{\frac{2}{\pi}} \times$$

$$\left\{ \sqrt{S} - \frac{2m\pi v\sqrt{\beta^2 n^2 - 1}}{Tc^2 \beta^2 n^2 \sqrt{S}} \right\} \frac{\cos [C]}{\left[(r' - a)^2 + z'^2\right]^{1/4}}$$

where:

$$C = \left[\frac{2m\pi}{T} (t - t_o) - S\sqrt{(r' - a)^2 + z'^2} + \frac{\pi}{4} \right]$$

When separated into individual components these become:

$$E''_r = \frac{2e\pi}{cT} \sqrt{\frac{2}{\pi}} \sum_m \frac{(r' - a)}{\left[(r' - a)^2 + z'^2\right]^{3/4}} \left\{ \frac{2m\pi\sqrt{\beta^2 n^2 - 1}}{Tc\beta^2 n^2 \sqrt{S}} - \frac{v}{c}\sqrt{S} \right\} \cos [C] \qquad (31)$$

$$E''_z = \frac{2e\pi}{cT}\sqrt{\frac{2}{\pi}}\sum_m \frac{z'}{\left[(r' - a)^2 + z'^2\right]^{3/4}}\left\{\frac{2m\pi\sqrt{\beta^2 n^2 - 1}}{Tc\beta^2 n^2\sqrt{S}} - \frac{v}{c}\sqrt{S}\right\}\cos[C] \tag{32}$$

$$E''_\phi = -\frac{4e\pi^2}{c^2T^2}\sum_m \sqrt{\frac{2}{\pi S}}\frac{m\left(1 - \frac{1}{\beta^2 n^2}\right)}{\left[(r' - a)^2 + z'^2\right]^{1/4}}\cos[C] \tag{33}$$

$$H''_r = \frac{2e\pi}{cT}\sqrt{\frac{2}{\pi}}\sum_m \frac{z'}{\left[(r' - a)^2 + z'^2\right]^{3/4}}\left\{\sqrt{S} - \frac{2m\pi v\sqrt{\beta^2 n^2 - 1}}{Tc^2\beta^2 n^2\sqrt{S}}\right\}\cos[C] \tag{34}$$

$$H''_z = -\frac{2e\pi}{cT}\sqrt{\frac{2}{\pi}}\sum_m \frac{(r' - a)}{\left[(r' - a)^2 + z'^2\right]^{3/4}}\left\{\sqrt{S} - \frac{2m\pi v\sqrt{\beta^2 n^2 - 1}}{Tc^2\beta^2 n^2\sqrt{S}}\right\}\cos[C] \tag{35}$$

These fields may be transformed to a reference frame where an observer sees the electron traveling in a circle by the inverse transformations:

$$\overline{E}''' = \overline{E}'' - \frac{\overline{V}}{c}\times\overline{H} \tag{36}$$

$$\overline{B}''' = \overline{B}'' + \frac{\overline{V}}{c}\times\overline{E} \tag{37}$$

It is obvious that these transformation equations do not fit a transformation which is not linear; however, if the radius, a, is assumed large and radial acceleration is neglected, the transformation holds to a first order of approximation. To fit these equations to the problem, $\overline{V}$ must

become $a\alpha$ where α is the angular velocity in the ϕ' direction and is assumed constant. With these considerations, the components of the transformation become:

$$E_r''' = E_r'' - \frac{a\alpha}{c} H_z'' \tag{38}$$

$$E_\phi''' = E_\phi'' \tag{39}$$

$$E_z''' = E_z'' + \frac{a\alpha}{c} H_r'' \tag{40}$$

$$H_r''' = H_r'' + \frac{a\alpha}{c} E_z'' \tag{41}$$

$$H_z''' = H_z'' - \frac{a\alpha}{c} E_r'' \tag{42}$$

The field components in the primed coordinate system then become:

$$E_r''' = \frac{2e\pi}{cT}\sqrt{\frac{2}{\pi}} \sum_m \frac{(r' - a)}{\left[(r' - a)^2 + z'^2\right]^{3/4}} \left\{ \frac{2m\pi\sqrt{\beta^2 n^2 - 1}}{Tc\beta^2 n^2 \sqrt{S}} - \frac{v}{c}\sqrt{S} + \frac{a\alpha}{c}\sqrt{S} - \frac{2m\pi a\alpha v\sqrt{\beta^2 n^2 - 1}}{Tc^3\beta^2 n^2 \sqrt{S}} \right\} \cos\,[D] \tag{43}$$

$$E_z''' = \frac{2e\pi}{cT}\sqrt{\frac{2}{\pi}} \sum_m \frac{z'}{\left[(r' - a)^2 + z'^2\right]^{3/4}} \left\{ \frac{2m\pi\sqrt{\beta^2 n^2 - 1}}{Tc\beta^2 n^2 \sqrt{S}} - \frac{v}{c}\sqrt{S} + \frac{a\alpha}{c}\sqrt{S} - \frac{2m\ a\alpha v\sqrt{\beta^2 n^2 - 1}}{Tc^3\beta^2 n^2 \sqrt{S}} \right\} \cos\,[D] \tag{44}$$

$$E_\phi''' = -\frac{4e\pi^2}{c^2 T^2} \sum_m \sqrt{\frac{2}{\pi S}}\ \frac{m\left(1 - \frac{1}{\beta^2 n^2}\right)}{\left[(r' - a)^2 + z'^2\right]^{1/4}} \cos\,[D] \tag{45}$$

$$H_r''' = \frac{2e\pi}{cT}\sqrt{\frac{2}{\pi}}\sum_m \frac{z'}{\left[(r'-a)^2+z'^2\right]^{3/4}}\left\{\sqrt{S} - \frac{2m\pi v\sqrt{\beta^2 n^2 - 1}}{Tc^2\beta^2 n^2\sqrt{S}}\right.$$

$$\left. - \frac{a\alpha v}{c^2}\sqrt{S} + \frac{2m\pi a\alpha\sqrt{\beta^2 n^2 - 1}}{Tc^2\beta^2 n^2\sqrt{S}}\right\} \cos[D] \tag{46}$$

$$H_z''' = -\frac{2e\pi}{cT}\sqrt{\frac{2}{\pi}}\sum_m \frac{(r'-a)}{\left[(r'-a)^2+z'^2\right]^{3/4}}\left\{\sqrt{S} - \frac{2m\pi v\sqrt{\beta^2 n^2 - 1}}{Tc^2\beta^2 n^2\sqrt{S}}\right.$$

$$\left. - \frac{a\alpha v}{c^2}\sqrt{S} + \frac{2m\pi a\alpha\sqrt{\beta^2 n^2 - 1}}{Tc^2\beta^2 n^2\sqrt{S}}\right\} \cos[D] \tag{47}$$

with:

$$D = \left[\frac{2m\pi}{T}\left(t - \frac{\phi'}{\alpha}\right) - S\sqrt{(r'-a)^2 + z'^2} + \frac{\pi}{4}\right]$$

where

$$t_o \rightarrow \frac{\phi'}{\alpha} .$$

If $a\alpha$ is set equal to v, i. e., if the velocity in the z direction is equal to the tangential velocity in the ϕ' direction, then equations (43) through (47) reduce to:

$$E_r''' = \frac{2e\pi}{cT}\sqrt{\frac{2}{\pi}}\sum_m \frac{(r'-a)}{\left[(r'-a)^2+z'^2\right]^{3/4}} \times$$

$$\left\{\frac{2m\pi\sqrt{\beta^2 n^2 - 1}}{Tc\beta^2 n^2\sqrt{S}}(1-\beta^2)\right\} \cos[D] \tag{48}$$

$$E_z''' = \frac{2e\pi}{cT}\sqrt{\frac{2}{\pi}}\sum_m \frac{z'}{\left[(r'-a)^2+z'^2\right]^{3/4}} \times$$

$$\left\{\frac{2m\pi\sqrt{\beta^2 n^2 - 1}}{Tc\beta^2 n^2\sqrt{S}}(1-\beta^2)\right\} \cos[D] \tag{49}$$

$$E_z''' = \frac{4e\pi^2}{c^2T^2}\sqrt{\frac{2}{\pi}}\sum_m \frac{z'}{\left[(r'-a)^2+z'^2\right]^{3/4}} \times$$

$$\left\{\frac{m\sqrt{\beta^2n^2-1}}{\beta^2n^2\sqrt{S}}\right\}\cos[D] \tag{54}$$

$$E_\phi''' = -\frac{4e\pi^2}{c^2T^2}\sum_m\sqrt{\frac{2}{\pi S}}\;\frac{m\left(1-\frac{1}{\beta^2n^2}\right)}{\left[(r'-a)^2+z'^2\right]^{1/4}}\cos[D] \tag{55}$$

$$H_r''' = \frac{2e\pi}{cT}\sqrt{\frac{2}{\pi}}\sum_m \frac{z'\sqrt{S}}{\left[(r'-a)^2+z'^2\right]^{3/4}}\cos[D] \tag{56}$$

$$H_z''' = -\frac{2e\pi}{cT}\sqrt{\frac{2}{\pi}}\sum_m \frac{(r'-a)\sqrt{S}}{\left[(r'-a)^2+z'^2\right]^{3/4}}\cos[D] \tag{57}$$

$$D = \left[\frac{2m\pi}{T}\left(t-\frac{\phi'}{\alpha}\right) - S\sqrt{(r'-a)^2+z'^2}+\frac{\pi}{4}\right]$$

Since the fields are now represented by a Fourier series and ω has been redefined as $\frac{2m\pi}{T}$, S must also be redefined. Henceforth, S will be defined by the relation:

$$S^2 = \frac{4m^2\pi^2}{T^2a^2\alpha^2}\left(\beta^2n^2-1\right) \tag{58}$$

where β is defined by:

$$\beta = \frac{a\alpha}{c}\,. \tag{59}$$

Since the above series of transformations seem to have yielded correct relations for the fields, one immediately begins to wonder whether the direct coordinate transformation would not yield the same results. The direct coordinate transformation not only yields the same field expressions but also offers a proof of the validity of the resultant expressions. Here by direct coordinate transformation is meant the direct application of (17),

(18) and (19) to (20), (21) and (22) with modifications of the solution to fit the periodic nature of the problem. Of course, this argument could also be turned around and the transformation procedure involving the use of the auxiliary reference frame might be offered as the proof of correctness.

The radiation out of a toroid surrounding the electron path for one trip of the electron around the path may be obtained from the expression:

$$W = \frac{c}{4\pi} \int_o^T \int_A \overline{(\overline{E} \times \overline{H})} \cdot \overline{dA} dt \tag{60}$$

where A is the area of the surface, $\overline{dA}$ is the elemental vector area, and T is the period of the electron motion. The integration over T may be performed with the aid of the relation:

$$\int_o^T \cos\left(\frac{2m\pi t}{T} + \gamma\right) \cos\left(\frac{2p\pi t}{T} + \gamma\right) dt = \frac{T}{2}\, \delta_{mp} \tag{61}$$

where γ is an arbitrary angle and δ_{mp} is the Kroenecker delta. This yields the result:

$$W = \frac{e^2}{c^2} \frac{2\pi}{T^2} \sum_m \sum_p m\left(1 - \frac{1}{\beta^2 n^2}\right) \delta_{mp} \times$$

$$\int_A \left(\frac{z' \overline{a}'_z + (r' - a)\, \overline{a}'_r}{\left[(r' - a)^2 + z'^2\right]} \right) \cdot \overline{dA} \tag{62}$$

The surface integral is, perhaps, best evaluated separately. The parametric equations of the toroidal surface may be written as:

$$x = (a + b \cos \theta) \cos \phi$$
$$y = (a + b \cos \theta) \sin \phi$$
$$z = b \sin \theta \tag{63}$$

where a is the distance from the origin to the toroidal axis (electron path), b is the distance from the axis to the surface of the toroid, θ is the angle between the $r - \phi$ plane and a vector from the axis to the surface of the

toroid, and ϕ is the angular distribution in the $r - \phi$ plane. Then the elemental vector area may be written with:

$$\overline{dA} = \left[\frac{\partial(y,z)}{\partial(\phi,\theta)}\,\bar{i} + \frac{\partial(z,x)}{\partial(\phi,\theta)}\,\bar{j} + \frac{\partial(x,y)}{\partial(\phi,\theta)}\,\bar{k}\right] d\theta d\phi \tag{64}$$

which evaluates to the result:

$$\begin{aligned}\overline{dA} = \Big[&(ab\cos\theta + b^2\cos^2\theta)\cos\phi\,\vec{i}\\ &+(ab\cos\theta + b^2\cos^2\theta)\sin\phi\,\vec{j}\\ &+(a + b\cos\theta)b\sin\theta\,\vec{k}\Big]d\theta d\phi\end{aligned} \tag{65}$$

in rectangular coordinates.

The vector $\left(\frac{z'\bar{a}'_z + (r' - a)\bar{a}'_r}{\left[(r' - a)^2 + z'^2\right]}\right)$ may be written on the torus in rectangular coordinates as:

$$\left(\frac{z'\bar{a}'_z + (r' - a)\bar{a}'_r}{\left[(r' - a)^2 + z'^2\right]}\right) = \frac{\cos\theta\cos\phi}{b}\,\vec{i} + \frac{\cos\theta\sin\phi}{b}\,\vec{j} + \frac{\sin\theta}{b}\,\vec{k} \tag{66}$$

Then combining these two expressions, the surface integral may be written as:

$$\int_A \left(\frac{z'\bar{a}'_z + (r' - a)\bar{a}'_r}{\left[(r' - a)^2 + z'^2\right]}\right)\cdot\overline{dA} = \int_o^{2\pi}\int_o^{2\pi}\left[(a + b\cos\theta)\cos^2\theta + (a + b\cos\theta)\sin^2\theta\right]d\theta d\phi \tag{67}$$

which evaluates immediately to yield:

$$\int_A \left(\frac{z'\bar{a}'_z + (r' - a)\bar{a}'_r}{\left[(r' - a)^2 + z'^2\right]}\right)\cdot\overline{dA} = 4\pi^2 a \tag{68}$$

Equation (62) now takes the form:

$$W = \frac{e^2}{c^2} \frac{8\pi^3 a}{T^2} \sum_m \sum_p m\left(1 - \frac{1}{\beta^2 n^2}\right)\delta_{mp} \ . \tag{69}$$

The summation over p in (69) may be immediately removed along with the Knonecker delta and the radiation out of the toroid during one period of the electron motion becomes:

$$W = \frac{e^2}{c^2} \frac{8\pi^3 a}{T^2} \sum_m m\left(1 - \frac{1}{\beta^2 n^2}\right) . \tag{70}$$

It may be noticed that $2\pi a$ is exactly the path length for one period of the electron motion. Then if $2\pi a$ is set equal to ℓ, an expression for the radiation out of the toroid per unit path length may be written as:

$$\frac{dW}{d\ell} = \frac{e^2}{c^2} \frac{4\pi^2}{T^2} \sum_m m\left(1 - \frac{1}{\beta^2 n^2}\right) \tag{71}$$

by taking the derivative of (70) with respect to ℓ.

Equation (71) appears to "blow up" for large m; however, since m is a measure of the frequency of the radiation, a physical system will have a "cutoff" frequency. The "cutoff" may be introduced by the fact that the radiation must be limited to wavelengths which are greater than the classical diameter of the electron, or the "cutoff" may occur where $n(\omega) > \frac{1}{\beta}$ in the dielectric. In the first case, the cutoff will be governed by the expression:

$$\frac{\lambda}{2\pi} = d \tag{72}$$

where d is the classical diameter of the electron. Then since:

$$\frac{c}{n\lambda} = \frac{\omega}{2\pi} = \frac{m}{T}$$

m_{max} is from (72):

$$m_{max} = \frac{cT}{2\pi n d} \tag{73}$$

and the expression for the radiation becomes:

$$\frac{dW}{d\ell} = \frac{e^2}{c^2} \frac{4\pi^2}{T^2} \sum_{m=1}^{\frac{cT}{2\pi nd}} m\left(1 - \frac{1}{\beta^2 n^2}\right) \quad . \tag{74}$$

For the condition of cutoff due to the frequency dependence of the dielectric medium, the "cutoff" condition is:

$$n\left(\omega_{max}\right) > \frac{1}{\beta} \tag{75}$$

so that above $m = m_{max}$ there is no radiation and the expression of the radiation per unit path length becomes:

$$\frac{dW}{d\ell} = \frac{e^2}{c^2} \frac{4\pi^2}{T^2} \sum_{m=1}^{m_{max}} m\left(1 - \frac{1}{\beta^2 n^2}\right) \tag{76}$$

In both (74) and (76), the term in brackets is normally a very slowly changing function of frequency so that it may be removed from under the summation sign in most cases.

Equations (74) and (76) are very similar in form to the result of Frank and Tamm which is stated here from the Appendix:

$$\frac{dW}{d\ell} = \frac{e^2}{c^2} \int_{\beta n > 1} \omega d\omega \left(1 - \frac{1}{\beta^2 n^2}\right) \tag{77}$$

For the limit of $\frac{1}{T}$ small (T large), expressions (74) and (76) approach (77).

Applications

With some thought a number of device configurations can be envisioned which would employ an electron traveling in a circle. Two such configurations will be discussed.

Consider an electron beam in which the electrons are accelerated to a velocity near the velocity of light and shot between two closely spaced slabs of dielectric in a perpendicular magnetic field (Figure 5). Then as the beam enters the static magnetic field, it will be bent into a circular path. Since the electron must appear to be inside the medium for the Cerenkov effect to exist, the spacing between the slabs must be very close--of approximately the same size as the diameter of the electron beam.

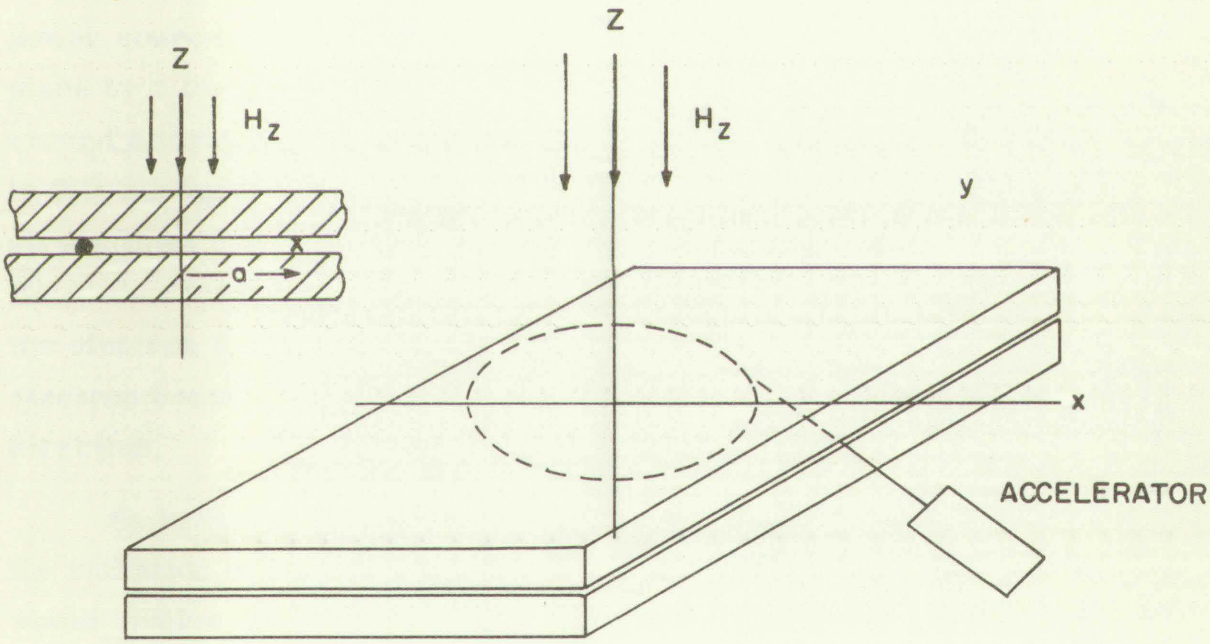

Figure 5. An Electron Constrained to a Circle by a Perpendicular Magnetic Field

In this configuration, the electron would continually lose energy through radiation and spiral into the center of the coordinate system. If, however, a slowly increasing magnetic field is applied in the z direction, the loss due to radiation can be compensated for and the electron will be held in the circular orbit and retain its constant tangential velocity.

There will also be a tendency for dispersion of the electron beam in the positive and negative z directions due to the charge repulsion between the electrons. It is believed that if the surface conductivity of the dielectric is made extremely low, a surface charge will be built up on the dielectric which will constrain the electrons to the r-ϕ plane. This surface charge will be established by electrons which are repelled by the beam and strike the surface. If the electron beam intensity is sufficiently low, it should be relatively easy to establish an equilibrium in the z direction.

Another configuration, such as Figure 6, might also be usable for a power source. In this case the electron beam is constrained to the r-ϕ plane by a toroidal magnetic field which is established by a helix wrapped around a toroidal piece of dielectric with a circular hole along the axis. In this configuration, the electron beam would be forced to follow the streamlines of the magnetic field. In addition to the constraining magnetic field, there will be a surface charge build-up which will aid in constraining the electron beam. If additional constraint were required to hold the electron beam in a circle, a magnetic field might also be added in the z direction.

As in the first example, the electrons will tend to slow down due to the radiation of energy. One proposal for maintaining the electron beam would involve bunching the beam and applying an a-c field to a helix wound on the toroid. This would have the effect of coupling energy into the beam from the traveling field similar to the method used in a linear accelerator.

Both of these devices would generate radiation in a band of frequencies. A cavity or some other tuning device would be necessary to restrict the output to a narrow range of frequencies. A Laser might be used to limit the output bandwidth and provide an increase in signal output; however, these devices operate at too low a power for present application.

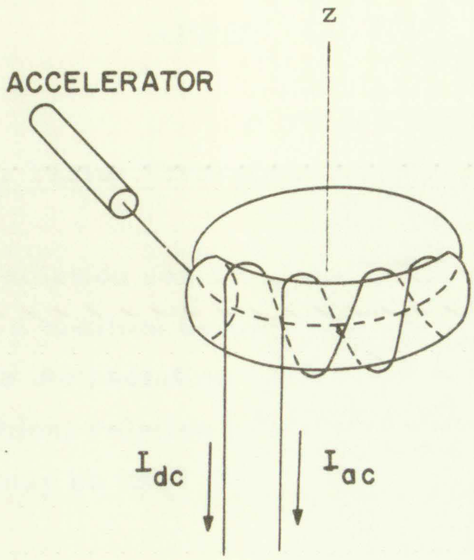

Figure 6. An Electron Constrained to a Circle by a Toroidal Magnetic Field

Summary

A first order theory has been developed for an electron traveling in a circle as an extension of the Frank and Tamm theory of Cerenkov radiation. This theory has resulted in an eigenfrequency spectrum for the radiated energy as opposed to the continuous spectrum for the Frank and Tamm model. It is believed that this theory can be applied to a practical microwave power source in the sub-millimeter region of the electromagnetic spectrum. Two devices which should be capable of applying this theory have been proposed with the hope that they might lead to a practical device in the future.

APPENDIX

Frank and Tamm Theory of Cerenkov Radiation

Since Cerenkov radiation can be considered to be caused by the instantaneous polarization of a medium by the passage of a charged particle with velocity $v > \frac{c}{n}$ and since the radiation from the medium can be treated macroscopically, the dinamical relation between the polarization, $\overline{P}$, and the electric intensity, $\overline{E}$, may be used (Sommerfield):

$$\frac{\partial^2 \overline{P}}{\partial t^2} + \sum_s \omega_s^2 \overline{P}_s = a \overline{E} \tag{78}$$

where ω_s are frequencies of the molecular oscillators of the medium. The field variables may be expanded in Fourier integrals as:

$$\overline{E} = \int_{-\infty}^{\infty} \overline{E}_\omega e^{i\omega t} d\omega$$

$$\overline{P} = \int_{-\infty}^{\infty} \overline{P}_\omega e^{i\omega t} d\omega \tag{79}$$

and the connection:

$$\overline{P}_\omega = (n^2 - 1) \overline{E}_\omega \tag{80}$$

may be obtained between $\overline{P}_\omega$ and $\overline{E}_\omega$ where n is the refractive index of the medium for a frequency ω.

From (79 and (80), Maxwells equations reduce to the form:

$$\overline{H}_\omega = \nabla \times \overline{A}_\omega \tag{81}$$

$$\overline{E}_\omega = -\text{grad}\,\phi_\omega - \frac{i\omega}{c}\overline{A}_\omega$$

$$= -\frac{ic}{\omega n^2}\nabla(\nabla \cdot \overline{A}_\omega) - \frac{i\omega}{c}\overline{A}_\omega \tag{82}$$

$$\nabla^2 \overline{A}_\omega + \frac{\omega^2 n^2}{c^2}\overline{A}_\omega = -\frac{4\pi}{c}\overline{J}_\omega \tag{83}$$

where $\mu = 1$ and the connection between the vector and scaler potentials is:

$$\nabla \cdot \overline{A}_\omega + \frac{i\omega}{c} n^2 \phi_\omega = 0 \tag{84}$$

If an electron, e, is moving through the medium along the z axis with a constant velocity, v, then the current density may be written as:

$$J_z = ev\,\delta(x)\,\delta(y)\,\delta(z - vt) \tag{85}$$

and the Fourier transform of J_z becomes:

$$J_z(\omega) = \frac{e}{2\pi} e^{-\frac{i\omega z}{v}} \delta(x)\,\delta(y) \tag{86}$$

This may be rewritten in cylindrical coordinates as:

$$J_z(\omega) = \frac{e}{4\pi^2 \rho} e^{-\frac{i\omega z}{v}} \delta(\rho) \tag{87}$$

Substituting (87) into (83) and setting:

$$A_z(\omega) = u(\rho)\,e^{-\frac{i\omega z}{v}}$$

$$A_\rho = A_\phi = 0 \tag{88}$$

the expression:

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + s^2 u = -\frac{e}{\pi c \rho}\delta(\rho) \tag{89}$$

is obtained for u where:

$$S^2 = \frac{\omega^2}{v^2}(\beta^2 n^2 - 1) = -\sigma \tag{90}$$

From (89) it is obvious that u is a cylindrical function which satisfies the Bessel equation:

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + S^2 u = 0 \tag{91}$$

at all points except the point $\rho = 0$. To find the condition on u at $\rho = 0$, replace the right-hand side of (89) with a function, f, such that:

$$f = -\frac{2e}{\pi c \rho_o^2} \quad \text{if } \rho < \rho_o$$

$$f = 0 \qquad \text{if } \rho > \rho_o \tag{92}$$

integrate the resultant equation over the circle of radius ρ_o, and take the limit as $\rho_o \longrightarrow 0$. It will be noticed here that the δ - function has been replaced by a function, f, which approaches it in the limit. Then the condition on u at $\rho = 0$ becomes:

$$\lim_{\rho \to 0}\left(\rho \frac{\partial u}{\partial \rho}\right) = -\frac{e}{\pi c} \tag{93}$$

Now two cases must be investigated--the case for $\beta n < 1$ and the case for $\beta n > 1$. For the case with $\beta n < 1$, the solution of (91) which meets the condition (93) and vanishes at infinity is:

$$u = \frac{ie}{2c} H_o^{(1)}(i\sigma\rho) \tag{94}$$

since $S^2 < 1$ and $\sigma^2 = -S^2 > 0$ where σ is a real quantity. Here $H_o^{(1)}$ is the Hankel function of the first kind.

If $\sigma\rho \gg 1$, then the asymptotic expansion of $H_o^{(1)}$ may be employed to obtain with (88) and (94):

$$A_z = \frac{e}{c}\int_{-\infty}^{\infty} \frac{e^{-\sigma\rho + i\omega\left(t - \frac{z}{v}\right)}}{\sqrt{2\pi\sigma\rho}}\, d\omega \tag{95}$$

From (95) it is seen that the field from an electron traveling with a velocity $v < \frac{c}{n}$ decreases exponentially with ρ so that there is no radiation.

For the case of $\beta n > 1$ in some frequency range, S is real and the solution of (91) which meets the condition (93) is

$$u = -\frac{ie}{2c} H_o^{(2)} (S\rho) \tag{96a}$$

for $\omega > 0$ and:

$$u = \frac{ie}{2c} H_o^{(1)} (S\rho) \tag{96b}$$

for $\omega < 0$. Here $H_o^{(1)}$ is the Hankel function of the first kind and $H_o^{(2)}$ is the Hankel function of the second kind. S is always taken positive.

If $S\rho >> 1$, the exponential values of the Hankel functions may be used and with (88) yield:

$\omega > 0$

$$A_z(\omega) = -\frac{e}{c\sqrt{2\pi S\rho}} \exp\left[i\omega\left(t - \frac{z}{v}\right) - i\left(S\rho - \frac{3\pi}{4}\right)\right] \tag{97a}$$

and:

$\omega < 0$

$$A_z(\omega) = -\frac{e}{c\sqrt{2\pi S\rho}} \exp\left[i\omega\left(t - \frac{z}{v}\right) + i\left(S\rho - \frac{3\pi}{4}\right)\right] \tag{97b}$$

Applying (81) to (97a) and (97b), expressions for the magnetic intensity are obtained as:

$\omega > 0$

$$H_\phi(\omega) = \left\{\frac{e\sqrt{S}}{c\sqrt{2\pi\rho}} - \frac{ie\rho^{-\frac{3}{2}}}{2c\sqrt{2\pi S}}\right\} \exp\left[i\omega\left(t - \frac{z}{v}\right) - i\left(S\rho - \frac{\pi}{4}\right)\right]$$

$\omega < 0$

$$H_\phi(\omega) = \left\{\frac{e\sqrt{S}}{c\sqrt{2\pi\rho}} - \frac{ie\rho^{-\frac{3}{2}}}{2c\sqrt{2\pi S}}\right\} \exp\left[i\omega\left(t - \frac{z}{v}\right) + i\left(S\rho - \frac{\pi}{4}\right)\right]$$

These expressions may be combined with the aide of

$$2\cos[X] = e^{iX} + e^{-iX}$$

$$2\,i\sin[X] = e^{iX} - e^{-iX} \tag{98}$$

by taking both expressions over the range $\omega > 0$ to obtain:

$$H_\phi = \frac{e}{c}\sqrt{\frac{2}{\pi\rho}} \int_{\omega>o} \sqrt{S}\cos[X]\, d\omega + \frac{e\rho^{-\frac{3}{2}}}{c\sqrt{2\pi}} \int_{\omega>o} \frac{\sin[X]}{\sqrt{S}}\, d\omega$$

$$[X] = \left[\omega\left(t - \frac{z}{v}\right) - S\rho + \frac{\pi}{4}\right] \tag{99}$$

The second integral may be neglected at far distances which gives the result:

$$H_\phi = \frac{e}{c}\sqrt{\frac{2}{\pi\rho}} \int_{\omega>o} \sqrt{S}\cos[X]\, d\omega$$

$$[X] = \left[\omega\left(t - \frac{z}{v}\right) - S\rho + \frac{\pi}{4}\right] \tag{100}$$

The expressions for the electric intensities, E_z and E_ρ may be calculated from (82) and (97). An intermediate form of these expressions will result which must be defined over the two regions of ω as in the intermediate result obtained for the magnetic intensity. The two regions of the intermediate forms for the electric intensities may be combined by the same method as was used to obtain (99). A factor appears in the result which can be neglected as $\rho \longrightarrow \infty$ in the same fashion as was done to obtain (100). Then the resultant expressions for the electric intensity are:

$$E_z = -\frac{e}{c^2}\sqrt{\frac{2}{\pi\rho}} \int_{\omega>o} \left(1 - \frac{1}{\beta^2 n^2}\right) \frac{\cos[X]}{\sqrt{S}}\,\omega\, d\omega \tag{101}$$

$$E_\rho = \frac{e}{c^2}\sqrt{\frac{2}{\pi\rho}} \int_{\omega>0} \frac{\sqrt{\beta^2 n^2 - 1}}{\beta^2 n^2 \sqrt{S}} \cos\,[X]\ \omega\, d\omega \tag{102}$$

The range of integration is positive here and restricted to the frequency range where $\beta n(\omega) \geq 1$.

The total energy radiated by the electron through a cylinder of length ℓ, where the axis of the cylinder coincides with the line of motion of the electron, may be calculated from:

$$W = 2\pi\rho\ell \int_{-\infty}^{\infty} \frac{c}{4\pi} \left[\overline{E}\ \overline{H}\right]_\rho dt \tag{103}$$

With the help of the relation:

$$\int_{-\infty}^{\infty} \cos(\omega t + \alpha) \cos(\omega' t + \beta)\, dt = \pi\delta(\omega - \omega') \tag{104}$$

where α and β are arbitrary angles, Equation (103) may be applied to the field expressions to obtain:

$$W = \frac{e^2 \ell}{c^2} \int_{\beta n > 1} \left(1 - \frac{1}{\beta^2 n^2}\right) \omega\, d\omega \tag{105}$$

as the total Cerenkov radiation from an electron traveling in a straight line. The Cerenkov radiation per unit path length can then be written as:

$$\frac{dW}{d\ell} = \frac{e^2}{c^2} \int_{\beta n > 1} \left(1 - \frac{1}{\beta^2 n^2}\right) \omega\, d\omega\,. \tag{106}$$

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