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A Study of an Inverse Feed-Back, Low Frequency, Low-Pass Filter

Ralph A. Nobles

NOBLES — LOW-PASS FILTER

A STUDY OF AN INVERSE FEED-BACK, LOW FREQUENCY, LOW-PASS FILTER

By

Ralph A. Nobles

A thesis submitted in
partial fulfillment of the
requirements for the Degree of
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This thesis, directed and approved by the candidate's committee, has been accepted by the Graduate Committee of the University of New Mexico in partial fulfillment of the requirements for the degree of

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By

Ralph A. Nobles

Thesis committee

Roy Thomas

CHAIRMAN

Victor H. Regener

John G. Breiland

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INTRODUCTION

In connection with other research work in the Physics Department at the University of New Mexico it became of interest to make a detailed study of a resistance-capacitance coupled low-pass filter which incorporated inverse feed-back.

In order to determine the effect of the inverse feed-back on the filters, calculations are made on circuits with and without feed-back. Response curves are calculated for step-wave inputs and for sine-wave inputs, and these calculated curves are then compared with experimental response curves.

CHAPTER I

CALCULATION OF RESPONSE CURVES OF A TWO SECTION R-C SERIES COUPLED FILTER WITHOUT FEED-BACK

The circuit diagram of the two section resistance-capacitance series coupled circuit is given in Fig. 1.

Fig. 1

Two section filter without feed-back

INTRODUCTION

In connection with other research work in the Physics Department at the University of New Mexico it became of interest to make a detailed study of a resistance-capacitance coupled band-pass filter with incorporated inverse feedback.

In order to determine the effect of the inverse feed-back on the filters, calculations are made on circuits with and without feed-back. Response curves are calculated for step-function inputs and for sine-wave inputs, and these calculated curves are then compared with experimental response curves.

CHAPTER I

CALCULATION OF RESPONSE CURVE OF A TWO SECTION R-C SERIES COUPLED FILTER WITHOUT FEED-BACK

The circuit diagram of the two section resistance-capacitance series coupled circuit is given in Fig. 1.

Fig. 1

Two section filter without feed-back.

In the following calculation and in all subsequent calculations in this paper, square upper case letters will stand for the Laplace transform[1] of the corresponding lower case letter. Thus E_0 will be the Laplace transform of $\mathcal{E}_0$, I_0 the transform of $\mathcal{I}_0$ etc. The symbol Z will be used as the parameter in the Laplace transforms.

Applying Kirchhoff's laws[2] to the elementary circuits in Fig. 2

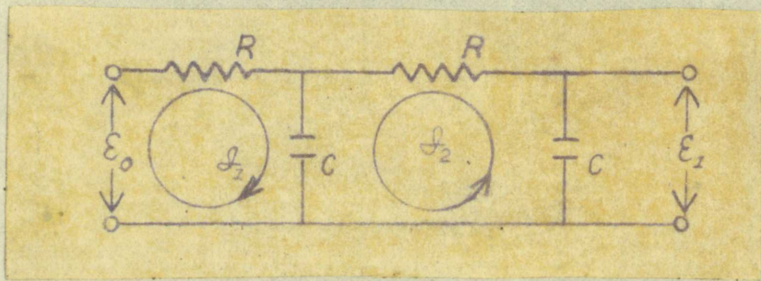

Fig. 2

we obtain the following transform equations:

(1.1) $E_0 = I_1 r + \dfrac{I_1 + I_2}{cZ}$

(1.2) $0 = I_2 r + \dfrac{I_1 + I_2}{cZ} + \dfrac{I_2}{cZ}$

(1.3) $E_1 = \dfrac{I_2}{cZ}$

Solving these equations for E_1 in terms of E_0 gives:

(1.4) $E_1 = \dfrac{E_0}{c^2 r^2 Z^2 + 3rcZ + 1}$

In the actual circuits, r was 10^7 ohms and c was 1/2 microfarad, so rc = 5, and $r^2c^2 = 25$. Thus (1.4) becomes:

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1 R. V. Churchill, Modern Operational Mathematics in Engineering, New York: McGraw Hill, 1944, Chapters I, II, III.

2 L. Page and N.I. Adams, Principles of Electricity, New York: D. Van Nostrand, 1931, p. 173.

In the following exposition and in all subsequent calculations in this paper, capital upper case letters will stand for the Laplace transform[1] of the corresponding lower case letter. Thus E_0 will be the transform of e_0; I_0 the transform of i_0 etc. The symbol z will be used as the parameter in the Laplace transform.

Applying Kirchhoff's laws[2] to the circuit shown in Fig. 2,

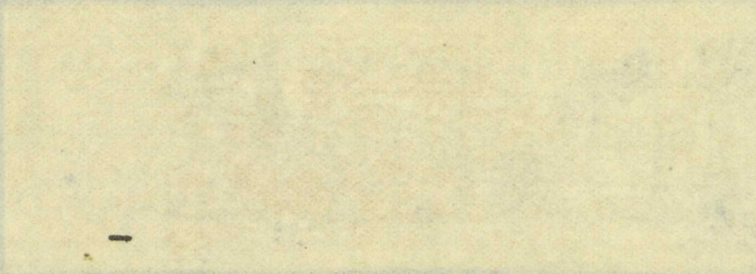

Fig. 2

we obtain the following transform equations:

$$(1.1)\quad E_0 = I_1 r + \frac{I_1 - I_2}{cz}$$

$$(1.2)\quad 0 = I_2 r + \frac{I_2 - I_1}{cz} + \frac{I_2}{cz}$$

$$(1.3)\quad E_1 = \frac{I_2}{cz}$$

Solving these equations for E_1 in terms of E_0 gives:

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In the actual circuit, r was 10^6 ohms and c was 1/2 microfarad, so rc = 1/2, and r^2c^2 = 1/4. Thus (1.4) becomes

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[1] R. V. Churchill, Modern Operational Mathematics in Engineering, New York: McGraw Hill, 1944, Chapters 1, 2, 3.

[2] L. Page and N.I. Adams, Principles of Electricity, New York: D. Van Nostrand, 1931, p. 372.

(1.5) $$E_1 = \frac{E_o}{25Z^2 + 15Z + 1} \cong \frac{E_o}{25(Z+0.524)(Z+0.076)}$$

The Laplace transform of a step input, Fig. 2, beginning at t = 0 and having an amplitude of $\mathcal{E}$ is:

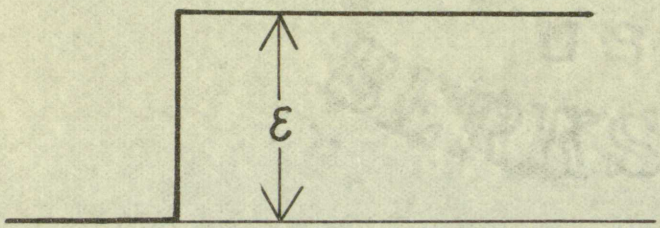

Fig. 3

(1.6) $$E_o = \int_0^\infty e^{-zt} f(t)\,dt = \int_0^\infty e^{-zt}\mathcal{E}\,dt = \frac{-e^{-zt}\mathcal{E}}{Z}\Bigg|_0^\infty = \frac{\mathcal{E}}{Z},$$

where

$f(t) = \mathcal{E}$ for $t > 0$, and $f(t) = 0$ for $t < 0$.

Now by the use of the inversion theorem we get:

(1.7) $$\mathcal{E}_1 = \frac{1}{2\pi i}\int_C \frac{E_o e^{zt}\,dZ}{25(Z+0.524)(Z+0.076)}$$

where E_o is the Laplace transform of the input wave, which in this case is given by equation 1.6. On substitution of this value for E_o in relation 1.7 we get:

(1.8) $$\mathcal{E}_1 = \frac{\mathcal{E}}{25\cdot 2\pi i}\int_C \frac{e^{zt}\,dZ}{Z(Z+0.524)(Z+0.076)}$$

The integral in equation 1.8 has simple poles at Z = 0, Z = -.524, and Z = -.076 (See Fig. 4).

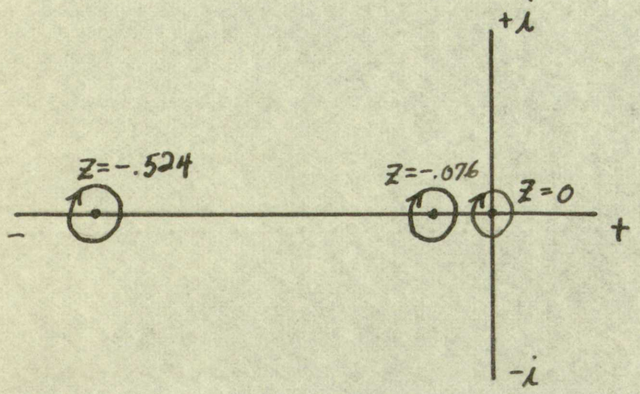

Fig. 4

Poles of the integral in equation 1.8

Evaluation of the expression at these poles gives:

$$(1.9)\quad \mathcal{E}_1 = \frac{\mathcal{E}}{25}\left[\frac{1}{(0.524)(0.076)} + \frac{e^{-.524t}}{(-.524)(-.524+.076)} + \frac{e^{-.076t}}{(-.076)(-.076+.524)}\right]$$

This reduces to:

$$(1.10)\quad \mathcal{E}_1 = \frac{\mathcal{E}}{25}\left[25.1 + 4.28\, e^{-.524t} - 29.4\, e^{-.076t}\right]$$

Equation 1.10 is then the response of a two section, R-C series coupled filter without feed-back to a step-wave input.

Table 1

Tabulation of $\mathcal{E}_1$ as a function of t

t (sec)	$\mathcal{E}_1$
1	0.009 $\mathcal{E}$
5	0.218 $\mathcal{E}$
10	0.452 $\mathcal{E}$
20	0.747 $\mathcal{E}$
30	0.900 $\mathcal{E}$
40	0.948 $\mathcal{E}$
50	0.980 $\mathcal{E}$
60	0.992 $\mathcal{E}$
70	1.00 $\mathcal{E}$

See Fig. 19 for a plot of Table 1.

In the case of a sine wave beginning at t = 0 and having an amplitude of $\mathcal{E}$ and a frequency of $\frac{\omega}{2\pi}$ the Laplace transform is given by:

$$(1.11) \quad E_0 = \int_0^\infty e^{-zt}(\mathcal{E}\sin\omega t)\,dt = \frac{\mathcal{E}}{2i}\int_0^\infty e^{-zt}\left(e^{+i\omega t} - e^{-i\omega t}\right)dt$$

$$= \frac{\mathcal{E}}{2i}\left[\left.\frac{e^{(-z+i\omega)t}}{-z+i\omega}\right|_0^\infty + \left.\frac{e^{-(z+i\omega)t}}{z+i\omega}\right|_0^\infty\right] = \frac{\mathcal{E}}{2i}\left[\frac{1}{z-i\omega} - \frac{1}{z+i\omega}\right]$$

$$= \frac{\mathcal{E}\omega}{z^2+\omega^2}$$

If nowin equation 1.7, which was obtained by the use of the inversion theorem, we substitute the Laplace transform of a sine wave (equation 1.11) instead of that of a step-wave we get:

$$(1.12) \quad \mathcal{E}_1 = \frac{1}{2\pi i}\int_c \frac{\omega\,\mathcal{E}\,e^{zt}\,dz}{(z^2+\omega^2)(z+.524)(z+.076)\,25}$$

$$= \frac{\mathcal{E}\omega}{25\cdot 2\pi i}\int_c \frac{e^{zt}\,dz}{(z+i\omega)(z-i\omega)(z+.524)(z+.076)}$$

This expression has simple poles at $Z = -i\omega$, $Z = \omega i$, $Z = -.524$ and $Z = -.076$.

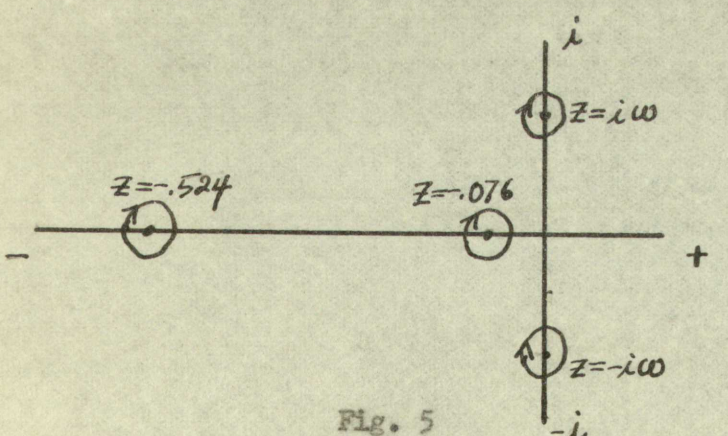

Fig. 5

Poles of the integral in equation 1.12

Evaluation of the integral at these poles gives:

$$(1.13)\quad \mathcal{E}_1 = \frac{\mathcal{E}\omega}{25}\left[\frac{e^{-i\omega t}}{(-2i\omega)(-i\omega t+.524)(-i\omega+.076)} + \frac{e^{i\omega t}}{(2i\omega)(i\omega+.524)(i\omega+.076)}\right]$$

$$+\ \frac{\mathcal{E}\omega}{25}\left[\text{transient terms}\right]$$

For t >> 1 sec the transient terms no longer give a contribution and equation 1.13 becomes simply:

$$(1.14)\quad \mathcal{E}_1 = \frac{\mathcal{E}\omega}{25}\left[\frac{e^{-i\omega t}}{(-2i\omega)(-i\omega+.524)(-i\omega+.076)} + \frac{e^{i\omega t}}{(2i\omega)(i\omega+.524)(i\omega+.076)}\right]$$

This expression reduces to:

$$(1.15)\quad \mathcal{E}_1 = \frac{\mathcal{E}}{25}\left[\frac{-.6\omega}{(.0398-\omega^2)^2+.36\omega^2}\cos\omega t + \frac{.0398-\omega^2}{(.0398-\omega^2)^2+.36\omega^2}\sin\omega t\right]$$

$$= \frac{\mathcal{E}}{25}\left[A\cos\omega t + B\sin\omega t\right] = \frac{\mathcal{E}}{25}\left(\sqrt{A^2+B^2}\right)\left[\sin(\omega t+\delta_\omega)\right]$$

where $A = \dfrac{-.6\omega}{(.0398-\omega^2)^2+.36\omega^2}$, $B = \dfrac{.0398-\omega^2}{(.0398-\omega^2)^2+.36\omega^2}$

$$\delta_\omega = \tan^{-1}\frac{A}{B}$$

The frequency response in terms of the input amplitude is given by the quantity $\sqrt{A^2 + B^2}\,/25$ in equation 1.15.

Table 2

Tabulation of the frequency response of equation 1.15

ω	A	B	$\sqrt{A^2+B^2}/25$
10^{-2}	-3.75	24.8	1.01
10^{-1}	-13.4	6.6	0.595
10^{0}	$-.47$	$-.75$	3.5×10^{-2}
10^{1}	-6×10^{-4}	-10^{-2}	$\sim10^{-4}$
10^{2}	~ 0	-10^{-4}	~ 0

For a plot of the values of the amplitude of equation 1.15 as a function of ω see Fig. 20.

CHAPTER II

CALCULATION OF RESPONSE CURVES OF A TWO SECTION R-C SERIES COUPLED FILTER WITH INVERSE FEED-BACK

The circuit diagram of a two section resistance-capacitance filter with inverse feed-back is given in Fig. 6.

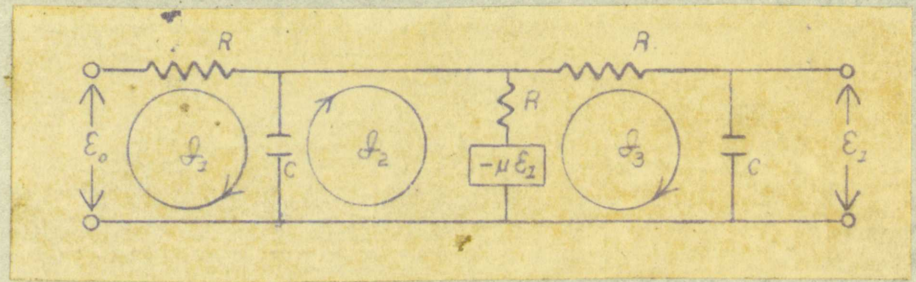

Fig. 6

Two section filter with inverse feed-back

Applying Kirchhoff's laws to the elementary circuits in Fig. 6, we obtain the following transform equations:

$$E_o = I_1 r + \frac{I_1 - I_2}{cz} \tag{2.1}$$

$$\mu E_1 = \frac{I_2 - I_1}{cz} + (I_2 - I_3) r \tag{2.2}$$

$$-\mu E_1 = (I_3 - I_2) r + I_3 r + \frac{I_3}{cz} \tag{2.3}$$

$$E_1 = \frac{I_3}{cz} \tag{2.4}$$

Solving equations 2.1 through 2.4 for E_1 in terms of E_o gives:

$$E_1 = \frac{E_o}{r^2c^2z^2 + 4rcz + \mu + 2} \tag{2.5}$$

Now by the use of the inversion theorem and the Laplace transform for a step-wave input (Equation 1.6) we get:

$$(2.6)\quad \mathcal{E}_1 = \frac{1}{2\pi i}\int_c \frac{E_0 e^{zt}\,dz}{r^2c^2z^2+4rcz+\mu+2} = \frac{\mathcal{E}}{2\pi i}\int_c \frac{e^{zt}\,dz}{z(r^2c^2z^2+4rcz+\mu+2}$$

$$= \frac{\mathcal{E}}{2\pi i}\int_c \frac{e^{zt}\,dz}{r^2c^2 z\left(z+\frac{2+\sqrt{2-\mu}}{cr}\right)\left(z-\frac{-2+\sqrt{2-\mu}}{cr}\right)}$$

Here, as in the previous case, r = 10^7 ohms, C = 1/2 microfarad, and in addition μ = 10. Thus as before rC = 5, and r^2C^2 = 25. Equation 2.6 then becomes:

$$(2.7)\quad \mathcal{E}_1 = \frac{\mathcal{E}}{25\cdot 2\pi i}\int_c \frac{e^{zt}\,dz}{\left(z+\frac{2+i2\sqrt{2}}{5}\right)\left(z-\frac{-2+i2\sqrt{2}}{5}\right)z}$$

The integral in equation 2.7 has simple poles at Z = 0, Z = $-\frac{2+i2\sqrt{2}}{5}$, and Z = $\frac{-2+2\sqrt{2}i}{5}$.

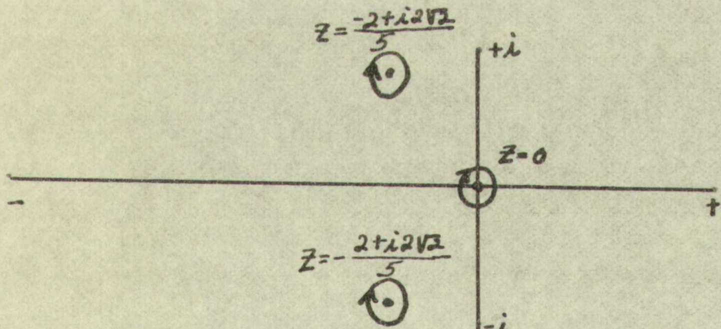

Fig. 7

Poles of the integral in equation 2.7

Evaluation of the integral at these poles gives:

$$(2.8)\quad \mathcal{E}_1 = \mathcal{E}\left[\frac{1}{12} + \frac{e^{-\frac{2+2\sqrt{2}i}{5}t}}{-16+8\sqrt{2}i} + \frac{e^{\frac{-2+2\sqrt{2}i}{5}t}}{-16-8\sqrt{2}i}\right]$$

By algebraic manipulations equation 2.8 can be put in the form:

$$(2.9)\qquad \mathcal{E}_1 = \frac{\mathcal{E}}{12}\left[1 - e^{-.4t}\left(\cos 0.565t + 0.707 \sin 0.565t\right)\right]$$

Equation 2.9 is then the analytical expression for the response of a two section filter with inverse feed-back to a step-wave input.

Table 3

Tabulation of the values of $\mathcal{E}_1$ as a function of t.

T (sec)	12 $\mathcal{E}_1$
0	0 $\mathcal{E}$
0.5	0.054 $\mathcal{E}$
1	0.18 $\mathcal{E}$
1.5	0.345 $\mathcal{E}$
2	0.522 $\mathcal{E}$
3	0.828 $\mathcal{E}$
4	1.018 $\mathcal{E}$
5	1.103 $\mathcal{E}$
6	1.104 $\mathcal{E}$
7	1.073 $\mathcal{E}$
8	1.036 $\mathcal{E}$
10	0.993 $\mathcal{E}$

For a plot of the values in Table 3 see Fig. 19.

If we substitute the Laplace transform of a sine-wave for E_1 in Equation 2.5 and then use the inversion theorem we get:

$$(2.10)\qquad \mathcal{E}_1 = \frac{1}{2\pi i}\int_c \frac{E_0 e^{zt}\,dz}{c^2r^2\left(z + \frac{2+2\sqrt{2}i}{cr}\right)\left(z - \frac{-2+2\sqrt{2}i}{cr}\right)}$$

$$= \frac{\mathcal{E}\omega}{2\pi i}\int_c \frac{e^{zt}\,dz}{25(z^2+\omega^2)\left(z + \frac{2+2\sqrt{2}i}{5}\right)\left(z - \frac{-2+2\sqrt{2}i}{5}\right)}$$

$$= \frac{\mathcal{E}\omega}{25\cdot 2\pi i}\int_c \frac{e^{zt}\,dz}{(z+i\omega)(z-i\omega)\left(z+\frac{2+2\sqrt{2}i}{5}\right)\left(z-\frac{-2+2\sqrt{2}i}{5}\right)}$$

The integral in equation 2.10 has simple poles at $Z = i\omega$, $Z = -i\omega$, $Z = -\frac{2+2\sqrt{2}i}{5}$, and $Z = \frac{-2+2\sqrt{2}i}{5}$.

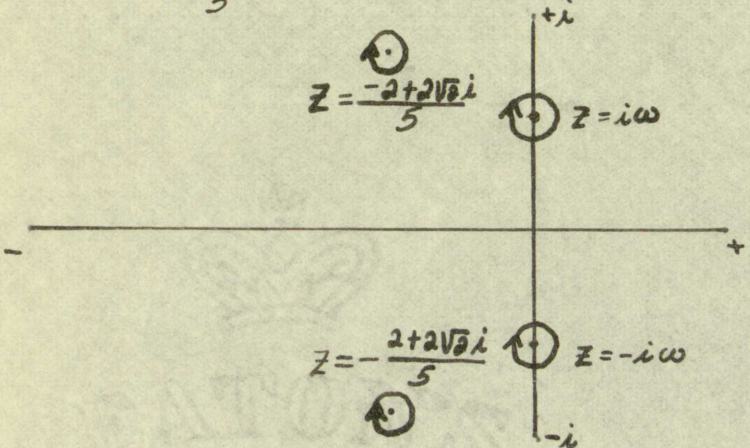

Fig. 8

Poles of the integral in equation 2.10

Evaluation of the integral at these poles gives:

$$(2.11)\quad \mathcal{E}_1 = \frac{\mathcal{E}\omega}{25}\left[\frac{e^{-i\omega t}}{(-2i\omega)\left(-i\omega+\frac{2+2\sqrt{2}i}{5}\right)\left(-i\omega+\frac{2-2\sqrt{2}i}{5}\right)}\right.$$

$$+\frac{e^{i\omega t}}{(2i\omega)\left(i\omega+\frac{2+2\sqrt{2}i}{5}\right)\left(i\omega+\frac{2-2\sqrt{2}i}{5}\right)}+\frac{e^{\frac{-2-2\sqrt{2}i}{5}t}}{\left(\frac{-2-2\sqrt{2}i}{5}+i\omega\right)\left(\frac{-2-2\sqrt{2}i}{5}-i\omega\right)\left(\frac{-2-2\sqrt{2}i}{5}+\frac{2-2\sqrt{2}i}{5}\right)}$$

$$\left.+\frac{e^{\frac{-2+2\sqrt{2}i}{5}t}}{\left(\frac{-2+2\sqrt{2}i}{5}+i\omega\right)\left(\frac{-2+2\sqrt{2}i}{5}-i\omega\right)\left(\frac{-2+2\sqrt{2}i}{5}+\frac{2+2\sqrt{2}i}{5}\right)}\right]$$

For $t \gg 1$ equation 2.11 reduces to:

$$(2.12) \quad \mathcal{E}_1 = \frac{\mathcal{E}\omega}{25}\Bigg[\frac{e^{-i\omega t}}{(-2i\omega)\left(-i\omega + \frac{2+2\sqrt{2}i}{5}\right)\left(-i\omega + \frac{2-2\sqrt{2}i}{5}\right)}$$

$$+ \frac{e^{i\omega t}}{(2i\omega)\left(i\omega + \frac{2+2\sqrt{2}i}{5}\right)\left(i\omega + \frac{2-2\sqrt{2}i}{5}\right)}\Bigg]$$

Equation 2.12 can further be reduced to:

$$(2.13) \quad \mathcal{E}_1 = \frac{\mathcal{E}}{25}\left[-A\cos\omega t - B\sin\omega t\right] = \frac{\mathcal{E}}{25}\sqrt{A^2+B^2}\left[-\sin(\omega t + \delta_\omega)\right]$$

$$= \frac{\mathcal{E}}{25}\sqrt{A^2+B^2}\left[\sin(\omega t + \delta_\omega + \pi)\right]$$

where $A = \dfrac{\frac{16\omega}{5}}{(2\omega^2 - \frac{24}{25})^2 + \frac{64}{25}\omega^2}$, $B = \dfrac{2(2\omega^2 - \frac{24}{25})}{(2\omega^2 - \frac{24}{25})^2 + \frac{64}{25}\omega^2}$,

and $\delta_\omega = \tan^{-1}\dfrac{A}{B}$

The frequency response of equation 2.13 is of course given by the factor $\sqrt{A^2+B^2}/25$.

Table 4

Frequency response of equation 2.13 as a function of ω

ω (rad/sec)	$\sqrt{A^2+B^2}/25$
0.01	8.4×10^{-2}
0.1	8.3×10^{-2}
1.0	4.2×10^{-2}
2	1.04×10^{-2}
10	4×10^{-4}
100	0

A plot of the frequency response of equation 2.13 is given in Fig. 20.

The phase shift in equation 2.13 follows in Table 5.

Table 5

Phase shift of equation 2.13 as a function of ω

ω (rad/sec)	A/B	$\tan^{-1}\frac{A}{B}$
10^{-1}	$-.155$	$171^\circ\ 11'$
1	1.54	57°
2	.455	$24^\circ\ 28'$
10^1	0	0
10^2	0	0

See Fig. 21 for a plot of the phase shift of equation 2.13.

CHAPTER III

CALCULATION OF RESPONSE CURVES OF FOUR SECTION FILTER WITHOUT FEED-BACK

The circuit diagram of a four section resistance-capacitance series coupled circuit without feed-back is given in Fig. 9.

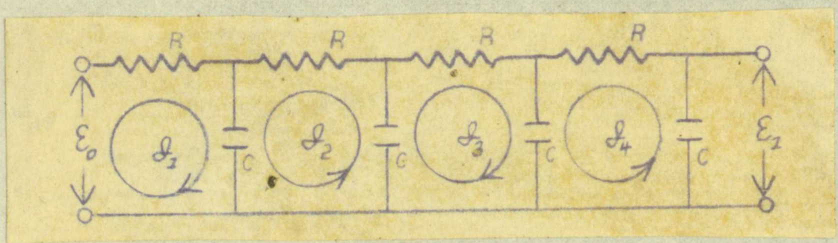

Fig. 9

Four section R-C series coupled filter

Analysis of the circuit in Fig. 9 leads to the following five transform equations:

(3.1) $$E_0 = I_1 r + \frac{I_1 + I_2}{cz}$$

(3.2) $$0 = I_2 r + \frac{I_2 + I_1}{cz} + \frac{I_2 + I_3}{cz}$$

(3.3) $$0 = I_3 r + \frac{I_3 + I_2}{cz} + \frac{I_3 + I_4}{cz}$$

(3.4) $$0 = I_4 r + \frac{I_4 + I_3}{cz} + \frac{I_4}{cz}$$

(3.5) $$E_1 = \frac{I_4}{cz}$$

Solving these equations for E_1 in terms of E_0 gives:

(3.6) $$E_1 = \frac{-E_0}{a^4 - a^3 - 3a^2 + 2a + 1}$$

where $a = rcz + 2$

and since here also rc = 5, (see Chapter I, page 2) we have:

(3.7) $a = 5Z + 2$

Substitution of this value for a in equation 3.6 gives:

(3.8) $$E_1 = \frac{-E_0}{625Z^4 + 875Z^3 + 375Z^2 + 50Z + 1}$$

The denominator in equation 3.8 can be factored approximately by the use of Horner's method.[3] On factoring, equation 3.8 becomes:

(3.9) $$E_1 \cong \frac{-E_0}{625(Z+.024)(Z+.225)(Z+.72)(Z+.43)}$$

Now by using the Laplace transform of a step-wave input from equation 1.6 and equation 3.9 in the inversion theorem we get:

(3.10) $$\mathcal{E}_1 = \frac{-\mathcal{E}}{625 \cdot 2\pi i} \int_c \frac{e^{zt}\,dz}{Z(Z+.024)(Z+.225)(Z+.72)(Z+.43)}$$

This expression has simple poles at Z = 0, Z = -0.024, Z = -0.225, Z = -0.72, and Z = -0.43.

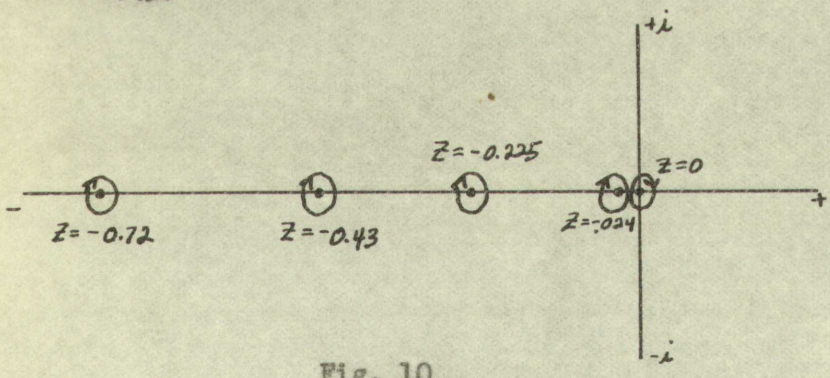

Fig. 10

Poles of the ~~integral~~ integrand in equation 3.10

3 E. J. Wilczynski and H. E. Slaught, College Algebra, New York: Allyn and Bacon, 1916, p. 164.

Evaluating the integral in equation 3.10 by the theory of residues we get:

$$(3.11)\quad \mathcal{E}_1 = \frac{-\mathcal{E}}{625}\left[\frac{1}{(.024)(.225)(.43)(.72)} + \frac{e^{-.024t}}{(-.024)(-.024+.225)(-.024+.43)(-.024+.72)}\right.$$

$$+ \frac{e^{-.225t}}{(-.225)(-.225+.024)(-.225+.43)(-.225+.72)} + \frac{e^{-.72t}}{(-.72)(-.72+.024)(-.72+.225)(-.72+.43)}$$

$$\left. + \frac{e^{-.43t}}{(-.43)(-.43-.024)(-.43+.225)(-.43+.72)}\right]$$

This reduces to:

$$(3.12)\quad \mathcal{E}_1 \cong \frac{-\mathcal{E}}{625}\left[597 - 716\,e^{-.024t} + 213\,e^{-.225t} + 13.25\,e^{-.72t} - 101\,e^{-.43t}\right]$$

Table 6

Tabulation of the values of equation 3.12 as a function of t

t (sec)	$\mathcal{E}_1$
10	$0.067\,\mathcal{E}$
20	$0.248\,\mathcal{E}$
30	$0.392\,\mathcal{E}$
50	$0.609\,\mathcal{E}$
100	$0.851\,\mathcal{E}$
150	$0.924\,\mathcal{E}$
200	$0.946\,\mathcal{E}$

The fact that the asymptotic value of $\mathcal{E}_1$ is $0.957\,\mathcal{E}$ instead of $1.000\,\mathcal{E}$ is due to inaccuracy in factoring the denominator of equation 3.8.

For a plot of equation 3.12 see Fig. 22. Equation 3.12 is the analytical expression for the response of a four section filter, without feed-back, to a step-wave input.

The analytical expression for the response of the filter to a sine-wave input is given by using the Laplace transform from equation 1.11 together with relation 3.8 in the inversion theorem, giving:

$$(3.13) \qquad \mathcal{E}_1 = \frac{\mathcal{E}\omega}{2\pi i \cdot 625} \int_c \frac{e^{zt}\, dz}{(z+i\omega)(z-i\omega)(z+.024)(z+.225)(z+.72)(z+.43)}$$

This expression has simple poles at $Z = -i\omega$, $Z = i\omega$, $Z = -.024$, $Z = -.225$, $Z = -.72$, and $Z = -.43$.

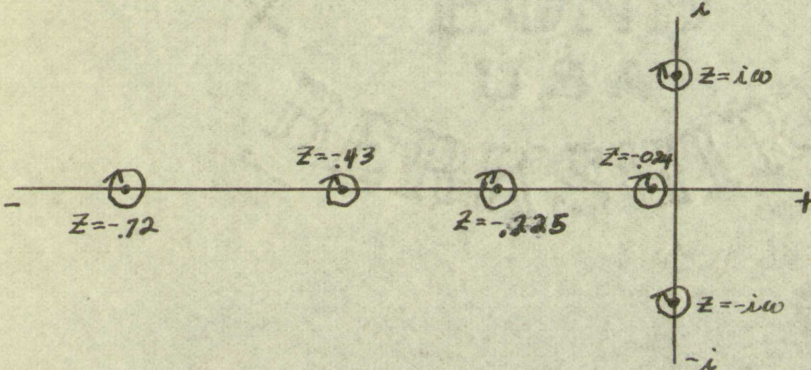

Fig. 11

Poles of the integral in equation 3.13

Evaluation of the integral by residue theorem gives:

$$(3.14) \qquad \mathcal{E}_1 = \frac{\mathcal{E}\omega}{625}\Bigg[\frac{e^{-i\omega t}}{(-2i\omega)(-i\omega+.024)(-i\omega+.225)(-i\omega+.43)(-i\omega+.72)}$$

$$+ \frac{e^{i\omega t}}{(2i\omega)(i\omega+.024)(i\omega+.225)(i\omega+.43)(i\omega+.72)}$$

$$+ \frac{e^{-.024t}}{(-.024+i\omega)(-.024-i\omega)(-.024+.225)(-.024+.43)(-.024+.72)}$$

$$+ \frac{e^{-.225t}}{(-.225+i\omega)(-.225-i\omega)(-.225+.024)(-.225+.43)(-.225+.72)}$$

$$+ \frac{e^{-.43t}}{(-.43+i\omega)(-.43-i\omega)(-.43+.024)(-.43+.225)(-.43+.72)}$$

$$+ \frac{e^{-.72t}}{(-.72+i\omega)(-.72-i\omega)(-.72+.024)(-.72+.225)(-.72+.43)}\Bigg]$$

For t >> 1, that is after the transients have died out, equation 3.14 reduces to:

(3.15)
$$\mathcal{E}_1 = \frac{\mathcal{E}}{625}\left[A \cos \omega t + B \sin \omega t\right]$$

$$= \frac{\mathcal{E}}{625}\sqrt{A^2+B^2}\left[\sin(\omega t + \delta_\omega)\right]$$

where
$$A = \frac{(1.4\omega^3 - 0.083\omega)}{\omega^8 + .76\omega^6 + .131\omega^4 + .0049\omega^2 + .0000029}$$

$$B = \frac{(\omega^4 - .6\omega^2 + .0017}{\omega^8 + .76\omega^6 + .131\omega^4 + .0049\omega^2 + .0000029}$$

and
$$\delta_\omega = \tan^{-1}\frac{A}{B}$$

The frequency response of equation 3.15 is of course given by the factor $\sqrt{A^2+B^2}$

Table 7

Tabulation of the frequency dependent amplitude of equation 3.15

ω (rad/sec)	A	B	$\sqrt{A^2 + B^2}$
10^{-2}	-2.45×10^2	4.83×10^2	5.4×10^2
10^{-1}	-1.06×10^2	$-.66 \times 10^2$	1.24×10^2
1	0.695	.21	.725
10^1	1.32×10^{-6}	10^{-4}	$\sim 10^{-4}$
10^2	0	10^{-8}	0
10^{-3}	-2.86×10^1	5.87×10^2	5.87×10^2

See Fig. 23 for a plot of the frequency response of equation 3.15.

CHAPTER IV

CALCULATION OF RESPONSE CURVES OF A FOUR SECTION FILTER WITH FEED-BACK

The circuit diagram of the four section resistance-capacitance series coupled filter with inverse feed-back is given in Fig. 12.

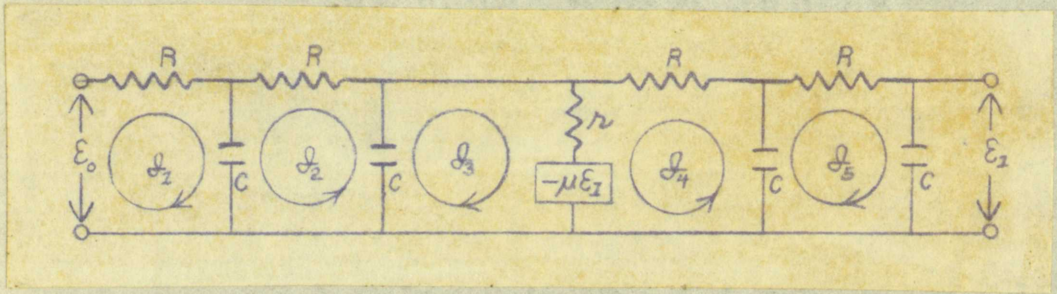

Fig. 12

Four section inverse feed-back filter

Analysis of the circuit in Fig. 12 leads to the following set of transform equations:

(4.1) $$E_0 = I_1 R + \frac{I_1 + I_2}{cZ}$$

(4.2) $$0 = I_2 R + \frac{I_1 + I_2}{cZ} + \frac{I_2 + I_3}{cZ}$$

(4.3) $$\mu E_1 = (I_3 + I_4) r + \frac{I_3 + I_2}{cZ}$$

(4.4) $$\mu E_1 = (I_4 + I_3) r + I_4 R + \frac{I_4 + I_5}{cZ}$$

(4.5) $$0 = I_5 R + \frac{I_5 + I_4}{cZ} + \frac{I_5}{cZ}$$

(4.6) $$E_1 = \frac{I_5}{cZ}$$

The values of the constants used in the experimental set-up on this filter were as follows: $R = 10^7$ ohms, $c = 1/2$ microfarad, $\mu = 10$ and $r = 2 \times 10^7$ ohms. The resistance r was always chosen equal to $n \times 10^7$ ohms, where n is one-half of the number of filter sections. This was done in order to match the impedance of the filter sections to the feed-back section.

On solving equations 4.1 through 4.6 for E_1 in terms of E_0 one gets:

$$(4.7) \qquad E_1 = \frac{2E_0}{2(RcZ)^4 + 15(RcZ)^3 + 35(RcZ)^2 - \mu(RcZ) - 2\mu + 4}$$

Since $Rc = 5$ and $\mu = 10$, equation 4.7 becomes:

$$(4.8) \qquad E_1 = \frac{2E_0}{1250 Z^4 + 1875 Z^3 + 875 Z^2 + 185 Z + 24}$$

Factoring the denominator of 4.8 by Horner's Method gives:

$$(4.9) \qquad E_1 = \frac{E_0}{625(Z+.429)(Z+.850)(Z+.1105+i.2012)(Z+.1105-i.2012)}$$

The analytic expression for the response of the filter to a step-wave input is obtained by using the transform of a step-wave from equation 1.6 and equation 4.9 in the inversion theorem, giving:

$$(4.10) \qquad \mathcal{E}_1 = \frac{\mathcal{E}}{625 \cdot 2\pi i} \int_c \frac{e^{Zt}\, dZ}{Z(Z+.429)(Z+.850)(Z+.1105+i.2012)(Z+.1105-i.2012)}$$

This expression has simple poles at $Z = 0$, $Z = -.850$, $Z = -.429$, $Z = -.1105 - i.2012$, and $Z = -.1105 + i.2012$.

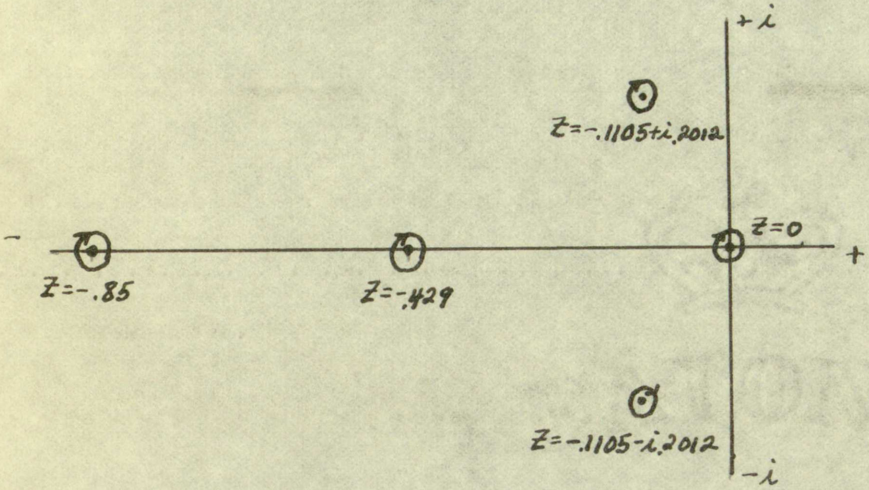

Fig. 13

Poles of the integral in equation 4.10

Evaluation of the integral at these poles gives:

(4.11)
$$\mathcal{E}_1 = \frac{\mathcal{E}}{625}\left[\frac{1}{(.429)(.850)(.1105+i.2012)(.1105-i.2012)}\right.$$

$$+\frac{e^{-.429t}}{(-.429)(-.429+.850)(-.429+.1105+i.2012)(-.429+.1105-i.2012)}$$

$$+\frac{e^{-.850t}}{(-.850)(-.850+.429)(-.850+.1105+i.2012)(-.850+.1105-i.2012)}$$

$$+\frac{e^{(-.1105-i.2012)t}}{(-.1105-i.2012)(-.1105-i.2012+.429)(-.1105-i.2012+.850)(-.1105-i.2012+.1105-i.2012)}$$

$$\left.+\frac{e^{(-.1105+i.2012)t}}{(-.1105+i.2012)(-.1105+i.2012+.429)(-.1105+i.2012+.850)(-.1105+i.2012+.1105+i.2012)}\right]$$

This reduces to:

$$(4.12) \quad \mathcal{E}_1 = \frac{\mathcal{E}}{625}\left[52.1 - 39.05\, e^{-.429t} + 4.76\, e^{-.85t}\right]$$

$$+ \frac{\mathcal{E}}{625}\, e^{-.1105t}\left[-17.92 \cos .2012t - 73.15 \sin .2012t\right]$$

Equation 4.12 is the analytic expression for the reaction of a four section filter with feed-back to a step-wave input.

Table 8

Tabulation of the values of equation 4.12

t (sec)	625 $\mathcal{E}_1$
0	$-0.22\ \mathcal{E}$
1	$-0.14\ \mathcal{E}$
2	$-0.68\ \mathcal{E}$
3	$2.44\ \mathcal{E}$
5	$13.48\ \mathcal{E}$
10	$64.2\ \mathcal{E}$
15	$107.0\ \mathcal{E}$
20	$117.9\ \mathcal{E}$
30	$104.2\ \mathcal{E}$
50	$104.6\ \mathcal{E}$

See Fig. 22 for a plot of the values of equation 4.12.

The response of this filter to a sine-wave input is obtained by using the transform of a sine-wave from equation 1.11 and relation 4.9 in the inversion theorem, giving:

$$(4.13) \quad \mathcal{E}_1 = \frac{\mathcal{E}\omega}{625 \cdot 2\pi i}\int_C \frac{e^{zt}\,dz}{(z+i\omega)(z-i\omega)(z+.429)(z+.85)(z+.1105+.2012i)(z+.1105-i.2012)}$$

This expression has simple poles at $Z = -i\omega$, $Z = i\omega$, $Z = -.429$, $Z = -.85$, $Z = -.1105 - i.2012$, and $Z = -.1105 + i.2012$.

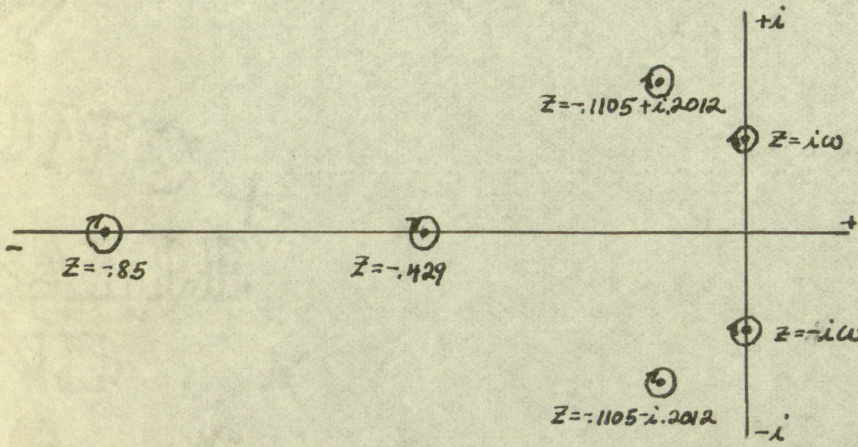

Fig. 14

Poles of the integral in equation 4.13

Evaluation of the integral at these poles gives:

(4.14) $$\mathcal{E}_1 = \frac{\mathcal{E}}{625}\left[\frac{(1.5\omega^3-.1475\omega)2\cos\omega t+(\omega^4+0.0191-0.699\omega^2)2\sin\omega t}{(1.5\omega^3-.1475\omega)^2+(\omega^4-0.699\omega^2+0.0194)^2}\right.$$

$$+\ 33.55\,\frac{\omega e^{-.429t}}{\omega^2+.184} - 8.13\,\frac{\omega e^{-.85t}}{\omega^2+.7225}$$

$$\left.2\omega e^{-.1105t}\left\{\frac{(.00584-.085\omega^2)2\cos .2012t+(.00158+\omega^2 0.0786)2\sin .2012t}{(.00584-.085\omega^2)^2+(.00158+\omega^2 0.0786)^2}\right\}\right]$$

For $t \gg 1$ equation 4.14 reduces to:

(4.15) $$\mathcal{E}_1 = \frac{\mathcal{E}}{625}\left[A\cos\omega t + B\sin\omega t\right]$$

$$= \frac{\mathcal{E}}{625}\sqrt{A^2+B^2}\left[\sin(\omega t+\delta\omega)\right]$$

where
$$A = \frac{3\omega^3 - .295\omega}{(1.5\omega^3 - .1475\omega)^2 + (\omega^4 - .699\omega^2 + .0191)^2}$$

$$B = \frac{2\omega^4 + 0.0382 - 1.398\omega^2}{(1.5\omega^3 - .1475\omega)^2 + (\omega^4 - .699\omega^2 + .0191)^2}$$

$$\delta_\omega = \tan^{-1}\frac{A}{B}$$

Table 9

Tabulation of the frequency dependent factor in equation 4.15

ω	$\sqrt{A^2+B^2}$
10^{-2}	105.3
10^{-1}	100.0
1.0	1.44
2.0	0.124
10^{1}	2.02×10^{-4}
10^{2}	2×10^{-8}

See Fig. 23 for a plot of the values of equation 4.15.

CHAPTER V

CONSIDERATION OF THE PROBLEM OF THE RESPONSE OF A 2N SECTION RESISTANCE-CAPACITANCE SERIES COUPLED FILTER WITH INVERSE FEED-BACK

The circuit diagram of a 2N section resistance-capacitance series coupled filter is given in Fig. 15.

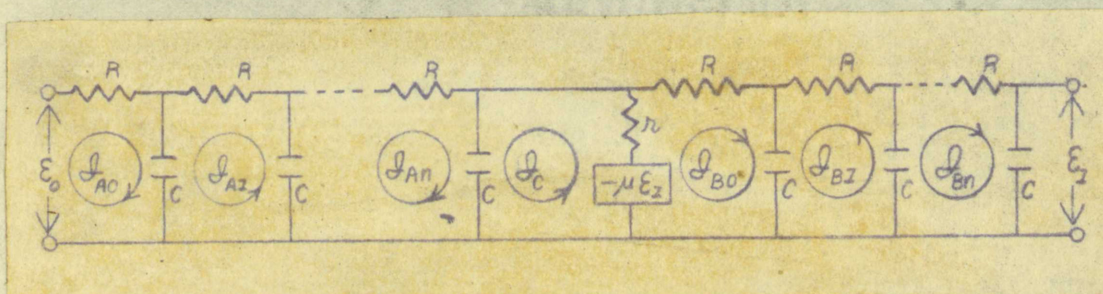

Fig. 15

2N section filter with inverse feed-back

Analysis of the circuit in Fig. 15 leads to the following set of transform equations:

(5.1) $$E_0 = I_{A0}\left(R+\frac{1}{cz}\right) + \frac{I_{A1}}{cz}$$

(5.2) $$0 = I_{A1}\left(R+\frac{2}{cz}\right) + I_{A0}\frac{1}{cz} + I_{A2}\frac{1}{cz}$$

(5.3) $$0 = I_{A(n-1)}\left(\frac{1}{cz}\right) + I_{An}\left(R+\frac{2}{cz}\right) + \frac{I_c}{cz}$$

(5.4) $$\mu E_1 = I_c\left(r+\frac{1}{cz}\right) + I_{B0}r + \frac{I_{An}}{cz}$$

(5.5) $$\mu E_1 = I_c R + I_{B0}\left(2R+\frac{1}{cz}\right) + \frac{I_{B1}}{cz}$$

(5.6) $$0 = I_{B1}\left(R+\frac{2}{cz}\right) + I_{B0}\frac{1}{cz} + \frac{I_{B2}}{cz}$$

(5.7) $$0 = \frac{I_{B(n-1)}}{cz} + I_{Bn}\left(R+\frac{2}{cz}\right)$$

(5.8) $$E_1 = \frac{I_{Bn}}{cz}$$

Now one can set[4] $I_{An} = Ae^{n\theta} + Be^{-n\theta}$ and $I_{Bn} = Ce^{n\theta} + De^{-n\theta}$ where A, B, C, and D are arbitrary independent constants so that $I_{A0} = A + B$, $I_{AK} = Ae^{K\theta} + Be^{-K\theta}$, $I_{B0} = C + D$, $I_{BK} = Ce^{K\theta} + De^{-K\theta}$. Substituting then the values of the I's above in equation 5.2 we get:

(5.9) $$0 = (Ae^{\theta} + Be^{-\theta})\left(R+\frac{2}{cz}\right) + (A+B)\frac{1}{cz} + (Ae^{2\theta} + Be^{-2\theta})\frac{1}{cz}$$

Equation 5.6 can be rewritten as:

(5.10) $$0 = \left[e^{\theta}\left(R+\frac{2}{cz}\right) + \frac{1}{cz} + e^{2\theta}\frac{1}{cz}\right]A + \left[e^{-\theta}\left(R+\frac{2}{cz}\right) + \frac{1}{cz} + e^{-2\theta}\frac{1}{cz}\right]B$$

Since A and B are arbitrary independent constants,

(5.11) $$e^{\theta}\left(R+\frac{2}{cz}\right) + \frac{1}{cz} + \frac{e^{2\theta}}{cz} = 0$$

This can be written as:

(5.12) $$\theta = \operatorname{arc\,cosh}\left(-\frac{cRz+2}{2}\right)$$

A similar argument using equation 5.6 instead of equation 5.2 will likewise lead to equation 5.12, indicating that only one θ is needed in the solution.

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[4] H. S. Carslaw and J. C. Jaeger, Operational Methods in Applied Mathematics, London: Oxford Press, 1941, p. 42-43.

Rewriting equations 5.1 through 5.8 in terms of the exponentials gives:

(5.13) $$E_o = A\left(R + \frac{1}{cz} + \frac{e^{\theta}}{cz}\right) + B\left(R + \frac{1}{cz} + \frac{e^{-\theta}}{cz}\right)$$

(5.14) $$0 = A\left(\frac{e^{(n-1)\theta}}{cz} + e^{n\theta}R + \frac{2e^{n\theta}}{cz}\right) + B\left(\frac{e^{-(n-1)\theta}}{cz} + e^{-n\theta}R + \frac{2e^{-n\theta}}{cz}\right)$$

(5.15) $$0 = A\frac{e^{n\theta}}{cz} + B\frac{e^{-n\theta}}{cz} + Cr + Dr + I_c\left(r + \frac{1}{cz}\right) - \mu E_1$$

(5.16) $$0 = C\left(2R + \frac{1}{cz} + \frac{e^{\theta}}{cz}\right) + D\left(2R + \frac{1}{cz} + e^{-\theta}\right) + I_c R - \mu E_1$$

(5.17) $$0 = C\left(\frac{e^{(n-1)\theta}}{cz} + Re^{n\theta} + \frac{2e^{n\theta}}{cz}\right) + D\left(\frac{e^{-(n-1)\theta}}{cz} + e^{-n\theta}R + \frac{2e^{-n\theta}}{cz}\right)$$

(5.18) $$0 = C\frac{e^{n\theta}}{cz} + D\frac{e^{-n\theta}}{cz} - E_1$$

This is a set of six linear algebraic equations with constant coefficients, in six unknowns, A, B, C, D, I, and E_1. Hence one can solve for E_1 in terms of E_o. However, the form of this solution is so complicated as to preclude the comparison with experimental results. For that reason it was decided not to complete these calculations in this paper.

CHAPTER VI

EXPERIMENTAL INVESTIGATION OF TWO AND FOUR-SECTION FILTERS WITH AND WITHOUT INVERSE FEED-BACK

Two and four-section filters having circuit constants equal to those in the circuits in Chapters I through V were investigated for step-wave inputs. A diagram of the inverse feed-back amplifier[5] and the filter sections is given in Fig. 18. For investigation of filters without inverse feed-back switch SW was opened; for inverse feed-back it was closed.

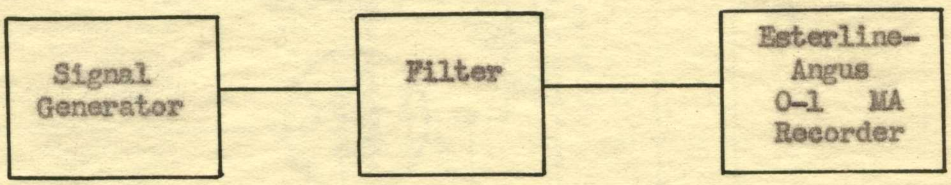

Fig. 16

Block diagram of the experimental set-up

The signal generator for the step-wave input signal was merely a battery and a DPDT switch.

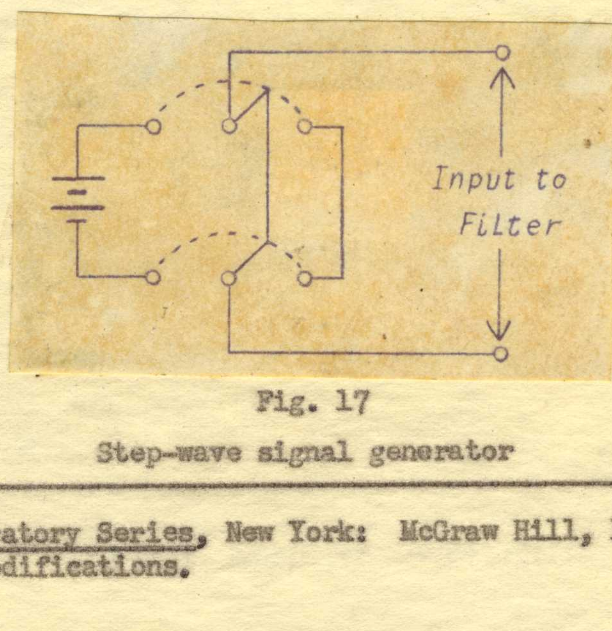

Fig. 17

Step-wave signal generator

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[5] Radiation Laboratory Series, New York: McGraw Hill, 1948, Vol. 18, p. 486, with modifications.

CHAPTER V

EXPERIMENTAL INVESTIGATION OF TWO RC LOW-FREQUENCY FILTERS WITH AND WITHOUT INVERSE FEED-BACK

Two and four-section filters having circuit constants equal to those in the circuits in chapter [illegible] were [illegible] for step-wave inputs. A diagram of the inverse feed-back [illegible] and the filter sections is given in Fig. [illegible]. Responses of filters without inverse feed-back [illegible] switch [illegible] was closed.

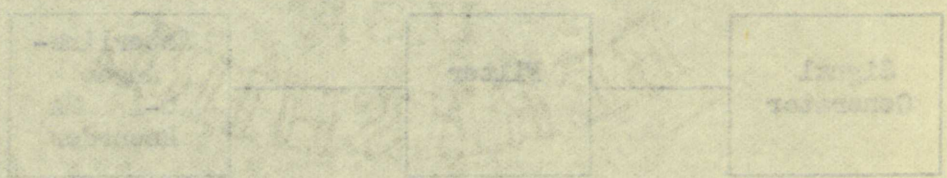


Fig. 16

Block diagram of the experimental set-up

The signal generator for the step-wave input signal was simply a battery and a DPDT switch.

Fig. 17

Step-wave signal generator

5 Radiation Laboratory Series, New York: McGraw-Hill, 1949, Vol. 19, p. 136, with modifications.

A plot of the experimental data obtained for a step-wave input signal into a two section filter without feed-back is given in Fig. 19, and that for a two section filter with feed-back is also plotted in Fig. 19. Experimental response data of four section filters with and without inverse feed-back are given in Fig. 22.

A plot of the experimental data obtained for a step-wave input signal into a two section filter without feed-back is given in Fig. 18, and that for a two section filter with feed-back is also plotted in Fig. 19. Experimental response data of four section filters with and without inverse feed-back are given in Fig. 21.

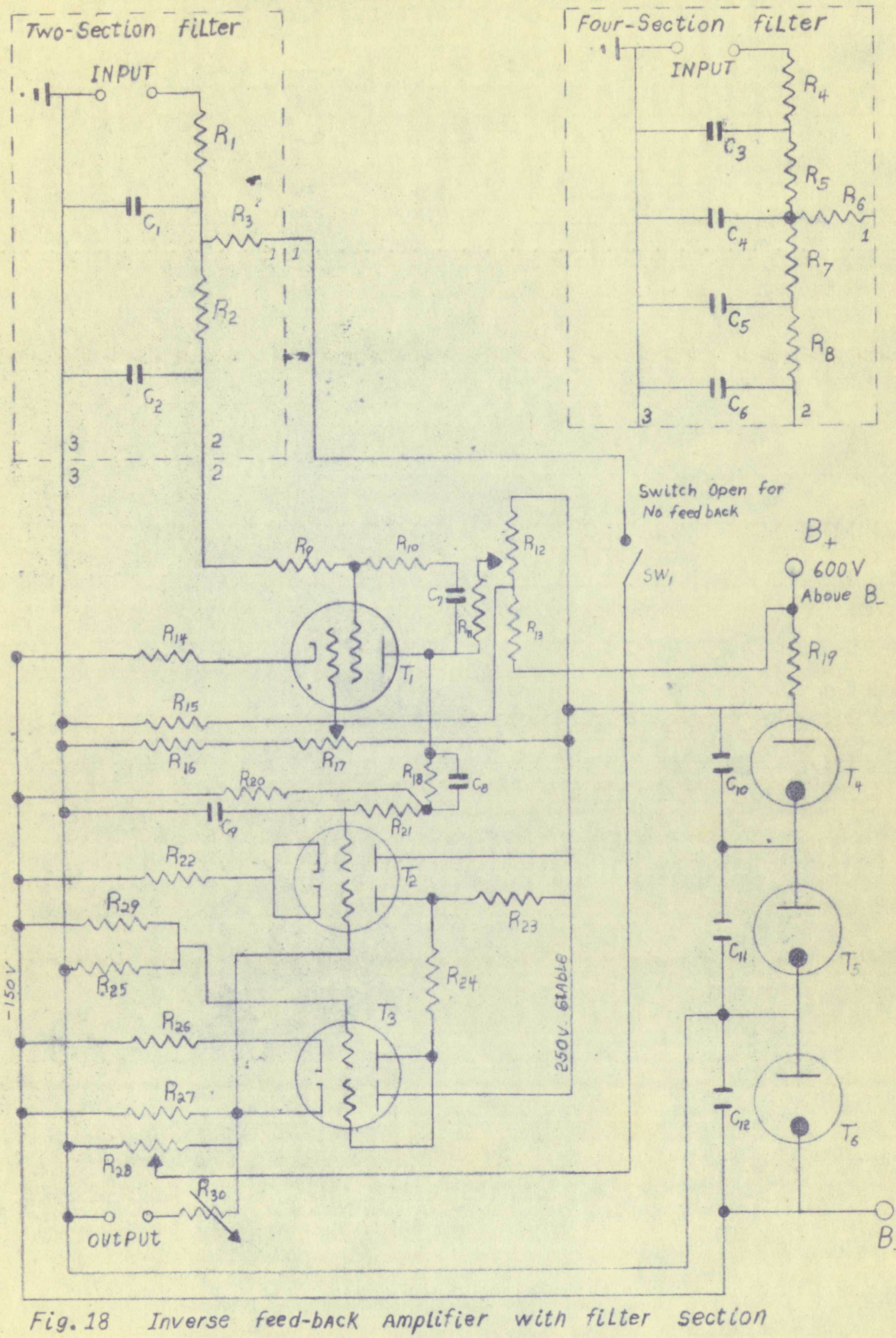
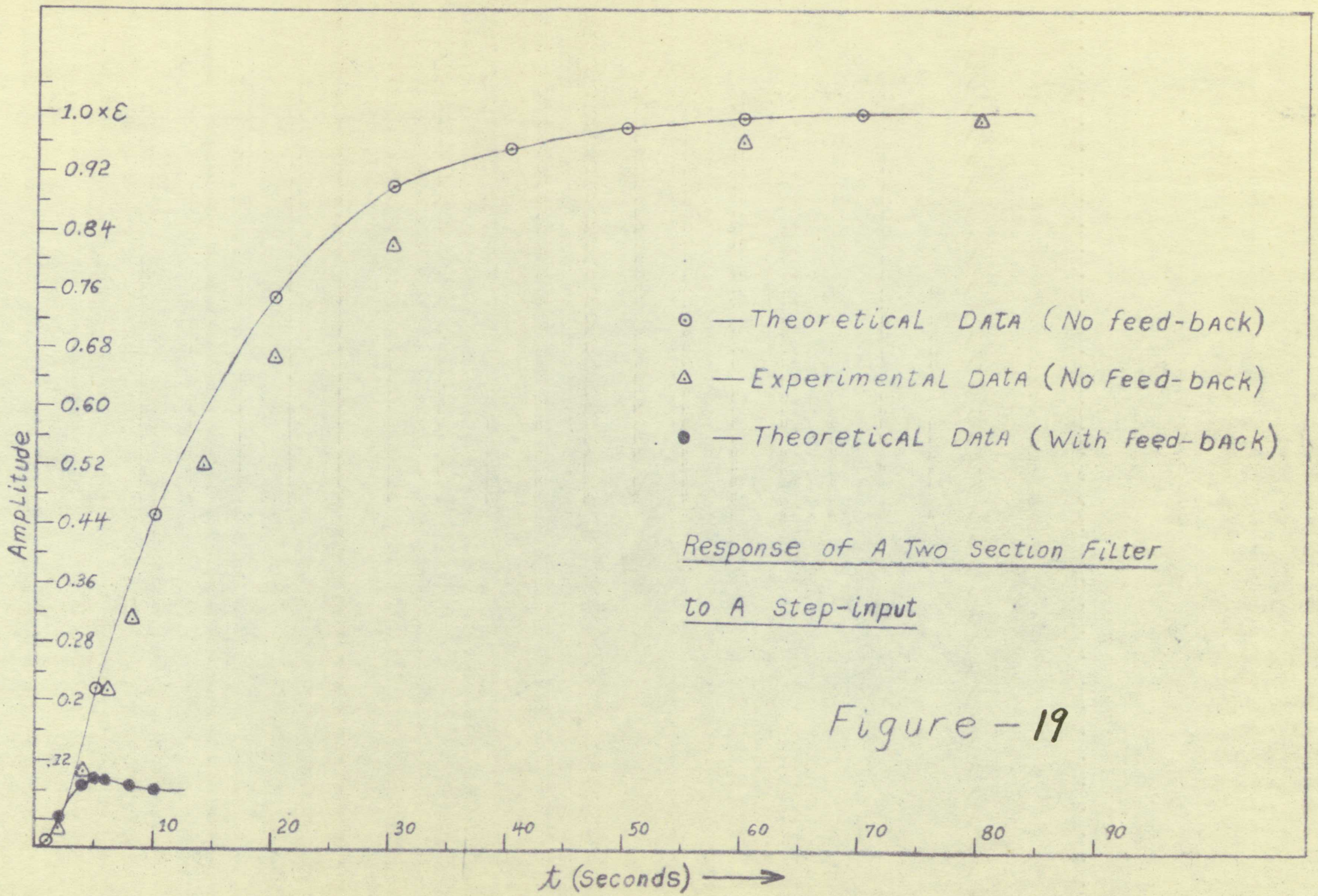


Fig. 18 Inverse feed-back Amplifier with filter section

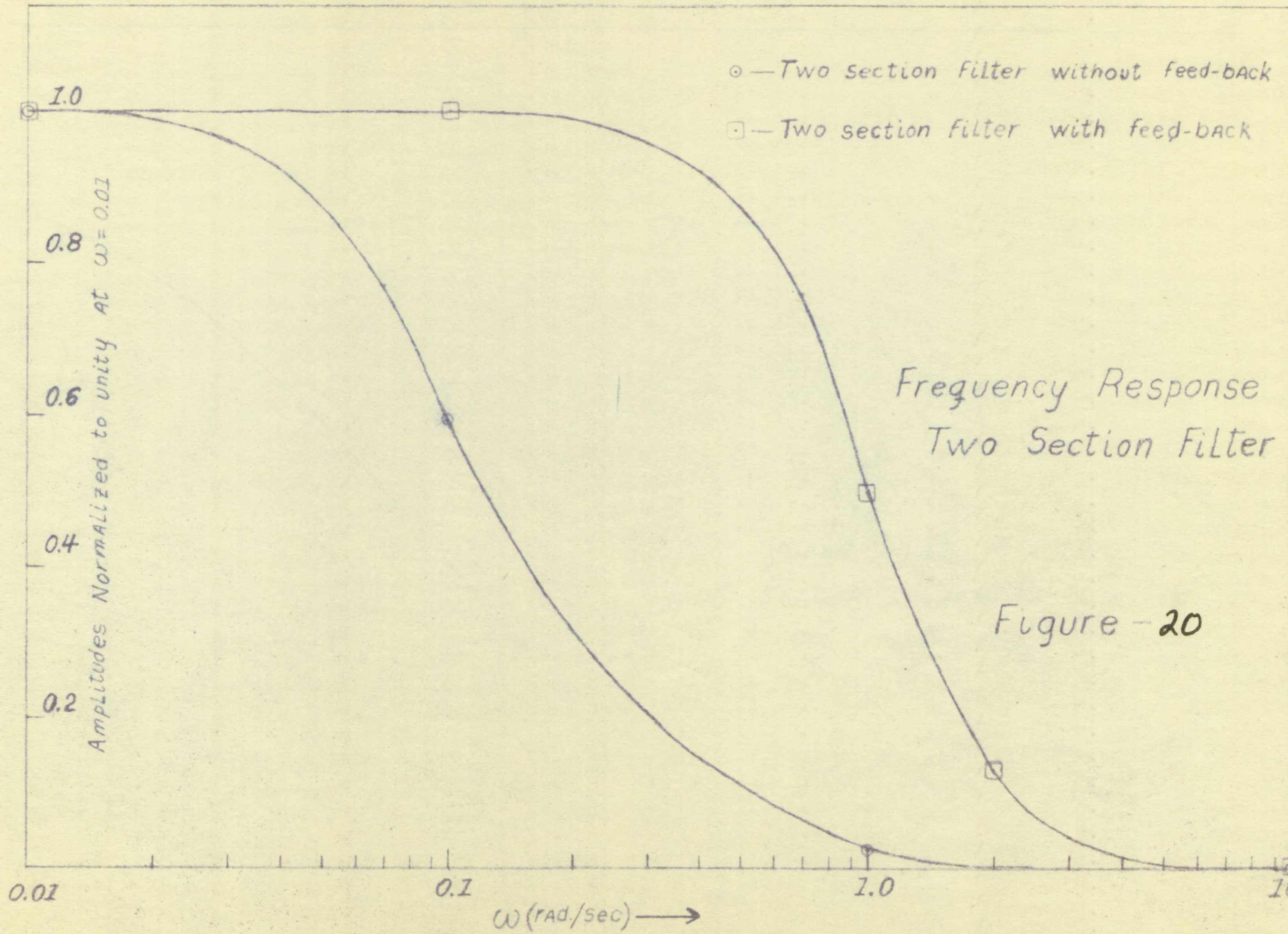
CIRCUIT CONSTANTS FOR FEED-BACK AMPLIFIER (Fig. 8)

R1,2,3,4,5,7,8,	10M	½W
R6	20M	½W
R 10,9,27,25,29	100K	1W
R 14	200K	1W
R 11,22	150K	1W
R 12	50K	1W
R 13	33K	1W
R 19	4K	5W
R 20,14	47K	1W
R 18,24,26,16	1M	1w
R 21	4.7M	½w
R 23	570K	1w
R 30	10K	Pot.
R 28	1M	Pot.
R 17	2K	Pot.
C 1,2,3,4,5,6,	½MF.	400V.
C7,8	100MMF	Mica
C9	0.03MF	400V.
C 10,11	8 MF	800V.
C 12	8 MF.	450V.
T1	6v6	
T2,3	6SL7	
T4,6	VR 150	
T5	VR105	



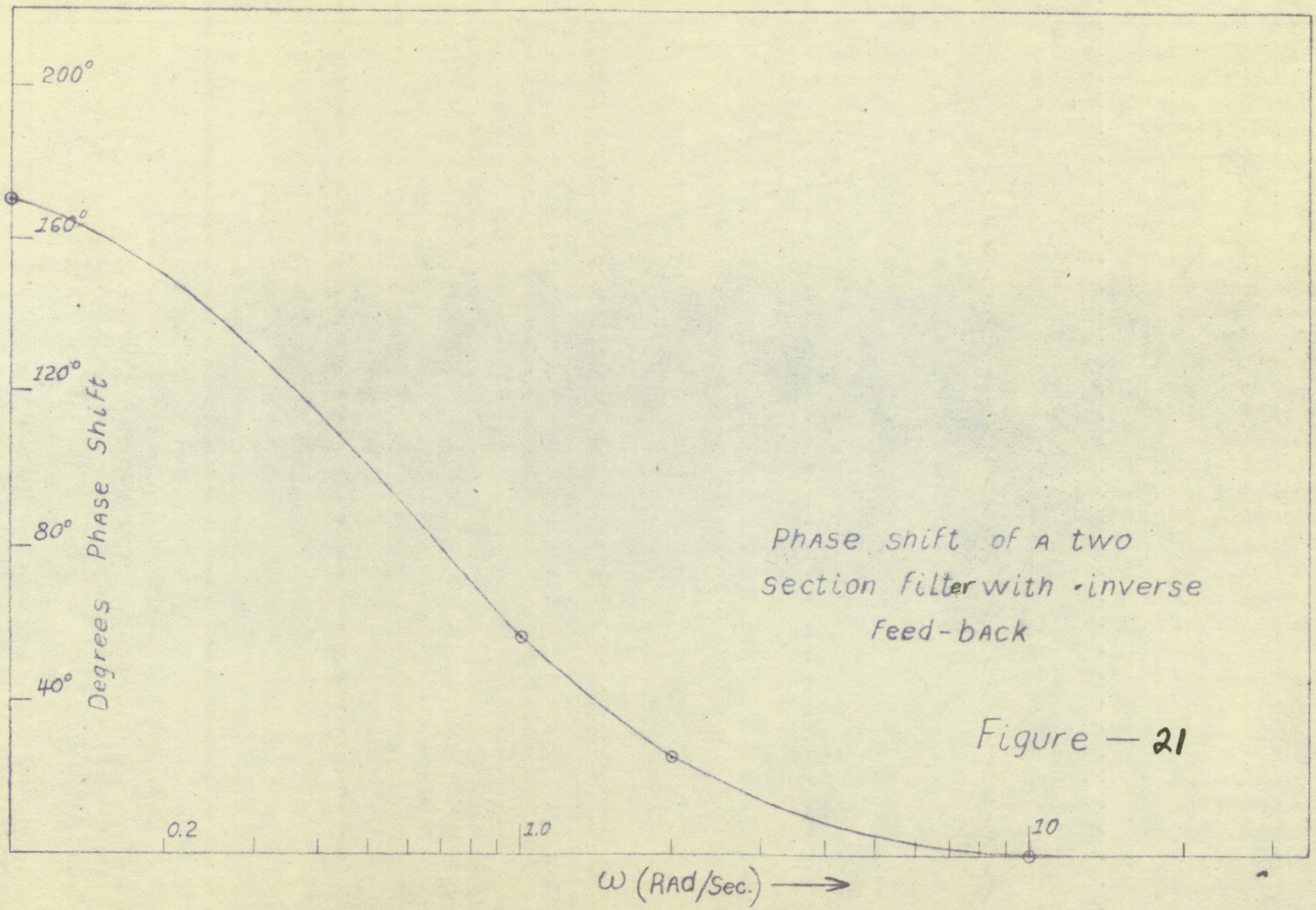
Response of A Two Section Filter to A Step-input

Figure - 19



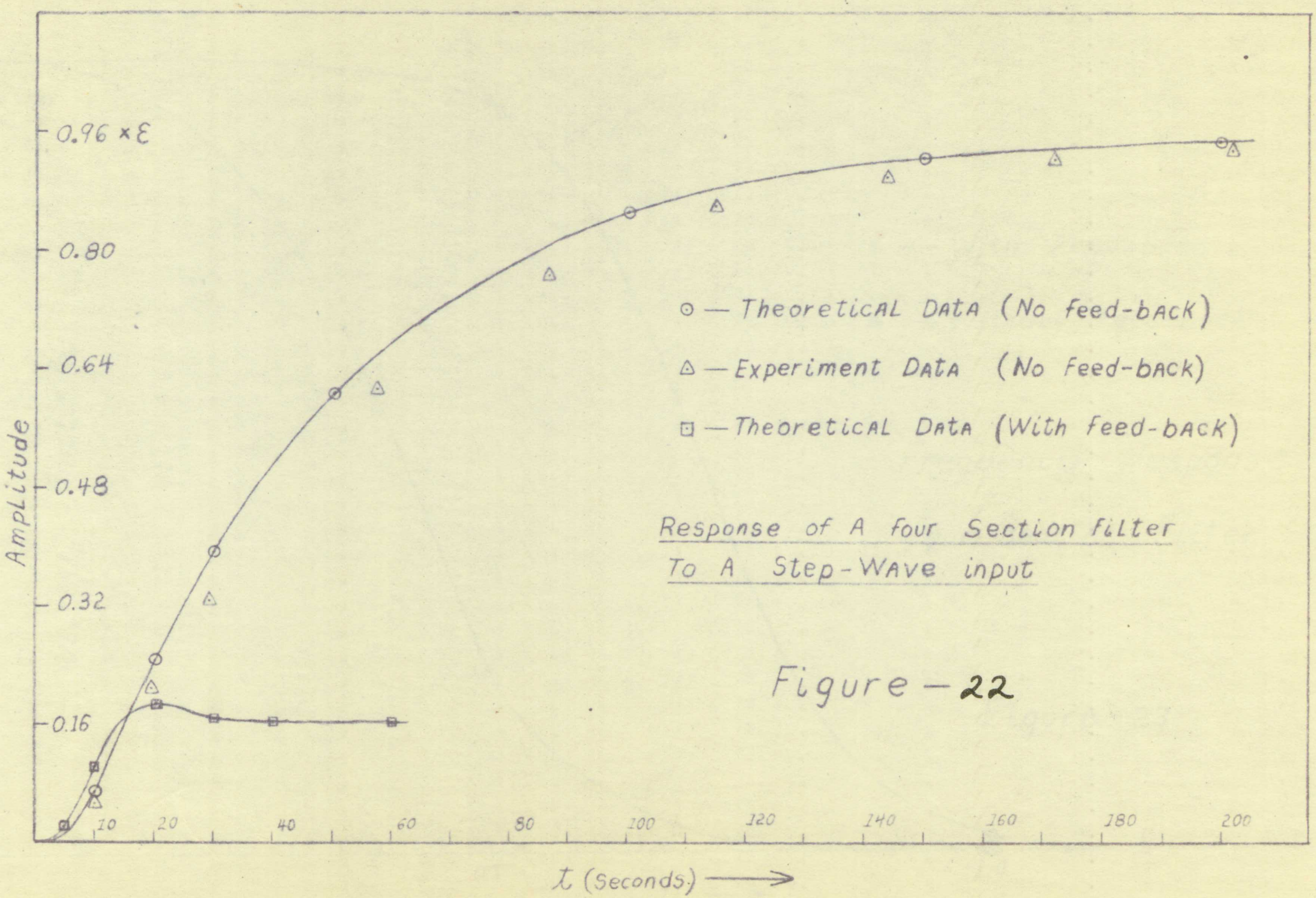
Frequency Response
Two Section Filter

Figure - 20



Phase shift of A two section filter with inverse Feed-back

Figure — 21



Response of A four Section filter
To A Step-Wave input

Figure — 22

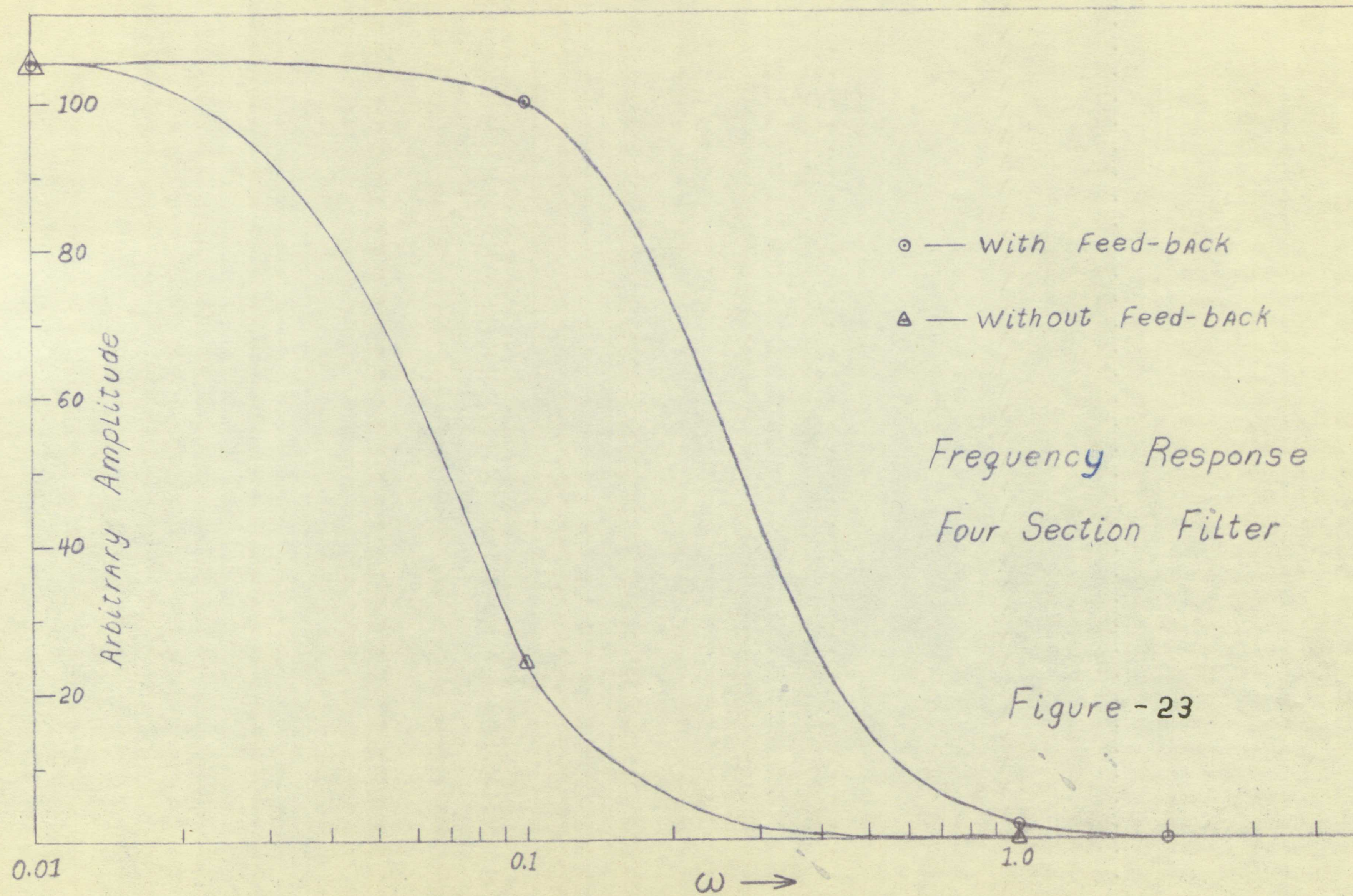

100
80
60
40
20
Arbitrary Amplitude
0.01
0.1
1.0
ω →
⊙ — With Feed-back
△ — Without Feed-back
Frequency Response
Four Section Filter

Figure - 23

CONCLUSIONS

The experimental data agree very well with the theoretical curves. The slight shift of the experimental points to the right of the theoretical step-wave response curves is probably due to stray capacitances that were not included in the theoretical calculations. The inclusion of inverse feed-back speeds up the reaction time of the filters to a step-wave input by about a factor of ten, and causes the voltage to overshoot its final equilibrium value. The cut-off frequency of the filters with inverse feed-back is about a factor of 5 to 10 higher than the same filter without feed-back.

The most valuable characteristic of filters with negative feed-back, however, is to be seen in the greatly improved sharpness of the cut-off in their frequency response curves.

ACKNOWLEDGMENTS

I am indebted to Professor Victor H. Regener, as it was he who first suggested a thorough investigation of these circuits. Much of the circuitry used in the experimental part of the investigation was designed and built by Professor Regener. I am also indebted to Professor Roy Thomas for many valuable discussions and a continued interest in the work. I am also grateful to him for a clear presentation of the theory of Laplace Transformations.

[illegible]

The experimental data agree very well with the theoretical curves. The slight shift of the experimental points to the right of the theoretical step-wave response curves is probably due to stray capacitances that were not included in the theoretical calculations. The insertion of inverse feed-back speeds up the response time of the filters to a step-wave input by about a factor of ten, the output voltage rising faster toward its final equilibrium value. The cut-off frequency of the filters with inverse feed-back is about a factor of 5 to 10 higher than the same filters without feed-back.

The most valuable characteristic of filters with negative feed-back, however, is to be seen in the greatly improved sharpness of the cut-off in their frequency response curves.

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D.T.