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Utilization of the Relationship Between the Complex Immittance and the Indicial Immittance Functions for the Synthesis of Two Terminal Networks

Julian T. Lefler

SYNTHESIS OF TWO TERMINAL NETWORKS - LEFLER

UTILIZATION OF THE RELATIONSHIP BETWEEN THE COMPLEX IMMITTANCE AND THE INDICIAL IMMITTANCE FUNCTIONS FOR THE SYNTHESIS OF TWO TERMINAL NETWORKS

By

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THE PROBLEM AND ITS ANALYSIS

I. BACKGROUND

Historically, the mathematical procedures involved in the design of an electrical network, the terminal characteristics of which comply with a given rational immittance (impedance or admittance) function of the complex frequency variable, have attracted the attention of many investigators. As a result of their research, a number of such procedures have been developed which, for the most part, have been documented in the literature of mathematics, physics, and engineering. The first major accomplishment in the field came in 1924 when Foster developed and published his Reactance Theorem [9]. From the time of Foster's original contribution until the present, progress in this field has been such that there now exist at least ten well-known, as well as several less well-known methods for the physical realization of networks [8].

Regardless of the fact that there are presently a number of diverse mathematical approaches to this particular type of synthesis problem, there is a continuing need for the development of additional methods. Each different approach, by reason of its particular mode of operation upon the complex immittance function will inherently yield network topological characteristics and circuit parameter values peculiar to the method; or will resolve functions not readily handled by other techniques; or will be less intricate mathematically than

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Regardless of the fact that there are presently a number of diverse mathematical approaches to this particular type of synthesis problem, there is a continuing need for the development of additional methods. Each different approach, by reason of its particular mode of operation upon the complex immittance function will generate its own network topological characteristics and circuit parameter values peculiar to the method or will realize functions not readily handled by other techniques, or will be more direct mathematically than

analogous procedures. These characteristics are of importance in the field of network design in that they provide for a wider area of choice in the selection of the type and configuration of a network, as well as a less restricted range of circuit element values for the fulfillment of a given set of design conditions.

II. THE PROBLEM

It was the purpose of this study (1) to develop a new method of network synthesis from a specified complex immittance function; and (2) to demonstrate the application of the method by means of its use in the solution of representative synthesis problems. This work deals only with the synthesis of two terminal networks containing linear, passive, lumped-constant elements, from given rational immittance functions of the complex frequency variable.

Not considered as a part of this work are (1) the development of the immittance function from frequency-response specifications (the approximation problem); or (2) the treatment of active, non-linear, or time varying network elements.

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A review of the existing frequency domain network realization techniques indicates that there is one point in common to all presently documented methods [4], [7]. Each depends upon the decomposition of the

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III. ANALYSIS OF THE PROBLEM

A review of the existing frequency domain network realization techniques indicates that there is one point in common to all presently documented methods [4], [7]. Each depends upon the decomposition of the

complex immittance function, which is expressed as a rational function of p, the complex frequency variable. Regardless of the method used to accomplish such decomposition, the end result of the procedure is an equivalent rational expression for the complex immittance in which p appears to the first or zero powers only; the arrangement of the decomposed function being such that the variable and constant terms are recognizable as network elements in their proper phasor relationship with respect to the network configuration.

In the broadest of terms, most of the presently available synthesis procedures of this character may be thought of as adaptations and extensions of either the Foster or the Cauer methods. It is of course true that each such adaptation has broadened the field of application of the technique involved, thereby increasing the scope of its practical application. The basic Foster synthesis method involves the decomposition of the complex immittance function by means of a partial fraction expansion. When the impedance function is so expanded, a network consisting of a group of anti-resonant circuits in parallel is specified. This network type, illustrated in Figure 1, is known as the First Foster Canonical Form.

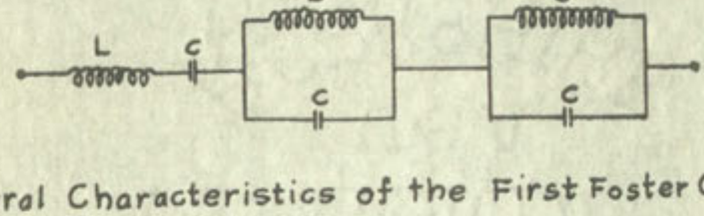

Fig.1.- General Characteristics of the First Foster Canonical Form

If, on the other hand, the given admittance function is so decomposed, a parallel group of resonant circuits is specified as illustrated by Figure 2. This configuration is known as the Second Foster Canonical Form.

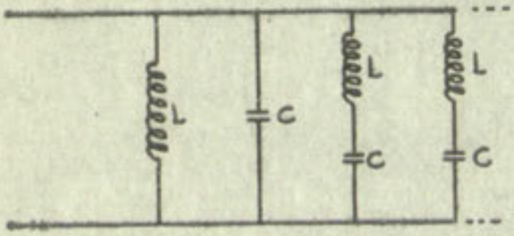

Fig.2.- General Characteristics of the Second Foster Canonical Form

The partial fraction decomposition process as used in the Foster technique is the familiar algebraic method whereby a given rational fraction may be decomposed into the sum of so-called partial fractions, the denominator of each such partial fraction being a factor of the denominator of the original fraction. As an example, assume that it is required to synthesize the network defined by

$$\dot{Z}(p) = \frac{12p^4 + 12p^2 + 1}{6p^3 + 3p}$$

The function $\dot{Z}(p)$ may be decomposed into a sum of partial fractions by the following procedure:

$$\frac{12p^4 + 12p^2 + 1}{6p(p + \frac{1}{2})} = \frac{Ap^2 + B}{6p} + \frac{Cp}{p^2 + \frac{1}{2}}$$

$$12p^4 + 12p^2 + 1 = Ap^4 + \tfrac{1}{2}Ap^2 + Bp^2 + \tfrac{1}{2}B + 6Cp^2$$

$$A = 12;\ B = 2;\ C = \tfrac{2}{3}$$

$$\frac{12p^4 + 12p^2 + 1}{6p^3 + 3p} = \frac{12p^2 + 2}{6p} + \frac{\frac{2}{3}p}{p^2 + \frac{1}{2}} = 2p + \frac{1}{3p} + \frac{1}{\frac{3p}{2} + \frac{3}{4p}}$$

Here the term 2p is recognizable as a series inductance of value 2; the term 1/3p as a series capacitance of value 3; and the term

$$\frac{1}{3p/2 + 3/4p}$$

as a parallel L-C group in series with the above single elements wherein the capacitance has a value of 3/2 and the inductance has a value of 4/3. The First Foster Canonical Form is thus specified. The Second Foster Canonical Form would be obtained in like manner by a partial fraction expansion of the function

$$\dot{Y}(p) = \frac{6p^3 + 3p}{12p^4 + 12p^2 + 1}$$

The first and second Cauer Canonical Forms are specified by the results of a continuing fraction-type decomposition of the given immittance function. These canonical forms are of a ladder type configuration as illustrated by Figure 3 and Figure 4.

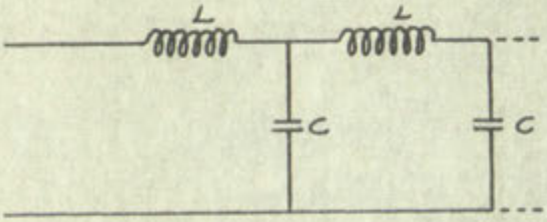

Fig. 3.- General Characteristics of the First Cauer Canonical Form

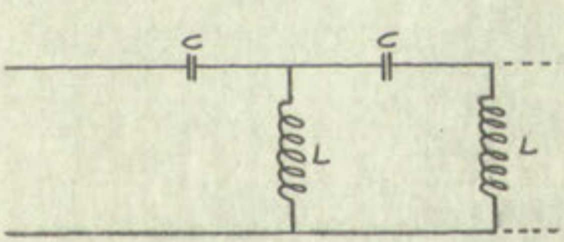

Fig 4.- General Characteristics of the Second Cauer Canonical Form

The continuing fraction decomposition of the Cauer technique consists of repeatedly performing the divisions indicated by the fraction bars until the exponents of the variable have been reduced to unity or zero values and the function is of a form compatible with the physical form of a ladder network. The following example illustrates the technique:

$$\dot{Z}(p) = \frac{12p^4 + 12p^2 + 1}{6p^3 + 3p} =$$

$$2p + \frac{6p^2+1}{6p^3+3p} \quad = \quad 2p + \cfrac{1}{p + \cfrac{1}{3p + \cfrac{1}{2p}}}$$

Here the quantities 2p and 3p are recognizable as specifying inductive impedances of values 2 and 3 respectively and the quantities p and 2p are likewise recognizable as specifying capacitive admittances of values 1 and 2 respectively. The network specified is therefore of the First Cauer Canonical Form. If the function $\dot{Z}(p)$ were to be written in terms of the ascending rather than the descending powers of the variable in this example, and the indicated division operations performed in the manner illustrated above, the Second Cauer Canonical Form would be specified.

Developments of the type discussed here, following after the basic Foster and Cauer work have all been based on the fundamental idea of an immittance function decomposition process restricted entirely to an operation within the frequency domain. Research into the literature of circuit theory fails to indicate that this somewhat artificial barrier has previously been penetrated with the specific object in view of the widening of the scope of network synthesis from a specified complex immittance.

Since the efforts of many investigators have been directed, over a period of more than thirty years, toward the development of additional decomposition techniques the conclusion was reached that, in all probability, at least a majority of the total possible methods of this sort

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had been thought out, developed, and documented. As a consequence, this study was directed along the lines of the identification of network elements and network configuration by means of circuit response to certain excitation functions.

It is a well-known fact that the solution of the differential equation of a network contains, in an easily identifiable form, the values of the network parameters either as coefficients, or as exponents of the Naperian base. For instance, the discharging curve of a capacitor through a resistor is expressed by $q(t) = CV_0 \epsilon^{-t/RC}$ where q(t) is the charge on the capacitor at any particular time t; V_0 is the voltage across the capacitor due to its charge at time $t = 0$; and the constants C and R as coefficient and exponents are the exact capacitance and resistance of the circuit elements. Likewise, the current rise in an inductance in series with a resistance is expressed as $i(t) = E/R(1-\epsilon^{-Rt/L})$ where i(t) is the magnitude of the current in the circuit at any particular time t; E is the magnitude of the driving voltage at time t; and the constants R and L as coefficient and exponents are the exact inductance and resistance of the circuit elements. Both of the above equations presuppose quiescent initial conditions within the circuit.

The integro-differential equation of a circuit or network does not lend itself to writing in the defining form of a response function thus:

$$\text{Response} = \frac{\text{effect}}{\text{cause}}$$

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The integro-differential equation of a linear network does not lend itself to writing in the defining form of a response function that

since either the current or the voltage parameter will appear as a derivative or as an integral, with respect to time. The solution of such equations however, may be written in response form. For instance, in the example above, of the current rise in an inductance in series with a resistance there is given, in the case of the original differential equation:

$$e = Ri + L\, {}^{di}/_{dt} \tag{1}$$

The method for the solution of equation (1) by transformation calculus, assuming quiescent initial conditions is:

$$E/p = RI(p) + pLI(p) \tag{2}$$

$$E/p = (R + pL)\, I(p) \tag{3}$$

$$I(p) = \frac{E}{p(pL+R)} = \frac{E}{L} \cdot \frac{1}{p(p + R/L)} \tag{4}$$

$$i = \frac{e}{L} \cdot \frac{(1 - \epsilon^{-Rt/L})}{R/L} \tag{5}$$

In this series of equations, either the Laplace transformed original equation or the solution to the original equation could be written in response form, thus:

$$\frac{I(p)}{E} = \frac{1}{p(pL+R)} \tag{6}$$

and

$$i/e = 1/R\,(1 - \epsilon^{-Rt/L}) \tag{7}$$

where equations (6) and (7) are related through the inverse Laplace transform, and where both equations represent the current response of an R-L shunt arm to an excitation function of voltage.

It is apparent that if the term e of equation (1) is arbitrarily made to be a unit-step function of voltage at time $t = 0$, the response of this R-L network, as indicated by equation (7) will be:

$$A(t) = i/_{u(t)=1} = 1/_R(1 - \epsilon^{-Rt/L}) \tag{8}$$

in which A(t) is the indicial admittance of the network and is defined as the time function of current entering the network in response to the application of a unit-step function of voltage U(t) to the network terminals at time $t = 0$. Here it is noted that the response function A(t) may be obtained from an originally specified immittance function $\dot{Y}(p)$ or $\dot{Z}(p)$ since, considering that $e = u(t)$, equation (4) is of the form:

$$I(p) = \frac{E = 1}{p(pL + R)} = \frac{1}{p\dot{Z}(p)} = \frac{\dot{Y}(p)}{p} \tag{9}$$

hence:

$$A(t) = i = \mathcal{L}^{-1}\left[1/_{p\dot{Z}(p)}\right] = \mathcal{L}^{-1}\left[\dot{Y}(p)/_p\right] \tag{10}$$

Analysis of equations (8) and (10) leads to the conclusion that if the inverse Laplace transform is applied to the function $\dot{Y}(p)/p$ of an inductance-resistance shunt arm, the resulting function A(t) will

clearly indicate the values of the resistance and the inductance elements comprising the shunt arm by means of a process of the comparison of like parts of identities. This same type of discussion could be carried out for other types of shunt arms than the R-L type used here. In fact, Chapter II will develop a complete synthesis method based on the utilization of the indicial admittance function as obtained from a specified complex immittance function.

The reasoning outlined above is equally applicable to the definition of network elements and configuration by means of the response function J(t), the indicial impedance. This function is defined as the time function of voltage produced at the terminals of a network in response to a unit-step function of current entering the network at time $t = 0$. Chapter III will develop the theory and procedures for this method of synthesis.

Since the synthesis methods to be developed in this work are based upon the obtaining of the response functions A(t) or J(t) from a given specification of complex immittance, there may arise the semantic question of whether such procedures are to be considered as a frequency domain or conversely, as a time domain synthesis. It is pointed out that the function of immittance to which the unknown network must conform is a function of frequency. Further, the functions A(t) and J(t) will be used only to define the constants R, L, and C which constants exist simultaneously and maintain their integrity in both the frequency and time domains. The fact that A(t) and J(t) are functions of the time variable is purely incidental since this variable will not, at any time,

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play an active role in the procedure. It only appears in the equations, when they are used as hereinafter described, for the purpose of mathematical rigor. The synthesis methods as described in this work should therefore be classified solely as a frequency domain synthesis. Accordingly, no relaxation of the general principle that the immittance function to be synthesized must be a positive real function, involving Hurwitz polynomials, is to be expected.

CHAPTER II

CURRENT RESPONSE AS A TOOL FOR SYNTHESIS

I. THE THEORY

In order to develop the theory and technique for the current response method of synthesis, the starting point will be to assume the general R-L-C network of Figure 5.

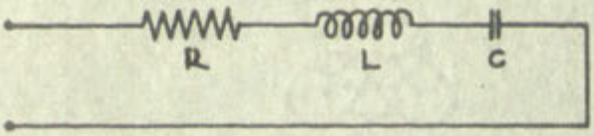

Fig. 5.- General R-L-C Network

The voltage e_0 is specified to be any constant voltage and the current i_0 is the response to the excitation e_0 applied to the network terminals. Under these conditions the differential equation of the network of Figure 5 is:

$$e_0 = Ri_0 + L\, {di_0}/{dt} + 1/C \int i_0\, dt \qquad (1)$$

Since e_0 is a constant and if the Laplace transform is applied to both sides of equation (1) there is given:

$$E_0/p = RI_0(p) + pLI_0(p) - i_0(0+) + 1/pC\,[I_0(p) + Q_0] \qquad (2)$$

where p is the complex frequency variable, $p = \sigma + j\omega$. Now assume that the initial conditions are quiescent and that the excitation is applied at the time $t = 0$. Under such conditions equation (2)

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reduces to:

$$\frac{E_0}{p} = RI_0(p) + pLI_0(p) + \frac{I_0(p)}{pC} \tag{3}$$

$$\frac{E_0}{p} = I_0(p)\left[R + pL + \frac{1}{pC}\right] \tag{4}$$

$$I_0(p) = \frac{E_0}{p\left[R + pL + \frac{1}{pC}\right]} \tag{5}$$

$$\frac{I_0(p)}{E_0} = \frac{1}{p\left[R + pL + \frac{1}{pC}\right]} \tag{6}$$

In equation (4), (5), and (6) the term in the square brackets is recognizable as the complex impedance $\dot{Z}(p)$ of the network and the left hand member of equation (6) is in the form of a response divided by an excitation. The right hand member of equation (6) is therefore a true response function. It is obvious that this response function may be written as:

$$\frac{1}{p\left[R + pL + \frac{1}{pC}\right]} = \frac{1}{p\dot{Z}_0(p)} = \frac{\dot{Y}_0(p)}{p} \tag{7}$$

If the excitation e_0 is assigned the value unity and in fact, to be more specific, if e_0 is made to be a unit-step function of voltage at time $t = 0$, it is evident that the response of the network is a function of current only, thus:

$$\frac{I_0(p)}{1} = \frac{1}{p\dot{Z}_0(p)} = \frac{\dot{Y}_0(p)}{p} \tag{8}$$

If now the inverse Laplace transform is applied to both sides of equation (8) there is obtained:

$$A(t) = i_0 = \mathcal{L}^{-1}\left[\frac{Y_0(p)}{p}\right] \tag{9}$$

Here, equation (9) defines the Indicial Admittance of the network of Figure 5 and represents, as a function of the time variable, the current response of the network to the excitation of a unit-step function of voltage applied to the network terminals at time $t = 0$.

It is clear that the network of Figure 5 may be considered to be a shunt network composed of a shunt arm of admittance $1/\dot{Z}_0(p)$ across the network terminals, through which a current $\dot{i}_0$ flows as a response to the excitation $e_0 = 1 = U(t)$. Now suppose that any number of additional shunt arms of admittance $\dot{Y}_1(p)$, $\dot{Y}_2(p)$, $\dot{Y}_3(p)$, $\cdots\cdots$ are added to the network as shown in Figure 6.

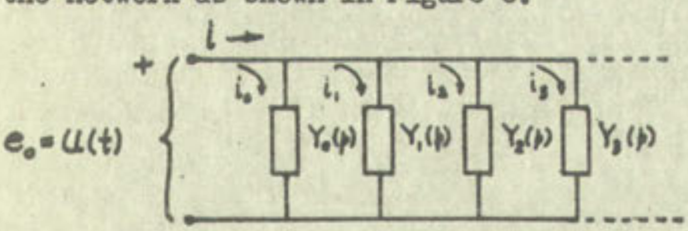


Fig. 6.- Shunt - type Network

Under these circumstances, each shunt arm will be excited by the same function of voltage U(t) and each will admit its individual current response as a part of i, the total response. The total current response i will then be:

$$i = \mathcal{L}^{-1}\left[\frac{Y_0(p)}{p}\right] + \mathcal{L}^{-1}\left[\frac{Y_1(p)}{p}\right] + \mathcal{L}^{-1}\left[\frac{Y_2(p)}{p}\right] + \cdots\cdots \tag{10}$$

and so on for as many shunt arms as the network contains. It is apparent that equation (10) may be written as:

$$i = \mathcal{L}^{-1}\left[\frac{Y_0(p) + Y_1(p) + Y_2(p) + \cdots\cdots}{p}\right] = \mathcal{L}^{-1}\left[\frac{Y_{TOTAL}(p)}{p}\right] \qquad (11)$$

by reason of the definition of the inverse Laplace transform, namely:

$$\mathcal{L}^{-1}[F(p)] \triangleq 1/2\pi j \oint F(p)\,\epsilon^{pt}\,dp \qquad (12)$$

Hence:

$$\begin{aligned}
&1/2\pi j\left[\oint (Y_0(p)/p)\epsilon^{pt}dp + \oint (Y_1(p)/p)\epsilon^{pt}dp + \cdots\cdots\right] = \\
&1/2\pi j\left[\oint (Y_0(p)/p)\epsilon^{pt}dp + (Y_1(p)/p)\epsilon^{pt}dp + \cdots\cdots\right] = \\
&1/2\pi j\left[\oint \left(Y_0(p)/p + Y_1(p)/p + \cdots\right)\epsilon^{pt}dp\right] = \\
&1/2\pi j \oint (Y_{TOTAL}(p)/p)\epsilon^{pt}dp = \\
&\mathcal{L}^{-1}\left[Y_{TOT}(p)/p\right] = A(t) \qquad (13)
\end{aligned}$$

And finally;

$$\begin{aligned}
&\mathcal{L}^{-1}\left[Y_{TOT}(p)/p\right] = 1/2\pi j \oint (Y_{TOT}(p)/p)\epsilon^{pt}dp = \\
&\quad \Sigma\,\text{Res.}\left[(Y_{TOT}(p)/p)\epsilon^{pt};\ \text{poles of } (Y_{TOT}(p)/p)\epsilon^{pt}\right] \qquad (14)
\end{aligned}$$

The mathematical configuration of $\mathcal{L}^{-1}[Y_{TOT}(p)/p]$ will be that of a sum of terms, as is indicated by the right hand side of equation (14). Each residue of the sum will represent the individual current response of a portion of the network to the excitation U(t) and, under such conditions, the physical configuration of a network consisting of a number

of shunt arms across the network terminals will be compatible with the mathematical configuration of the response function.

The inverse transform of the function $Y_{TOT}(p)/p$ will invariably be functions of the resistance, inductance, and/or capacitance parameters of a network yielding the response, in a form which will positively identify the elements of the network in their proper configuration except in the case where the network contains a pure capacitive shunt arm, which case will be treated later.

As an example serving to illustrate the method, assume the simple impedance function:

$$\dot{Z}(p) = \frac{p+1}{p+3} \tag{15}$$

which by reason of its pole-zero configuration is known to represent an R-L network. The function Y(p)/p is then:

$$\frac{\dot{Y}(p)}{p} = \frac{p+3}{p(p+1)} = \frac{p+3}{p^2+p} \tag{16}$$

$$A(t) = \mathcal{L}^{-1}\left[\frac{\dot{Y}(p)}{p}\right] = \mathcal{L}^{-1}\left[\frac{p+3}{p^2+p}\right] = \frac{1}{2\pi j}\oint \frac{p+3}{p^2+p}\,\epsilon^{pt}\,dp \tag{17}$$

$$= \sum \text{Res.}\left[\frac{p+3}{p^2+p}\,\epsilon^{pt}\right] = \left.\frac{(p+3)\,\epsilon^{pt}}{d/dp\,(p^2+p)}\right|_{\text{poles of } \frac{\dot{Y}(p)}{p}\epsilon^{pt}} \tag{18}$$

$$= \left. \frac{(p+3)\,\epsilon^{pt}}{2p+1} \right|_{p=0,-1} = 3 - 2\,\epsilon^{-t} \tag{19}$$

It is known that the indicial admittance of a pure resistance R is 1/R, and further that such response of a resistance R_L and an inductance L_R, in series is $1/R_L\,(1 - \epsilon^{-R_L t/L_R})$. If now the left side of equation (19) is rewritten:

$$3 - 2\,\epsilon^{-t} = 1 + 2(1 - \epsilon^{-(1)t}) \tag{20}$$

there is given the indicial admittance a network consisting of a shunt arm of pure resistance in which, by comparison with the known response functions stated in the foregoing paragraph:

$$\frac{1}{R} = 1 \tag{21}$$

$$R = 1 \tag{22}$$

in parallel with a shunt arm consisting of a resistance and inductance where, again by comparison:

$$\frac{1}{R_L} = 2 \tag{23}$$

$$R_L = \frac{1}{2} \tag{24}$$

and:

$$\frac{R_L}{L_R} = \frac{1/2}{L_R} = 1 \tag{25}$$

$$L_R = 1/2 \tag{26}$$

A physical realization of the original function:

$$\dot{Z}(p) = \frac{p + 1}{p + 3} \tag{15}$$

would therefore be the network shown in Figure 7. There are of course an infinite number of networks which will yield the basic responses of equation (16) and equation (19).

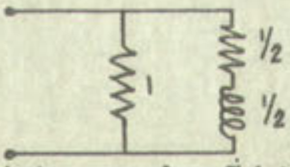


Fig. 7.- Shunt Network for $\dot{Z}(p) = (p+1)/(p+3)$

In the practical application of the response synthesis technique, the formal procedures of equations (17) and (18) would be unnecessary since they express only the step by step method of obtaining the inverse Laplace transform of $\dot{Y}(p)/p$. Actually, equation (19) can be written directly from equation (16).

It is now obvious that it will only be necessary to develop the indicial admittances of all possible combinations of R, L, and C elements in shunt arms, in order to be able to synthesize the class of admittance functions under consideration. Such a development is made

in Section III of this Chapter.

II. THE POLES OF THE ADMITTANCE FUNCTION

Before proceeding further with the development of the synthesis method, it is expedient to examine the poles of the admittance function. Since $\dot{Y}(p)$ for a network of normal complexity will be a rational fraction of the general form

$$\dot{Y}(p) = \frac{N(p)}{D(p)} \tag{27}$$

the general form of the corresponding current response function will be

$$\frac{\dot{Y}(p)}{p} = \frac{N(p)}{pD(p)} \tag{28}$$

The evaluation of the residues of the function

$$F(p) = \frac{P(p)}{Q(p)} = \frac{\dot{Y}(p)}{p}\epsilon^{pt} = \frac{N(p)}{pD(p)}\epsilon^{pt} \tag{29}$$

depends upon the location of the poles of F(p). Hence, it may be stated that the poles of F(p) are related, through the inverse transform, to the indicial admittance functions. It is apparent that there will always be a pole of F(p) at $p = 0$ due to the multiplier p of pD(p). If now a network containing all possible types of linear passive element shunt arms is assumed, its general configuration will be as shown in Figure 8.

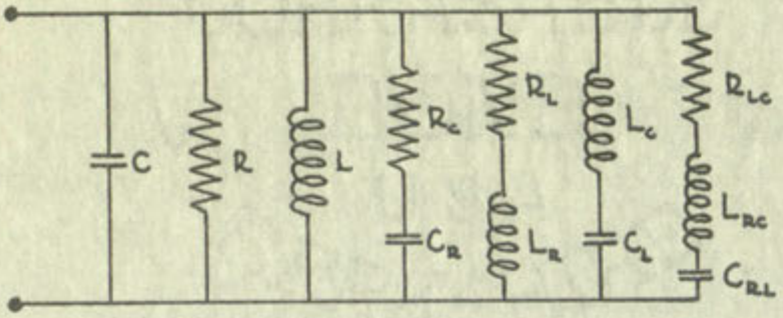


Fig.8.- The All-Shunt Type Network

This type of network will be referred to as an all-shunt network. From Figure 8, it may be seen that the complex admittance of this general form of network is:

$$Y(p) = \frac{pC}{1} + \frac{1}{R} + \frac{1}{pL} + \frac{1}{R_c + \frac{1}{pC_R}} + \frac{1}{pL_R + R_L} + \frac{1}{pL_c + \frac{1}{pC_L}} + \frac{1}{pL_{Rc} + R_{Lc} + \frac{1}{pC_{RL}}} = \quad (30)$$

$$\frac{pC}{1} + \frac{1}{R} + \frac{1}{pL} + \frac{pC_R}{pR_cC_R + 1} + \frac{1}{pL_R + R_L} + \frac{pC_L}{p^2L_cC_L + 1} + \frac{pC_{RL}}{p^2L_{Rc}C_{RL} + pR_{Lc}C_{RL} + 1} \quad (31)$$

for which the denominator D(p) of the combined terms of (31) is:

$$D(p) = (1)(R)(pL)(pR_cC_R + 1)(pL_R + R_L)(p^2L_cC_L + 1)(p^2L_{Rc}C_{RL} + pR_{Lc}C_{RL} + 1) \quad (32)$$

In such a network, each shunt arm establishes a separate and distinct factor of D(p) and each such factor except (1) and (R) will establish a pole or poles of F(p) in (29) at various values of the variable p. Also, as has been pointed out previously, there will always be a pole of F(p) at $p = 0$. It is evident also that if an all-shunt network contains a shunt arm consisting of an element of pure inductance only, the factors (p) and (pL) of pD(p) will act together to yield a second order pole at $p = 0$.

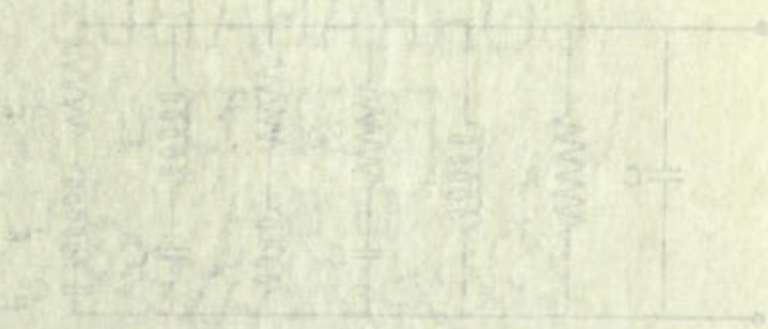


Fig. 8 - The All-Shunt Type Network

This type of network will be [illegible]. [illegible] Figure 8, it may be seen that the complex admittance of this general form of network is:

$$Y(p) = \frac{pC}{1} + \frac{1}{R} + \frac{1}{pL} + \frac{1}{R_2 + \frac{1}{pC_2}} + \frac{1}{pL_3 + R_3} + \frac{1}{pL_4 + \frac{1}{pC_4}} + \frac{1}{pL_5 + R_5 + \frac{1}{pC_5}} \qquad (29)$$

$$\frac{pC}{1} + \frac{1}{R} + \frac{1}{pL} + \frac{pC_2}{pR_2C_2 + 1} + \frac{1}{pL_3 + R_3} + \frac{pC_4}{p^2L_4C_4 + 1} + \frac{pC_5}{p^2L_5C_5 + pR_5C_5 + 1} \qquad (30)$$

for which the denominator D(p) of the combined terms of (30) is:

$$D(p) = (1)(R)(pL)(pR_2C_2 + 1)(pL_3 + R_3)(p^2L_4C_4 + 1)(p^2L_5C_5 + pR_5C_5 + 1) \qquad (31)$$

In such a network, each shunt arm contributes a separate and distinct factor of D(p) and each such factor except (1) and (R) will establish a pole or poles of F(p) in (29) at various values of the variable p. Also, as has been pointed out previously there will be a pole of F(p) at $p = \infty$. It is evident also that if an all-shunt network contains a shunt arm consisting of an element of pure inductance only, the factors (p) and (pL) of pD(p) will act to make F(p) have a second order pole at $p = 0$.

The factor $(p^2 L_{RC} C_{RL} + p R_{LC} C_{RL} + 1)$ of D(p), generated by the R-L-C arm may yield a second order pole at some negative real value of $p \neq 0$ when this factor is determined by network element values which result in its being algebraically, a perfect square. Or, it may yield two first order poles at values of p which are real, negative, and unequal; or it may yield two first order poles which are complex conjugates, in the left half p-plane. The conditions under which such poles occur are developed in Section III of this Chapter. Since this factor of D(p) will yield, under the influence of various values of elements, three distinct types of pole configuration in F(p), it may be postulated that an R-L-C shunt arm may have any one of three different forms of current response to the voltage unit-step function. This postulate is indeed true as will be shown in Section III of this Chapter. The characteristic of the R-L-C shunt arm, of multiple response, does not in any way impair the usefulness of the response synthesis technique.

Further examination of the factors of D(p) shows that poles of F(p) of order greater than the second will not occur. This fact is of considerable importance, since the evaluation of residues of poles of order greater than the second is quite laborious.

III. THE INDICIAL ADMITTANCE OF STANDARD TYPES OF SHUNT ARMS

As was shown in Section I of this Chapter, the function $Y(p)/p$ represents the current response, of a network having an admittance Y(p), to unity voltage excitation. Further, the inverse Laplace transform of $Y(p)/p$, the indicial admittance of the network, will yield

The factor $(p^2 L_{n4} C_{n4} + pR_{n4} C_{n4} + 1)$ of $D(p)$, generated by the R-L-C arm may yield a second order pole at some negative real value of $p \neq 0$ when this factor is determined by network elements which result in its being algebraically a perfect square. Or, it may yield two first order poles at values of p which are real, negative, and unequal; or it may yield two first order poles which are complex conjugates, in the left half p-plane. The conditions under which such poles occur are developed in Section III of this Chapter. Since this factor of D(p) will yield, under the influence of various values of elements, three distinct types of pole configuration in F(p), it may be postulated that an R-L-C shunt arm may have any one of three different forms of current response to the voltage unit-step function. This postulate is indeed true as will be shown in Section III of this Chapter. The characteristic of the R-L-C shunt arm, of multiple response, does not in any way impair the usefulness of the response synthesis techniques.

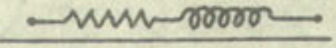
Further examination of the factors of D(p) shows that poles of F(p) of order greater than the second will not occur. This fact is of considerable importance, since the evaluation of residues of poles of order greater than the second is quite laborious.

III. THE INITIAL ADMITTANCE OF STANDARD TYPES OF SHUNT ARMS

As was shown in Section I of this Chapter, the function $Y(p)/p$ represents the current response of a network having an admittance, Y(p), to unity voltage excitation. Further, the inverse-Laplace transform of $Y(p)/p$, the initial admittance of the network, will yield

directly the R, L, and C parameters of the network from which the response is obtained (except in those cases in which the network contains a pure capacitive shunt arm, which case will require special treatment). Therefore if the indicial admittances of all possible combinations of R, L, and C as shunt arms are developed and tabulated, such a tabulation will serve as a means for identifying the parameters (except in the case of the pure capacitive arm) of the all-shunt network defined by a given admittance function. It is of course basic to the method, as well as all other synthesis methods, that the given admittance function be capable of synthesis in the configuration required.

Figure 8 shows that the indicial admittance functions for a maximum of seven standard forms of shunt arms will be required for development. Accordingly these seven standard forms of shunt arms are treated below.

1. Resistance - Inductance 

$$\frac{Y(p)}{p} = \frac{1}{p(pL+R)} = \frac{1}{pL(p+R/L)} \tag{33}$$

$$\frac{P(p)}{Q(p)} = \frac{Y(p)}{p}\epsilon^{pt} = \frac{\epsilon^{pt}}{pL(p+R/L)} \tag{34}$$

$P(p)/Q(p)$ has poles at $p = 0, -R/L$

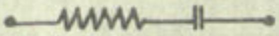
$$\frac{P(p)}{Q'(p)} = \frac{\epsilon^{pt}}{2pL+R} \tag{35}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=0} = \frac{1}{R} \tag{36}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-R/L} = -\frac{\epsilon^{-Rt/L}}{R} \tag{37}$$

$$\sum \text{Res.}\left[\frac{P(p)}{Q(p)} \quad 0, -R/L\right] = \frac{1}{R} - \frac{1}{R}\epsilon^{-Rt/L} \tag{38}$$

$$A(t) = \mathcal{L}^{-1}\left[\frac{Y(p)}{p}\right] = \frac{1}{R}\left[1 - \epsilon^{-Rt/L}\right] \tag{39}$$

2. Resistance - Capacitance 

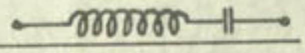
$$\frac{Y(p)}{p} = \frac{1}{p(R+\frac{1}{pC})} = \frac{1}{pR+\frac{1}{C}} = \frac{1}{R(p+\frac{1}{RC})} \tag{40}$$

$$\frac{P(p)}{Q(p)} = \frac{Y(p)}{p}\epsilon^{pt} = \frac{\epsilon^{pt}}{R(p+\frac{1}{RC})} \tag{41}$$

$P(p)/Q(p)$ has a pole at $p = -1/RC$

$$\frac{P(p)}{Q'(p)} = \frac{\epsilon^{pt}}{R} \tag{42}$$

$$A(t) = \mathcal{L}^{-1}\left[\frac{Y(p)}{p}\right] = \left.\frac{P(p)}{Q'(p)}\right|_{p=-1/RC} = \frac{1}{R}\epsilon^{-t/RC} \tag{43}$$

3. Inductance - Capacitance 

$$\frac{Y(p)}{p} = \frac{1}{p(pL+\frac{1}{pC})} = \frac{pC}{p(p^2LC+1)} = \frac{1}{L(p^2+\frac{1}{LC})} \tag{44}$$

$$\frac{P(p)}{Q(p)} = \frac{Y(p)}{p}\epsilon^{pt} = \frac{\epsilon^{pt}}{L(p^2+\frac{1}{LC})} \tag{45}$$

$P(p)/Q(p)$ has poles at $p = \pm j\sqrt{1/LC}$

$$\frac{P(p)}{Q'(p)} = \frac{\varepsilon^{pt}}{2pL} \tag{46}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=j\sqrt{1/LC}} = \frac{\varepsilon^{jt\sqrt{1/LC}}}{2jL\sqrt{1/LC}} = \frac{\varepsilon^{jt\sqrt{1/LC}}}{2j\sqrt{L/C}} \tag{47}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-j\sqrt{1/LC}} = -\frac{\varepsilon^{-jt\sqrt{1/LC}}}{2jL\sqrt{1/LC}} = -\frac{\varepsilon^{-jt\sqrt{1/LC}}}{2j\sqrt{L/C}} \tag{48}$$

$$A(t) = \mathcal{L}^{-1}\left[\frac{Y(p)}{p}\right] = \frac{1}{\sqrt{L/C}}\left[\sin\sqrt{1/LC}\,(t)\right] = \sqrt{C/L}\,\sin\frac{t}{\sqrt{LC}} \tag{49}$$

4. Resistance - Inductance - Capacitance

$$\frac{Y(p)}{p} = \frac{1}{p(R + pL + \frac{1}{pC})} = \frac{1}{L\left(p + R/2L + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)\left(p + R/2L - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)} \tag{50}$$

(a.) For $\dfrac{R^2}{4L^2} = \dfrac{1}{LC}$ (51)

$$\frac{Y(p)}{p} = \frac{1}{L(p + R/2L)^2} \tag{52}$$

$$\frac{P(p)}{Q(p)} = \frac{Y(p)}{p}\varepsilon^{pt} = \frac{\varepsilon^{pt}}{L(p + R/2L)^2} \tag{53}$$

$P(p)/Q(p)$ has a second order pole at $p = -R/2L$

$$\left.\begin{aligned} P(p) &= \varepsilon^{pt} \\ P'(p) &= t\,\varepsilon^{pt} \\ Q'(p) &= 2pL + R \\ Q''(p) &= 2L \\ Q'''(p) &= 0 \end{aligned}\right|_{p=-R/2L} \begin{aligned} &= \varepsilon^{-Rt/2L} \\ &= t\,\varepsilon^{-Rt/2L} \\ &= 0 \\ &= 2L \\ &= 0 \end{aligned} \tag{54}$$

$$\frac{6P'Q'' - 2PQ'''}{3(Q'')^2} = \frac{6(t\,\varepsilon^{-Rt/2L})(2L)}{12L^2} = \frac{t\,\varepsilon^{-Rt/2L}}{L} \tag{55}$$

$$\frac{R}{2L} = \frac{1}{\sqrt{LC}} \tag{56}$$

$$A(t) = \mathcal{L}^{-1}\left[\frac{Y(p)}{p}\right] = \frac{t}{L}\,\varepsilon^{-Rt/2L} = \frac{t}{L}\,\varepsilon^{-\sqrt{1/LC}\,t} \tag{57}$$

(b) For $\dfrac{R^2}{4L^2} > \dfrac{1}{LC}$ (58)

$$\frac{Y(p)}{p} = \frac{1}{L\left(p + R/2L + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)\left(p + R/2L - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)} \tag{59}$$

$$K \triangleq \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

$$\frac{P(p)}{Q(p)} = \frac{Y(p)}{p}\,\varepsilon^{pt} = \frac{\varepsilon^{pt}}{L(p + R/2L + K)(p + R/2L - K)} \tag{60}$$

$P(p)/Q(p)$ has poles at $p = (-R/2L + K), (-R/2L - K)$

$$\frac{P(p)}{Q'(p)} = \frac{\epsilon^{pt}}{2pL+R} \tag{61}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-R/2L+k} = \frac{\epsilon^{(-R/2L+k)t}}{2kL} \tag{62}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-R/2L-k} = -\frac{\epsilon^{(-R/2L-k)t}}{2kL} \tag{63}$$

$$A(t) = \mathcal{L}^{-1}\left[\frac{Y(p)}{p}\right] = \frac{\epsilon^{-Rt/2L}}{kL}\left[\frac{\epsilon^{kt} - \epsilon^{-kt}}{2}\right] = \frac{\epsilon^{-Rt/2L}}{kL}\sinh(kt) \tag{64}$$

(C) For $\frac{1}{LC} > \frac{R^2}{4L^2}$

$$\frac{Y(p)}{p} = \frac{1}{L\left(p + R/2L + j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right)\left(p + R/2L - j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right)} \tag{65}$$

$$\beta \triangleq \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

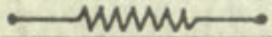
$$\frac{P(p)}{Q(p)} = \frac{Y(p)}{p}\epsilon^{pt} = \frac{\epsilon^{pt}}{L(p + R/2L + j\beta)(p + R/2L - j\beta)} \tag{66}$$

$$\frac{P(p)}{Q'(p)} = \frac{\epsilon^{pt}}{2pL+R} \tag{67}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-R/2L+j\beta} = \frac{\epsilon^{(-R/2L+j\beta)t}}{2j\beta L} \tag{68}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-R/2L-j\beta} = -\frac{\epsilon^{(-R/2L-j\beta)t}}{2j\beta L} \tag{69}$$

$$A(t) = \mathcal{L}^{-1}\left[\frac{Y(p)}{p}\right] = \frac{\epsilon^{-Rt/2L}}{\beta L}\left[\frac{\epsilon^{j\beta t} - \epsilon^{-j\beta t}}{2j}\right] = \frac{\epsilon^{-Rt/2L}}{\beta L}\sin(\beta t) \tag{70}$$

5. *Pure Resistance* 

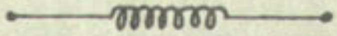
$$\frac{Y(p)}{p} = \frac{1}{pR} \tag{71}$$

$$\frac{P(p)}{Q(p)} = \frac{Y(p)}{p}\,\epsilon^{pt} = \frac{\epsilon^{pt}}{pR} \tag{72}$$

$$\frac{P(p)}{Q'(p)} = \frac{\epsilon^{pt}}{R} \tag{73}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=0} = \frac{1}{R} \tag{74}$$

$$A(t) = \mathcal{L}^{-1}\left[\frac{Y(p)}{p}\right] = \frac{1}{R} \tag{75}$$

6. *Pure Inductance* 

$$\frac{Y(p)}{p} = \frac{1}{p^2L} \tag{76}$$

$$\frac{P(p)}{Q(p)} = \frac{Y(p)}{p}\,\epsilon^{pt} = \frac{\epsilon^{pt}}{p^2L} \tag{77}$$

$P(p)/Q(p)$ has a second order pole at $p = 0$

$$\left.\begin{array}{l} P(p) = \epsilon^{pt} \\ P'(p) = t\,\epsilon^{pt} \\ Q'(p) = 2pL \\ Q''(p) = 2L \\ Q'''(p) = 0 \end{array}\right|_{p=0} \begin{array}{l} = 1 \\ = t \\ = 0 \\ = 2L \\ = 0 \end{array} \tag{78}$$

$$\frac{6P'Q''-2PQ'''}{3(Q'')^2} = \frac{(6)(t)(2L)}{12L^2} = \frac{t}{L} \tag{79}$$

$$A(t) = \mathcal{L}^{-1}\left[\frac{Y(p)}{p}\right] = \frac{t}{L} \tag{80}$$

7. Pure Capacitance

$$\frac{Y(p)}{p} = C \tag{81}$$

$$\frac{P(p)}{Q(p)} = \frac{Y(p)}{p}\epsilon^{pt} = C\epsilon^{pt} \tag{82}$$

$$\frac{C}{2\pi j}\oint \epsilon^{pt}\,dp = C\delta(t) \tag{83}$$

$$A(t) = \mathcal{L}^{-1}\left[\frac{Y(p)}{p}\right] = C\delta(t) \tag{84}$$

As indicated previously, the significance of the current response $C\delta(t)$ of a pure capacitive shunt arm must receive special consideration. The unit impulse $\delta(t)$ is defined as a function whose value is zero every where except in an arbitrarily small interval around $t = 0$ where it becomes infinite in such a way that:

$$\int_{-a}^{+b} \delta(t)\,dt = 1 \qquad 0 < a, b < \infty \tag{85}$$

Graphically, the impulse function of current which is of interest here is shown in Figure 9.

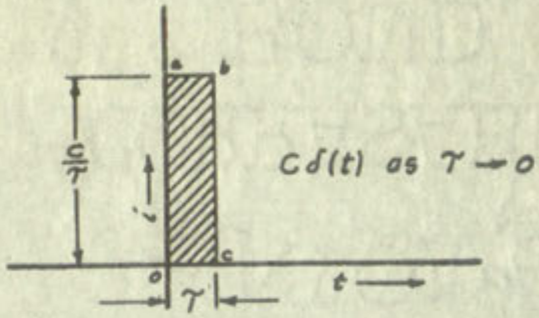


Fig. 9.- Impulse Function of Strength C

One interpretation of the physical situation existing when an all shunt network, containing a pure, uncharged capacitive arm is excited by a unit-step function of voltage at time $t = 0$ is as follows: First, since the capacitor is uncharged, the impressed voltage of unity sees a short circuit and since mathematically, the voltage remains at unity value, the response is a current which rises from zero value to infinity in zero time (theoretically). This current rise is represented by the line segment o-a of Figure 9. Second, the current response is maintained at infinite value for an infinitesimal time τ, at the end of which interval the capacitor reaches full charge. The point in time at which the capacitor reaches full charge is indicated in Figure 9 as the point b. Third, since the capacitor is at this point fully charged, it presents a back e.m.f. exactly equal and opposite to the exciting voltage and hence its current response immediately becomes zero as indicated in the figure by the line segment b-c. It is important to note that the time interval $\tau \to 0$ is infinitesimally small in comparison with any other time parameters of the network with which the pure capacitive

shunt arm may be associated.

To continued the interpretation on the same basis as outlined in the preceding paragraph, it is apparent that the admittance of the pure capacitive shunt arm is infinite until the impulse function of current $C\delta(t)$ reaches the point b of Figure 9 in time, while the admittances of the other shunt arms of the network have finite values during this interval. Hence under these conditions the total current response of the network, for the interval $\tau \to 0$ will be the current response of the pure capacitive shunt arm. This response, infinite current for an infinitesimal period of time is identified by the indicial admittance function $\mathcal{L}^{-1}[Y(p)/p]$ of the network as being identically equal to zero, since this function deals with finite intervals of time rather than with infinitesimals.

At the time the current impulse $C\delta(t)$ returns to zero (point C of Figure 9), the admittance of the pure capacitive shunt arm is zero hence its current response is zero and the indicial admittance of the network becomes the sum of the indicial admittances of those arms of the network which are not purely capacitive.

From the above discussion it may be stated that a pure capacitive arm of an all-shunt network actually acts as a switch to apply the unit-step function of voltage to the remainder of the network at a time $t = (0 + \tau)$. However, since the unit-step function of voltage is thus applied to the portion of the network capable of yielding a true indicial admittance at time $t = (0 + \tau)$, the effect is simply to establish a new zero point in time insofar as the time function current

response of the whole network is concerned. As a corollary to the above, it may be stated that $\mathcal{L}^{-1}\left[Y(p)/p\right]$ for an all-shunt network containing a pure capacitive shunt arm defines the current response of the network to a unit-step function of voltage only when consideration is given to the fact that the pure capacitive element is fully charged at time $t = 0$. The value of such a capacitive element will therefore not be defined by the indicial admittance function $\mathcal{L}^{-1}\left[Y(p)/p\right]$ when the network contains other element types in addition to the pure capacitive shunt arm.

The indicial admittance functions developed above are tabulated in Table I. This table has been used in the solution of the all-shunt network realization problems given as examples of the response synthesis in Appendix A. There now remains the development of a method for the identification of a pure capacitive shunt arm associated with a general R-L-C network. Such a development is made in Part IV of this Chapter.

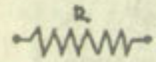
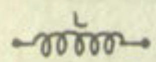
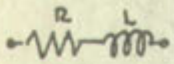
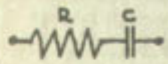
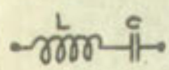
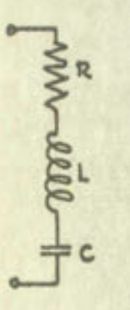
response of the whole network is concerned. As a corollary to the above, it may be stated that $\mathcal{L}^{-1}\left[Y(s)/s \right]$ at [illegible] for an all-shunt network containing a pure capacitive shunt arm [illegible] the current response of the network to a unit-step function of $v(t)$ are only step [illegible] given to the fact that the pure capacitive arm [illegible] is charged at time $t = 0$. The value of such a capacitive element will therefore not be defined by the initial admittance function $\mathcal{L}^{-1}\left[Y(s)/s \right]$ when the network contains other elements [illegible] the pure capacitive shunt arm.

The initial admittances [illegible] are tabulated in Table I. This table has been used in the solution of the all-shunt network [illegible] problems given as examples of the response synthesis in Appendix A. There now remains the development of a method for the identification of a pure capacitive shunt arm associated with a general R-L-C network. Such a development is made in Part IV of this Chapter.

TABLE I

THE INDICIAL ADMITTANCE OF VARIOUS STANDARD

SHUNT ARM CONFIGURATIONS

	$\frac{1}{R}$	
	$\frac{t}{L}$	
	$C\,\delta(t)$	ALONE
	0	ASSOCIATED WITH GENERAL NETWORK
	$\frac{1}{R}\left(1-\varepsilon^{-Rt/L}\right)$	
	$\frac{1}{R}\varepsilon^{-t/RC}$	
	$\sqrt{\frac{C}{L}}\,\sin\frac{t}{\sqrt{LC}}$	
	$\frac{t}{L}\varepsilon^{-Rt/2L} = \frac{t}{L}\varepsilon^{-t\sqrt{1/LC}}$	$\frac{R^2}{4L^2} = \frac{1}{LC}$
	$\frac{\varepsilon^{-Rt/2L}}{KL}\sinh(Kt)\,;\ K \triangleq \sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}$	$\frac{R^2}{4L^2} > \frac{1}{LC}$
	$\frac{\varepsilon^{-Rt/2L}}{\beta L}\sin(\beta t)\,;\ \beta \triangleq \sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}$	$\frac{1}{LC} > \frac{R^2}{4L^2}$

IV. IDENTIFICATION OF THE PURE CAPACITIVE SHUNT ARM BY MEANS OF THE ADMITTANCE FUNCTION Y(p)

Since the indicial admittance of the pure capacitive shunt arm is zero when it is associated with the general R-L-C network, provision must be made for its identification and evaluation in order to preserve the effectiveness of the current response synthesis method. Fortunately, if the function $\dot{Y}(p)$ is capable of being synthesized in an all-shunt configuration, it will also inherently provide the means for identifying and evaluating a pure capacitive shunt arm, merely by an inspection of its mathematical configuration, when and if the requirement for such a shunt arm exists.

The general form of the networks under consideration consists of all possible configurations of R, L, and C elements in shunt across the network terminals thus:

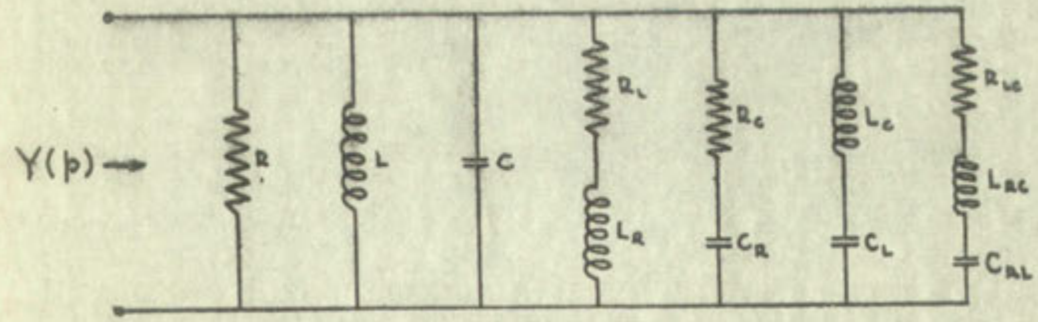


Fig.10.- General Form of All-Shunt Network

for which:

$$Y(p) = \frac{1}{R} + \frac{1}{pL} + \frac{pC}{1} + \frac{1}{pL_R + R_L} + \frac{1}{R_C + \frac{1}{pC_R}} + \frac{1}{pL_C + \frac{1}{pC_L}} + \frac{1}{R_{LC} + pL_{RC} + \frac{1}{pC_{RL}}} \qquad (86)$$

$$Y(p) = \frac{1}{R} + \frac{1}{pL} + \frac{pC}{1} + \frac{1}{pL_R + R_L} + \frac{pC_R}{pR_cC_R + 1} + \frac{pC_L}{p^2L_cC_L + 1} + \frac{pC_{RL}}{p^2L_{RC}C_{RL} + pR_{LC}C_{RL} + 1} \tag{87}$$

Combining the terms of (87) over the common denominator gives:

$$Y(p) = [(pL)(pL_R + R_L)(pR_cC_R + 1)(p^2L_cC_L + 1)(p^2L_{RC}C_{RL} + pR_{LC}C_{RL} + 1)] / D(p) \tag{88a}$$

$$+ [(R)(pL_R + R_L)(pR_cC_R + 1)(p^2L_cC_L + 1)(p^2L_{RC}C_{RL} + pR_{LC}C_{RL} + 1)] / D(p) \tag{88b}$$

$$+ [(pC)(R)(pL)(pL_R + R_L)(pR_cC_R + 1)(p^2L_cC_L + 1)(p^2L_{RC}C_{RL} + pR_{LC}C_{RL} + 1)] / D(p) \tag{88c}$$

$$+ [(R)(pL)(pR_cC_R + 1)(p^2L_cC_L + 1)(p^2L_{RC}C_{RL} + pR_{LC}C_{RL} + 1)] / D(p) \tag{88d}$$

$$+ [(pC_R)(R)(pL)(pL_R + R_L)(p^2L_cC_L + 1)(p^2L_{RC}C_{RL} + pR_{LC}C_{RL} + 1)] / D(p) \tag{88e}$$

$$+ [(pC_L)(R)(pL)(pL_R + R_L)(pR_cC_R + 1)(p^2L_{RC}C_{RL} + pR_{LC}C_{RL} + 1)] / D(p) \tag{88f}$$

$$+ [(pC_{RL})(R)(pL)(pL_R + R_L)(pR_cC_R + 1)(p^2L_cC_L + 1)] / D(p) \tag{88g}$$

Where $D(p)$ is the common denominator:

$$D(p) = (R)(pL)(pL_R + R_L)(pR_cC_R + 1)(p^2L_cC_L + 1)(p^2L_{RC}C_{RL} + pR_{LC}C_{RL} + 1) \tag{88h}$$

If equation (88) is expanded to obtain the first few terms involving the highest powers of p in Y(p), there is obtained:

$$Y(p) = \frac{(L L_R R_c C_R L_c C_L L_{RC} C_{RL})\, p^7 + \text{powers of } p \text{ of lower order}}{(R L L_R R_c C_R L_c C_L L_{RC} C_{RL})\, p^7 + \text{powers of } p \text{ of lower order}} \tag{88a'}$$

$$+ \frac{(R L_R R_c C_R L_c C_L L_{RC} C_{RL})\, p^6 + \text{powers of } p \text{ of lower order}}{(R L L_R R_c C_R L_c C_L L_{RC} C_{RL})\, p^7 + \text{powers of } p \text{ of lower order}} \tag{88b'}$$

$$Y(p) = \frac{1}{R} + \frac{1}{pL} + \frac{pC}{1} + \frac{1}{pL_s + R_s} + \frac{pC_a}{pR_aC_a + 1} + \frac{pC_L}{p^2L_LC_L + 1} + \frac{pC_{ac}}{p^2L_{ac}C_{ac} + pR_{ac}C_{ac} + 1}$$

Combining the terms of (57) over the common denominator gives

$$Y(p) = \left[(pL)(pL_s + R_s)(pR_aC_a + 1)(p^2L_LC_L + 1)(p^2L_{ac}C_{ac} + pR_{ac}C_{ac} + 1)\right] / D(p) \quad (58a)$$

$$+ \left[(R)(pL_s + R_s)(pR_aC_a + 1)(p^2L_LC_L + 1)(p^2L_{ac}C_{ac} + pR_{ac}C_{ac} + 1)\right] / D(p) \quad (58b)$$

$$+ \left[(pC)(R)(pL)(pL_s + R_s)(pR_aC_a + 1)(p^2L_LC_L + 1)(p^2L_{ac}C_{ac} + pR_{ac}C_{ac} + 1)\right] / D(p) \quad (58c)$$

$$+ \left[(R)(pL)(pR_aC_a + 1)(p^2L_LC_L + 1)(p^2L_{ac}C_{ac} + pR_{ac}C_{ac} + 1)\right] / D(p) \quad (58d)$$

$$+ \left[(pC_a)(R)(pL)(pL_s + R_s)(p^2L_LC_L + 1)(p^2L_{ac}C_{ac} + pR_{ac}C_{ac} + 1)\right] / D(p) \quad (58e)$$

$$+ \left[(pC_L)(R)(pL)(pL_s + R_s)(pR_aC_a + 1)(p^2L_{ac}C_{ac} + pR_{ac}C_{ac} + 1)\right] / D(p) \quad (58f)$$

$$+ \left[(pC_{ac})(R)(pL)(pL_s + R_s)(pR_aC_a + 1)(p^2L_LC_L + 1)\right] / D(p) \quad (58g)$$

Where D(p) is the common denominator.

$$D(p) = (R)(pL)(pL_s + R_s)(pR_aC_a + 1)(p^2L_LC_L + 1)(p^2L_{ac}C_{ac} + pR_{ac}C_{ac} + 1) \quad (58h)$$

If equation (58) is expanded to obtain the first few terms involving the highest powers of p in Y(p), there is obtained

$$Y(p) = \frac{(L\,L_sR_aC_aL_LC_LL_{ac}C_{ac})\,p^7 + \text{powers of } p \text{ of lower order}}{(RLL_sR_aC_aL_LC_LL_{ac}C_{ac})\,p^6 + \text{powers of } p \text{ of lower order}} \quad (59a)$$

$$+ \frac{(RL_sR_aC_aL_LC_LL_{ac}C_{ac})\,p^6 + \text{powers of } p \text{ of lower order}}{(RLL_sR_aC_aL_LC_LL_{ac}C_{ac})\,p^6 + \text{powers of } p \text{ of lower order}} \quad (59b)$$

$$+ \frac{(CRLL_R R_C C_R L_C C_L L_{RC} C_{RL})\,p^8 + \text{powers of } p \text{ of lower order}}{(RLL_R R_C C_R L_C C_L L_{RC} C_{RL})\,p^7 + \text{powers of } p \text{ of lower order}} \qquad (88c')$$

+ Other terms of the form $N(p)/D(p)$ involving powers of p in $N(p)$ of order ≤ 7

Analyzing term (88c′) of the above expansion:

$$\frac{(CRLL_R R_C C_R L_C C_L L_{RC} C_{RL})\,p^8 + \cdots\cdots}{(RLL_R R_C C_R L_C C_L L_{RC} C_{RL})\,p^7 + \cdots\cdots} \qquad (88c')$$

and recalling that this particular term was generated by the effect of a pure capacitive reactance in shunt with other shunt arms containing all possible basic series combinations of R, L, and C, which combinations in shunt generated terms $N(p)/D(p)$ in which p appeared in powers of order ≤ 7 , it is possible to state:

<u>Criteria</u> <u>A</u>. If an all-shunt network contains a shunt arm having only pure capacitive reactance, then $Y(p)$ for such network must necessarily be of the form:

$$Y(p) = \frac{a_1 p^m + a_2 p^{m-1} + a_3 p^{m-2} + \cdots\cdots + a_m p + a_{m+1}}{b_1 p^{m-1} + b_2 p^{m-2} + b_3 p^{m-3} + \cdots\cdots + b_{m-1} p + b_m} \qquad (89)$$

and that $C = a_1 / b_1$ is the value of the pure capacitive reactance in shunt.

Further analysis of equation (88) leads to the basis of:

<u>Criteria B</u>. If an all shunt network does not contain a shunt arm having only pure capacitive reactance then Y(p) for such network must necessarily be of either the form:

$$Y(p) = \frac{a_1 p^m + a_2 p^{m-1} + a_3 p^{m-2} + \cdots\cdots + a_m p + a_{m+1}}{b_1 p^m + b_2 p^{m-1} + b_3 p^{m-2} + \cdots\cdots + b_m p + b_{m+1}} \tag{90}$$

or the form:

$$Y(p) = \frac{a_1 p^{m-1} + a_2 p^{m-2} + a_3 p^{m-3} + \cdots\cdots + a_{m-1} p + a_m}{b_1 p^m + b_2 p^{m-1} + b_3 p^{m-2} + \cdots\cdots + b_m p + b_{m+1}} \tag{91}$$

In equations (89), (90), and (91) of Criteria A and B it is pointed out that the conditions for physical realizability dictate that all a_m and b_m be equal to or greater than zero. Further, since these equations represent the complex admittance function in its most general form, the various a_m and b_m may certainly be equal to zero. For instance, in the lossless network case, alternate powers of p are always absent in both N(p) and D(p) signifying that the a_m and b_m coefficients of such missing terms are identically equal to zero.

and that $C = ...$ is the value of the pure capacitive reactance in shunt.

Further analysis of equation (68) leads to the next at.

Criteria B: If small shunt network does not contain a shunt arm having only pure capacitive reactance then Y(p) for such network must necessarily be of either the form

$$Y(p) = \frac{a_1 p^m + a_2 p^{m-1} + a_3 p^{m-2} + \cdots + a_m p + a_{m+1}}{b_1 p^m + b_2 p^{m-1} + b_3 p^{m-2} + \cdots + b_m p + b_{m+1}}$$

or the form

$$Y(p) = \frac{a_1 p^{m-1} + a_2 p^{m-2} + a_3 p^{m-3} + \cdots + a_{m-1} p + a_m}{b_1 p^m + b_2 p^{m-1} + b_3 p^{m-2} + \cdots + b_m p + b_{m+1}} \quad (3)$$

In equations (69), (70), and (71) of Criteria A and B it is pointed out that the conditions for physical realizability dictate that all a_m and b_m be equal to or greater than zero. Further, since these equations represent the complex admittance function in its most general form, the various a_m and b_m may certainly be equal to zero. For instance, in the lossless network case, alternate powers of p are always absent in both A(p) and B(p) signifying that the a_m and b_m coefficients of such missing terms are identically equal to zero.

V. OUTLINE OF THE PROCEDURE FOR CURRENT RESPONSE SYNTHESIS

The method for the synthesis of two terminal networks, utilizing the response function A(t) as obtained by a transformation of an originally specified complex immittance function, has now been completely developed. The step-by-step processes, as set forth in the first four sections of this Chapter, are enumerated below.

1. Examine the given function Y(p) to determine the applicability of either Criteria A or Criteria B. If Criteria A applies, determine the numerical value of the capacitance in the pure capacitive shunt arm by dividing the coefficient of the highest power of p in N(p) by the coefficient of the highest power of p in D(p).

2. Form the response function $^{Y(p)}/p$.

3. Multiply the response function $^{Y(p)}/p$ by the exponential ε^{pt} to form the function $P(p)/Q(p) = \frac{Y(p)}{p}\varepsilon^{pt}$

4. Determine the poles of $^{P(p)}/Q(p)$ by finding the zeros of Q(p).

5. Determine the residues of $^{P(p)}/Q(p)$. This may be accomplished, if there is no pole of order greater than unity by evaluating the function $^{P(p)}/Q'(p)$ at the values of p which cause such poles. If second order poles exist the residues due to such poles may be determined by means of

V. OUTLINE OF THE PROCEDURE FOR TRANSIENT RESPONSE SYNTHESIS

The method for the synthesis of two terminal networks, utilizing the response function $A(t)$ as obtained by a transformation of an arbitrarily specified complex impedance function, has now been completely developed. The step-by-step procedures, as set forth in the first four sections of this Chapter, are summarized below.

1. Examine the given function $Z(p)$ to determine the applicability of either Criteria A or Criteria B. If Criteria B applies, determine the magnitude value of the capacitance in the pure capacitive shunt arm by dividing the coefficient of the highest power of p in $Q(p)$ by the coefficient of the highest power of p in $P(p)$.

2. Form the response function $A(p)/p$.

3. Multiply the response function $A(p)/p$ by the exponential e^{pt} to form the function $P(p)/Q(p) = [illegible] e^{pt}$

4. Determine the poles of $P(p)/Q(p)$ by finding the zeros of $Q(p)$.

5. Determine the residues of $P(p)/Q(p)$. This may be accomplished, if there is no pole of order greater than unity, by evaluating the function $P(p)/Q'(p)$ at the values of p which cause such poles. If second order poles exist, the residues due to such poles may be determined by means of

the formula:

$$R = \frac{6P'Q'' - 2PQ'''}{3(Q'')^2}$$

wherein the derivitives are evaluated at the value of p which establishes the second order pole.

6. Sum the residues. This sum will be the indicial admittance of the network sought.

7. Compare the terms of the indicial admittance function thus obtained with the standard forms of indicial admittance of Table I. By equating like terms of the identities and solving for the various network parameters, an all-shunt network equivalent of the given function Y(p) can be realized.

To illustrate the application of the method outlined above, a representative problem is solved immediately below. Several other problems demonstrating the application of the current response synthesis technique are solved in Appendix A.

Example Problem:

$$\dot{Y}(p) = \frac{p^3 + 42p^2 + 90p + 100}{100p^2 + 200p}$$

Type A. $C = 1/100$

$$\frac{\dot{Y}(p)}{p} = \frac{p^3 + 42p^2 + 90p + 100}{100p^2(p+2)}$$

$$\frac{P(p)}{Q(p)} = \frac{\dot{Y}(p)}{p}\,\epsilon^{pt} = \frac{(p^3 + 42p^2 + 90p + 100)\,\epsilon^{pt}}{100p^3 + 200p^2}$$

$$P(0) = 100$$

$$P'(0) = 100t + 90$$

$$Q''(0) = 400$$

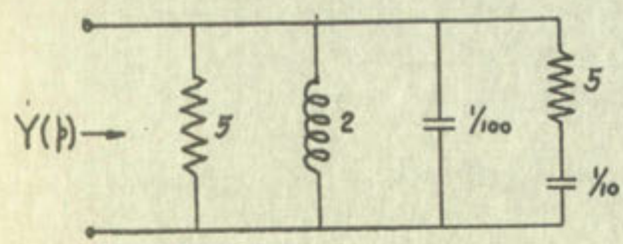
$$Q'''(0) = 600$$

$$\frac{6P'Q'' - 2PQ'''}{3(Q'')^2} = \frac{t}{2} + \frac{1}{5}$$

$$\frac{P(p)}{Q'(p)} = \frac{(p^3 + 42p^2 + 90p + 100)\,\epsilon^{pt}}{300p^2 + 400p}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-2} = \frac{1}{5}\,\epsilon^{-2t}$$

$$A(t) = 1/5 + t/2 + 1/5\,\epsilon^{-2t}$$



CHAPTER III

NETWORK SYNTHESIS BY MEANS OF VOLTAGE RESPONSE

I. THEORY

The theory underlying the method of network synthesis by means of the comparison of the voltage response functions of an unknown network with standard types of voltage response functions is almost identical to the current response theory of Chapter II. In the voltage case, the starting point is again a general R-L-C network as shown in Figure 11.

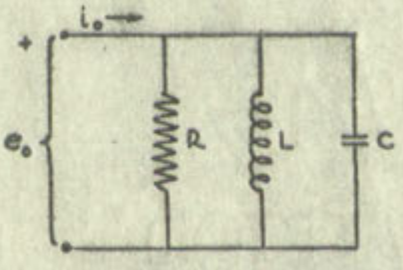


Fig. 11.- General Form of R-L-C Network

Here, if i_o is considered to be the excitation and e_o the response, the differential equation of the circuit will be:

$$i_o = \frac{e_o}{R} + \frac{1}{L}\int e_o dt + C\frac{de_o}{dt} \tag{1}$$

If i_o is taken as a unit step function of current at time $t = 0$ and if quiescent initial conditions are specified, the application of the Laplace transform to both sides of equation (1) results in:

$$\frac{1}{p} = \frac{E_o(p)}{R} + \frac{1}{L}\left[\frac{E_o(p)}{p}\right] + C\left[\,pE_o(p)\right] \tag{2}$$

CHAPTER III

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1. THEORY

The theory underlying the method of network synthesis by means of the comparison of the voltage response functions of an unknown network with standard types of voltage response functions is almost identical to the current response theory of Chapter II. In the voltage case, the starting point is again a general R-L-C network in shown in Figure 1.

Fig. 1.—General form of R-L-C Networks

Here, if i_o is considered to be the excitation and e_o the response, the differential equation of the circuit will be:

$$i_o = \frac{e_o}{R} + \frac{1}{L}\int e_o dt + C\frac{de_o}{dt} \qquad (1)$$

If i_o is taken as a unit step function of current at time t = 0 and if quiescent initial conditions are specified, the application of the Laplace transform to both sides of equation (1) results in:

$$\frac{1}{p} = \frac{E_o(p)}{R} + \frac{1}{L}\left[\frac{E_o(p)}{p}\right] + C\left[pE_o(p)\right] \qquad (2)$$

$$\frac{1}{p} = E_0(p)\left[\frac{1}{R} + \frac{1}{pL} + pC\right] = \dot{Y}_0(p)\,E_0(p) \tag{3}$$

$$E_0(p) = \frac{1}{p\dot{Y}_0(p)} = \frac{\dot{Z}_0(p)}{p} \tag{4}$$

In equation (4), $E_0(p)$ is the voltage response to the excitation of a unit step function of current exciting a general impedance $\dot{Z}(p)$.

If now the inverse Laplace transform is applied to both sides of equation (4) there is given:

$$e_0 = J(t) = \mathcal{L}^{-1}\left[\frac{\dot{Z}(p)}{p}\right] \tag{5}$$

where e_0 is the indicial impedance $J(t)$ of the network of Figure 11.

The network of Figure 11 may be considered to be a series network consisting of a single impedance $\dot{Z}_0(p) = (pLR)/(p^2RLC + pL + R)$ in series with the network terminals. In this case, the voltage response e_0 will appear across $\dot{Z}_0(p)$. If now any number of additional impedances, $\dot{Z}_1(p)$, $\dot{Z}_2(p)$, $\dot{Z}_3(p)$, $\cdots\cdots$ are placed in series with each other and also with $\dot{Z}_0(p)$ as shown in Figure 12, each such impedance will be excited by the same unit step function of current which enters the network at time $t = 0$, and each will contribute

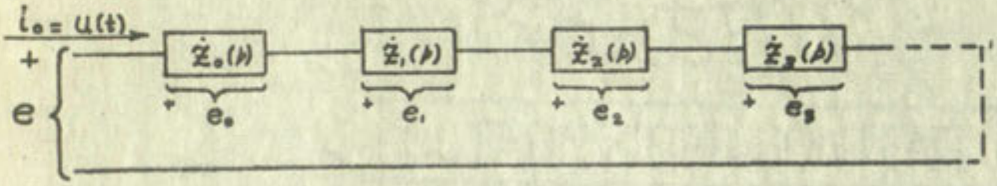


Fig. 12.- Network of Series Impedances

its share of the total voltage response $e = J(t)$. This total voltage response will be:

$$e = \mathcal{L}^{-1}\left[\frac{\dot{Z}_0(p)}{p}\right] + \mathcal{L}^{-1}\left[\frac{\dot{Z}_1(p)}{p}\right] + \mathcal{L}^{-1}\left[\frac{\dot{Z}_2(p)}{p}\right] + \cdots\cdots \tag{6}$$

which, as was shown in Chapter II can be written as:

$$e = \mathcal{L}^{-1}\left[\frac{\dot{Z}_{TOTAL}(p)}{p}\right] \tag{7}$$

Here, again as before, the right hand side of equation (7) will be of the form of a sum of terms resulting from the evaluation of the residues of $\left[\dot{Z}_{TOT}(p)\,\epsilon^{pt}\right] / p$ at each of its poles. Each complete term of the sum will represent the indicial impedance of one of the impedance groups comprising the network the consequently the sum of all such terms will be the total indicial impedance of the network. The mathematical form of equation (7) is therefore compatible with the physical configuration of the network of Figure 12.

The inverse transform of $\dot{Z}_{TOT}(p)/p$ will always be functions of the R, L, and C parameters of a network yielding the voltage

response e, in a form which will identify their value and phasor connotation within the network. However, as in the case of current response, it will be found that the indicial impedance will fail to identify one type of pure circuit element, except in the degenerate case where that particular element comprises the total network. As might be expected, the indicial impedance fails to identify a series element of pure inductance. The identification of the pure inductance, in order to complete the synthesis procedure, will be discussed in Section III of this Chapter.

As a simple example to illustrate synthesis by means of voltage response, the same function of impedance that was used to demonstrate current response synthesis in Chapter II will be used. Thus,

$$\dot{Z}(p) = \frac{p+1}{p+3} \tag{8}$$

$$\frac{\dot{Z}(p)}{p} = \frac{p+1}{p^2+3p} \tag{9}$$

$$J(t) = \mathcal{L}^{-1}\left[\frac{\dot{Z}(p)}{p}\right] =$$

$$\left.\frac{(p+1)\,\epsilon^{pt}}{2p+3}\right|_{p=0,-3} = \frac{1}{3} + \frac{2}{3}\,\epsilon^{-3t} \tag{10}$$

It will be shown in Section III of this Chapter that the indicial impedance of a pure resistance R is R itself and further, that the indicial impedance of a resistance R_L in parallel with an inductance L_R is $R_L\,\epsilon^{-R_L t/L_R}$. By comparing these standard response

response e, in a form which will identify their value and place connection within the network. However, as in the case of current response, it will be found that the indicial impedance will fail to identify one type of pure circuit element, except in the degenerate case where that particular element comprises the total network. As might be expected, the indicial impedance fails to identify a series element of pure inductance. The identification of the pure inductance, in order to complete the synthesis procedure, will be discussed in Section 2.4 of this Chapter.

As a simple example to illustrate synthesis by means of voltage response, the same function of impedance that was used to demonstrate current response synthesis in Chapter II will be used. Thus,

$$Z(p) = \frac{p+1}{p+3} \qquad (47)$$

$$\frac{Z(p)}{p} = \frac{p+1}{p^2+3p} \qquad (48)$$

$$J(t) = \mathcal{L}^{-1}\left[\frac{Z(p)}{p}\right] =$$

$$\left.\frac{(p+1)e^{pt}}{2p+3}\right|_{p=0,-3} = \frac{1}{3} + \frac{2}{3}e^{-3t} \qquad (50)$$

It will be shown in Section 2.3 of this Chapter that the indicial impedance of a pure resistance R is R itself, and further, that the indicial impedance of a resistance R_1 in parallel with an inductance L_a is $R_1 e^{-R_1 t/L_a}$, denoting these elements' response

functions with the response of the unknown networks given by equation (10) there is obtained

$$R = \frac{1}{3} \tag{11}$$

$$R_L = \frac{2}{3} \tag{12}$$

$$\frac{R_L}{L_a} = \frac{2/3}{L_a} = 3 \tag{13}$$

$$L_a = \frac{2}{9} \tag{14}$$

A physical realization of the impedance function of equation (8) is therefore:

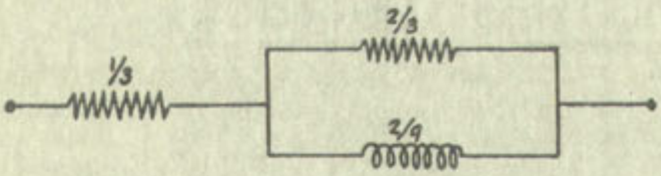


Fig. 13.- All-Series Network for $\dot{Z}(p) = (p+1)/(p+3)$

Additional examples, of a more comprehensive nature, of synthesis by means of voltage response techniques are given at the end of this Chapter and in Appendix B.

II. THE POLES OF THE IMPEDANCE FUNCTION

As was the case in the current response synthesis method, the poles of the immittance function play an important role in the voltage response method. The impedance function $\dot{Z}(p)$, when it is of normal

complexity will be of the form $\dot{Z}(p) = N(p)/D(p)$ and hence the voltage response to a unit-step function of current will be of the form:

$$\frac{\dot{Z}(p)}{p} = \frac{N(p)}{p\,D(p)} \tag{15}$$

The process of obtaining the indicial impedance of the unknown networks represented by the given impedance function $\dot{Z}(p)$ involves, as before, the application of the inverse Laplace transform to $N(p)/pD(p)$ and hence the evaluation of the residues of the function $[N(p)\epsilon^{pt}]/pD(p)$ at its singular points. These singular points again as before, will exist only at the poles of $N(p)/pD(p)$ which, in turn, are generated only by the zeros of $pD(p)$ since $N(p)$ and ϵ^{pt} are analytic everywhere in the finite p-plane. It is immediately apparent that there will always be a pole of $[N(p)\epsilon^{pt}]/pD(p)$ at $p = 0$ and that the other poles of this function will be established by the factors of $D(p)$.

The general form of the network under consideration consists of impedance groups in series with the network terminals. These impedance groups consist, as is shown in Figure 14, of all possible parallel combinations of R, L, and C elements, thus yielding in the general network case all possible phasor relationships consistent with the restrictions placed on the network configuration by the voltage response concept.

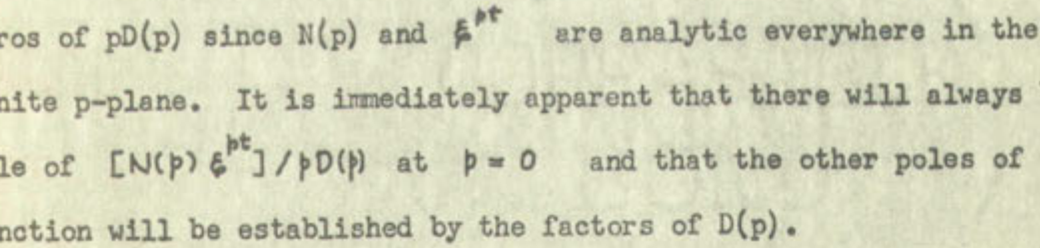


Fig. 14.- General Form of the All-Series Network

complexity will be of the form $Z(p) = N(p)/D(p)$ and hence its voltage response to a unit-step function of current will be of the form

$$\frac{Z(p)}{p} = \frac{N(p)}{p\,D(p)} \qquad (19)$$

The process of obtaining the indicial impedance of the network represented by the given impedance function Z(p) involves, as before, the application of the inverse Laplace transform to N(p)/pD(p) and hence the evaluation of the residues of the function $[N(p)e^{pt}]/pD(p)$ at its singular points. These singular points are seen to be [illegible] only at the poles of N(p)/pD(p), which, in turn, are generated only by the zeros of pD(p) since N(p) and e^{pt} are analytic everywhere in the finite p-plane. It is immediately apparent that there will always be a pole of $[N(p)e^{pt}]/pD(p)$ at $p = 0$ and that the other poles of this function will be established by the zeros of D(p).

The general form of the network under [illegible] consists of impedance groups in series with the network terminals. These impedance groups consist, as is shown in Figure 14, of all possible parallel combinations of R, L, and C elements. Thus, including in the general network case all possible [illegible] restrictions placed on the network configuration by the [illegible]

Fig. 14 - General Form of the [illegible] Series Network

This type of network will be referred to as an all-series configuration. From Figure 14 it may be seen that the impedance Z(p) of the all-series network is:

$$\dot{Z}(p) = R + pL + \frac{1}{pC} + \frac{1}{\frac{1}{R_L} + \frac{1}{pL_R}} + \frac{1}{\frac{1}{R_C} + pC_R} + \frac{1}{\frac{1}{pL_C} + pC_L} + \frac{1}{\frac{1}{R_{LC}} + \frac{1}{pL_{RC}} + pC_{RL}} \qquad (16)$$

$$= R + pL + \frac{1}{pC} + \frac{pR_L L_R}{pL_R + R_L} + \frac{R_C}{pR_C C_R + 1} + \frac{pL_C}{p^2 L_C C_L + 1} + \frac{pR_{LC} L_{RC}}{p^2 R_{LC} L_{RC} C_{RL} + pL_{RC} + R_{LC}} \qquad (17)$$

for which the denominator D(p) of the combined terms of equation (17) is:

$$D(p) = (pC)(pL_R + R_L)(pR_C C_R + 1)(p^2 L_C C_L + 1)(p^2 R_{LC} L_{RC} C_{RL} + pL_{RC} + R_{LC}) \qquad (18)$$

In a network of this character it may be seen that each impedance group, except for the pure resistance and the pure inductive reactance, will establish a separate and distinct factor of D(p). Each such factor, not equal to a constant only, will establish the location of a zero of D(p) and consequently the location of a pole of N(p)/D(p). Since each complete term of the indicial impedance of the network depends upon the evaluation of a residue of $[\dot{Z}(p)\epsilon^{pt}]/p$ at a pole or poles established by a factor of pD(p), it may be stated that each impedance group of the all-series network will establish a separate and complete term in the sum representing the indicial impedance function. Any pure resistance element (group) present will be accounted for by the pole at $p = 0$ created by the multiplier p of pD(p). However, since the pure inductance element of the general all-series configura-

This type of network will be referred to as an all-series configuration. From Figure 14 it may be seen that the impedance $Z(p)$ of the all-series network is:

$$Z(p) = R + pL + \frac{1}{pC} + \frac{1}{\frac{1}{R_a} + \frac{1}{pL_a}} + \frac{1}{\frac{1}{R_b} + pC_b} + \frac{1}{\frac{1}{pL_c} + pC_c} + \frac{1}{\frac{1}{R_d} + \frac{1}{pL_d} + pC_d} \qquad (16)$$

$$= R + pL + \frac{1}{pC} + \frac{pR_aL_a}{pL_a + R_a} + \frac{R_b}{pR_bC_b + 1} + \frac{pL_c}{p^2L_cC_c + 1} + \frac{pR_dL_d}{p^2R_dL_dC_d + pL_d + R_d} \qquad (17)$$

for which the denominator $D(p)$ of the component terms of equation (17) is

$$D(p) = (pC)(pL_a + R_a)(pR_bC_b + 1)(p^2L_cC_c + 1)(p^2R_dL_dC_d + pL_d + R_d) \qquad (18)$$

In a network of this character it may be seen that each impedance group, except for the pure resistance and the pure inductive reactance, will establish a separate and distinct factor of $D(p)$. Each such factor, not equal to a constant only, will establish the location of a zero of $D(p)$ and consequently the location of a pole of $N(p)/pD(p)$. Since each complete term of the indicial impedance of the network depends upon the evaluation of a residue of $[F(p)e^{pt}]$ at a particular pole or poles established by a factor of $pD(p)$, it may be stated that each impedance group of the all-series network will contribute a separate and complete term in the sum representing the indicial impedance function. Any pure resistance element (group) present will be accounted for by the pole at $p = 0$ created by the multiplier p of $pD(p)$, however, since the pure inductance elements in the general all-series configuration

tion does not create a factor in D(p), its presence in a network will not be accounted for in the indicial impedance function. The lack of response of the pure inductance will be treated in Section IV of this Chapter.

The factor $(p^2 R_{LC} L_{RC} C_{RL} + p L_{RC} + R_{LC})$ of D(p) which is generated by the R-L-C impedance group may give rise to three different types of pole configurations in N(p)/D(p). First, it may generate a second order pole at a value of p which is wholly real and negative. Second, it may generate two first order poles which are wholly real and negative, and third, it may produce two first order poles which are complex conjugates lying in the left half p-plane. As was the case for the R-L-C group in the current response technique of Chapter II, it may therefore be expected that the R-L-C group of the all-series network will produce three distinct types of indicial impedance functions. The interrelationship of the R, L, and C elements comprising the group will determine the type of response generated. As will be demonstrated by the solution of representative problems, the fact of the existence of three distinct types of indicial impedance functions for the general R-L-C group does not impair the effectiveness of the synthesis method.

Examination of D(p) in its factored form indicates that, as was the case for the all-shunt network, poles of order higher than the second will not occur. Hence, all of the procedures involved in the voltage response synthesis will be similar to those for the current response synthesis. In fact the only difference will be in the physical configuration of the two networks, and in the fact that each method will

tion does not create a factor in D(p); its presence in a network will not be accounted for in the inductance impedance functions. The [illegible] response of the pure inductance will be treated in Section IV of this Chapter.

The factor $(p^2 [illegible] + R_{[illegible]})$ [illegible] of D(p) which in generated by the R-L-C impedance group may give rise to three different types of pole configurations in [illegible](p)/[illegible](p). First, it may generate a second order pole at a value of p which is wholly real and negative. Second, it may generate two first order poles which are wholly real and negative, and third, it may produce two first order poles which are complex conjugates lying in the left half p-plane. As was the case for the R-L-C group in the current response technique of Chapter II, it may therefore be expected that the R-L-C group of the all-shunt network will produce three distinct types of inductance impedance functions. The interrelationship of the R, L, and C elements comprising the group will determine the type of response generated. As will be demonstrated by the solution of representative problems, the fact of the existence of three distinct types of inductance impedance functions for the general R-L-C group does not impair the effectiveness of the synthesis method.

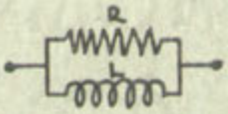
Examination of D(p) in its factored form indicates that, as was the case for the all-shunt network, poles of order higher than the second will not occur. Hence, all of the procedures developed in the voltage response synthesis will be similar to those for the current response synthesis. In fact the only difference will be in the structural configuration of the two networks, and in the fact that each method will

require its individual tabulation of standard response functions.

III. THE INDICIAL IMPEDANCE OF STANDARD TYPES OF IMPEDANCE GROUPS

In order to effectively synthesize a network of the all-series class by voltage response methods, it is necessary to develop and tabulate the indicial impedance functions for all of the impedance groups shown in Figure 14. These standard voltage response functions are developed below and the results thereof are tabulated in Table II.

1. Resistance - Inductance



$$\frac{\dot{Z}(p)}{p} = \frac{1}{p\left(\frac{1}{R}+\frac{1}{pL}\right)} = \frac{R}{p+\frac{R}{L}} \tag{19}$$

$$\frac{P(p)}{Q(p)} = \frac{Z(p)}{p}\epsilon^{pt} = \frac{R\epsilon^{pt}}{p+\frac{R}{L}} \tag{20}$$

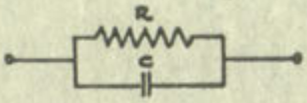
$P(p)/Q(p)$ has first order pole at $p = -R/L$

$$\frac{P(p)}{Q'(p)} = R\epsilon^{pt} \tag{21}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-R/L} = R\epsilon^{-Rt/L} \tag{22}$$

$$J(t) = \mathcal{L}^{-1}\left[\frac{\dot{Z}(p)}{p}\right] = R\epsilon^{-Rt/L} \tag{23}$$

2. Resistance - Capacitance



$$\frac{\dot{Z}(p)}{p} = \frac{1}{p(\frac{1}{R} + pC)} = \frac{1}{pC(p + \frac{1}{RC})} \tag{24}$$

$$\frac{P(p)}{Q(p)} = \frac{\dot{Z}(p)}{p}\epsilon^{pt} = \frac{\epsilon^{pt}}{pC(p + \frac{1}{RC})} \tag{25}$$

$P(p)/Q(p)$ has first order poles at $p = 0, -1/RC$

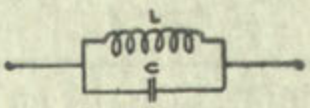
$$\frac{P(p)}{Q'(p)} = \frac{\epsilon^{pt}}{2pC + \frac{1}{R}} \tag{26}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=0} = \frac{1}{1/R} = R \tag{27}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-1/RC} = \frac{\epsilon^{-t/RC}}{-1/R} = -R\epsilon^{-t/RC} \tag{28}$$

$$J(t) = \mathcal{L}^{-1}\left[\frac{\dot{Z}(p)}{p}\right] = R - R\epsilon^{-t/RC} = R(1 - \epsilon^{-t/RC}) \tag{29}$$

3. Inductance - Capacitance



$$\frac{\dot{Z}(p)}{p} = \frac{1}{p(\frac{1}{pL} + pC)} = \frac{1}{C(p^2 + \frac{1}{LC})} \tag{30}$$

$$\frac{P(p)}{Q(p)} = \frac{\dot{Z}(p)}{p}\epsilon^{pt} = \frac{\epsilon^{pt}}{C(p^2 + \frac{1}{LC})} \tag{31}$$

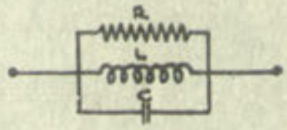
$P(p)/Q(p)$ has first order poles at $p = \pm j\sqrt{1/LC}$

$$\frac{P(p)}{Q'(p)} = \frac{\epsilon^{pt}}{2pC} \tag{32}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=j\sqrt{1/LC}} = \frac{\epsilon^{jt\sqrt{1/LC}}}{2jC\sqrt{1/LC}} = \frac{\epsilon^{jt\sqrt{1/LC}}}{2j\sqrt{C/L}} \tag{33}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-j\sqrt{1/LC}} = -\frac{\epsilon^{-jt\sqrt{1/LC}}}{2j\sqrt{C/L}} \tag{34}$$

$$J(t) = \mathcal{L}^{-1}\left[\frac{\dot{Z}(p)}{p}\right] = \sqrt{\frac{L}{C}}\;\sin t\sqrt{\frac{1}{LC}} \tag{35}$$

Resistance - Inductance - Capacitance 

$$\frac{Z(p)}{p} = \frac{1}{p\left(\frac{1}{R}+\frac{1}{pL}+pC\right)} = \frac{1}{C\left(p+\frac{1}{2RC}+\sqrt{\frac{1}{4R^2C^2}-\frac{1}{LC}}\right)\left(p+\frac{1}{2RC}-\sqrt{\frac{1}{4R^2C^2}-\frac{1}{LC}}\right)} \tag{36}$$

(a.) For $\frac{1}{4R^2C^2} = \frac{1}{LC}$:

$$\frac{Z(p)}{p} = \frac{1}{C\left(p+\frac{1}{2RC}\right)^2} \tag{37}$$

$$\frac{P(p)}{Q(p)} = \frac{\dot{Z}(p)}{p}\epsilon^{pt} = \frac{\epsilon^{pt}}{C\left(p+\frac{1}{2RC}\right)^2} \tag{38}$$

$P(p)/Q(p)$ has a second order pole at $p = -1/2RC$

$$\left.\begin{array}{ll} P(p) = \epsilon^{pt}/C & = 1/C\,(\epsilon^{-t/2RC}) \\ P'(p) = t/C\,(\epsilon^{pt}) & = t/C\,(\epsilon^{-t/2RC}) \\ Q'(p) = 2p + 1/RC & = 0 \\ Q''(p) = 2 & = 2 \\ Q'''(p) = 0 & = 0 \end{array}\right|_{p=-1/2RC} \tag{39}$$

$$\frac{6P'Q'' - 2PQ'''}{3(Q'')^2} = \frac{(6)(t/C\,\epsilon^{-t/2RC})(2)}{12} = \frac{t}{C}\,\epsilon^{-t/2RC} \tag{40}$$

$$J(t) = \mathcal{L}^{-1}\;\frac{\dot{Z}(p)}{p} = \frac{t}{C}\epsilon^{-t/2RC} = \frac{t}{C}\epsilon^{-t/\sqrt{LC}} \tag{41}$$

(b.) For $\frac{1}{4R^2C^2} > \frac{1}{LC}$:

$$\frac{\dot{Z}(p)}{p} = \frac{1}{C\left(p + \frac{1}{2RC} + \sqrt{\frac{1}{4R^2C^2} - \frac{1}{LC}}\right)\left(p + \frac{1}{2RC} - \sqrt{\frac{1}{4R^2C^2} - \frac{1}{LC}}\right)} \qquad (42)$$

$$K \triangleq \sqrt{\frac{1}{4R^2C^2} - \frac{1}{LC}}$$

$$\frac{P(p)}{Q(p)} = \frac{\dot{Z}(p)}{p}\,\varepsilon^{pt} = \frac{\varepsilon^{pt}}{C\left(p + \frac{1}{2RC} + K\right)\left(p + \frac{1}{2RC} - K\right)} \qquad (43)$$

$P(p)/Q(p)$ has first order poles at $p = -\left[\frac{1}{2RC} + K\right], -\left[\frac{1}{2RC} - K\right]$

$$\frac{P(p)}{Q'(p)} = \frac{\varepsilon^{pt}}{2pC + 1/R} \qquad (44)$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p = -\frac{1}{2RC} + K} = \frac{\varepsilon^{-t/2RC}\,\varepsilon^{Kt}}{2KC} \qquad (45)$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p = -\frac{1}{2RC} - K} = -\frac{\varepsilon^{-t/2RC}\,\varepsilon^{Kt}}{2KC} \qquad (46)$$

$$J(t) = \mathcal{L}^{-1}\left[\frac{Z(p)}{p}\right] = \frac{\varepsilon^{-t/2RC}}{KC}\left[\frac{\varepsilon^{Kt} - \varepsilon^{-Kt}}{2}\right] = \frac{\varepsilon^{-t/2RC}}{KC}\sinh(Kt) \qquad (47)$$

(c.) For $\frac{1}{LC} > \frac{1}{4R^2C^2}$:

$$\frac{\dot{Z}(p)}{p} = \frac{1}{C\left(p + \frac{1}{2RC} + j\sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}\right)\left(p + \frac{1}{2RC} - j\sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}\right)} \tag{48}$$

$$\beta \triangleq \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}$$

$$\frac{P(p)}{Q(p)} = \frac{\dot{Z}(p)}{p}\varepsilon^{pt} = \frac{\varepsilon^{pt}}{C\left(p + \frac{1}{2RC} + j\beta\right)\left(p + \frac{1}{2RC} - j\beta\right)} \tag{49}$$

$P(p)/Q(p)$ has first order poles at $p = -\left[\frac{1}{2RC} + j\beta\right], -\left[\frac{1}{2RC} - j\beta\right]$

$$\frac{P(p)}{Q'(p)} = \frac{\varepsilon^{pt}}{2pC + 1/R} \tag{50}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p = -\frac{1}{2RC} - j\beta} = -\frac{\varepsilon^{-t/2RC}\,\varepsilon^{-j\beta t}}{2j\beta C} \tag{51}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p = -\frac{1}{2RC} + j\beta} = \frac{\varepsilon^{-t/2RC}\,\varepsilon^{j\beta t}}{2j\beta C} \tag{52}$$

$$J(t) = \mathcal{L}^{-1}\left[\frac{Z(p)}{p}\right] = \frac{\varepsilon^{-t/2RC}}{\beta C}\left[\frac{\varepsilon^{j\beta t} - \varepsilon^{-j\beta t}}{2j}\right] = \frac{\varepsilon^{-t/2RC}}{\beta C}\sin(\beta t) \tag{53}$$

5. Pure Resistance

$$\frac{\dot{Z}(p)}{p} = \frac{R}{p} \tag{54}$$

$$\frac{P(p)}{Q(p)} = \frac{\dot{Z}(p)}{p}\,\epsilon^{pt} = \frac{R\,\epsilon^{pt}}{p} \tag{55}$$

$P(p)/Q(p)$ has a first order pole at $p = 0$

$$\frac{P(p)}{Q'(p)} = R\,\epsilon^{pt} \tag{56}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=0} = R \tag{57}$$

$$J(t) = \mathcal{L}^{-1}\left[\frac{\dot{Z}(p)}{p}\right] = R \tag{58}$$

6. Pure Capacitance

$$\frac{\dot{Z}(p)}{p} = \frac{1}{p^2 C} \tag{59}$$

$$\frac{P(p)}{Q(p)} = \frac{\dot{Z}(p)}{p}\,\epsilon^{pt} = \frac{\epsilon^{pt}}{p^2 C} \tag{60}$$

$P(p)/Q(p)$ has a second order pole at $p = 0$

$$\left.\begin{aligned} P(p) &= \epsilon^{pt} \\ P'(p) &= t\,\epsilon^{pt} \\ Q'(p) &= 2pC \\ Q''(p) &= 2C \\ Q'''(p) &= 0 \end{aligned}\;\right|_{p=0} \begin{aligned} &= 1 \\ &= t \\ &= 0 \\ &= 2C \\ &= 0 \end{aligned} \tag{61}$$

$$\frac{6P'Q'' - 2PQ'''}{3(Q'')^2} = \frac{6(t)(2C)}{12C^2} = \frac{t}{C} \tag{62}$$

$$J(t) = \mathcal{L}^{-1}\left[\frac{\dot{Z}(p)}{p}\right] = \frac{t}{C} \tag{63}$$

Pure Inductance

$$\frac{\dot{Z}(p)}{p} = \frac{pL}{p} = L \tag{64}$$

$$J(t) = \mathcal{L}^{-1}\left[\frac{\dot{Z}(p)}{p}\right] = \frac{L}{2\pi j}\oint \epsilon^{pt}\, dp = L\delta(t) \tag{65}$$

As set forth above, the significance of the voltage response of the pure inductive element in series, when the excitation is a unit step function of current, will require specific interpretation. Considering the pure inductive element alone across the network terminals at time $t = 0$ when the current unit-step is applied, it is apparent that this element will offer infinite impedance to the flow of current for a period of zero time. This comes about by reason of the fact that the current rises from zero to unity value in zero time as shown in Figure 15.

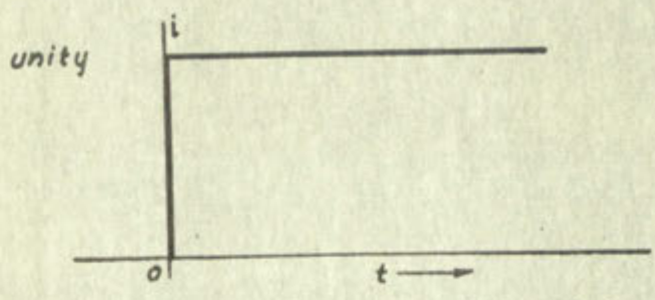


Fig.15.- Unit-Step Function of Current

From Figure 15 it may be seen that the rate of change of current, from zero current at $t = 0$ to unity current at the same time $t = 0$, is in fact infinite. The voltage drop across the pure inductive element during this zero interval, $L\ di/dt$, is therefore (theoretically) infinite. Hence the impedance of the element itself must be infinite for the zero interval of time. At the time the current reaches unity value and a measurable time period begins, the current remains at an unchanging, constant value of unity and hence the voltage drop across the element, $L\ di/dt$ equals zero for any measurable time interval past $t = 0$. The impedance of the pure inductive element then for any time not equal to zero is identically zero. From this discussion, it is seen that the voltage drop across a pure inductive element, and consequently the impedance of the element, subscribes exactly to the definition of the impulse function $L\,\delta(t)$. Obviously, at any time greater than zero the voltage response of this element type to a unit-step function of current will be zero, hence the indicial impedance, $\mathcal{L}^{-1}\left[\dot{Z}(p)/p\right]$ for a pure inductance under these conditions is zero.

It will now be necessary to develop a method for the identification of the pure inductive element, in series with other impedance groups in order to complete the voltage response synthesis method. This development is the subject of Section IV of this Chapter.

IV. IDENTIFICATION OF THE PURE INDUCTIVE SERIES ELEMENT BY MEANS OF THE IMPEDANCE FUNCTION $\dot{Z}(p)$

The given impedance function $\dot{Z}(p)$ from which the all-series net-

From Figure 15 it may be seen that the rate of change of current, from zero current at $t = 0$ to unity current at the same time $t = 0$, is in fact infinite. The voltage drop across the pure inductive element during this zero interval, $L\,di/dt$, is therefore (theoretically) infinite. Hence the impedance of the element itself is infinite for the zero interval of time. At the time the current reaches unity value and a measurable time period begins, the current remains at an unchanging, constant value of unity and hence the voltage drop across the element, $L\,di/dt$ equals zero for any measurable time interval past $t = 0$. The impedance of the pure inductive element then for any time not equal to zero is identically zero. From this discussion, it is seen that the voltage drop across a pure inductive element, and consequently the impedance of the element, coincides exactly to the definition of the impulse function $L\,\delta(t)$. Obviously, at any time greater than zero the voltage response of this element type to a unit-step function of current will be zero, hence the indicial impedance, [illegible] for a pure inductance under these conditions is zero.

It will now be necessary to develop the method for the identification of the pure inductive element, in series with other impedance groups in order to complete the voltage response synthesis method. This development is the subject of Section IV of this chapter.

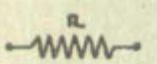
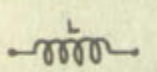
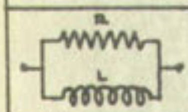
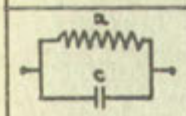
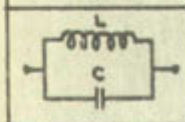
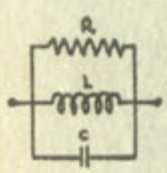
IV. IDENTIFICATION OF THE PURE INDUCTIVE ELEMENT BY MEANS OF THE INDICIAL IMPEDANCE A(t)

The given impedance function $Z(p)$ for which the indicial [illegible]

TABLE II

THE INDICIAL IMPEDANCE OF VARIOUS STANDARD

SERIES GROUP CONFIGURATIONS

	R	
	$L\,\delta(t)$	ALONE
	0	ASSOCIATED WITH GENERAL NETWORK
	$\frac{t}{C}$	
	$R\varepsilon^{-Rt/L}$	
	$R(1-\varepsilon^{-t/RC})$	
	$\sqrt{\frac{L}{C}}\sin\frac{t}{\sqrt{LC}}$	
	$\frac{t}{C}\varepsilon^{-t/2RC}=\frac{t}{C}\varepsilon^{-t/\sqrt{LC}}$	$\frac{1}{4R^2C^2}=\frac{1}{LC}$
	$\frac{\varepsilon^{-t/2RC}}{KC}\sinh(Kt);\ K\triangleq\sqrt{\frac{1}{4R^2C^2}-\frac{1}{LC}}$	$\frac{1}{4R^2C^2}>\frac{1}{LC}$
	$\frac{\varepsilon^{-t/2RC}}{\beta C}\sin(\beta t);\ \beta\triangleq\frac{1}{LC}-\frac{1}{4R^2C^2}$	$\frac{1}{LC}>\frac{1}{4R^2C^2}$

work is to be synthesized may be made to specify the value of the pure inductive series element, when such an element is required, that the indicial impedance function fails to specify. The method used to accomplish this result is similar to that used for the identification of the pure capacitive arm in the current response synthesis and consists simply of the inspection of the immittance function. By the reasoning and procedure set forth in Section IV of Chapter II it may be shown that the presence of a pure inductive series element in the general, all-series network will cause the highest power of p in N(p) of $\dot{Z}(p)$ to be of order one greater than the highest power of p in D(p). By the same type reasoning and procedure it may be shown that the numerical value of such a pure inductance will be expressed by a_1 / b_1 where a_1 and b_1 are the coefficients of the highest powers of p in N(p) and D(p) respectively. Since the mathematical procedures involved in making this identification and evaluation are identical to those of Section IV Chapter II, they will not be repeated here.

In accordance with the foregoing paragraph it is evident that criteria, similar to that established for the current response case, may be established for the voltage response synthesis. These criteria are:

Criteria A. If an all-series network contains a series element of pure inductive reactance, then Z(p) for such a network must necessarily be of the form:

$$\dot{Z}(p) = \frac{a_1 p^m + a_2 p^{m-1} + a_3 p^{m-2} + \cdots + a_m p + a_{m+1}}{b_1 p^{m-1} + b_2 p^{m-2} + b_3 p^{m-3} + \cdots + b_{m-1} p + b_m} \tag{66}$$

and $L = a_1 / b_1$ is the value of the inductive reactance in series with the remainder of the network, and

<u>Criteria</u> <u>B</u>. If an all-series network does not contain a pure inductive reactance in series with the remainder of the network, then $\dot{Z}(p)$ for such a network must necessarily be of the form:

$$\dot{Z}(p) = \frac{a_1 p^m + a_2 p^{m-1} + a_3 p^{m-2} + \cdots + a_m p + a_{m+1}}{b_1 p^m + b_2 p^{m-1} + b_3 p^{m-2} + \cdots + b_m p + b_{m+1}} \tag{67}$$

or of the form:

$$\dot{Z}(p) = \frac{a_1 p^{m-1} + a_2 p^{m-2} + a_3 p^{m-3} + \cdots + a_{m-1} p + a_m}{b_1 p^m + b_2 p^{m-1} + b_3 p^{m-2} + \cdots + b_m p + b_{m+1}} \tag{68}$$

Here again, as in the case of Criteria A and Criteria B for the current response case, all a_m and b_m must be greater than or equal to zero for physical realizability. However, the various a_m and b_m may be equal to zero as dictated by the network giving rise to the impedance $\dot{Z}(p)$.

V. OUTLINE OF THE PROCEDURE FOR VOLTAGE RESPONSE SYNTHESIS

In accordance with the first four sections of this Chapter, the step-by-step procedures for voltage response synthesis method are:

1. Examine the given function $\dot{Z}(p)$ to determine the applicability of either Criteria A or Criteria B. If Criteria A

and $L = a_1/b_1$ is the value of the inductive reactance in series with the remainder of the network; and

Criteria 3. If an all-series network does not contain a pure inductive reactance in series with the remainder of the network, then $\dot{Z}(p)$ for such a network must necessarily be of the form:

$$\dot{Z}(p) = \frac{a_1 p^m + a_2 p^{m-1} + a_3 p^{m-2} + \cdots + a_m p + a_{m+1}}{b_1 p^m + b_2 p^{m-1} + b_3 p^{m-2} + \cdots + b_m p + b_{m+1}} \qquad (13A)$$

or of the form:

$$\dot{Z}(p) = \frac{a_1 p^{m-1} + a_2 p^{m-2} + a_3 p^{m-3} + \cdots + a_{m-1} p + a_m}{b_1 p^m + b_2 p^{m-1} + b_3 p^{m-2} + \cdots + b_m p + b_{m+1}} \qquad (13B)$$

Here again, as in the case of Criteria 1 and Criteria 2 for the current response case, all a_m and b_m must be greater than or equal to zero for physical realizability. However, the various a_m and b_m may be equal to zero as dictated by the network giving rise to the impedance $\dot{Z}(p)$.

V. OUTLINE OF THE PROCEDURE FOR VOLTAGE RESPONSE SYNTHESIS

In accordance with the first four sections of this Chapter, the step-by-step procedure for voltage response synthesis method is:

1. Examine the given function $\dot{Z}(p)$ to determine the applicability of either Criteria 2 or Criteria 3. If Criteria

applies determine the value of the pure inductance in series with the remainder of the network by dividing the coefficient of the highest power of p in N(p) by the coefficient of the highest power of p in D(p).

2. Form the response function $\dot{Z}(p)/p$.

3. Multiply the response function $\dot{Z}(p)/p$ by the exponential ϵ^{pt} to form the function $P(p)/Q(p) = [\dot{Z}(p)/p]\,\epsilon^{pt}$

4. Determine the poles of $P(p)/Q(p)$ by finding the zeros of $Q(p)$.

5. Determine the values of the residues of $P(p)/Q(p)$. This may be accomplished, if there are no poles of $P(p)/Q(p)$ of order greater than the first, by evaluating the function $P(p)/Q'(p)$ at the values of p which cause such poles. If second order poles exist, the residues due to such poles may be evaluated by means of the formula:

$$R = \frac{6P'Q'' - 2PQ'''}{3(Q'')^2}$$

wherein the derivitives are evaluated at the value of p which establishes the second order pole.

6. Sum the residues. This sum will be the indicial impedance function of the network sought.

7. Compare the terms of the indicial impedance function thus obtained with the standard forms of indicial impedance of Table II. By equating like terms of the identities and solving for the various network parameters, an all-series network equivalent of the given function $\dot{Z}(p)$ may be realized.

A representative problem, illustrating the voltage response synthesis method is solved immediately below. Additional examples of the method of voltage response synthesis are given in Appendix B.

Example problem.

$$\dot{Z}(p) = \frac{2p^3 + 1001p^2 + 1600p + 50000}{p^2 + 500p}$$

Type A. L = 2

$$\frac{\dot{Z}(p)}{p} = \frac{2p^3 + 1001p^2 + 1600p + 50000}{p^3 + 500p^2}$$

$$\frac{P(p)}{Q(p)} = \frac{\dot{Z}(p)}{p}\epsilon^{pt} = \frac{(2p^3 + 1001p^2 + 1600p + 50000)\,\epsilon^{pt}}{p^2(p+500)}$$

$$P(0) = 50000$$

$$P'(0) = 50000t + 1600$$

$$Q''(0) = 1000$$

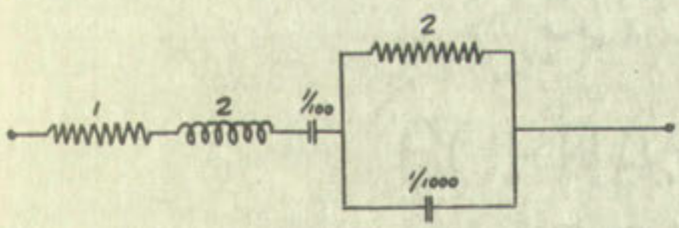
$$Q'''(0) = 6$$

$$\frac{6P'Q'' - 2PQ'''}{3(Q'')^2} = \frac{t}{1/100} + 3$$

$$\frac{P(p)}{Q'(p)} = \frac{(2p^3 + 1001p^2 + 1600p + 50000)\,\epsilon^{pt}}{3p^2 + 1000p}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-500} = -2\,\epsilon^{-500t}$$

$$J(t) = 3 + \frac{t}{1/100} - 2\,\epsilon^{-500t} = 1 + \frac{t}{1/100} + 2(1 - \epsilon^{-500t})$$



CHAPTER IV

THE APPLICATION OF THE RESPONSE SYNTHESIS TECHNIQUE TO THE EQUIVALENT CIRCUIT PROBLEM

I. REVIEW OF THE PROBLEM AND THE TECHNIQUE

A problem of major importance in the field of network theory involves the procedures for the design of a variety of networks, each having the same immittance function as that of a given parent network. It is an accepted fact that if an immittance function is capable of physical realization in one configuration, it is also capable of realization in an infinite number of equivalent networks.

One of the most noteworthy of the attributes of the response synthesis technique is its inherent ability to specify, in a simple and straightforward manner, an infinite number of networks equivalent to certain classes of parent networks. A second important characteristic is that, within limits, the exact value of a network element or elements may be pre-selected for incorporation in the equivalent network. This second characteristic of the response synthesis is not inherent to other known synthesis techniques.

II. APPLICATION OF THE RESPONSE SYNTHESIS TECHNIQUE

The response synthesis method for the realization of equivalent networks has for its starting point a network of either the all-shunt or the all-series configuration, the element values of which are known

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The response synthesis method for the generation of equivalent networks has for its starting point a network of either the all-shunt or the all-series configuration, the element values of which are known

constants. As the first step in the development of networks equivalent to the parent network, the sum of terms representing the total indicial immittance of the parent network is written by reference to Table I or Table II as may be appropriate. The total response function is then manipulated algebraically in such a manner that the value of the sum remains unchanged while some of the terms of the sum are adjusted in value and/or character and hence connote shunt arms or series impedance groups of a different nature from those of the parent network. Each such manipulation will then result in the specification of a network equivalent in indicial immittance and hence in complex immittance to the parent network and with network elements which have changed in value and also in impedance character.

It is pointed out that the response functions for inductance-capacitance, pure inductance, pure capacitance, and two of the three types of response of the resistance-inductance-capacitance shunt arms or series impedance groups apparently do not lend themselves to the manipulations referred to above. Consequently, when these types of arms or groups are present in the parent network, they will appear unchanged in all networks derived by these procedures as equivalent to the parent. As a result of this condition, the only response functions which will be dealt with in the specification of equivalent networks are those of pure resistance, resistance-inductance, resistance-capacitance, and resistance-inductance-capacitance wherein the response is a function of the hyperbolic sine.

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There are several conditions upon the relationship between the

values of network elements which must be fulfilled if the indicial immittance functions are to be capable of specifying networks equivalent to a given parent network. These conditions are brought about by the requirement that the equivalent networks be physically realizable. It is believed that a set of rules enumerating these conditions would serve no useful purpose inasmuch as the response statements may be so easily manipulated to determine physical realizability of the equivalent network that the memorizing of, or reference to such rules would represent a more cumbersome procedure. There are several types of response manipulation possible which will be demonstrated.

Since the theory involved in the application of the equivalent circuit specification technique is identical to that of response synthesis as presented in Chapters II and III, the following simple examples will be sufficient to demonstrate the application of the method. As a first example, assume the R-L network of Figure 16.

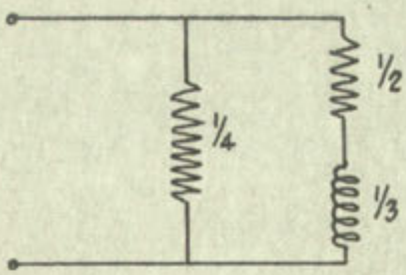


Fig.16.- R-L Parent Network

Reference to Table I shows that the response function for this network is:

$$\Delta(t) = 4 + 2(1 - \varepsilon^{-3t/2}) \tag{1}$$

$$= 6 - 2\varepsilon^{-3t/2} \tag{2}$$

$$= 6 - 3\varepsilon^{-3t/2} + \varepsilon^{-3t/2} \tag{3}$$

$$= 3 + 3(1 - \varepsilon^{-3t/2}) + \varepsilon^{-3t/2} \tag{4}$$

Statement (4), by reference to Table I, yields the equivalent network of Figure 17.

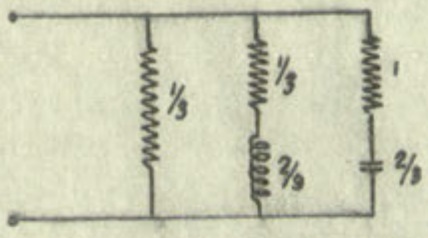


Fig. 17. R-L-C Network Equivalent to Parent Network

A second manipulation of the basic response function on (2) yields:

$$\Delta(t) = 6 - 2\varepsilon^{-3t/2} \tag{2}$$

$$= 6 - 4\varepsilon^{-3t/2} + 2\varepsilon^{-3t/2} \tag{5}$$

$$= 2 + 4(1 - \varepsilon^{-3t/2}) + 2\varepsilon^{-3t/2} \tag{6}$$

Here, statement (6) specifies the equivalent network of Figure 18.

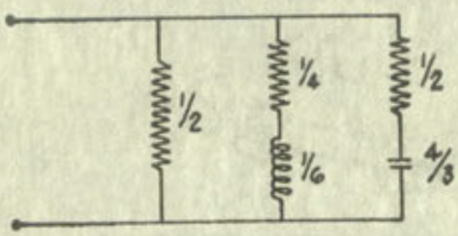


Fig. 18 R-L-C Network Equivalent to Parent Network

It is obvious that the pure resistive arm may be eliminated entirely by the manipulation:

$$6 - 2\varepsilon^{-3t/2} \tag{2}$$

$$6 - 6\varepsilon^{-3t/2} + 4\varepsilon^{-3t/2} \tag{7}$$

$$6(1 - \varepsilon^{-3t/2}) + 4\varepsilon^{-3t/2} \tag{8}$$

Wherein statement (8) specifies the equivalent network of Figure 19.

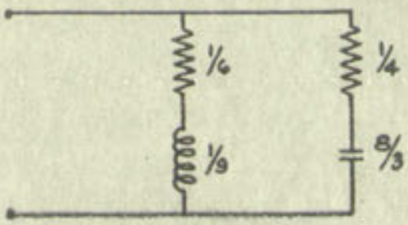


Fig. 19.- R-L-C Network Equivalent to Parent Network

All of the networks, Figures 16 through 19 have the admittance function:

$$\dot{Y}(p) = \frac{8p + 18}{2p + 3} \tag{9}$$

It is apparent that the process outlined above may be applied an infinite number of times and that each new manipulation will yield a network equivalent to the parent network.

To illustrate a requirement of the method that an element arm or series group consisting of a pure resistance be present in order that the response function be capable of this type of manipulation, assume the parent network of Figure 20.

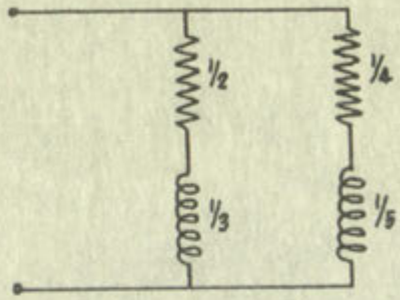


Fig. 20. - Parent Network Lacking a Pure Resistance Arm

The indicial admittance of this network is:

$$A(t) = 2(1-\epsilon^{-3t/2}) + 4(1-\epsilon^{-5t/4}) \tag{10}$$

$$= 6 - 2\epsilon^{-3t/2} - 4\epsilon^{-5t/4} \tag{11}$$

It is apparent that statement (11) is not capable of the type of manipulation demonstrated above, except to the form of statement (10). However, the network of Figure 20 may be resynthesized to the all-series configuration thus:

$$\dot{Y}(p) = \frac{1}{p/3 + 1/2} + \frac{1}{p/5 + 1/4} \tag{12}$$

$$= \frac{6}{2p+3} + \frac{20}{4p+5} \tag{13}$$

$$= \frac{24p + 30 + 40p + 60}{8p^2 + 22p + 15} \tag{14}$$

$$\dot{Z}(p) = \frac{8p^2 + 22p + 15}{64p + 90} \tag{15}$$

$$\frac{P(p)}{Q(p)} = \frac{\dot{Z}(p)}{p}\,\epsilon^{pt} = \frac{(8p^2+22p+15)\,\epsilon^{pt}}{64p^2+90p} \tag{16}$$

$$\frac{P(p)}{Q'(p)} = \frac{(8p^2+22p+15)\,\epsilon^{pt}}{128p+90} \tag{17}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=0} = \frac{1}{6} \tag{18}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-45/32} = \frac{1}{768}\,\epsilon^{-45t/32} \tag{19}$$

Yielding the all-series equivalent to Figure 20.

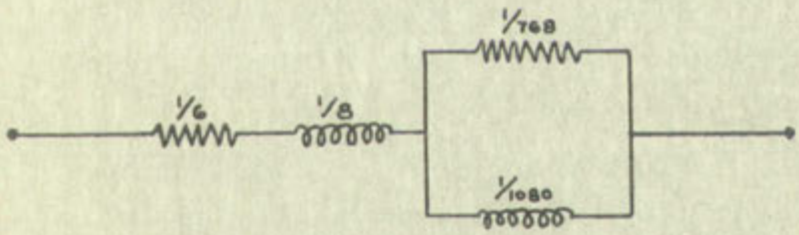


Fig. 21.- All-Series Equivalent of Network of Fig. 20

The network of Figure 21 has the response function:

$$J(t) = \frac{1}{6} + \frac{1}{768}\,\epsilon^{-45t/32} \tag{20}$$

which function may be manipulated to produce an infinite number of equivalent networks. To avoid manipulation with fractional coefficients, it is possible to multiply the whole response function by a factor which will clear all fractions, perform the manipulation, and then divide by the previous multiplier to obtain the new form of the response function,

thus:

$$768\left[\frac{1}{6} + \frac{1}{768}\epsilon^{-45t/32}\right] = \tag{21}$$

$$128 + \epsilon^{-45t/32} = \tag{22}$$

$$128 + 10\epsilon^{-45t/32} - 9\epsilon^{-45t/32} = \tag{23}$$

$$119 + 10\epsilon^{-45t/32} + 9(1 - \epsilon^{-45t/32}) \tag{24}$$

Hence the indicial impedance of a network equivalent to both Figures 20 and 21 will be:

$$J(t) = \frac{119}{768} + \frac{10}{768}\epsilon^{-45t/32} + \frac{9}{768}(1 - \epsilon^{-45t/32}) \tag{25}$$

which function yields the network of Figure 22 which has the same

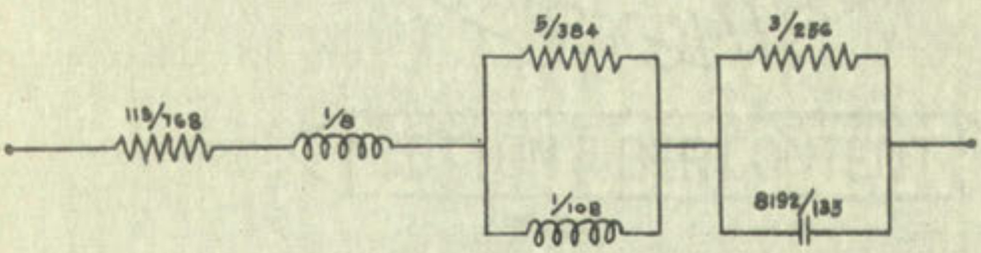


Fig. 22.- Network Equivalent to those of Fig. 20 and Fig. 21

immittance functions as the networks of Figure 20 and Figure 21. An infinite number of manipulations similar to that of statement (25) may be carried out to specify various network equivalents.

The response function for the network of Figure 20 is capable of still another type of manipulation not involving resynthesis. For example, starting with statement (11) there is given:

$$A(t) = 6 - 2\epsilon^{-3t/2} - 4\epsilon^{-5t/4} \tag{11}$$

$$= 6 + 2\epsilon^{-5t/4} - 2\epsilon^{-3t/2} - 6\epsilon^{-5t/4} \tag{26}$$

$$= 2\epsilon^{-11t/8}\left(\epsilon^{t/8} - \epsilon^{-t/8}\right) + 6\left(1 - \epsilon^{-5t/4}\right) \tag{27}$$

$$= \frac{\epsilon^{-11t/8}}{1/4}\left[\frac{\epsilon^{t/8} - \epsilon^{-t/8}}{2}\right] + 6\left(1 - \epsilon^{-5t/4}\right) \tag{28}$$

$$= \frac{\epsilon^{-11t/8}}{1/4}\sinh(t/8) + 6\left(1 - \epsilon^{-5t/4}\right) \tag{29}$$

Here by reference to Table I, statement (29) specifies the all-shunt network of Figure 23.

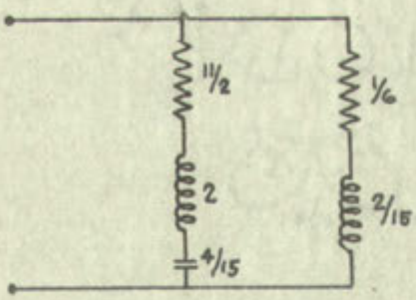


Fig. 23.- Network Equivalent to Network of Fig. 20

It is apparent that the basic response statement (11) is capable of an infinite number of rearrangements of the type used to specify the equivalent network of Figure 23. An example of such further manipulation is:

$$\Delta(t) = 6 - 2\epsilon^{-3t/2} - 4\epsilon^{-5t/4} \tag{11}$$

$$= 6 + \epsilon^{-5t/4} - \epsilon^{-3t/2} - 5\epsilon^{-5t/4} - \epsilon^{-3t/2} \tag{30}$$

$$= \epsilon^{-11t/8}(\epsilon^{t/8} - \epsilon^{-t/8}) + 5(1 - \epsilon^{-5t/4}) + (1 - \epsilon^{-3t/2}) \tag{31}$$

$$= \frac{\epsilon^{-11t/8}}{1/2}\left[\frac{\epsilon^{t/8} - \epsilon^{-t/8}}{2}\right] + 5(1 - \epsilon^{-5t/4}) + (1 - \epsilon^{-3t/2}) \tag{32}$$

$$= \frac{\epsilon^{-11t/8}}{1/2}\sinh(t/8) + 5(1 - \epsilon^{-5t/4}) + (1 - \epsilon^{-3t/2}) \tag{33}$$

Thus specifying the network of Figure 24 equivalent to that of Figure 20.

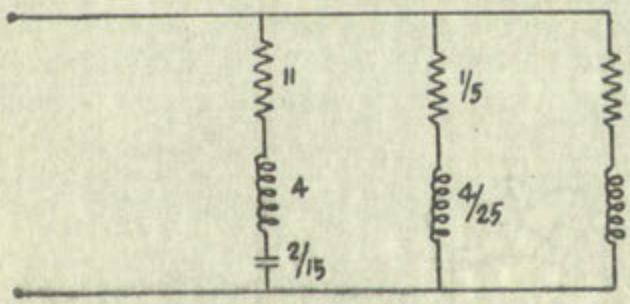


Fig. 24.- Network Equivalent to Network of Fig.20

It is also apparent that a parent network containing an R-L-C shunt arm or series group may be converted to networks containing not more than two elements per impedance unit by a reversal of the process demonstrated by statements (30) through (33) provided,

1. That in the R-L-C unit, $R^2/4L^2 > 1/LC$ for the all-shunt case, and $1/4R^2C^2 > 1/LC$ for the all-series case in order that the unit response will be a

function of the hyperbolic sine, and

2. There is included in the parent network a unit of pure resistance sufficiently large to subsidize the R-L unit which will always be produced.

III. PRESPECIFICATION OF NETWORK ELEMENTS

As mentioned earlier in this Chapter, the response synthesis technique possesses the important attribute, present in no other known synthesis method, of allowing for the pre-selection, within limits, of the values of network elements for incorporation in an equivalent network. The process for such pre-selection involves only the determination, based on physical realizability criteria, of the allowable limits on the ranges of the values of the various network elements. The evaluation of the limits involves only the determination of those conditions which will insure that the total network response function does not specify a network which would require the use of a non-physical element or elements. As an example, assume the network of Figure 25.

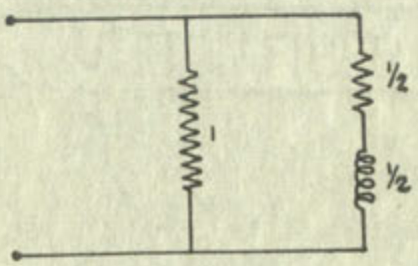


Fig. 25.- R-L Network for Demonstration of Element Prespecification

For this network, having a complex admittance of:

$$\dot{Y}(p) = \frac{p+3}{p+1} \tag{34}$$

and an indicial admittance of:

$$A(t) = 1 + 2(1-\epsilon^{-t}) \tag{35}$$

$$= 3 - 2\epsilon^{-t} \tag{36}$$

it is apparent that the response, contributed by the pure resistance arm, of value unity is at its maximum value since any manipulation of the indicial admittance within the bounds established by physical realizability can only serve to reduce this value by the use of part or all of the amount to subsidize an increasing coefficient of an R-L response term. Thus,

$$A(t) = 1 + 2(1-\epsilon^{-t}) \tag{37}$$

$$= 3 - 2\epsilon^{-t} \tag{38}$$

$$= 3 - 2.5\epsilon^{-t} + 0.5\epsilon^{-t} \tag{39}$$

$$= 0.5 + 2.5(1-\epsilon^{-t}) + 0.5\epsilon^{-t} \tag{40}$$

wherein the response $0.5\,\epsilon^{-t}$ creates an R-C shunt arm. Therefore,

$$\frac{1}{R} \leqslant 1 \tag{41}$$

$$R \geqslant 1 \tag{42}$$

Now consider the response of the R-L arm of the network which has the standard form of:

$$\frac{1}{R_L}\left(1 - \epsilon^{-R_L t/L_R}\right) \tag{43}$$

From statement (36) it is apparent that $1/R_L$ cannot be greater in value than 3 since at this point the whole of the pure resistance response will be reduced to zero. Therefore:

$$1/R_L \leqslant 3 \tag{44}$$

$$R_L \geqslant 1/3 \tag{45}$$

It is also apparent from statement (35) that by breaking the expression for the response of an R-L arm into the sum of two parts, thus specifying in the network two parallel R-L shunt arms, the numerical value of one component of the sum may be made as small as may be desired. Hence:

$$1/R_L \geqslant 0 \tag{46}$$

$$R_L \leqslant \infty \tag{47}$$

Then, from equations (45) and (47) there is given:

$$1/3 \leq R_L \leq \infty \tag{48}$$

In statement (43) the exponential $\varepsilon^{-t} = \varepsilon^{-(1)t} = \varepsilon^{-R_L t/L_R}$ constitutes the statement

$$\frac{R_L}{L_R} = 1 \tag{49}$$

$$L_R = R_L \tag{50}$$

Hence from equations (48) and (50) there is given:

$$1/3 \leq L_R \leq \infty \tag{51}$$

The coefficient $1/R_c$ of the R-C arm response is zero when the network is in its minimal form as specified by statement (35). The maximum value of $1/R_c$ is unity and occurs when all of the pure resistance response, unity, is used to increase the coefficient of the R-L response term with a consequent release of a $+(1)\,\varepsilon^{-t}$ to keep the total response at its specified value. Therefore

$$0 \leq \frac{1}{R_c} \leq 1 \tag{52}$$

$$1 \leq R_c \leq \infty \tag{53}$$

The exponential of the R-C response term $\epsilon^{-t} = \epsilon^{-(1)t} = \epsilon^{-t/R_c C_R}$ implies the statement

$$\frac{1}{R_c C_R} = 1 \tag{54}$$

$$R_c = \frac{1}{C_R} \tag{55}$$

Therefore from the inequality (53)

$$1 \leq 1/C_R \leq \infty \tag{56}$$

$$0 \leq C_R \leq 1 \tag{57}$$

Collecting and tabulating the results of (42) through (57) gives:

1. $1 \leq R \leq \infty$
2. $1/3 \leq R_L \leq \infty$
3. $1/3 \leq L_R \leq \infty$
4. $1 \leq R_c \leq \infty$
5. $0 \leq C_R \leq 1$

Element values outside the above five ranges will cause the resulting equivalent network to be non-physical.

To illustrate the element pre-specification technique the following element values, lying within the ranges above, will be pre-specified and a group of networks, each equivalent to the parent network of Figure 19 and each containing one of the pre-specified elements, will be designed.

1. $R = 6$

2. $R_L = 5/12$

3. $L_R = 7/18$

4. $R_c = 1000$

5. $C_R = 1/100$

The basic response statement for the parent network of Figure 19 is:

$$A(t) = 3 - 2\epsilon^{-t} \tag{36}$$

For $R = 6$:

$$A(t) = 1/6 + 17/6 - 12/6\, \epsilon^{-t} \tag{58}$$

$$= 1/6 + 17/6 - 17/6\, \epsilon^{-t} + 5/6\, \epsilon^{-t} \tag{59}$$

$$= 1/6 + 17/6(1 - \epsilon^{-t}) + 5/6\, \epsilon^{-t} \tag{60}$$

Wherein (60) specifies the network of Figure 26.

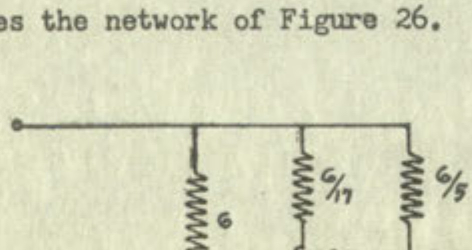


Fig. 26. Equivalent Network with $R = 6$

For $R_L = 5/12$:

$$A(t) = 3 - {}^{12}/_{5}\,\epsilon^{-t} + {}^{2}/_{5}\,\epsilon^{-t} \tag{61}$$

$$= {}^{3}/_{5} + {}^{12}/_{5}(1 - \epsilon^{-t}) + {}^{2}/_{5}\,\epsilon^{-t} \tag{62}$$

Wherein (62) specifies the network of Figure 27.

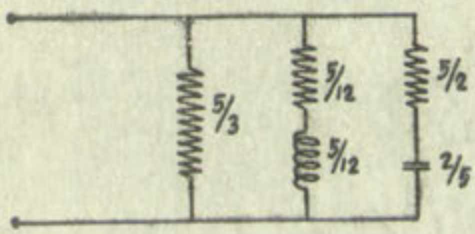


Fig. 27. Equivalent Network with $R_L = 5/12$

For $L_R = 7/18$:

$$A(t) = 3 - {}^{18}/_{7}\,\epsilon^{-t} + {}^{4}/_{7}\,\epsilon^{-t} \tag{63}$$

$$= {}^{3}/_{7} + {}^{18}/_{7}(1 - \epsilon^{-t}) + {}^{4}/_{7}\,\epsilon^{-t} \tag{64}$$

Wherein (64) specifies the network of Figure 28.

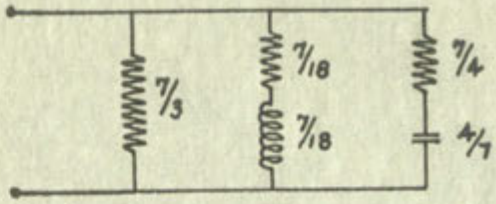


Fig. 28. Equivalent Network with $L_R = 7/18$

For $R_c = 1000$:

$$A(t) = 3 - \frac{2001}{1000}\epsilon^{-t} + \frac{1}{1000}\epsilon^{-t} \tag{65}$$

$$= \frac{999}{1000} + \frac{2001}{1000}(1 - \epsilon^{-t}) + \frac{1}{1000}\epsilon^{-t} \tag{66}$$

Wherein (66) specifies the network of Figure 29.

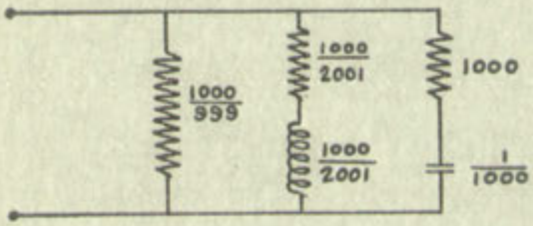


Fig. 29. Equivalent Network with $R_c = 1000$

For $C_R = 1/100$:

$$A(t) = 3 - \frac{201}{100}\epsilon^{-t} + \frac{1}{100}\epsilon^{-t} \tag{67}$$

$$= \frac{99}{100} + \frac{201}{100}(1 - \epsilon^{-t}) + \frac{1}{100}\epsilon^{-t} \tag{68}$$

Wherein (68) specifies the network of Figure 30.

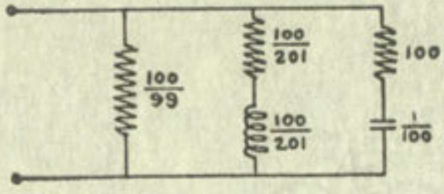


Fig. 30. Equivalent Network with $C_R = 1/100$

The circumstance, in the above example of a one to one numerical relationship between R_L and L_R and the reciprocal relationship between the values of R_c and C_R is by no means characteristic of the method. In the simple example used, for the reason of the ease with which the various processes can be followed, the exponent unity of the Naperian base is the cause of the relationships noted above. Generally, if the exponent of the exponential is α , the relationship between R_L and L_R will be:

$$R_L/L_R = \alpha \tag{69}$$

$$R_L = \alpha L_R \tag{70}$$

and between R_c and C_R will be:

$$1/R_cC_R = \alpha \tag{71}$$

$$R_c = 1/\alpha C_R \tag{72}$$

Since nothing in the development of the limits on the range of the individual network elements depended upon the location within their own ranges of the other network elements, it may be possible to prespecify more than one element for inclusion in an equivalent network. The ability to prespecify one element in the network is assured by keeping that element value within the limits required for physical realizability. On the other hand, the ability to prespecify more than one element is dependent upon the availability of such additional pure resistance response as may be required to subsidize the additional R-L and/or R-C arm or group responses generated by the incorporation of the addition-

ally specified elements. Hence, since the ability to pre-specify more than one element is dependent upon the parameters of the parent network, each such problem will of necessity require individual evaluation. Such evaluation may be performed by assuming that it is possible to incorporate the desired number of pre-specified elements; expand the response function to allow for their incorporation; and finally evaluate the expanded function against physical realizability criteria. If the expanded response function is non-physically realizable, one or more of the additional pre-specified elements must be dropped from consideration.

As an example of the process assume the network of Figure 25, the basic current response function for which was shown to be $3 - 2\epsilon^{-t}$ Further, assume that it is desired to pre-specify all five of the basic network elements at the previously used within-range values of:

1. $R = 6$
2. $R_L = 5/12$
3. $L_R = 7/18$
4. $R_C = 1000$
5. $C_R = 1/100$

The indicial admittance of the parent network is accordingly expanded, step-by-step to include each of the above element values, thus:

$$3 - 2\epsilon^{-t} = \tag{36}$$

$$1/6 + 17/6 - 2\epsilon^{-t} = \tag{73}$$

$$1/6 + 17/6 - 12/5\,\epsilon^{-t} + 2/5\,\epsilon^{-t} \tag{74}$$

$$\tfrac{1}{6} + \tfrac{17}{6} - \tfrac{12}{5}\epsilon^{-t} - \tfrac{18}{7}\epsilon^{-t} + \tfrac{104}{35}\epsilon^{-t} = \tag{75}$$

$$\tfrac{1}{6} + \tfrac{17}{6} - \tfrac{12}{5}\epsilon^{-t} - \tfrac{18}{7}\epsilon^{-t} + \tfrac{1}{1000}\epsilon^{-t} + \tfrac{20793}{7000}\epsilon^{-t} = \tag{76}$$

$$\tfrac{1}{6} + \tfrac{17}{6} - \tfrac{12}{5}\epsilon^{-t} - \tfrac{18}{7}\epsilon^{-t} + \tfrac{1}{1000}\epsilon^{-t} + \tfrac{1}{100}\epsilon^{-t} + \tfrac{20723}{7000}\epsilon^{-t} = \tag{77}$$

$$\tfrac{1}{6} + \tfrac{595}{210} - \tfrac{504}{210}\epsilon^{-t} - \tfrac{540}{210}\epsilon^{-t} + \tfrac{1}{1000}\epsilon^{-t} + \tfrac{1}{100}\epsilon^{-t} + \tfrac{20723}{7000}\epsilon^{-t} \tag{78}$$

Inspection of the second, third, and fourth terms of (78) shows that there is not sufficient pure resistance response to subsidize the response requirements of both $R_L = 5/12$ and $L_R = 7/18$ hence either R_L or L_R must be dropped from consideration as a prespecified element. Even the dropping of $R = 6$ from consideration will not provide the additional response necessary. Hence assume that it is decided to sacrifice the prespecification of $R_L = 5/12$. Statement (78) would then be adjusted to be:

$$A(t) = \tfrac{1}{6} + \tfrac{595}{210} - \tfrac{540}{210}\epsilon^{-t} + \tfrac{1}{1000}\epsilon^{-t} + \tfrac{1}{100}\epsilon^{-t} + \tfrac{3923}{7000}\epsilon^{-t} \tag{79}$$

$$= \tfrac{1}{6} + \tfrac{55}{210} + \tfrac{540}{210}(1 - \epsilon^{-t}) + \tfrac{1}{1000}\epsilon^{-t} + \tfrac{1}{100}\epsilon^{-t} + \tfrac{3923}{7000}\epsilon^{-t} \tag{80}$$

$$= \tfrac{1}{6} + \tfrac{11}{42} + \tfrac{18}{7}(1 - \epsilon^{-t}) + \tfrac{1}{1000}\epsilon^{-t} + \tfrac{1}{100}\epsilon^{-t} + \tfrac{3923}{7000}\epsilon^{-t} \tag{81}$$

The response function (81) yields the network of Figure 31 which

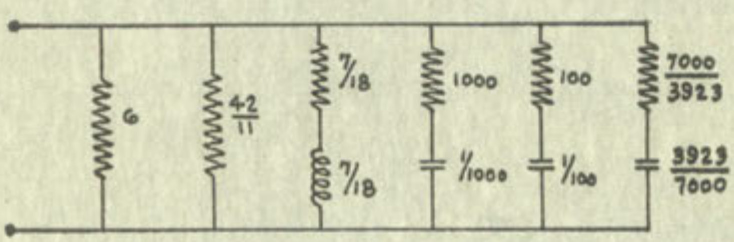


Fig. 31. Equivalent Network with Four Prespecified Elements

incorporates four of the five elements originally desired for prespecification and has the required admittance $\dot{Y}(p) = (p+3)/(p+1)$. There are, of course, an infinite number of rearrangements of the equivalent network of Figure 31 made possible by further manipulation of the response statement (81).

The techniques used for the prespecification of equivalent network elements using the all-shunt configuration are equally applicable for the all-series type networks. Further example involving equivalent networks and the prespecification of elements therefor are given in Appendix C.

CHAPTER V

SUMMARY AND CONCLUSION

The foregoing chapters have developed the theory and demonstrated the application of the response synthesis procedure. The development has been limited to those concepts dealing with two-terminal, lumped constant networks resembling in configuration the First and Second Foster Canonical Forms.

The theory underlying the synthesis method shows that a physical realization of a given rational immittance function may be developed by constructing a network in segments, in such a way that the current or voltage response of each segment will add to the response of each other segment to yield the known total response of the network required. Standard response functions for all possible segment types consistent with the network configurations used have been calculated and tabulated. The use of these tables reduces the work and time involved in the procedure.

The total response of the unknown network is derived by Laplace transform methods from the given immittance function of the complex frequency variable. However, the procedure for the transformation from the frequency domain to the response in the time domain is so standardized that a knowledge of the Laplace transformation calculus is not required for the application of the method. A step-by-step procedure has been outlined in a manner that the synthesis method may be used by persons whose knowledge of mathematics does not extend beyond the

algebra of complex numbers together with a rudimentary grasp of the differential calculus to the extent of ability to apply the rules for obtaining the first, second, and third derivitives of simple functions.

The response synthesis method has been shown to possess the unique features of specifying the parameters of networks equivalent in immittance to a parent network and, in addition, allowing for the pre-specification of the values of network elements for incorporation in equivalent networks. The mathematical labor involved in the accomplishment of these highly important aspects of the method is surprisingly moderate.

It is felt that the possibilities of the response synthesis techniques are by no means limited to the development which has been made herein. Further work involving the application of response functions for the synthesis of multi terminal pair networks, transfer functions, and networks of other configurations than the two types developed in this work are needed in order to fully utilize the possibilities of the method initially developed in this work.

APPENDIX A

SYNTHESIS OF TWO-TERMINAL NETWORKS BY CURRENT RESPONSE METHODS

Six problems, representative of the current response synthesis procedure are solved in this Appendix. These problems involve R-L, R-C, L-C, and R-L-C networks, all of which are synthesized in the all-shunt configuration characteristic of the current response synthesis technique. The multiple response which is characteristic of the R-L-C shunt arm is illustrated in problems (5) and (6). Four of the problems solved illustrate the Type A admittance function wherein the identification and evaluation of the pure capacitive shunt arm is made directly from the defining complex admittance function.

$$(1.)\quad \dot{Y}(p) = \frac{3p^3 + 168p^2 + 1380p + 2400}{10p^2 + 160p + 600}$$

Type A; $C = 3/10$

$$\frac{\dot{Y}(p)}{p} = \frac{3p^3 + 168p^2 + 1380p + 2400}{10p(p+6)(p+10)}$$

$$\frac{P(p)}{Q(p)} = \frac{Y(p)}{p}\epsilon^{pt} = \frac{3p^3 + 168p^2 + 1380p + 2400}{10p^3 + 160p^2 + 600p}\epsilon^{pt}$$

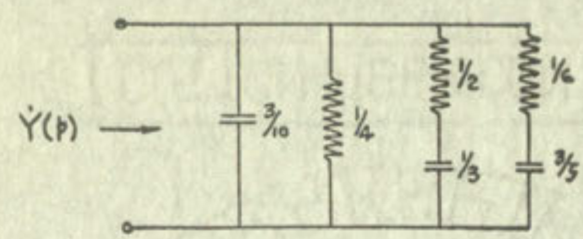
$$\frac{P(p)}{Q'(p)} = \frac{(3p^3 + 168p^2 + 1380p + 2400)\epsilon^{pt}}{30p^2 + 320p + 600}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=0} = \frac{2400}{600} = 4$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-6} = \frac{-480}{-240}\epsilon^{-6t} = 2\epsilon^{-6t}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-10} = \frac{2400}{400}\epsilon^{-10t} = 6\epsilon^{-10t}$$

$$A(t) = 4 + 2\epsilon^{-6t} + 6\epsilon^{-10t}$$



$$(2.)\quad \dot{Y}(p) = \frac{20p^3 + 129p^2 + 206p + 24}{2p^3 + 10p^2 + 12p}$$

$$\frac{\dot{Y}(p)}{p} = \frac{20p^3 + 129p^2 + 206p + 24}{2p^2(p+2)(p+3)}$$

$$\frac{P(p)}{Q(p)} = \frac{\dot{Y}(p)}{p}\epsilon^{pt} = \frac{20p^3 + 129p^2 + 206p + 24}{2p^4 + 10p^3 + 12p^2}\epsilon^{pt}$$

$$\left.\begin{array}{l} P(p) = (20p^3 + 129p^2 + 206p + 24)\,\epsilon^{pt} \\ P'(p) = t(20p^3 + 129p^2 + 206p + 24)\,\epsilon^{pt} + (60p^2 + 258p + 206)\,\epsilon^{pt} \\ Q'(p) = 8p^3 + 30p^2 + 24p \\ Q''(p) = 24p^2 + 60p + 24 \\ Q'''(p) = 48p + 60 \end{array}\right|_{p=0} \begin{array}{l} = 24 \\ = 24t + 206 \\ \\ = 24 \\ = 60 \end{array}$$

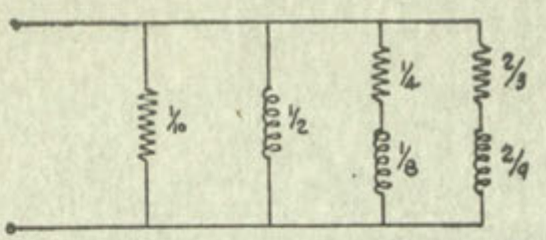
$$\frac{6P'Q'' - 2PQ'''}{3(Q'')^2} = 2t + \frac{31}{2}$$

$$\frac{P(p)}{Q'(p)} = \frac{(20p^3 + 129p^2 + 206p + 24)\,\epsilon^{pt}}{8p^3 + 30p^2 + 24p}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-2} = -\frac{32}{8}\epsilon^{-2t} = -4\epsilon^{-2t}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-3} = -\frac{27}{18}\epsilon^{-3t} = -\frac{3}{2}\epsilon^{-3t}$$

$$A(t) = 31/2 + 2t - 4\epsilon^{-2t} - \tfrac{3}{2}\epsilon^{-3t} = 10 + 2t + 4(1 - \epsilon^{-2t}) + 3/2\,(1 - \epsilon^{-3t})$$



$$(3.)\quad \dot{Y}(p) = \frac{640p^3 + 3320p^2 + 4647p + 2520}{480p^3 + 1818p^2 + 1680p}$$

$$\frac{\dot{Y}(p)}{p} = \frac{640p^3 + 3320p^2 + 4647p + 2520}{2p^2(16p+35)(15p+24)}$$

$$\frac{P(p)}{Q(p)} = \frac{\dot{Y}(p)}{p}\epsilon^{pt} = \frac{640p^3 + 3320p^2 + 4647p + 2520}{480p^4 + 1818p^3 + 1680p^2}\epsilon^{pt}$$

$$\left.\begin{aligned} P'(p) &= t(640p^3+3320p^2+4647p+2520)\epsilon^{pt} + (1920p^2+6640p+4647)\epsilon^{pt} \\ Q''(p) &= 5760p^2 + 10908p + 3360 \\ Q'''(p) &= 11520p + 10908 \end{aligned}\right|_{p=0} \begin{aligned} &= 2520t + 4647 \\ &= 3360 \\ &= 10908 \end{aligned}$$

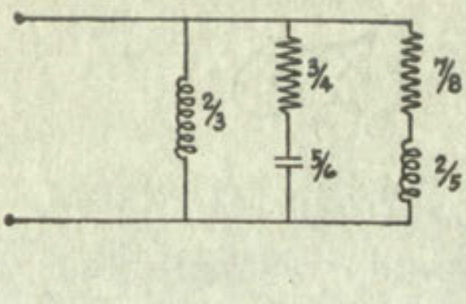
$$\frac{6P'Q'' - 2PQ'''}{3(Q'')^2} = \frac{3t}{2} + \frac{8}{7}$$

$$\frac{P(p)}{Q'(p)} = \frac{(640p^3 + 3320p^2 + 4647p + 2520)\epsilon^{pt}}{1920p^3 + 5454p^2 + 3360p}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-35/16} = -\frac{8}{7}\epsilon^{-35t/16}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-24/15} = \frac{4}{3}\epsilon^{-8t/5}$$

$$A(t) = 8/7 + 3t/2 + 4/3\,\epsilon^{-8t/5} - 8/7\,\epsilon^{-35t/16} = 3t/2 + 4/3\,\epsilon^{-8t/5} + 8/7\left(1 - \epsilon^{-35t/16}\right)$$



$$(4)\quad \dot{Y}(p) = \frac{p^6 + 26p^4 + 107p^2 + 90}{2p^5 + 16p^3 + 30p}$$

Type A; $C = 1/2$

$$\frac{\dot{Y}(p)}{p} = \frac{p^6 + 26p^4 + 107p^2 + 90}{2p^2(p^2+5)(p^2+3)}$$

$$\frac{P(p)}{Q(p)} = \frac{\dot{Y}(b)}{p}\epsilon^{pt} = \frac{(p^6 + 26p^4 + 107p^2 + 90)\epsilon^{pt}}{2p^6 + 16p^4 + 30p^2}$$

$$\left.\begin{aligned} P'(p) &= t(p^6 + 26p^4 + 107p^2 + 90)\epsilon^{pt} + (6p^5 + 104p^3 + 214p)\epsilon^{pt} \\ Q''(p) &= 60p^4 + 192p^2 + 60 \\ Q'''(p) &= 240p^3 + 384p \end{aligned}\right|_{p=0} \begin{aligned} &= 90t \\ &= 60 \\ &= 0 \end{aligned}$$

$$\frac{6P'Q'' - 2PQ'''}{3(Q'')^2} = 3t$$

$$\frac{P(p)}{Q'(p)} = \frac{(p^6 + 26p^4 + 107p^2 + 90)\epsilon^{pt}}{12p^5 + 64p^3 + 60p}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=j\sqrt{5}} = \frac{4}{j2\sqrt{5}}\epsilon^{j\sqrt{5}t}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-j\sqrt{5}} = -\frac{4}{j2\sqrt{5}}\epsilon^{-j\sqrt{5}t}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=j\sqrt{3}} = \frac{2}{j2\sqrt{3}}\epsilon^{j\sqrt{3}t}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-j\sqrt{3}} = -\frac{2}{j2\sqrt{3}}\,\varepsilon^{-j\sqrt{3}}$$

$$A(t) = 3t + \frac{4}{\sqrt{5}}\left[\frac{\varepsilon^{j\sqrt{5}t} - \varepsilon^{-j\sqrt{5}t}}{2j}\right] + \frac{2}{\sqrt{3}}\left[\frac{\varepsilon^{j\sqrt{3}t} - \varepsilon^{-j\sqrt{3}t}}{2j}\right]$$

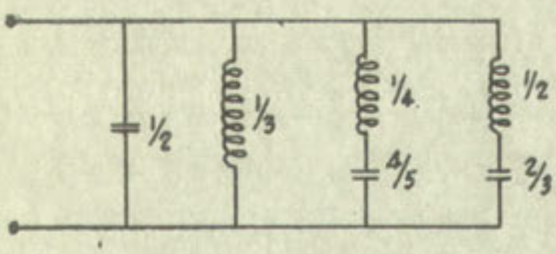
$$\doteq 3t + \frac{4}{\sqrt{5}}\sin(\sqrt{5}\,t) + \frac{2}{\sqrt{3}}\sin(\sqrt{3}\,t)$$

$$= \frac{t}{1/3} + \sqrt{\frac{16}{5}}\sin\left[\frac{t}{\sqrt{1/5}}\right] + \sqrt{\frac{4}{3}}\sin\left[\frac{t}{\sqrt{1/3}}\right]$$

$$\begin{cases} C_1/L_1 = 16/5 \\ L_1C_1 = 1/5 \end{cases} \qquad \begin{cases} C_2/L_2 = 4/3 \\ L_2C_2 = 1/3 \end{cases}$$

$$L_1 = 1/4 \qquad L_2 = 1/2$$

$$C_1 = 4/5 \qquad C_2 = 2/3$$



(5) $$\dot{Y}(p) = \frac{30p^4 + 104p^3 + 173p^2 + 154p + 84}{42p^3 + 28p^2 + 56p}$$

Type A; $C = 30/42 = 5/7$

$$\frac{\dot{Y}(p)}{p} = \frac{30p^4 + 104p^3 + 173p^2 + 154p + 84}{14p^2(3p^2 + 2p + 4)}$$

$$= \frac{30p^4 + 104p^3 + 173p^2 + 154p + 84}{42p^2(p + 1/3 + j\,1/3\sqrt{11})(p + 1/3 - j\,1/3\sqrt{11})}$$

$$\frac{P(p)}{Q(p)} = \frac{\dot{Y}(p)}{p}\,\varepsilon^{pt} = \frac{30p^4 + 104p^3 + 173p^2 + 154p + 84}{42p^4 + 28p^3 + 56p^2}\,\varepsilon^{pt}$$

$P(0) = 84$

$P'(0) = 84t + 154$

$Q''(0) = 112$

$Q'''(0) = 168$

$$\frac{6P'Q'' - 2PQ'''}{3(Q'')^2} = \frac{3t}{2} + 2 = \frac{t}{2/3} + 2$$

$$\frac{P(p)}{Q'(p)} = \frac{30p^4 + 104p^3 + 173p^2 + 154p + 84}{168p^3 + 84p^2 + 112p}\,\varepsilon^{pt}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p = -1/3 + j\,1/3\sqrt{11}} = \frac{\varepsilon^{-1/3}\,\varepsilon^{j\,1/3\sqrt{11}}}{j2\sqrt{11}}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p = -1/3 - j\,1/3\sqrt{11}} = -\frac{\varepsilon^{-1/3}\,\varepsilon^{-j\,1/3\sqrt{11}}}{j2\sqrt{11}}$$

$$A(t) = 2 + \frac{t}{2/3} + \frac{\varepsilon^{-1/3 t}}{\sqrt{11}}\left[\frac{\varepsilon^{j 1/3\sqrt{11}\, t} - \varepsilon^{-j 1/3\sqrt{11}\, t}}{2j}\right]$$

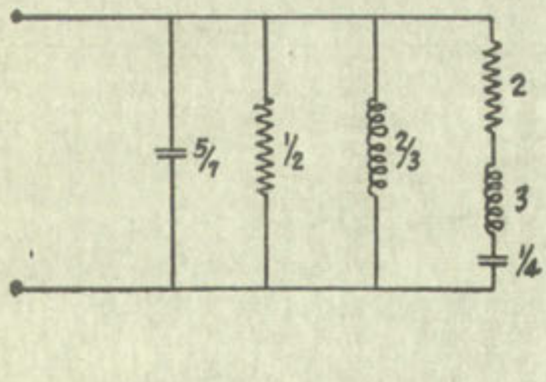
$$= 2 + \frac{t}{2/3} + \frac{\varepsilon^{-1/3 t}}{\sqrt{11}} \sin\left[(1/3\sqrt{11})t\right]$$

$$\begin{cases} R/2L = 1/3 \\ L\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}} = \sqrt{11} \\ \sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}} = \dfrac{1}{3}\sqrt{11} \end{cases}$$

$R = 2$

$L = 3$

$C = 1/4$



(6.) $\dot{Y}(p) = \dfrac{5p^5 + 70p^4 + 271p^3 + 410p^2 + 209p}{5p^4 + 70p^3 + 265p^2 + 380p + 180}$

Type A; $C = 5/5 = 1$

$$\frac{\dot{Y}(p)}{p} = \frac{5p^5 + 70p^4 + 271p^3 + 410p^2 + 209p}{5p(p+1)(p+2)^2(p+9)}$$

$$\frac{P(p)}{Q(p)} = \frac{\dot{Y}(p)}{p}\epsilon^{pt} = \frac{(5p^5 + 70p^4 + 271p^3 + 410p^2 + 209p)\,\epsilon^{pt}}{5p^5 + 70p^4 + 265p^3 + 380p^2 + 180p}$$

$P(-2) = 14\epsilon^{-2t}$

$P'(-2) = 14t\epsilon^{-2t} - 19\epsilon^{-2t}$

$Q''(-2) = 140$

$Q'''(-2) = -570$

$$\frac{6P'Q'' - 2PQ'''}{3(Q'')^2} = \frac{t}{5}\epsilon^{-2t}$$

$$\frac{P(p)}{Q'(p)} = \frac{(5p^5 + 70p^4 + 271p^3 + 410p^2 + 209p)\,\epsilon^{pt}}{25p^4 + 280p^3 + 795p^2 + 760p + 180}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=0} = 0$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-1} = \frac{\epsilon^{-t}}{8}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-9} = \frac{-\epsilon^{-9t}}{8}$$

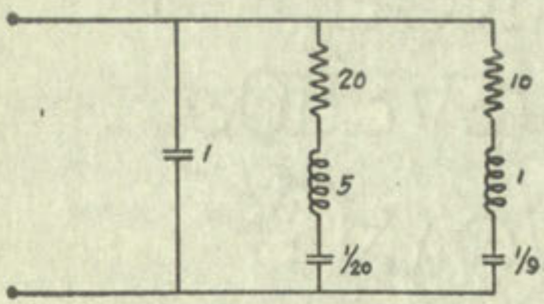
$$A(t) = \frac{t}{5}\varepsilon^{-2t} + \frac{\varepsilon^{-t} - \varepsilon^{-9t}}{8}$$

$$= \frac{t}{5}\varepsilon^{-2t} + \frac{\varepsilon^{-5t}}{4}\left[\frac{\varepsilon^{4t} - \varepsilon^{-4t}}{2}\right]$$

$$= \frac{t}{5}\varepsilon^{-2t} + \frac{\varepsilon^{-5t}}{4}\sinh(4t)$$

$$\begin{cases} L = 5 \\ R/2L = 2 \\ \sqrt{\dfrac{1}{LC}} = 2 \end{cases} \qquad \begin{cases} R/2L = 5 \\ L\sqrt{\dfrac{R^2}{4L^2} - \dfrac{1}{LC}} = 4 \\ \sqrt{\dfrac{R^2}{4L^2} - \dfrac{1}{LC}} = 4 \end{cases}$$

$R = 20$ | $R = 10$
$L = 5$ | $L = 1$
$C = 1/20$ | $C = 1/9$



APPENDIX B

SYNTHESIS OF TWO-TERMINAL NETWORKS BY VOLTAGE RESPONSE METHODS

Four problems, illustrating the voltage response synthesis technique are solved in this appendix. The networks obtained, in the all-series configuration, are all of the R-L-C type. They illustrate configurations of single elements in series with various R-C, R-L, and R-L-C series groups. Three of the problems illustrate the Type A impedance function wherein the identification and evaluation of the pure inductive series element is made directly from the defining complex impedance function.

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(1.) $$\dot{Z}(p) = \frac{2p^3 + 41p^2 + 34p + 80}{p^2 + 20p}$$

Type A; L = 2

$$\frac{\dot{Z}(p)}{p} = \frac{2p^3 + 41p^2 + 34p + 80}{p^2(p + 20)}$$

$$\frac{P(p)}{Q(p)} = \frac{\dot{Z}(p)}{p}\,\epsilon^{pt} = \frac{(2p^3 + 41p^2 + 34p + 80)\,\epsilon^{pt}}{p^3 + 20p^2}$$

$P(0) = 80$

$P'(0) = 80t + 34$

$Q''(0) = 40$

$Q'''(0) = 6$

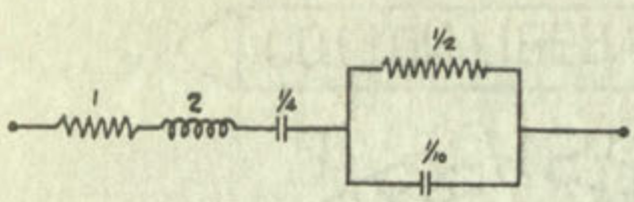
$$\frac{6P'Q'' - 2PQ'''}{3(Q'')^2} = 4t + 3/2$$

$$\frac{P(p)}{Q'(p)} = \frac{(2p^3 + 41p^2 + 34p + 80)\,\epsilon^{pt}}{3p^2 + 40p}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-20} = -\frac{1}{2}\,\epsilon^{-20t}$$

$$J(t) = 3/2 + 4t - 1/2\,\epsilon^{-20t}$$

$$= 1 + \frac{t}{1/4} + 1/2(1 - \epsilon^{-20t})$$



(2) $\dot{Z}(p) = \dfrac{2p^3 + 46p^2 + 253p + 160}{p^2 + 23p + 120}$

Type A; L = 2

$$\frac{\dot{Z}(p)}{p} = \frac{2p^3 + 46p^2 + 253p + 160}{p(p+8)(p+15)}$$

$$\frac{P(p)}{Q(p)} = \frac{\dot{Z}(p)}{p}\,\epsilon^{pt} = \frac{(2p^3 + 46p^2 + 253p + 160)\,\epsilon^{pt}}{p^3 + 23p^2 + 120p}$$

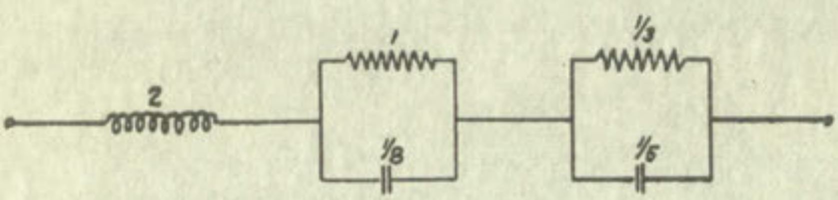
$$\frac{P(p)}{Q'(p)} = \frac{2p^3 + 46p^2 + 253p + 160)\,\epsilon^{pt}}{3p^2 + 46p + 120}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=0} = \frac{4}{3}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-8} = -\epsilon^{-8t}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-15} = -\frac{1}{3}\,\epsilon^{-15t}$$

$$J(t) = 4/3 - \epsilon^{-8t} - 1/3\,\epsilon^{-15t} = (1 - \epsilon^{-8t}) + 1/3\,(1 - \epsilon^{-15t})$$



(3) $\dot{Z}(p) = \dfrac{7p^2 + 45p + 60}{2p^2 + 10p + 12}$

$$\frac{\dot{Z}(p)}{p} = \frac{7p^2 + 45p + 60}{2p(p+2)(p+3)}$$

$$\frac{P(p)}{Q(p)} = \frac{\dot{Z}(p)}{p}\,\varepsilon^{pt} = \frac{(7p^2 + 45p + 60)\,\varepsilon^{pt}}{2p^3 + 10p^2 + 12p}$$

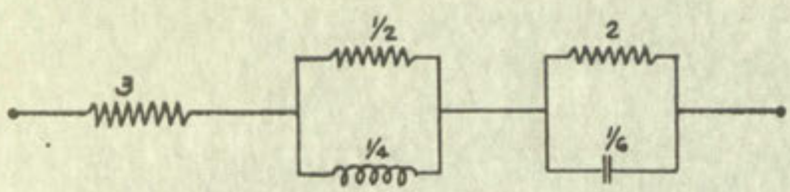
$$\frac{P(p)}{Q'(p)} = \frac{(7p^2 + 45p + 60)\,\varepsilon^{pt}}{6p^2 + 20p + 12}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=0} = 5$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-2} = \frac{1}{2}\,\varepsilon^{-2t}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-3} = -2\,\varepsilon^{-3t}$$

$$J(t) = 5 + \tfrac{1}{2}\varepsilon^{-2t} - 2\varepsilon^{-3t} = 3 + \tfrac{1}{2}\varepsilon^{-2t} + 2(1 - \varepsilon^{-3t})$$



$$(4)\quad \dot{Z}(p) = \frac{p^6 + 6p^5 + 40p^4 + 158p^3 + 351p^2 + 520p + 300}{2p^5 + 12p^4 + 32p^3 + 52p^2 + 30p}$$

Type A ; L = ½

$$\frac{\dot{Z}(p)}{p} = \frac{p^6 + 6p^5 + 40p^4 + 158p^3 + 351p^2 + 520p + 300}{2p^2(p+1)(p+3)(p+1+j2)(p+1-j2)}$$

$$\frac{P(p)}{Q(p)} = \frac{Z(p)}{p}\,\varepsilon^{pt} = \frac{(p^6 + 6p^5 + 40p^4 + 158p^3 + 351p^2 + 520p + 300)\,\varepsilon^{pt}}{2p^6 + 12p^5 + 32p^4 + 52p^3 + 30p^2}$$

$P(0) = 300$

$P'(0) = 300t + 520$

$Q''(0) = 60$

$Q'''(0) = 312$

$$\frac{6P'Q'' - 2PQ'''}{3(Q'')^2} = 10t$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-1} = \frac{1}{2}\,\varepsilon^{-t}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-3} = -\frac{1}{2}\,\varepsilon^{-3t}$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-1+j2} = -\frac{j}{4}\,\varepsilon^{-t}\,\varepsilon^{j2t}$$

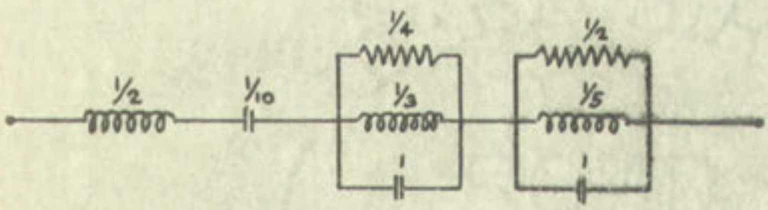
$$\left.\frac{P(p)}{Q'(p)}\right|_{p=-1-j2} = \frac{j}{4}\,\varepsilon^{-t}\,\varepsilon^{-j2t}$$

$$J(t) = \frac{t}{1/10} + \frac{1}{2}\epsilon^{-t} - \frac{1}{2}\epsilon^{-3t} + \frac{\epsilon^{-t}\epsilon^{j2t}}{j4} - \frac{\epsilon^{-t}\epsilon^{-j2t}}{j4}$$

$$= \frac{t}{1/10} + \frac{\epsilon^{-t} - \epsilon^{-3t}}{2} + \frac{\epsilon^{-t}}{2}\left[\frac{\epsilon^{j2t} - \epsilon^{-j2t}}{2j}\right]$$

$$\doteq \frac{t}{1/10} + \epsilon^{-2t}\left[\frac{\epsilon^{t} - \epsilon^{-t}}{2}\right] + \frac{\epsilon^{-t}}{2}\sin(2t)$$

$$= \frac{t}{1/10} + \epsilon^{-2t}\sinh(t) + \frac{\epsilon^{-t}}{2}\sin(2t)$$

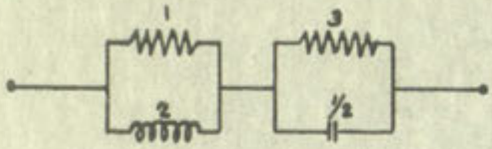


APPENDIX C

SOLUTION OF PROBLEMS INVOLVING EQUIVALENT NETWORKS AND PRESPECIFICATION OF NETWORK ELEMENTS

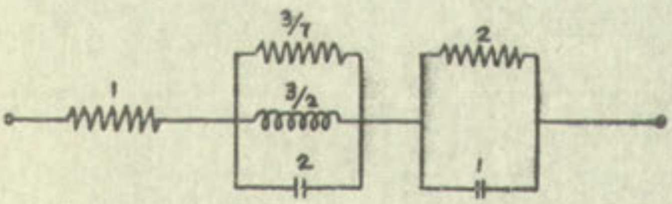
Two problems illustrating the specification of equivalent networks in the all-series configuration and the pre-specification of network elements for the all-series network are solved in this Appendix. Problem (2) illustrates the ease with which the response synthesis procedure handles the problem of network element prespecification.

(1.)



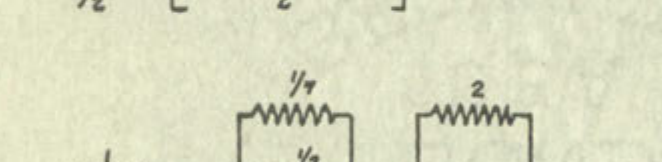
Parent Network

$$
\begin{aligned}
J(t) &= \epsilon^{-t/2} + 3(1 - \epsilon^{-2t/3}) \\
&= \epsilon^{-t/2} + 3 - 3\epsilon^{-2t/3} \\
&= 3 + 3\epsilon^{-t/2} - 3\epsilon^{-2t/3} - 2\epsilon^{-t/2} \\
&= 1 + 3\epsilon^{-7t/12}(\epsilon^{t/12} - \epsilon^{-t/12}) + 2 - 2\epsilon^{-t/2} \\
&= 1 + \frac{\epsilon^{-7t/12}}{1/6}\left[\frac{\epsilon^{t/12} - \epsilon^{-t/12}}{2}\right] + 2(1 - \epsilon^{-t/2})
\end{aligned}
$$



Equivalent Network

$$
\begin{aligned}
J(t) &= \epsilon^{-t/2} + 3 - 3\epsilon^{-2t/3} \\
&= 3 + \epsilon^{-t/2} - \epsilon^{-2t/3} - 2\epsilon^{-2t/3} \\
&= 1 + \epsilon^{-7t/12}(\epsilon^{t/12} - \epsilon^{-t/12}) + 2(1 - \epsilon^{-2t/3}) \\
&= 1 + \frac{\epsilon^{-7t/12}}{1/2}\left[\frac{\epsilon^{t/12} - \epsilon^{-t/12}}{2}\right] + 2(1 - \epsilon^{-2t/3})
\end{aligned}
$$



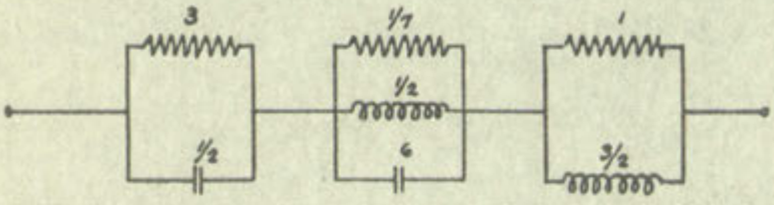
Equivalent Network

$$J(t) = \varepsilon^{-t/2} + 3 - 3\varepsilon^{-2t/3}$$

$$= 3 - 3\varepsilon^{-2t/3} + \varepsilon^{-t/2} - \varepsilon^{-2t/3} + \varepsilon^{-2t/3}$$

$$= 3(1 - \varepsilon^{-2t/3}) + \varepsilon^{-7t/12}(\varepsilon^{t/12} - \varepsilon^{-t/12}) + \varepsilon^{-2t/3}$$

$$= 3(1 - \varepsilon^{-2t/3}) + \frac{\varepsilon^{-7t/12}}{1/2}\left[\frac{\varepsilon^{t/12} - \varepsilon^{-t/12}}{2}\right] + \varepsilon^{-2t/3}$$



Equivolent Network

(2) Design an all-series type, two terminal network having a complex impedance of $\dot{Z}(p) = 60 + j0$ at all frequencies and containing a three element R-L-C group in which $R_{LC} = 5$, $L_{RC} = 75/2$, and $C_{RL} = 1/3$.

For the R-L-C group:

$$K = \sqrt{\frac{1}{4R^2C^2} - \frac{1}{LC}} = \sqrt{\frac{1}{4 \cdot 25 \cdot 1/9} - \frac{1}{75/2 \cdot 1/3}} = \sqrt{\frac{9}{100} - \frac{8}{100}} = \frac{1}{10}$$

$$KC = (1/10)(1/3) = 1/30$$

$$\frac{1}{2RC} = \frac{1}{10/3} = 3/10$$

$$\frac{\epsilon^{-t/2RC}}{KC} \sinh(Kt) = \frac{\epsilon^{-3t/10}}{1/30}\left[\frac{\epsilon^{t/10} - \epsilon^{-t/10}}{2}\right] =$$

$$15\,\epsilon^{-3t/10}(\epsilon^{t/10} - \epsilon^{-t/10}) = 15\,\epsilon^{-t/5} - 15\,\epsilon^{-2t/5}$$

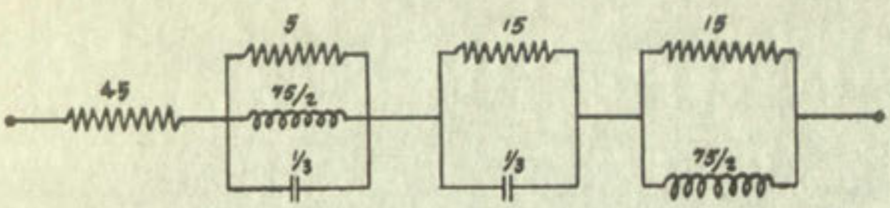
For the total network:

$$J(t) = 60 =$$

$$60 + (15\,\epsilon^{-t/5} - 15\,\epsilon^{-2t/5}) - 15\,\epsilon^{-t/5} + 15\,\epsilon^{-2t/5} =$$

$$45 + (15\,\epsilon^{-t/5} - 15\,\epsilon^{-2t/5}) + 15 - 15\,\epsilon^{-t/5} + 15\,\epsilon^{-2t/5} =$$

$$45 + \frac{\epsilon^{-3t/10}}{1/30}\sinh(t/10) + 15(1 - \epsilon^{-t/5}) + 15\,\epsilon^{-2t/5}$$



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