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Upon the Asymptotic Representation of Certain Entire Functions in Distant Portions of the Plane

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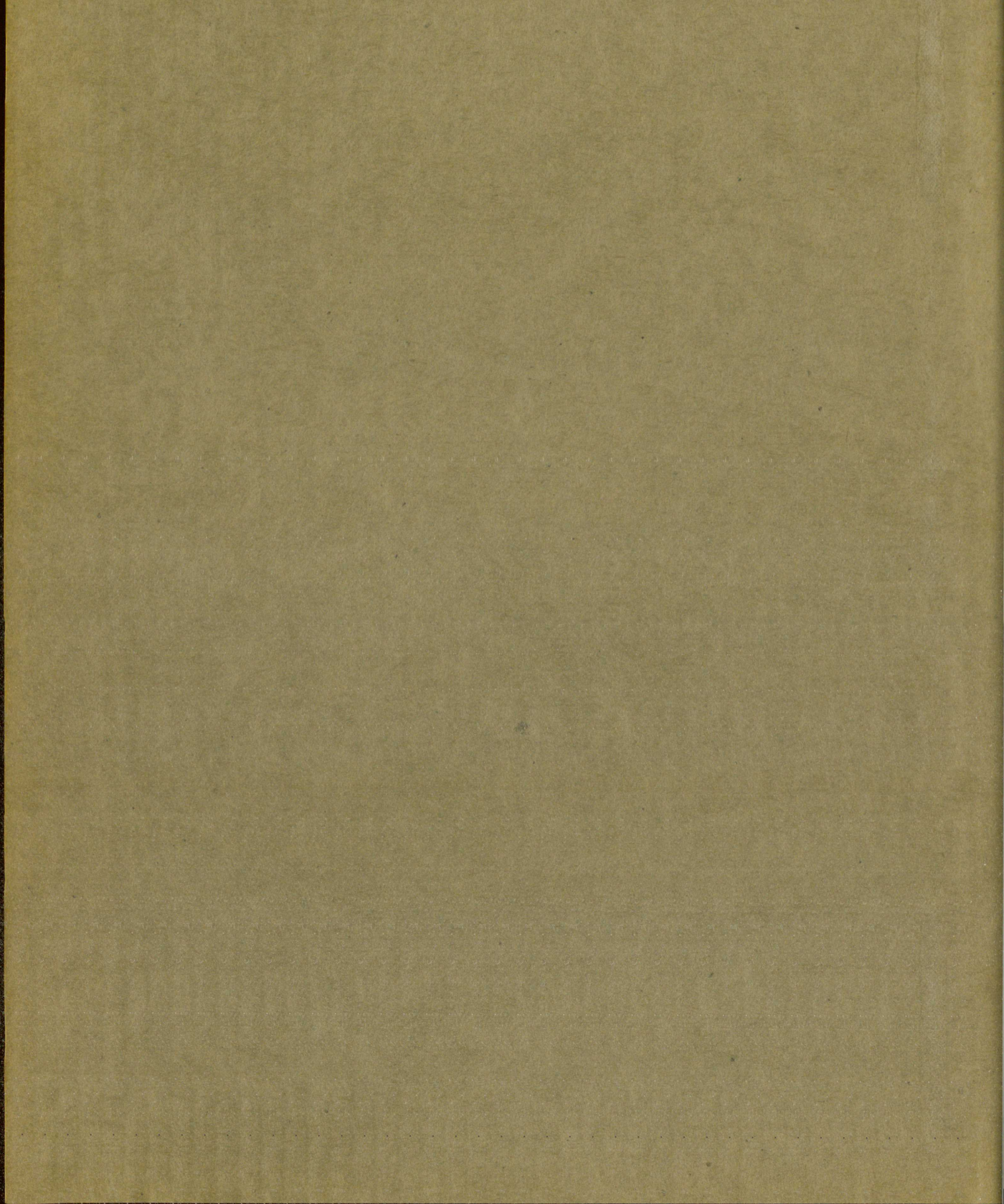


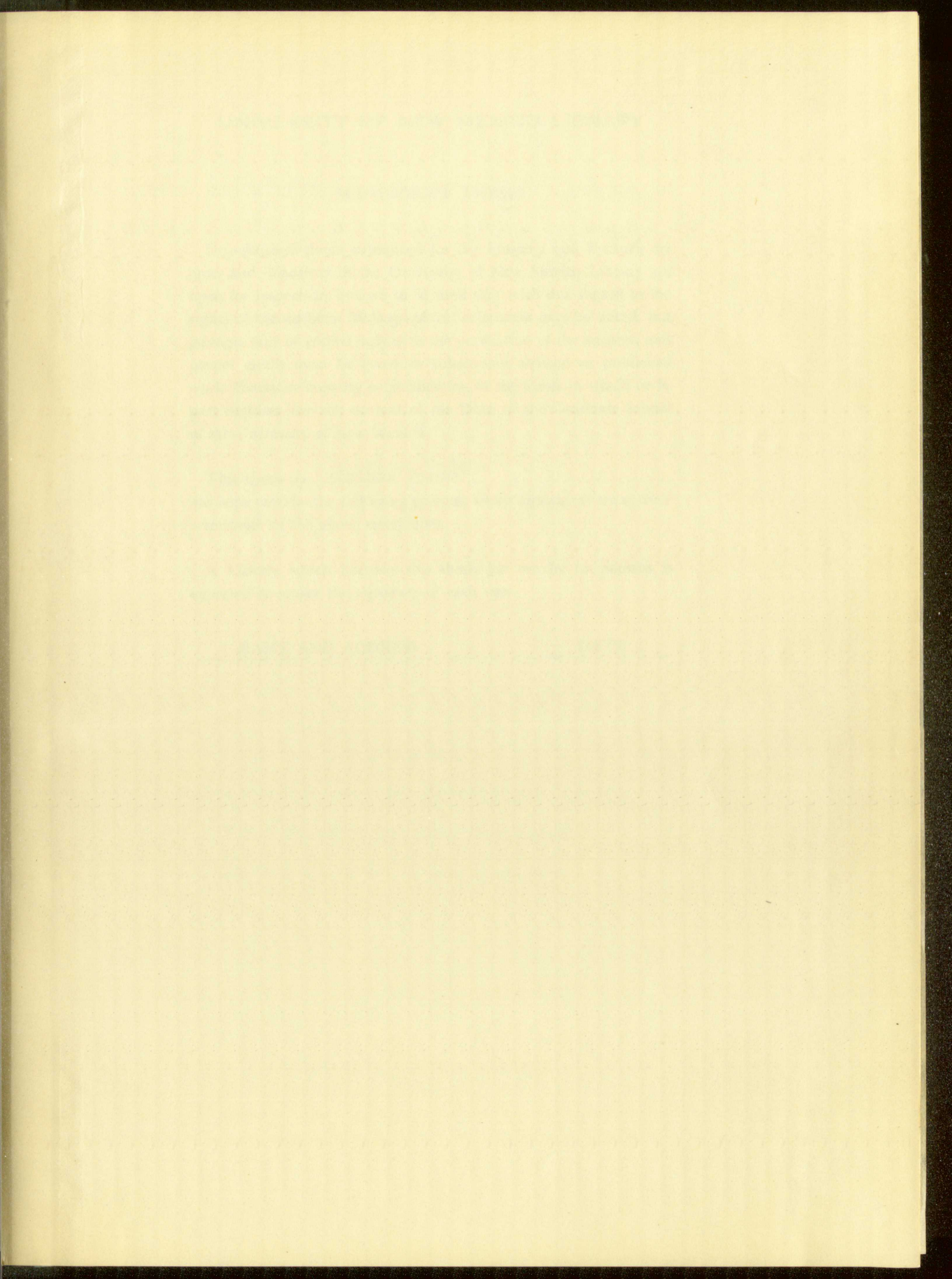
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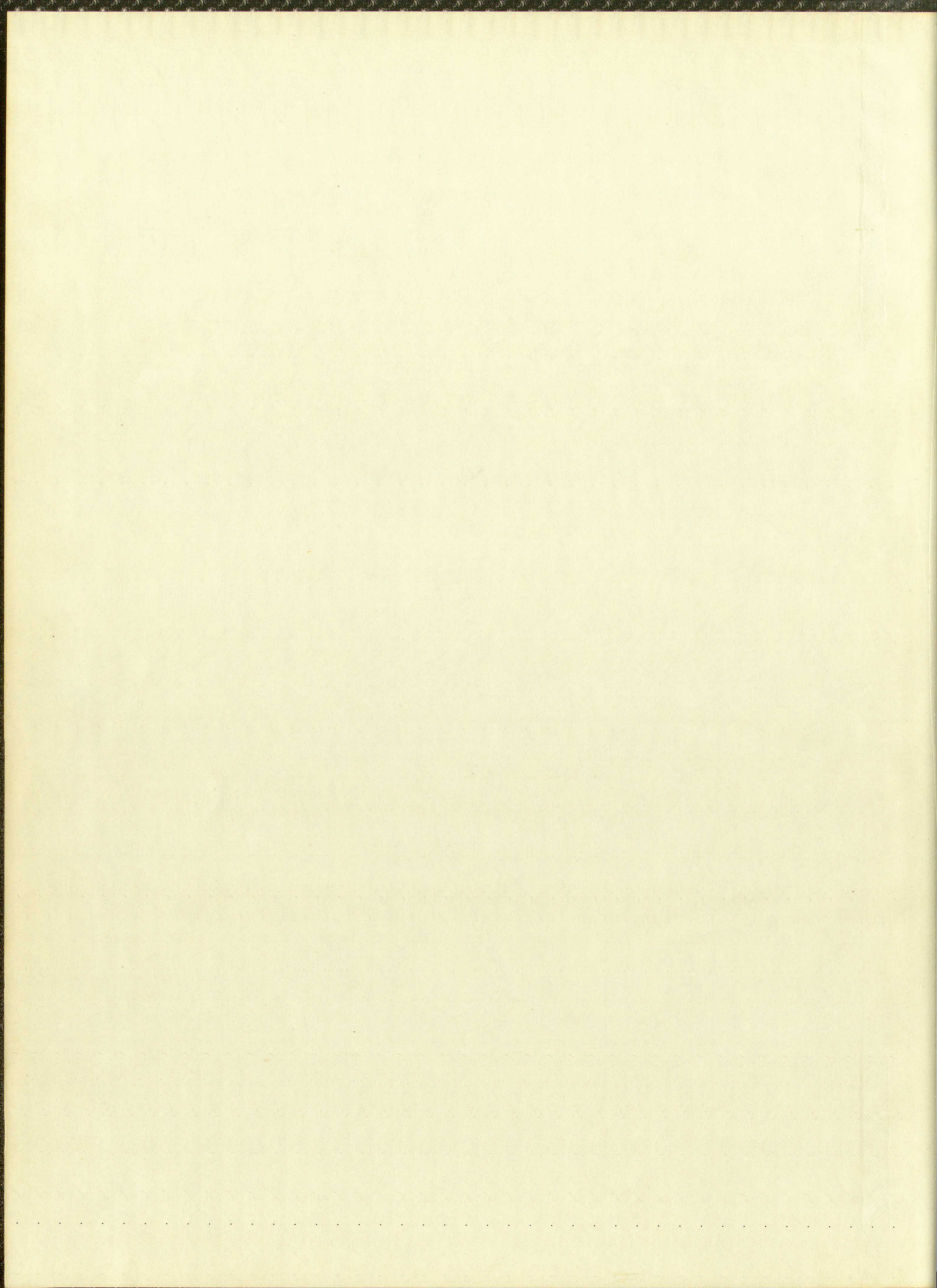
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UPON THE ASYMPTOTIC REPRESENTATION OF CERTAIN
ENTIRE FUNCTIONS IN DISTANT PORTIONS OF THE PLANE

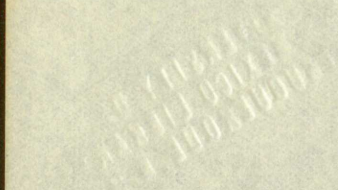
By

Abraham Franck

A Thesis

Submitted in partial fulfillment of the
Requirements for the Degree of
Master of Arts in Mathematics

University of New Mexico
1940



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May 21, 1940
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For the kindly guidance and constructive criticism which helped in the completion of this work, the author is deeply grateful to Dr. C. V. Newsom.

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TABLE OF CONTENTS

CHAPTER	PAGE
INTRODUCTION	i
I. FUNCTIONS OF THE BESSEL TYPE	1
II. FUNCTIONS OF THE DOUBLE BESSEL TYPE. . . .	14
BIBLIOGRAPHY	24

CHAPTER

INTRODUCTION

I. ...

II. ...

BIBLIOGRAPHY

INTRODUCTION

Probably the beginnings of the study of asymptotic series can be traced back to the seventeenth century mathematician, Leonard Euler. The first formal definition of an asymptotic representation was enunciated by Poincare, and has been adopted by later mathematicians.

Since the time of Poincare, notable advances have been made in this field by many men. The Calculus of Residues, as an instrument in the study, has been employed by Barnes, Ford, Newsom, and Van Engen.

The purpose of this paper is to study two particular entire functions which satisfy the conditions set up in a theorem due to Newsom. It is to be hoped that this report may be preliminary to the invention of a method which will lend itself toward the solution of certain general problems.

INTRODUCTION

Probably the beginning of the study of asymptotic series can be traced back to the eighteenth century mathematicians, Leonard Euler. The first formal definition of an asymptotic representation was introduced by Poincaré, and has been adopted by later mathematicians.

Since the time of Poincaré, notable advances have been made in this field by many men. The Calculus of Residues, as an instrument in the study, has been employed by Barnes, Ford, Newman, and Van Wijngaert. The purpose of this paper is to study two earlier entire functions which satisfy the conditions set up in a theorem due to Newman. It is to be hoped that this report may be preliminary to the discovery of a method which will lead itself toward the solution of certain general problems.

CHAPTER I

FUNCTIONS OF THE BESSEL TYPE

Recently Newsom proved a theorem pertaining to the behavior of certain entire functions in distant portions of the plane, namely,

Theorem. Let it be assumed that the coefficient $g(n)$ occurring in the general term of the power series,

$$(1) \quad \sum_{n=0}^{\infty} g(n) z^n \quad ; \text{ radius of convergence } = \infty$$

may be regarded as a function $g(w)$ of the complex variable $w = x + iy$ and as such satisfies the following two conditions:

- (a) is single valued and analytic throughout the finite w -plane,
- (b) is such that for all values of x and y one may write

$$(2) \quad |g(x+iy)| < K e^{k\pi|y|}$$

where K is a constant independent of x and y , and k is any given positive integer.

Then the function $f(z)$ defined by series (1) when considered for all values of z satisfying the

THEOREM 1 Let $f(x)$ be a function defined on the interval $[a, b]$ and let ξ be a point in this interval. Then the function $f(x)$ is continuous at ξ if and only if

for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all x in the interval $(\xi - \delta, \xi + \delta)$ we have $|f(x) - f(\xi)| < \epsilon$. This condition is satisfied if and only if the function $f(x)$ has a unique limit as x approaches ξ . In this case the limit is called the limit of the function at ξ and is denoted by $\lim_{x \rightarrow \xi} f(x)$.

$$\lim_{x \rightarrow \xi} f(x) = L \quad (1)$$

where L is a real number. This condition is satisfied if and only if the function $f(x)$ has a unique limit as x approaches ξ . In this case the limit is called the limit of the function at ξ and is denoted by $\lim_{x \rightarrow \xi} f(x)$.

(a) The function $f(x)$ is continuous at ξ if and only if the function $f(x)$ has a unique limit as x approaches ξ . In this case the limit is called the limit of the function at ξ and is denoted by $\lim_{x \rightarrow \xi} f(x)$.

(b) The function $f(x)$ is continuous at ξ if and only if the function $f(x)$ has a unique limit as x approaches ξ . In this case the limit is called the limit of the function at ξ and is denoted by $\lim_{x \rightarrow \xi} f(x)$.

$$|f(x) - f(\xi)| < K \quad (2)$$

where K is a constant. This condition is satisfied if and only if the function $f(x)$ has a unique limit as x approaches ξ . In this case the limit is called the limit of the function at ξ and is denoted by $\lim_{x \rightarrow \xi} f(x)$.

Thus the function $f(x)$ is continuous at ξ if and only if the function $f(x)$ has a unique limit as x approaches ξ . In this case the limit is called the limit of the function at ξ and is denoted by $\lim_{x \rightarrow \xi} f(x)$.

condition, $-\pi < \arg \pm z < \pi$, may be expressed in the form

$$(3) \quad f(z) = \int_{-l-\frac{1}{2}}^{\infty} \left\{ g(x) [\pm z]^x \frac{\sin k\pi x}{\sin \pi x} \right\} dx \\ - \sum_{m=-l}^{-1} g(m) z^m + f_k(l, z);$$

wherein l is any arbitrary positive integer and the upper or lower of the signs \pm is to be taken according as k is odd or even. Moreover, the expression $f_k(l, z)$ is such that

$$(4) \quad \lim_{|z| \rightarrow \infty} z^l f_k(l, z) = 0,$$

irrespective of the value chosen for l .¹

The factor of negative unity associated with z in the integrand is taken as $\cos \pi + i \sin \pi$ in the first part of this study. Comments are made later in regard to the use of other possible representations of -1 .

This paper is concerned with the behavior of

$$f(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{\Gamma(n+k_1) \Gamma(n+k_2)}$$

1

C.V. Newsom, "On the Character of Certain Entire Functions in Distant Portions of the Plane," American Journal of Mathematics, Vol. LX, No. 3, (July, 1938), pp. 561-572.

when $|z|$ becomes large. Thus, the coefficient $g(n)$ takes the form

$$g(n) = \frac{1}{\Gamma(n+k_1)\Gamma(n+k_2)}$$

where k_1 and k_2 are complex constants, and z^2 plays the same role as z in the theorem above. It is apparent that the theorem is applicable to this case if k is chosen as 2.²

It follows that

$$(5) \quad f(z) = \int_{-l-\frac{1}{2}}^{\infty} \left\{ g(x) [z^2 (\cos \pi x + i \sin \pi x)]^x \frac{\sin 2\pi x}{\sin \pi x} \right\} dx \\ - \sum_{m=-l}^{-1} g(m) z^{2m} + f(l, z);$$

where $f(l, z)$ possesses the property (4), and $-\pi < \arg z < 0$.

The integral involved in this relationship may be rewritten as

$$(6) \quad \int_{-l-\frac{1}{2}}^{\infty} \left\{ g(x) (\cos \pi x + i \sin \pi x) (2 \cos \pi x) z^{2x} \right\} dx \\ = \int_{-l-\frac{1}{2}}^{\infty} \left\{ g(x) z^{2x} [1 + (\cos 2\pi x + i \sin 2\pi x)] \right\} dx$$

Note Walter B. Ford, The Asymptotic Developments of Functions Defined by MacLaurin Series, (Ann Arbor, Michigan, University of Michigan Press, 1936), p. 61.

When $1 \leq k$ becomes large, the form

$$f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{ixt} dt$$

where A and B are constants, and

the same rule as in the case of $f(x)$.

that the theorem is valid for $f(x)$ as well as for $f(x)$ as in the case of $f(x)$.

It follows that

$$(a) \quad f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{ixt} dt$$

$$f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{ixt} dt$$

where $f(x)$ is a function of x .

$$-H < \pi < H$$

The integral $\int_{-\pi}^{\pi} f(t) e^{ixt} dt$ is a function of x .

be written as

$$(b) \quad \int_{-\pi}^{\pi} f(t) e^{ixt} dt = \int_{-\pi}^{\pi} f(t) e^{ixt} dt$$

$$= \int_{-\pi}^{\pi} f(t) e^{ixt} dt$$

For $f(x)$ is a function of x and $f(x)$ is a function of x .

$$= \int_{-l-\frac{1}{2}}^{\infty} \{g(x)[z^{2x} + z'^{2x}]\} dx,$$

where $\arg z' = \arg z + \pi$.

In view of the nature of $g(x)$, the existence of the integrals,

$$\int_{-l-\frac{1}{2}}^{\infty} g(x) z^{2x} dx \quad \text{and} \quad \int_{-l-\frac{1}{2}}^{\infty} g(x) z'^{2x} dx,$$

is readily demonstrable for any finite z .

Thus

$$(7) \quad f(z) = \int_{-l-\frac{1}{2}}^{\infty} \frac{z^{2x} dx}{\Gamma(x+k_1)\Gamma(x+k_2)} + \int_{-l-\frac{1}{2}}^{\infty} \frac{z'^{2x} dx}{\Gamma(x+k_1)\Gamma(x+k_2)} \\ - \sum_{m=-l}^{-1} g(m) z^{2m} + f(l, z);$$

where $f(l, z)$ has the property (4), $-\pi < \arg z < 0$, and $\arg z' = \arg z + \pi$.

The first integral of (7) may be re-expressed as

$$(8) \quad \int_{-l-\frac{1}{2}}^{\infty} \left[\frac{\Gamma(2x+2p)}{\Gamma(x+k_1)\Gamma(x+k_2)} \cdot \frac{z^{2x}}{\Gamma(2x+2p)} \right] dx \\ = \frac{2^{2p-1}}{\sqrt{\pi}} \int_{-l-\frac{1}{2}}^{\infty} \left[\frac{\Gamma(x+p)\Gamma(x+p+\frac{1}{2})u^{2x}}{\Gamma(x+k_1)\Gamma(x+k_2)\Gamma(2x+2p)} \right] dx,$$

if it is recalled that

$$\Gamma(2y) = \frac{2^{2y-1}}{\sqrt{\pi}} \Gamma(y)\Gamma(y+\frac{1}{2})$$

and if $u = 2z$. Of course, it should be noted at this point that $-\pi < \arg u < 0$. Upon making the transformation $x = \frac{1}{2}(x' + 1) - p$, and dropping the primes, the

$$= \int_{-\frac{1}{2}}^{\infty} \{g(x) - h(x)\} dx$$

where $g(x) = \frac{1}{2}x^2$ and

in view of the fact that

the integrals

$$\int_{-\frac{1}{2}}^{\infty} g(x) dx \quad \text{and} \quad \int_{-\frac{1}{2}}^{\infty} h(x) dx$$

is readily determined that

Thus

$$(7) \quad \{g(x) - h(x)\} = \frac{1}{2}x^2 - \frac{1}{2}x^2 = 0$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2}$$

where $\{g(x) - h(x)\}$ and

and $g(x) = \frac{1}{2}x^2$ and

The first integral

$$(8) \quad \int_{-\frac{1}{2}}^{\infty} \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_{-\frac{1}{2}}^{\infty} = \frac{1}{\frac{1}{2}} = 2$$

$$= \frac{1}{\sqrt{x}} \int_{-\frac{1}{2}}^{\infty} \frac{1}{x^{\frac{3}{2}}} dx = \left[-\frac{2}{\sqrt{x}} \right]_{-\frac{1}{2}}^{\infty} = \frac{2}{\sqrt{\frac{1}{2}}} = 2\sqrt{2}$$

It is readily seen

$$T(x) = \frac{1}{2}x^2$$

and $T(x) = \frac{1}{2}x^2$ and

point that $T(x) = \frac{1}{2}x^2$ and

then $x = \frac{1}{2}x^2 + \frac{1}{2}x^2 = x^2$

integral of (8) becomes

$$(9) \quad \frac{1}{2\sqrt{\pi}} \left(\frac{u}{2}\right)^{1-2p} \int_{-2l+2p-2}^{\infty} \frac{F(x) u^x dx}{\Gamma(x+1)}$$

where

$$F(x) = \frac{\Gamma\left(\frac{x+1}{2}\right) \Gamma\left(\frac{x+2}{2}\right)}{\Gamma\left(\frac{x+2k_1-2p+1}{2}\right) \Gamma\left(\frac{x+2k_2-2p+1}{2}\right)}.$$

Up to this point the constant p has been arbitrary. If p be assigned the value,

$$(10) \quad p = \frac{2k_1 + 2k_2 - 1}{4}$$

then it follows that

$$(11) \quad F(x) = 1 + \sum_{n=1}^s \frac{c_n}{(x+1)(x+2)\cdots(x+n)} + \frac{\delta(x, s)}{(x+1)(x+2)\cdots(x+s)},$$

where $\lim_{x \rightarrow \infty} \delta(x, s) = 0$.³

Therefore, the integral of (9) immediately takes the form,

$$(12) \quad \frac{1}{2\sqrt{\pi}} \left(\frac{u}{2}\right)^{1-2p} \left\{ \sum_{n=0}^s c_n \int_{-2l+2p-2}^{\infty} \frac{u^x dx}{\Gamma(x+n+1)} + \int_{-2l+2p-2}^{\infty} \frac{\delta(x, s) u^x dx}{\Gamma(x+s+1)} \right\},$$

³

Note Henry Van Engen, "Concerning Gamma Function Expansions," Tohoku Mathematical Journal, Vol. 45, Part I (September, 1938), pp. 124-129.

$$(9) \quad \frac{1}{2\sqrt{z}} \left(\frac{w}{z} \right) \left\{ \frac{F_1(w, z)}{F_2(w, z)} \right\}$$

where

$$F(w, z) = \frac{F_1(w, z)}{F_2(w, z)}$$

$$(10) \quad p = \frac{2\sqrt{z} - 1}{z}$$

then it follows that

$$(11) \quad F(z) = 1 + \sum_{k=1}^{\infty} \frac{F_k(z)}{F_0(z)}$$

where $F_k(z) = 0$
 $k \geq 2$

Therefore, we have
 the following

$$(12) \quad \frac{1}{2\sqrt{z}} \left(\frac{w}{z} \right) \left\{ \sum_{k=1}^{\infty} \frac{F_k(w, z)}{F_0(w, z)} \right\}$$

$$+ \sum_{k=1}^{\infty} \frac{F_k(w, z)}{F_0(w, z)}$$

It is easy to see that
 the above series is
 convergent for $|z| < 1$

where $c_0 = 1$, $\delta(x, s)$ has the property given in (11), and $-\pi < \arg u < 0$.

If a similar study be made upon the second integral of (7), the integral becomes

$$(13) \quad \frac{1}{2\sqrt{\pi}} \left(\frac{u'}{2}\right)^{1-2p} \left\{ \sum_{n=0}^s c_n \int_{-2l+2p-2}^{\infty} \frac{u'^x dx}{\Gamma(x+n+1)} + \int_{-2l+2p-2}^{\infty} \frac{\delta(x, s) u'^x dx}{\Gamma(x+s+1)} \right\},$$

where $c_0 = 1$, $\delta(x, s)$ has the property given in (11), and $\arg u' = \arg u + \pi$.

After making obvious transformations upon the integrals of (12) and (13), we may summarize the results by rewriting (7) as

$$(14) \quad \varphi(u) = \frac{1}{2\sqrt{\pi}} \left(\frac{u}{2}\right)^{1-2p} \left\{ \sum_{n=0}^s \frac{c_n}{u^n} \int_{-2l+2p+n-2}^{\infty} \frac{u^x dx}{\Gamma(x+1)} + \frac{1}{u^s} \int_{-2l+2p+s-2}^{\infty} \frac{\delta(x-s, s) u^x dx}{\Gamma(x+1)} \right\} + \frac{1}{2\sqrt{\pi}} \left(\frac{u'}{2}\right)^{1-2p} \left\{ \sum_{n=0}^s \frac{c_n}{u'^n} \int_{-2l+2p+n-2}^{\infty} \frac{u'^x dx}{\Gamma(x+1)} + \frac{1}{u'^s} \int_{-2l+2p+s-2}^{\infty} \frac{\delta(x-s, s) u'^x dx}{\Gamma(x+1)} \right\} - \sum_{m=-l}^{-1} g(m) \left(\frac{u}{2}\right)^{2m} + f_1(l, u);$$

wherein

$$(15) \quad \lim_{|u| \rightarrow \infty} u^l f_1(l, u) = 0, \quad \lim_{x \rightarrow \infty} \delta(x-s, s) = 0, \\ -\pi < \arg u < 0, \quad \text{and } \arg u' = \arg u + \pi.$$

where $\phi = 1, 2, \dots, n$ and $-\pi < \arg \phi < 0$.

If a similar result is obtained for $\phi = 1, 2, \dots, n$ then

$$(13) \quad \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z} dz = \sum_{\phi=1}^n \left\{ \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z} dz \right\} \quad \text{where } \phi = 1, 2, \dots, n$$

$$\int_{\gamma} \frac{f(z)}{z} dz = \int_{\gamma} \frac{f(z)}{z} dz$$

where $\phi = 1, 2, \dots, n$ and $-\pi < \arg \phi < 0$.

After making use of the result obtained in (13) and (14) we have

$$(14) \quad \phi(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z} dz = \sum_{\phi=1}^n \left\{ \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z} dz \right\}$$

$$= \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z} dz = \sum_{\phi=1}^n \left\{ \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z} dz \right\}$$

$$= \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z} dz = \sum_{\phi=1}^n \left\{ \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z} dz \right\}$$

wherein

$$(15) \quad \phi(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z} dz = \sum_{\phi=1}^n \left\{ \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z} dz \right\}$$

Employing an argument⁴ similar to that of Ford, let L represent the largest integer in $2l-2R(p)-n+2$, wherein l has been taken so large that $2l-2R(p)-n+2 > S$; $n = 0, 1, 2, \dots s$. Since

$$2l-2R(p)-n+2 \geq 2l-2R(p)-s+2$$

the desired condition will be realized if

$$(16) \quad l > R(p) + s - 1.$$

Under such a condition upon l as given above, and with the arguments of u and u' restricted as heretofore specified, it follows that

$$\int_{-2l+2p+n-2}^{\infty} \frac{u^x dx}{T(x+1)} = e^u - \Delta_n(u, L)$$

and

$$\int_{-2l+2p+n-2}^{\infty} \frac{u'^x dx}{T(x+1)} = e^{u'} - \Delta_n(u', L)$$

where

$$(17) \quad \lim_{|u| \rightarrow \infty} u^n \Delta_n(u, L) = 0 \quad \text{and} \quad \lim_{|u'| \rightarrow \infty} u'^n \Delta_n(u', L) = 0;$$

$n = 0, 1, 2, \dots s$. It is to be understood that the path of integration consists of the line drawn from the

Let \mathcal{L} be a linear operator on V and let \mathcal{L}^* be its adjoint. Then \mathcal{L}^* is defined by the relation

$$(\mathcal{L}^* f, g) = (f, \mathcal{L} g)$$

for all $f, g \in V$. The adjoint operator \mathcal{L}^* is also linear and satisfies $(\mathcal{L}^*)^* = \mathcal{L}$. If \mathcal{L} is self-adjoint, then $\mathcal{L} = \mathcal{L}^*$.

Under these conditions, the adjoint operator \mathcal{L}^* is also linear and satisfies $(\mathcal{L}^*)^* = \mathcal{L}$. If \mathcal{L} is self-adjoint, then $\mathcal{L} = \mathcal{L}^*$.

where \mathcal{L}^* is the adjoint operator of \mathcal{L} . The adjoint operator \mathcal{L}^* is also linear and satisfies $(\mathcal{L}^*)^* = \mathcal{L}$. If \mathcal{L} is self-adjoint, then $\mathcal{L} = \mathcal{L}^*$.

path of integration. The path of integration is a curve in the complex plane. The path of integration is a curve in the complex plane.

point $-2\ell+2p+n-2$ to infinity in the direction of the positive x -axis. We now have therefore,

$$\begin{aligned}
 (18) \quad \phi(u) = & \frac{e^u}{2\sqrt{\pi}} \left(\frac{u}{2}\right)^{1-2p} \sum_{n=0}^S \frac{c_n}{u^n} - \frac{1}{2\sqrt{\pi}} \left(\frac{u}{2}\right)^{1-2p} \sum_{n=0}^S \frac{c_n \Delta_n(u, L)}{u^n} \\
 & + \frac{1}{2\sqrt{\pi}} \left(\frac{u}{2}\right)^{1-2p} \frac{1}{u^s} \int_{-2\ell+2p+s-2}^{\infty} \frac{\delta(x-s, s) u^x dx}{\Gamma(x+1)} + \frac{e^{u'}}{2\sqrt{\pi}} \left(\frac{u'}{2}\right)^{1-2p} \sum_{n=0}^S \frac{c_n}{u'^n} \\
 & - \frac{1}{2\sqrt{\pi}} \left(\frac{u'}{2}\right)^{1-2p} \sum_{n=0}^S \frac{c_n \Delta_n(u', L)}{u'^n} + \frac{1}{2\sqrt{\pi}} \left(\frac{u'}{2}\right)^{1-2p} \frac{1}{u'^s} \int_{-2\ell+2p+s-2}^{\infty} \frac{\delta(x-s, s) u'^x dx}{\Gamma(x+1)} \\
 & - \sum_{m=-\ell}^{-1} g^{(m)} \left(\frac{u}{2}\right)^{2m} + \left\{ \right\}_1 (l, u).
 \end{aligned}$$

In studying this result it should be recalled that the properties (15) and (17) are applicable.

At this point in the analysis it is desirable to consider two sectors within the range of u , namely,

(Case I) $-\pi < \arg u < -\frac{\pi}{2}$ and (Case II) $-\frac{\pi}{2} < \arg u < 0$;

the situation where $\arg u = -\frac{\pi}{2}$ is excluded in this study.

Case I

The immediate problem in the consideration of this case pertains to the first integral of (18). This integral, ignoring the factor in front, may be readily changed to

points $-2\pi + 2\pi i$ and $2\pi + 2\pi i$ of the contour Γ are $z = 2\pi i$ and $z = -2\pi i$ respectively. The residues at these points are $\frac{1}{2\pi i}$ and $-\frac{1}{2\pi i}$ respectively. Hence

$$\begin{aligned} \int_{\Gamma} \frac{e^{iz}}{z} dz &= 2\pi i \left(\frac{1}{2\pi i} - \left(-\frac{1}{2\pi i}\right) \right) = 2\pi i \left(\frac{1}{2\pi i} + \frac{1}{2\pi i} \right) = 2\pi i \left(\frac{2}{2\pi i} \right) = 2\pi i \left(\frac{1}{\pi i} \right) = 2\pi i \left(-\frac{1}{\pi} \right) = -2 \\ \int_{\Gamma} \frac{e^{iz}}{z} dz &= \int_{-R}^{-r} \frac{e^{iz}}{z} dz + \int_{\gamma_r} \frac{e^{iz}}{z} dz + \int_r^R \frac{e^{iz}}{z} dz + \int_{\gamma_R} \frac{e^{iz}}{z} dz \end{aligned}$$

In passing to the limit $R \rightarrow \infty$ and $r \rightarrow 0$, the integrals over the large and small arcs γ_R and γ_r respectively, tend to zero. Hence

the integral over the real axis tends to $\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx$. Hence

$$(19) \quad \frac{1}{u} \int_{-2l+2p+s-1}^{\infty} \frac{\delta(x-s-1, s) u^x dx}{\Gamma(x)}$$

In view of the fact that $u = u'(\cos \pi + i \sin \pi)$, which implies that $0 < \arg u' < \frac{\pi}{2}$, the integral factor of (19) may be written as

$$(20) \quad I(u') = \int_{-2l+2p+s-1}^{\infty} \frac{P_1(x) u'^x dx}{\Gamma(x)} - i \int_{-2l+2p+s-1}^{\infty} \frac{P_2(x) u'^x dx}{\Gamma(x)}$$

where

$$P_1(x) = [\delta(x-s-1, s)] [\cos \pi x]$$

and

$$P_2(x) = [\delta(x-s-1, s)] [\sin \pi x].$$

Recalling relation (11), it is observed that

$$\delta(x, s) = A_1(x) + A_2(x) [(x+1)(x+2) \cdots (x+s)] \Gamma\left(\frac{x+1}{2}\right) \Gamma\left(\frac{x+2}{2}\right),$$

where $A_1(x)$ and $A_2(x)$ represent functions which are analytic in the finite portion of the plane. Thus

$$\delta(x, s) = A_1(x) + \frac{\sqrt{\pi} A_2(x) \Gamma(x+s+1)}{2^x},$$

since

$$\Gamma(2y) = \frac{2^{2y-1}}{\sqrt{\pi}} \Gamma(y) \Gamma\left(y + \frac{1}{2}\right)$$

and

$$\delta(x-s-1, s) = A_1(x-s-1) + \frac{\sqrt{\pi} A_2(x-s-1) \Gamma(x)}{2^{x-s-1}}$$

Hence it is readily observed that the integrands of

$$\frac{1}{4} \left(\frac{1}{\sqrt{1-x^2}} \right)$$

(19)

In this case, the function $f(x)$ is defined on the interval $(-\infty, \infty)$ and is continuous. The function $f(x)$ is defined by the equation

$$f(x) = \frac{1}{4} \left(\frac{1}{\sqrt{1-x^2}} \right)$$

where

$$f(x) = \frac{1}{4} \left(\frac{1}{\sqrt{1-x^2}} \right)$$

and

$$f(x) = \frac{1}{4} \left(\frac{1}{\sqrt{1-x^2}} \right)$$

Noting that

$$f(x) = \frac{1}{4} \left(\frac{1}{\sqrt{1-x^2}} \right)$$

where $f(x)$ is a function of x and $f(x)$ is a function of x

and $f(x)$ is a function of x

$$f(x) = \frac{1}{4} \left(\frac{1}{\sqrt{1-x^2}} \right)$$

since

$$f(x) = \frac{1}{4} \left(\frac{1}{\sqrt{1-x^2}} \right)$$

and

$$f(x) = \frac{1}{4} \left(\frac{1}{\sqrt{1-x^2}} \right)$$

Hence it is a function of x

(20) are bounded for any particular finite portion of the plane. Therefore

$$(21) \quad \lim_{|u| \rightarrow \infty} e^{-u} \int_{\alpha}^{\beta} \frac{P_1(x) u^x dx}{T(x)} = 0,$$

and

$$\lim_{|u| \rightarrow \infty} e^{-u} \int_{\alpha}^{\beta} \frac{P_2(x) u^x dx}{T(x)} = 0,$$

if α and β are chosen as any finite constants.

Also the integrals,

$$\int_0^{\infty} \frac{P_1(x) u^x dx}{T(x)} \quad \text{and} \quad \int_0^{\infty} \frac{P_2(x) u^x dx}{T(x)}$$

satisfy the conditions of a theorem due to Ford⁵ if $P_1(x)$ and $P_2(x)$ are properly defined in the vicinity of the origin; thus,

$$\lim_{|u| \rightarrow \infty} e^{-u} \int_0^{\infty} \frac{P_1(x) u^x dx}{T(x)} = 0,$$

and

$$\lim_{|u| \rightarrow \infty} e^{-u} \int_0^{\infty} \frac{P_2(x) u^x dx}{T(x)} = 0,$$

since $\lim_{x \rightarrow \infty} P_1(x) = 0$ and $\lim_{x \rightarrow \infty} P_2(x) = 0$.

5

Ford, op. cit., p. 38.

(12)

$$\frac{1000000}{1000000} = 1$$

and

$$\frac{1000000}{1000000} = 1$$

if x and y are

$$\frac{1000000}{1000000} = 1$$

entirely the same as the one in the previous section, and the only difference is that the order of the operations is reversed.

$$\frac{1000000}{1000000} = 1$$

and

$$\frac{1000000}{1000000} = 1$$

since the order of the operations is reversed, the result is the same as the one in the previous section.

$$\frac{1000000}{1000000} = 1$$

if x and y are

So, in summary,

$$(22) \quad \lim_{|u'| \rightarrow \infty} e^{-u'} I(u') = 0.$$

It is now possible to simplify relation (18) under the conditions of Case I as follows:

$$(23) \quad \begin{aligned} \varphi(u) &= \frac{e^u}{2\sqrt{\pi}} \left(\frac{u}{2}\right)^{1-2p} \sum_{n=0}^s \frac{c_n}{u^n} \\ &+ \frac{e^{u'}}{2\sqrt{\pi}} \left(\frac{u'}{2}\right)^{1-2p} \left[\sum_{n=0}^s \frac{c_n}{u'^n} + \frac{R(u', s)}{u'^s} \right], \end{aligned}$$

wherein $\lim_{|u'| = |u| \rightarrow \infty} R(u', s) = 0$, in view of the fact that

$$\begin{aligned} R(u', s) &= e^{-u'} \left[\left(\frac{u'}{2}\right)^{2p+s} \frac{I(u')}{u'} + \frac{1}{u'} \int_{-2l+2p+s-1}^{\infty} \frac{\delta(x-s-1, s) u'^x dx}{\Gamma(x)} \right. \\ &- \left(\frac{u'}{2}\right)^{2p-1} u'^s \sum_{n=0}^s \frac{c_n \Delta_n(u', L)}{u^n} - u'^s \sum_{n=0}^s \frac{c_n \Delta_n(u', L)}{u'^n} \\ &\left. - 2\sqrt{\pi} \left(\frac{u'}{2}\right)^{2p-1} u'^s \sum_{m=-1}^{-1} g^{(m)}\left(\frac{u'}{2}\right)^{2m} + 2\sqrt{\pi} \left(\frac{u'}{2}\right)^{2p-1} u'^s \left\{ (l, u) \right\} \right], \end{aligned}$$

where s is understood to be any arbitrarily large positive integer.

Using the symbol of asymptotic representation,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx \quad (22)$$

It is now easy to see that under the conditions

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^{\alpha} = \int_0^1 x^{\alpha} dx \quad (23)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^{\alpha} = \int_0^1 x^{\alpha} dx + \frac{1}{2\sqrt{n}}$$

where $\alpha > -1/2$ and $\alpha \neq -1/2$ then

$$R(n, \alpha) = \frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^{\alpha} - \int_0^1 x^{\alpha} dx = \frac{1}{2\sqrt{n}}$$

$$-\left(\frac{n}{2}\right)^{\alpha-1/2} \sum_{k=1}^n \frac{1}{k^{\alpha+1/2}} = \frac{1}{2\sqrt{n}}$$

$$-5\sqrt{n} \left(\frac{n}{2}\right)^{\alpha-1/2} \sum_{k=1}^n \frac{1}{k^{\alpha+1/2}} = \frac{1}{2\sqrt{n}}$$

where $\alpha > -1/2$ and $\alpha \neq -1/2$ then

positive integer

relation (23) may be written in the form,

$$(24) \quad \phi(u) \sim \frac{e^u}{2\sqrt{\pi}} \left(\frac{u}{2}\right)^{1-2\rho} \left[1 + \frac{c_1}{u} + \frac{c_2}{u^2} + \dots + \frac{c_n}{u^n} + \dots\right] \\ + \frac{e^{u'}}{2\sqrt{\pi}} \left(\frac{u'}{2}\right)^{1-2\rho} \left[1 + \frac{c_1}{u'} + \frac{c_2}{u'^2} + \dots + \frac{c_n}{u'^n} + \dots\right].$$

For Case II, where $-\frac{\pi}{2} < \arg u < 0$, the analysis is virtually identical with that employed in Case I. However, in considering Case II, the portion of (24) involving u becomes dominant instead of that involving u' as in the previous study. In truth, the result obtained in the second case is the same as (24).

Therefore, recalling that $u = 2z$ and that $u' = 2z' = -2z$, it follows that

$$(25) \quad f(z) \sim \frac{e^{zz} z^{\frac{3}{2} - (k_1 + k_2)}}{2\sqrt{\pi}} \left[1 + \frac{c_1}{2z} + \frac{c_2}{(2z)^2} + \dots + \frac{c_n}{(2z)^n} + \dots\right] \\ + \frac{e^{-2z} (-z)^{\frac{3}{2} - (k_1 + k_2)}}{2\sqrt{\pi}} \left[1 + \frac{c_1}{(-2z)} + \frac{c_2}{(-2z)^2} + \dots + \frac{c_n}{(-2z)^n} + \dots\right]$$

for the sectors, $-\pi < \arg z < -\frac{\pi}{2}$ and $-\frac{\pi}{2} < \arg z < 0$.

It is also a matter of interest to observe that the development (25) is also obtained for (Case I)

$0 < \arg z < \frac{\pi}{2}$ and (Case II) $\frac{\pi}{2} < \arg z < \pi$ if in equation

Equation (22) can be written as

$$\left[\frac{1}{2\sqrt{\pi}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \right] \left[\frac{1}{2\sqrt{\pi}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \right] \quad (24)$$

$$\left[\frac{1}{2\sqrt{\pi}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \right] \left[\frac{1}{2\sqrt{\pi}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \right] + \left[\frac{1}{2\sqrt{\pi}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \right] \left[\frac{1}{2\sqrt{\pi}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \right]$$

For Case II, where $\Delta x = \Delta y = \Delta z = \Delta t$, the two equations are virtually identical with the only difference being the sign of the second term. In equation (25), the second term is negative, while in equation (26), it is positive. This is due to the fact that in Case II, the second term is defined as the negative of the first term, while in Case I, it is defined as the positive of the first term.

Therefore, we can write

$$\left[\frac{1}{2\sqrt{\pi}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \right] \left[\frac{1}{2\sqrt{\pi}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \right] \quad (25)$$

$$\left[\frac{1}{2\sqrt{\pi}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \right] \left[\frac{1}{2\sqrt{\pi}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \right] + \left[\frac{1}{2\sqrt{\pi}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \right] \left[\frac{1}{2\sqrt{\pi}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \left(\frac{1}{2} \right)^{\frac{1}{2}} \right]$$

For the second case, where $\Delta x = \Delta y = \Delta z = \Delta t$, the two equations are virtually identical with the only difference being the sign of the second term. In equation (25), the second term is negative, while in equation (26), it is positive. This is due to the fact that in Case II, the second term is defined as the negative of the first term, while in Case I, it is defined as the positive of the first term.

(5) -1 be replaced by $\cos -\pi + i \sin -\pi$. The possibility of doing this follows as an immediate consequence of the fact that

$$\sum_{n=0}^{\infty} g(n) z^{2n} = \sum_{n=0}^{\infty} g(n) [z(\cos -\pi + i \sin -\pi)]^{2n}.$$

So in conclusion, we have obtained by quite a direct process the asymptotic development of $f(z)$ as defined by

$$f(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{\Gamma(n+k_1)\Gamma(n+k_2)}.$$

Moreover, the form (25) furnishes a satisfactory generalization of the classical representation of the Bessel function obtained by Hankel.

(3) - 1. The first of the two main results of this paper is the following theorem.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

So in the case of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ the sum is $\frac{\pi^2}{6}$.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Moreover, the first part of the theorem is a special case of a more general result of the author.

CHAPTER II

FUNCTIONS OF THE DOUBLE BESSEL TYPE

In the first chapter, our study was concerned with an $f(z)$ in which the $g(n)$ was the reciprocal of the product of two gamma functions. This chapter purports to generalize the previous study to an extent.

Suppose that

$$(1) \quad f(z) = \sum_{n=0}^{\infty} g(n) z^{4n}, \quad \text{rad. of conv.} = \infty;$$

where

$$g(n) = \frac{1}{\Gamma(n+k_1)\Gamma(n+k_2)\Gamma(n+k_3)\Gamma(n+k_4)}$$

and z^4 plays the same role as z in the theorem of Newsom.

It may readily be shown that $g(n)$ satisfies the conditions set up in the aforementioned theorem when $k = 3$.⁶

Thus, it follows that

$$(2) \quad f(z) = \int_{-\frac{1}{2}}^{\infty} \left\{ g(x) z^{4x} \frac{\sin 3\pi x}{\sin \pi x} \right\} dx - \sum_{n=-l}^{-1} g(n) z^{4n} + f(l, z);$$

where $f(l, z)$ has the property (4), (Chap. I), and

$$-\frac{\pi}{4} < \arg z < \frac{\pi}{4}.$$

Due to a well known trigonometric formula, the

THEORY OF THE ZETA FUNCTION

In the first chapter, we shall consider the function $\zeta(s)$ in which the ρ -values are the zeros of the function. We shall then generalize the previous results to the case of a general function.

Suppose that

$$(1) \quad \zeta(s) = \sum_{n=0}^{\infty} g(n) x^n$$

where

$$g(n) = \frac{1}{n!} \lim_{x \rightarrow 0} \frac{d^n \zeta(x)}{dx^n}$$

and x plays the same role as s in the previous chapter.

It may readily be seen that the conditions set up in the previous chapter are satisfied.

$$k = 2, \quad \sigma = 2.$$

It follows that

$$(2) \quad \zeta(s) = \sum_{n=0}^{\infty} g(n) x^n = \sum_{n=0}^{\infty} \frac{1}{n!} \lim_{x \rightarrow 0} \frac{d^n \zeta(x)}{dx^n} x^n$$

where $\zeta(s)$ has the properties

$$-\frac{\pi}{2} < \arg \zeta < \frac{\pi}{2}$$

One can well know that the function $\zeta(s)$ is not zero for $\sigma > 1$.

integral of (1) may be written as

$$(3) \quad \int_{-l-\frac{1}{2}}^{\infty} \{ g(x) z^{4x} (1 + 2 \cos 2\pi x) \} dx \\ = \int_{-l-\frac{1}{2}}^{\infty} \{ g(x) [z^{4x} + z'^{4x} + z''^{4x}] \} dx;$$

where $\arg z' = \arg z - \frac{\pi}{2}$, and $\arg z'' = \arg z + \frac{\pi}{2}$.

In view of the nature of $g(x)$, the existence of the integrals,

$$\int_{-l-\frac{1}{2}}^{\infty} g(x) z^{4x} dx, \quad \int_{-l-\frac{1}{2}}^{\infty} g(x) z'^{4x} dx, \quad \text{and} \quad \int_{-l-\frac{1}{2}}^{\infty} g(x) z''^{4x} dx$$

is readily demonstrable for any finite z .

Thus,

$$(4) \quad f(z) = \int_{-l-\frac{1}{2}}^{\infty} \frac{z^{4x}}{\Gamma(x+h_1)\Gamma(x+h_2)\Gamma(x+h_3)\Gamma(x+h_4)} dx + \int_{-l-\frac{1}{2}}^{\infty} \frac{z'^{4x}}{\Gamma(x+h_1)\Gamma(x+h_2)\Gamma(x+h_3)\Gamma(x+h_4)} dx \\ + \int_{-l-\frac{1}{2}}^{\infty} \frac{z''^{4x}}{\Gamma(x+h_1)\Gamma(x+h_2)\Gamma(x+h_3)\Gamma(x+h_4)} dx - \sum_{m=-l}^{-1} g(m) z^{4m} + f(l, z);$$

where $f(l, z)$ has the property (4), (Chap. I), and

$$-\frac{\pi}{4} < \arg z < \frac{\pi}{4}, \quad \arg z' = \arg z - \frac{\pi}{2}, \quad \arg z'' = \arg z + \frac{\pi}{2}.$$

Employing a procedure similar to the paragraph following (8), (Chap. I), the first integral of (4) may be re-expressed as

$$(5) \quad \frac{z^{2\rho_1+2\rho_2-2}}{\pi} \int_{-l-\frac{1}{2}}^{\infty} \frac{F_1(x) \cdot F_2(x) \cdot (z\bar{z})^{4x}}{\Gamma(2x+2\rho_1)\Gamma(2x+2\rho_2)} dx$$

integral of (1) is

$$(2) \int_{-\frac{1}{2}}^{\infty} \{g(x)z^{x+\frac{1}{2}}(x+\frac{1}{2})\} dx$$

$$= \int_{-\frac{1}{2}}^{\infty} \{g(x)z^{x+\frac{1}{2}}\} dx$$

where $g(x) = \frac{1}{x^2}$ if $x > 0$ and $g(x) = 0$ if $x < 0$.
 In view of (1) the integral

$$\int_{-\frac{1}{2}}^{\infty} g(x)z^{x+\frac{1}{2}} dx$$

is equal to

$$(4) \int_{-\frac{1}{2}}^{\infty} \frac{z^{x+\frac{1}{2}}}{x^2} dx = \int_{-\frac{1}{2}}^{\infty} \frac{z^{x+\frac{1}{2}}}{x^2} dx$$

$$+ \int_{-\frac{1}{2}}^{\infty} \frac{z^{x+\frac{1}{2}}}{x^2} dx$$

where $\int_{-\frac{1}{2}}^{\infty} \frac{z^{x+\frac{1}{2}}}{x^2} dx$ is the integral of $\frac{z^{x+\frac{1}{2}}}{x^2}$ over the interval $-\frac{1}{2} < x < \infty$.
 Putting $x = \frac{1}{2} + t$ we have

$$(5) \int_{-\frac{1}{2}}^{\infty} \frac{z^{x+\frac{1}{2}}}{x^2} dx = \int_0^{\infty} \frac{z^{t+\frac{1}{2}}}{(t+\frac{1}{2})^2} dt$$

where

$$F_1(x) = \frac{\Gamma(x+p_1)\Gamma(x+p_1+\frac{1}{2})}{\Gamma(x+h_1)\Gamma(x+h_2)}$$

$$F_2(x) = \frac{\Gamma(x+p_2)\Gamma(x+p_2+\frac{1}{2})}{\Gamma(x+h_3)\Gamma(x+h_4)}.$$

Using the identity, $\Gamma(2y) = \frac{2^{2y-1}}{\sqrt{\pi}} \Gamma(y)\Gamma(y+\frac{1}{2})$,
it is readily seen that (5) becomes

$$(6) \quad \frac{2^{2p_1+2p_2+4p_3-3}}{\pi^{3/2}} \int_{-1-\frac{1}{2}}^{\infty} \frac{F_1(x) \cdot F_2(x) \cdot F_3(x) (4z)^{4x}}{\Gamma(4x+4p_3)} dx,$$

where

$$F_3(x) = \frac{\Gamma(2x+2p_3)\Gamma(2x+2p_3+\frac{1}{2})}{\Gamma(2x+2p_1)\Gamma(2x+2p_2)}.$$

Upon making the transformations, $4x = x' - 4p_3 + 1$
and $z = u/4$, and dropping primes (6) becomes

$$(7) \quad \frac{2^{2p_1+2p_2}}{2^4 \pi^{3/2}} \left(\frac{u}{2}\right)^{1-4p_3} \int_{-4p_3-3}^{\infty} \frac{F_1'(x) \cdot F_2'(x) \cdot F_3'(x) u^x dx}{\Gamma(x+1)}$$

where $-\frac{\pi}{4} < \arg u < \frac{\pi}{4}$,

$$F_1'(x) = \frac{\Gamma(\frac{x}{4}+p_1-p_3+\frac{1}{4})\Gamma(\frac{x}{4}+p_1-p_3+\frac{3}{4})}{\Gamma(\frac{x}{4}-p_3+h_1+\frac{1}{4})\Gamma(\frac{x}{4}-p_3+h_2+\frac{1}{4})}$$

$$F_2'(x) = \frac{\Gamma(\frac{x}{4}+p_2-p_3+\frac{1}{4})\Gamma(\frac{x}{4}+p_2-p_3+\frac{3}{4})}{\Gamma(\frac{x}{4}-p_3+h_3+\frac{1}{4})\Gamma(\frac{x}{4}-p_3+h_4+\frac{1}{4})}$$

and

$$F_3'(x) = \frac{\Gamma(\frac{x+1}{2})\Gamma(\frac{x+2}{2})}{\Gamma(\frac{x}{2}+2p_1-2p_3+\frac{1}{2})\Gamma(\frac{x}{2}+2p_2-2p_3+\frac{1}{2})}.$$

Now it is possible to expand $F_1'(x)$ and $F_2'(x)$
in asymptotic series of the form

where

$$F(x) = \frac{f(x)}{g(x)}$$

$$F(x) = \frac{f(x)}{g(x)}$$

Using the chain rule, we have

it is readily seen that

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

(c)

where

$$F(x) = \frac{f(x)}{g(x)}$$

Upon setting $u = g(x)$, we have

and $u = g(x)$, and therefore

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

(v)

where $\frac{1}{g(x)} = \frac{1}{g(x)}$

$$F(x) = \frac{f(x)}{g(x)}$$

$$F(x) = \frac{f(x)}{g(x)}$$

$$F(x) = \frac{f(x)}{g(x)}$$

$$F(x) = \frac{f(x)}{g(x)}$$

and

$$F(x) = \frac{f(x)}{g(x)}$$

Now

in order to find

$$F_1'(x) \sim 1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots + \frac{a_n}{x^n} + \dots,$$

$$F_2'(x) \sim 1 + \frac{b_1}{x} + \frac{b_2}{x^2} + \dots + \frac{b_n}{x^n} + \dots,$$

if

$$p_1 = \frac{2k_1 + 2k_2 - 1}{4}$$

and

$$p_2 = \frac{2k_3 + 2k_4 - 1}{4} \quad 7$$

But due to the well known properties of asymptotic series, it follows that

$$(8) \quad F_1'(x) \cdot F_2'(x) = F_4(x) \sim 1 + \frac{d_1}{x} + \frac{d_2}{x^2} + \dots + \frac{d_n}{x^n} + \dots,$$

where $d_n = b_n + a_1 b_{n-1} + a_2 b_{n-2} + \dots + a_n$ 8

By similar analysis, it follows that

$$(9) \quad F_3'(x) \sim 1 + \frac{f_1}{x} + \frac{f_2}{x^2} + \dots + \frac{f_n}{x^n} + \dots$$

if
$$p_3 = \frac{4p_1 + 4p_2 - 1}{4} = \frac{2k_1 + 2k_2 + 2k_3 + 2k_4 - 3}{8}.$$

Thus, multiplying series (8) and (9), we find that

$$(10) \quad F_3'(x) \cdot F_4(x) = F(x) \sim 1 + \frac{g_1}{x} + \frac{g_2}{x^2} + \dots + \frac{g_n}{x^n} + \dots,$$

7

Van Engen, op. cit., p. 125.

8

Walter B. Ford, Studies on Divergent Series and Summability, (The MacMillan Company, New York, 1916) p. 24.

where $g_n = d_n + f_1 d_{n-1} + \dots + f_n$.

Since it is well known that a series of the form⁹
(10) may be written as a faculty series,

$$(11) \quad F(x) = 1 + \frac{c_1}{x+1} + \frac{c_2}{(x+1)(x+2)} + \dots + \frac{c_n + \delta(x,s)}{(x+1)(x+2)\dots(x+s)},$$

where $\lim_{x \rightarrow \infty} \delta(x,s) = 0$.

Hence, it is now possible to re-express (7) in the form

$$(12) \quad \frac{2^{2p_1+2p_2}}{2^4 \pi^{3/2}} \left(\frac{u}{2}\right)^{1-4p_3} \int_{-4\ell+4p_3-3}^{\infty} \frac{F(x) u^x dx}{\Gamma(x+1)}$$

where $-\frac{\pi}{4} < \arg u < \frac{\pi}{4}$, and $F(x)$ is given by (11).

It follows immediately that (12) takes the form,

$$(13) \quad \frac{2^{2p_1+2p_2}}{2^4 \pi^{3/2}} \left(\frac{u}{2}\right)^{1-4p_3} \left\{ \sum_{n=0}^s c_n \int_{-4\ell+4p_3-3}^{\infty} \frac{u^x dx}{\Gamma(x+n+1)} + \int_{-4\ell+4p_3-3}^{\infty} \frac{\delta(x,s) u^x dx}{\Gamma(x+s+1)} \right\}$$

where $c_0 = 1$, $\delta(x,s)$ has the property (11), and

$-\frac{\pi}{4} < \arg u < \frac{\pi}{4}$.

Similar studies upon the other two integrals of (4) give analogous representations. Hence, after making

where $\beta = \beta_1 + i\beta_2$

and β_1, β_2 are real numbers. (10) may be written as

$$F(x) = 1 + \frac{\beta}{x} + \frac{\beta^2}{x^2} + \dots \quad (11)$$

where $\beta(x) = \beta_1(x) + i\beta_2(x)$ and β_1, β_2 are functions of x .

the form

$$\left(\frac{2}{\pi} \right)^{1/2} \frac{1}{x^{1/2}} \left(\frac{2}{\pi} \right)^{1/2} \frac{1}{x^{1/2}} \quad (12)$$

where $\frac{1}{x} < \frac{1}{2}$ and $\frac{1}{x} > \frac{1}{2}$ in the following cases:

$$\left(\frac{2}{\pi} \right)^{1/2} \frac{1}{x^{1/2}} \left(\frac{2}{\pi} \right)^{1/2} \frac{1}{x^{1/2}} \quad (13)$$

$$+ \int_{-\infty}^{\infty} \frac{f(t) dt}{x-t}$$

where $\beta = \beta_1 + i\beta_2$ and β_1, β_2 are functions of x .

and for β_1, β_2 are functions of x .

Now we have

obvious transformations, we may write (4) as

$$\begin{aligned}
 (14) \quad \phi(u) = & \frac{2^{2p_1+2p_2}}{2^4 \pi^{3/2}} \left[\left(\frac{u}{2} \right)^{1-4p_3} \left\{ \sum_{n=0}^s \frac{c_n}{u^n} \int_{-4l+4p_3+n-3}^{\infty} \frac{u^x dx}{\Gamma(x+1)} \right. \right. \\
 & + \frac{1}{u^s} \int_{-4l+4p_3+s-3}^{\infty} \frac{\delta(x-s, s) u^x dx}{\Gamma(x+1)} \left. \right\} + \left(\frac{u'}{2} \right)^{1-4p_3} \left\{ \sum_{n=0}^s \frac{c_n}{u'^n} \int_{-4l+4p_3+n-3}^{\infty} \frac{u'^x dx}{\Gamma(x+1)} \right. \\
 & + \frac{1}{u'^s} \int_{-4l+4p_3+s-3}^{\infty} \frac{\delta(x-s, s) u'^x dx}{\Gamma(x+1)} \left. \right\} + \left(\frac{u''}{2} \right)^{1-4p_3} \left\{ \sum_{n=0}^s \frac{c_n}{u''^n} \int_{-4l+4p_3+n-3}^{\infty} \frac{u''^x dx}{\Gamma(x+1)} \right. \\
 & + \frac{1}{u''^s} \int_{-4l+4p_3+s-3}^{\infty} \frac{\delta(x-s, s) u''^x dx}{\Gamma(x+1)} \left. \right\} \Big] - \sum_{m=-l}^{-1} g(m) \left(\frac{u}{4} \right)^{4m} + f(l, u);
 \end{aligned}$$

wherein it is recalled that $-\frac{\pi}{4} < \arg u < \frac{\pi}{4}$, $\arg u' = \arg u - \frac{\pi}{2}$, and $\arg u'' = \arg u + \frac{\pi}{2}$.

Now let L be the largest integer in $4l - 4R(p) - n + 3$, where l has been taken so large that $4l - 4R(p) - n + 3 > 5$, $n = 0, 1, 2, \dots, s$. Since $4l - 4R(p) - n + 3 \geq 4l - 4R(p) - s + 3$, the desired condition is realized if

$$(15) \quad l > R(p) + \frac{s}{2} - \frac{3}{2}.$$

Under these conditions and those on u , u' , and u'' , it follows that

10

See the analysis in Chapter I following statement (15).

obvious transformation, we have the following

$$\begin{aligned}
 (14) \quad \phi(x) &= \frac{1}{2} \left\{ \frac{1}{x} + \frac{1}{x+1} \right\} \sum_{n=0}^{\infty} \frac{(-1)^n}{x^n} \\
 &+ \frac{1}{2} \left\{ \frac{1}{x} + \frac{1}{x+1} \right\} \sum_{n=0}^{\infty} \frac{(-1)^n}{x^n} \\
 &+ \frac{1}{2} \left\{ \frac{1}{x} + \frac{1}{x+1} \right\} \sum_{n=0}^{\infty} \frac{(-1)^n}{x^n}
 \end{aligned}$$

wherein it is assumed that $x \neq -1$.
 Let $n = 0, 1, 2, \dots$ and $x = -1 + \epsilon$,
 where ϵ is a small positive number, then
 the desired result is obtained.

$$(15) \quad \lim_{x \rightarrow -1} \phi(x) = \frac{1}{2}$$

Under these conditions, it is found that
 the following result is obtained:

$$(16) \quad \lim_{x \rightarrow -1} \phi(x) = \frac{1}{2}$$

$$\begin{aligned}
(16) \quad \phi(u) = & \frac{2^{2p_1+2p_2}}{2^4 \pi^{3/2}} \left[e^u \left(\frac{u}{2}\right)^{1-4p_3} \sum_{n=0}^s \frac{c_n}{u^n} - \left(\frac{u}{2}\right)^{1-4p_3} \sum_{n=0}^s \frac{c_n \Delta_n(u, L)}{u^n} \right. \\
& + \left(\frac{u}{2}\right)^{1-4p_3} \frac{1}{u^s} \int_{-4l+4p_3+s-3}^{\infty} \frac{\delta(x-s, s) u^x dx}{\Gamma(x+1)} + e^{u'} \left(\frac{u'}{2}\right)^{1-4p_3} \sum_{n=0}^s \frac{c_n}{u'^n} \\
& - \left(\frac{u'}{2}\right)^{1-4p_3} \sum_{n=0}^s \frac{c_n \Delta_n(u', L)}{u'^n} + \left(\frac{u'}{2}\right)^{1-4p_3} \frac{1}{u'^s} \int_{-4l+4p_3+s-3}^{\infty} \frac{\delta(x-s, s) u'^x dx}{\Gamma(x+1)} \\
& + e^{u''} \left(\frac{u''}{2}\right)^{1-4p_3} \sum_{n=0}^s \frac{c_n}{u''^n} - \left(\frac{u''}{2}\right)^{1-4p_3} \sum_{n=0}^s \frac{c_n \Delta_n(u'', L)}{u''^n} \\
& \left. + \left(\frac{u''}{2}\right)^{1-4p_3} \frac{1}{u''^s} \int_{-4l+4p_3+s-3}^{\infty} \frac{\delta(x-s, s) u''^x dx}{\Gamma(x+1)} \right] - \sum_{m=-l}^{-1} g(m) \left(\frac{u}{4}\right)^{4m} + f(l, u).
\end{aligned}$$

At this point it is desirable to divide the analysis into two cases, (Case I) $-\frac{\pi}{4} < \arg u < 0$ and (Case II) $0 < \arg u < \frac{\pi}{4}$. Employing an argument similar to that immediately following statement (19) in Chapter I, it is readily found that

$$\begin{aligned}
(17) \quad \phi(u) = & \frac{2^{2p_1+2p_2}}{2^4 \pi^{3/2}} \left[e^u \left(\frac{u}{2}\right)^{1-4p_3} \left\{ \sum_{n=0}^s \frac{c_n}{u^n} + \frac{R(u, s)}{u^s} \right\} \right. \\
& + e^{u'} \left(\frac{u'}{2}\right)^{1-4p_3} \sum_{n=0}^s \frac{c_n}{u'^n} + e^{u''} \left(\frac{u''}{2}\right)^{1-4p_3} \left\{ \sum_{n=0}^s \frac{c_n}{u''^n} + \frac{R(u'', s)}{u''^s} \right\} \left. \right],
\end{aligned}$$

$$\phi(n) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{z^{-n}}{1-z} dz = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \sum_{k=0}^{\infty} z^k z^{-n} dz = \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{k-n} dz$$

$$= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{k-n} dz = \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{k-n} dz$$

$$= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{k-n} dz = \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{k-n} dz$$

$$= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{k-n} dz = \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{k-n} dz$$

$$= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{k-n} dz = \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{k-n} dz$$

$$= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{k-n} dz = \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{k-n} dz$$

$$= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{k-n} dz = \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{k-n} dz$$

$$= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{k-n} dz = \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{k-n} dz$$

$$= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{k-n} dz = \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z^{k-n} dz$$

where

$$\begin{aligned}
 R(u, s) = & e^{-u} \left[-u^s \sum_{n=0}^s \frac{c_n \Delta_n(u, L)}{u^n} + \frac{1}{u} \int_{-4l+4p_3+s-2}^{\infty} \frac{\delta(x-s-l, s) u^x dx}{\Gamma(x)} \right. \\
 & - \left(\frac{u'}{u} \right)^{1-4p_3} u^s \sum_{n=0}^s \frac{c_n \Delta_n(u', L)}{u'^n} + \left(\frac{u'}{u} \right)^{1-4p_3-s} \frac{I(u)}{u'} \\
 & \left. + \frac{2^4 \pi^{3/2}}{2^{2p_1+2p_2}} \left(\frac{u}{2} \right)^{4p_3-1} u^s \left\{ -\sum_{m=2-l}^{-1} g(m) \left(\frac{u}{4} \right)^{4m} + f(l, u) \right\} \right],
 \end{aligned}$$

$$R(u'', s) = e^{-u''} \left[-u''^s \sum_{n=0}^s \frac{c_n \Delta_n(u'', L)}{u''^n} + \frac{1}{u''} \int_{-4l+4p_3+s-2}^{\infty} \frac{\delta(x-s-l, s) u''^x dx}{\Gamma(x)} \right],$$

$$I(u) = \int_{-4l+4p_3+s-2}^{\infty} \frac{\delta(x-s-l, s) u^x dx}{\Gamma(x)},$$

$$\lim_{|u| \rightarrow \infty} R(u, s) = 0, \text{ and } \lim_{|u'| \rightarrow \infty} R(u', s) = 0.$$

It is to be noted at this point that $\delta(x, s)$ and hence $\delta(x-s-l, s)$ can be treated as in the paragraphs following statement (20), (Chapter I). Of course, in this case $\delta(x, s)$ is found to be related to

$$R(x, z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} (x + iz e^{i\theta})}{1 - x e^{i\theta} - z e^{2i\theta}} d\theta$$

$$-\left(\frac{x}{z}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{x}{z}\right)^k \frac{d^k}{dx^k} \left(\frac{1}{1-x}\right)$$

$$+ \frac{2\pi}{z} \left(\frac{x}{z}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{x}{z}\right)^k \frac{d^k}{dx^k} \left(\frac{1}{1-x}\right)$$

$$R(x, z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} (x + iz e^{i\theta})}{1 - x e^{i\theta} - z e^{2i\theta}} d\theta$$

$$I(x) = \int_{-\infty}^{\infty} \frac{e^{ixt}}{1+t^2} dt$$

and hence \tilde{f} is a holomorphic function in the upper half-plane. It is also holomorphic in the lower half-plane, since $\tilde{f}(z) = \overline{f(\bar{z})}$. In this case \tilde{f} is holomorphic in the whole plane.

a number of similar functions which are all bounded for any finite portion of the plane. Thus, it follows that $\delta(x-s-l, s)$ must be bounded in any finite portion of the plane.

Thus, using the symbol of asymptotic representation, relation (15) may be written in the form,

$$\begin{aligned}
 (18) \quad \phi(u) \sim & \frac{2^{2p_1+2p_2}}{2^4 \pi^{3/2}} e^u \left(\frac{u}{2}\right)^{1-4p_3} \left[1 + \frac{c_1}{u} + \frac{c_2}{u^2} + \dots + \frac{c_n}{u^n} + \dots \right] \\
 & + \frac{2^{2p_1+2p_2}}{2^4 \pi^{3/2}} e^{u'} \left(\frac{u'}{2}\right)^{1-4p_3} \left[1 + \frac{c_1}{u'} + \frac{c_2}{u'^2} + \dots + \frac{c_n}{u'^n} + \dots \right] \\
 & + \frac{2^{2p_1+2p_2}}{2^4 \pi^{3/2}} e^{u''} \left(\frac{u''}{2}\right)^{1-4p_3} \left[1 + \frac{c_1}{u''} + \frac{c_2}{u''^2} + \dots + \frac{c_n}{u''^n} + \dots \right].
 \end{aligned}$$

Of course, it is to be observed that in this case, the portions involving u and u'' are dominant. If the same analysis is repeated for Case II, an identical representation is obtained, but the portions involving u and u' are now dominant.

Therefore, recalling that $u = 4z$, $u' = 4z' = -4iz$, $u'' = 4z'' = 4iz$, it follows that

a number of other cases, and for any finite number of cases, that $\delta x = 1$, and the number of cases of the plane.

Then, using the relation, relating the number of cases to the number of cases, we have

$$(13) \quad \phi(n) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{n}{t} \right)^s \left(\frac{1}{t} \right)^s dt$$

$$+ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{n}{t} \right)^s \left(\frac{1}{t} \right)^s dt$$

$$+ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{n}{t} \right)^s \left(\frac{1}{t} \right)^s dt$$

Of course, if n is a positive integer, then the number of cases is finite, and the number of cases is finite.

If the same number is repeated, then the number of cases is finite, and the number of cases is finite.

cal relations, and the number of cases is finite, and the number of cases is finite.

Therefore, the number of cases is finite, and the number of cases is finite.

II. The number of cases is finite, and the number of cases is finite.

$$\begin{aligned}
 (19) \quad f(z) \sim & \frac{e^{4z s_1 - (h_1 + h_2 + h_3 + h_4)} z}{2^{s_1/2} \pi^{3/2}} \left[1 + \frac{c_1}{4z} + \frac{c_2}{(4z)^2} + \dots + \frac{c_n}{(4z)^n} + \dots \right] \\
 & + \frac{e^{-4iz s_1 - (h_1 + h_2 + h_3 + h_4)} (-iz)}{2^{s_1/2} \pi^{3/2}} \left[1 + \frac{c_1}{-4iz} + \frac{c_2}{(-4iz)^2} + \dots + \frac{c_n}{(-4iz)^n} + \dots \right] \\
 & + \frac{e^{4iz s_1 - (h_1 + h_2 + h_3 + h_4)} (iz)}{2^{s_1/2} \pi^{3/2}} \left[1 + \frac{c_1}{4iz} + \frac{c_2}{(4iz)^2} + \dots + \frac{c_n}{(4iz)^n} + \dots \right]
 \end{aligned}$$

for the sectors $-\frac{\pi}{4} < \arg z < 0$ and $0 < \arg z < \frac{\pi}{4}$.

It is again a matter of interest to observe that the development (19) is also obtained for the sector $-\frac{3\pi}{4} < \arg z < -\frac{\pi}{4}$, if one considers the two distinct cases, (Case I) $-\frac{3\pi}{4} < \arg z < -\frac{\pi}{2}$, and (Case II) $-\frac{\pi}{2} < \arg z < -\frac{\pi}{4}$. The possibility of this new analysis follows from the fact that in relation (2) the factor of unity can be replaced by $\cos 2\pi + i \sin 2\pi$, inasmuch as

$$\sum_{n=0}^{\infty} g(n) z^{4n} = \sum_{n=0}^{\infty} g(n) [z(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})]^{4n}.$$

A similar analysis will show that (19) holds over

the sector $\frac{\pi}{4} < \arg z < \frac{3\pi}{4}$ if one considers (Case I)

$\frac{\pi}{4} < \arg z < \frac{\pi}{2}$ and (Case II) $\frac{\pi}{2} < \arg z < \frac{3\pi}{4}$. For this

study, the factor of unity is replaced by $\cos (-2\pi)$

$+i \sin (-2\pi)$ in relation (4).

$$(12) \quad \frac{1}{2} \left(\frac{1}{\sqrt{1-\frac{1}{2}}} - \frac{1}{\sqrt{1-\frac{1}{4}}} \right) = \frac{1}{2} \left(\frac{1}{\sqrt{\frac{1}{2}}} - \frac{1}{\sqrt{\frac{3}{4}}} \right) = \frac{1}{2} \left(\sqrt{2} - \frac{2}{\sqrt{3}} \right)$$

$$+ \frac{1}{2} \left(\frac{1}{\sqrt{1-\frac{1}{4}}} - \frac{1}{\sqrt{1-\frac{1}{9}}} \right) = \frac{1}{2} \left(\frac{1}{\sqrt{\frac{3}{4}}} - \frac{1}{\sqrt{\frac{8}{9}}} \right) = \frac{1}{2} \left(\frac{2}{\sqrt{3}} - \frac{3}{2\sqrt{2}} \right)$$

$$+ \frac{1}{2} \left(\frac{1}{\sqrt{1-\frac{1}{9}}} - \frac{1}{\sqrt{1-\frac{1}{16}}} \right) = \frac{1}{2} \left(\frac{1}{\sqrt{\frac{8}{9}}} - \frac{1}{\sqrt{\frac{15}{16}}} \right) = \frac{1}{2} \left(\frac{3}{2\sqrt{2}} - \frac{4}{\sqrt{15}} \right)$$

for the values $\frac{1}{2}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$ and $\frac{1}{25}$.

It is a well known fact that

the development of the function $\frac{1}{1-x}$ in powers of x is

cases, $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ for $|x| < 1$.

The possibility of this development is

fact, the function $\frac{1}{1-x}$ is analytic in the

region $|x| < 1$ and the series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } |x| < 1$$

the series $\sum_{n=0}^{\infty} x^n$ is convergent

$\frac{1}{4} < x < \frac{1}{2}$ and $\frac{1}{9} < x < \frac{1}{4}$ and

study, the function $\frac{1}{1-x}$ is analytic in the

region $|x| < 1$ and the series

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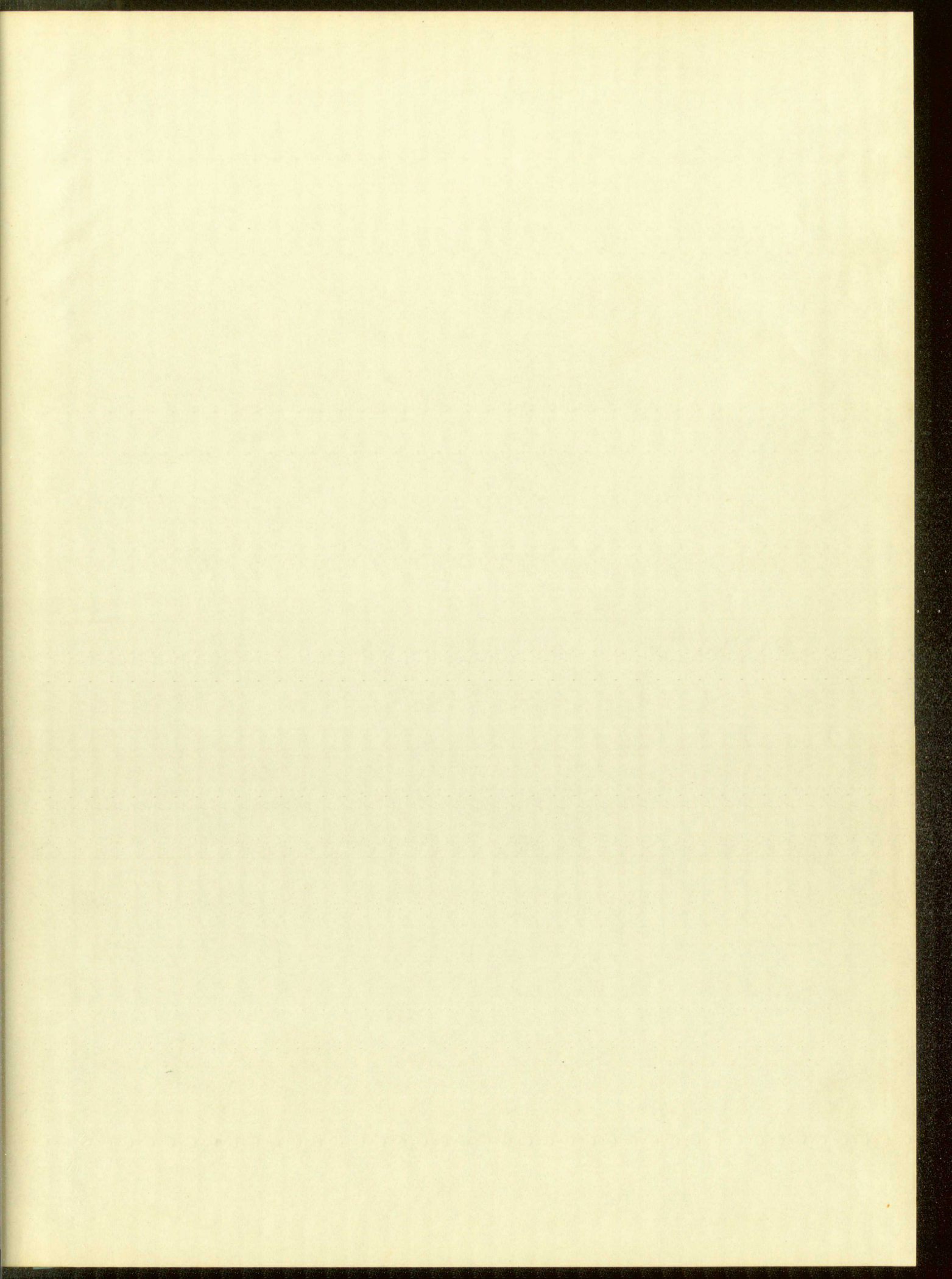
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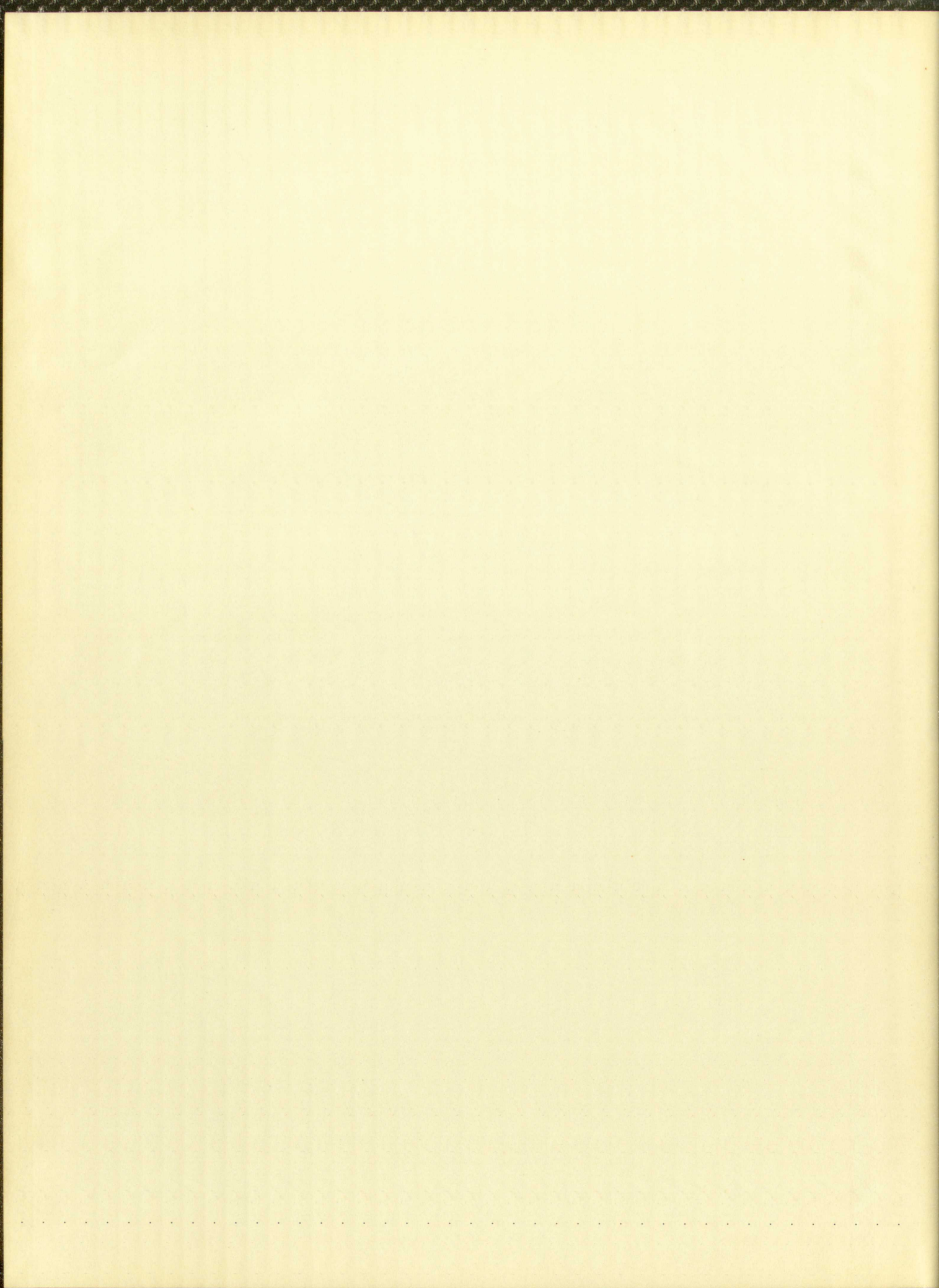
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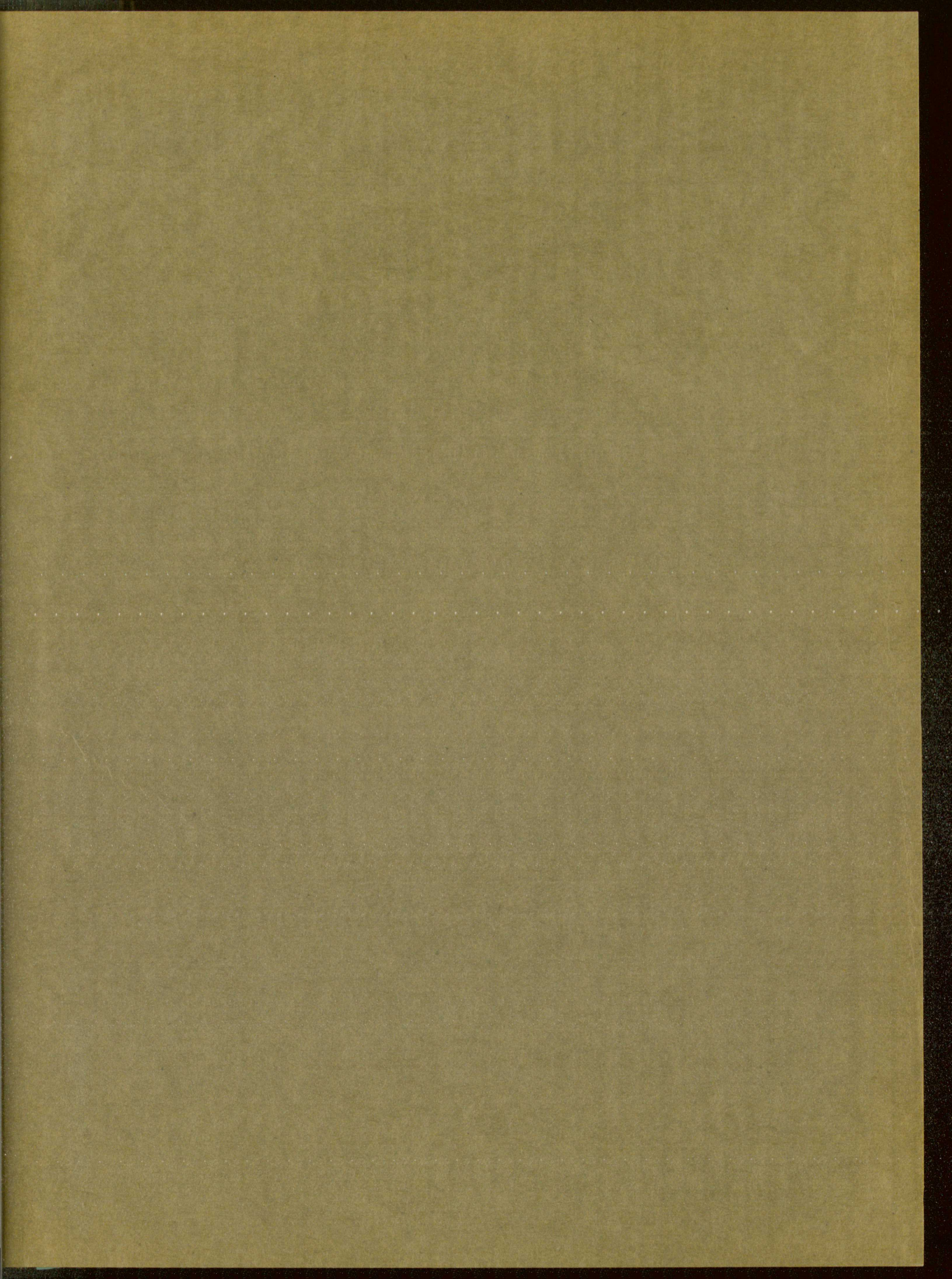
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