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Upon the Asymptotic Representation of Certain Entire Functions in Distant Portions of the Plane

Abraham Franck

FRANCK

FUNCTIONS

NAME AND ADDRESS	DATE

UPON THE ASYMPTOTIC REPRESENTATION OF CERTAIN ENTIRE FUNCTIONS IN DISTANT PORTIONS OF THE PLANE

By

Abraham Franck

A Thesis

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INTRODUCTION

Probably the beginnings of the study of asymptotic series can be traced back to the seventeenth century mathematician, Leonard Euler. The first formal definition of an asymptotic representation was enunciated by Poincare, and has been adopted by later mathematicians.

Since the time of Poincare, notable advances have been made in this field by many men. The Calculus of Residues, as an instrument in the study, has been employed by Barnes, Ford, Newsom, and Van Engen.

The purpose of this paper is to study two particular entire functions which satisfy the conditions set up in a theorem due to Newsom. It is to be hoped that this report may be preliminary to the invention of a method which will lend itself toward the solution of certain general problems.

INTRODUCTION

Probably the beginnings of the study of asymptotic series can be traced back to the seventeenth century mathematician, Leonard Euler. The first formal definition of an asymptotic representation was formulated by Poincare, and has been adopted by later mathematicians.

Since the time of Poincare, notable advances have been made in this field by many men. The Calculus of Residues, as an instrument in the study, has been employed by Barnes, Ford, Newsom, and Van Engen.

The purpose of this paper is to study two particular entire functions which satisfy the conditions set up in a theorem due to Newsom. It is to be hoped that this report may be preliminary to the invention of a method which will lend itself toward the solution of certain general problems.

CHAPTER I

FUNCTIONS OF THE BESSEL TYPE

Recently Newsom proved a theorem pertaining to the behavior of certain entire functions in distant portions of the plane, namely,

Theorem. Let it be assumed that the coefficient g(n) occurring in the general term of the power series,

$$(1) \qquad \sum_{n=0}^{\infty} g(n) z^{n} \quad ; \text{ radius of convergence } = \infty$$

may be regarded as a function g(w) of the complex variable w = x + iy and as such satisfies the following two conditions:

(a) is single valued and analytic throughout the finite w-plane,

(b) is such that for all values of x and y one may write

$$(2) \qquad |g(x+iy)| < K e^{k\pi|y|}$$

where K is a constant independent of x and y, and k is any given positive integer.

Then the function f(z) defined by series (1) when considered for all values of z satisfying the

CHAPTER I

FUNCTIONS [illegible]

Recently Newman proved a theorem regarding the behavior of certain entire functions in different portions of the plane, namely,

Theorem. Let it be assumed that the coefficients $g(n)$ occurring in the general term of the power series,

$$\sum_{n=0}^{\infty} g(n) z^n \quad , \text{ radius of convergence } = \infty \tag{1}$$

may be regarded as a function $g(w)$ of the complex variable $w = u + iv$ and as such satisfies the following two conditions:

(a) is single valued and analytic throughout the finite w-plane;

(b) is such that for all values of u and v one may write

$$|g(u+iv)| < K e^{k\pi|v|} \tag{2}$$

where K is a constant independent of u and v, and k is any given positive integer.

Then the function $f(z)$ defined by series (1) when considered for all values of z not [illegible]

condition, $-\pi < \arg \pm z < \pi$, may be expressed in the form

$$(3) \qquad f(z) = \int_{-l-\frac{1}{2}}^{\infty} \left\{ g(x)[\pm z]^x \frac{\sin k\pi x}{\sin \pi x} \right\} dx$$

$$- \sum_{m=-l}^{-1} g(m) z^m + \xi_k(l, z);$$

wherein l is any arbitrary positive integer and the upper or lower of the signs $\pm$ is to be taken according as k is odd or even. Moreover, the expression $\xi_k(l, z)$ is such that

$$(4) \qquad \lim_{|z| \to \infty} z^l \xi_k(l, z) = 0,$$

irrespective of the value chosen for l.[1]

The factor of negative unity associated with z in the integrand is taken as $\cos \pi + i \sin \pi$ in the first part of this study. Comments are made later in regard to the use of other possible representations of -1.

This paper is concerned with the behavior of

$$f(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{\Gamma(n+k_1)\Gamma(n+k_2)}$$

1

C.V. Newsom, "On the Character of Certain Entire Functions in Distant Portions of the Plane," <u>American Journal of Mathematics</u>, Vol. LX, No. 3, (July, 1938), pp. 561-572.

condition, $-\pi < \arg z < \pi$, may be expressed in the form

$$(3) \qquad f(z) = \int_{-l-\frac{1}{2}}^{\infty} \left\{ g(x)[\pm z]^x \frac{\sin k\pi x}{\sin \pi x} \right\} dx - \sum_{m=-l}^{-1} g(m) z^m + \xi_k(l,z),$$

wherein l is any arbitrary positive integer and the upper or lower of the signs $\pm$ is to be taken according as k is odd or even. Moreover, the remainder term $\xi_k(l,z)$ is such that

$$(4) \qquad \lim_{|z|\to\infty} z^l \xi_k(l,z) = 0,$$

irrespective of the value chosen for l.

The factor of negative unity associated with x in the integrand is taken as [illegible] in the first part of this study. Comments are made later in regard to the use of other possible exponential forms of -1.

This paper is concerned with the function[1]

$$f(z) = \sum_{m=0}^{\infty} \frac{z^{2m}}{\Gamma(m+b_1)\Gamma(m+b_2)}$$

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1 C.V. Newsom, "On the Character of Certain Entire Functions in Distant Portions of the Plane," American Journal of Mathematics, Vol. 60, No. 3 (July, 1938), pp. 561-572.

when $|z|$ becomes large. Thus, the coefficient g(n) takes the form

$$g(n) = \frac{1}{\Gamma(n+k_1)\Gamma(n+k_2)}$$

where k_1 and k_2 are complex constants, and z^2 plays the same role as z in the theorem above. It is apparent that the theorem is applicable to this case if k is chosen as 2.[2]

It follows that

$$(5) \qquad f(z) = \int_{-l-\frac{1}{2}}^{\infty} \left\{ g(x)\left[z^2(\cos\pi + i\sin\pi)\right]^x \frac{\sin 2\pi x}{\sin\pi x} \right\} dx - \sum_{m=-l}^{-1} g(m)\, z^{2m} + \xi(l,z);$$

where $\xi(l,z)$ possesses the property (4), and $-\pi < \arg z < 0$.

The integral involved in this relationship may be rewritten as

$$(6) \qquad \int_{-l-\frac{1}{2}}^{\infty} \left\{ g(x)(\cos\pi x + i\sin\pi x)(2\cos\pi x)\, z^{2x} \right\} dx$$

$$= \int_{-l-\frac{1}{2}}^{\infty} \left\{ g(x)\, z^{2x}\left[1 + (\cos 2\pi x + i\sin 2\pi x)\right] \right\} dx$$

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2 Note Walter B. Ford, The Asymptotic Developments of Functions Defined by MacLaurin Series, (Ann Arbor, Michigan, University of Michigan Press, 1936), p. 61.

$$= \int_{-\ell-\frac{1}{2}}^{\infty} \left\{ g(x)\left[z^{2x} + z'^{2x} \right] \right\} dx,$$

where $\arg z' = \arg z + \pi$.

In view of the nature of $g(x)$, the existence of the integrals,

$$\int_{-\ell-\frac{1}{2}}^{\infty} g(x) z^{2x} dx \quad \text{and} \quad \int_{-\ell-\frac{1}{2}}^{\infty} g(x) z'^{2x} dx,$$

is readily demonstrable for any finite z.

Thus

$$(7) \quad f(z) = \int_{-\ell-\frac{1}{2}}^{\infty} \frac{z^{2x} dx}{\Gamma(x+k_1)\Gamma(x+k_2)} + \int_{-\ell-\frac{1}{2}}^{\infty} \frac{z'^{2x} dx}{\Gamma(x+k_1)\Gamma(x+k_2)}$$
$$- \sum_{m=-\ell}^{-1} g(m) z^{2m} + \xi(\ell, z);$$

where $\xi(\ell, z)$ has the property (4), $-\pi < \arg z < 0$, and $\arg z' = \arg z + \pi$.

The first integral of (7) may be re-expressed as

$$(8) \quad \int_{-\ell-\frac{1}{2}}^{\infty} \left[\frac{\Gamma(2x+2p)}{\Gamma(x+k_1)\Gamma(x+k_2)} \cdot \frac{z^{2x}}{\Gamma(2x+2p)} \right] dx$$

$$= \frac{2^{2p-1}}{\sqrt{\pi}} \int_{-\ell-\frac{1}{2}}^{\infty} \left[\frac{\Gamma(x+p)\Gamma(x+p+\frac{1}{2})\, u^{2x}}{\Gamma(x+k_1)\Gamma(x+k_2)\Gamma(2x+2p)} \right] dx,$$

if it is recalled that

$$\Gamma(2y) = \frac{2^{2y-1}}{\sqrt{\pi}} \Gamma(y)\Gamma(y+\tfrac{1}{2})$$

and if $u = 2z$. Of course, it should be noted at this point that $-\pi < \arg u < 0$. Upon making the transformation $x = \frac{1}{2}(x' + 1) - p$, and dropping the primes, the

$$= \int_{-\frac{1}{2}-}^{\infty} \{g(x) [\,\ldots\,] \ldots$$

where arg x' = [illegible]

In view of the [illegible]

the integrals

$$\int_{-\frac{1}{2}-}^{\infty} g(x) \ldots dx \ldots$$

is readily [illegible]

Thus

(7) $f(z) = \int_{-\frac{1}{2}-}^{\infty} \frac{z^{-x}}{\Gamma(x+\alpha_1)\Gamma(x+\alpha_2)} \ldots$

$- \sum \ldots$

where $f(\ldots, z)$ [illegible]

and arg z' = arg $z + \pi$.

The first integral [illegible]

(8) $\int_{-\frac{1}{2}-}^{\infty} \left[\frac{\Gamma(\ldots)}{\Gamma(x+\alpha_1)\Gamma(x+\alpha_2)} \ldots \right] \ldots$

$$= \frac{2^{2\rho-1}}{\sqrt{\pi}} \int_{-\frac{1}{2}-}^{\infty} \left[\ldots \right] \ldots$$

if it is recalled that

$$\Gamma(2y) = \frac{2^{2y-1}}{\sqrt{\pi}} \Gamma(y) \Gamma(y+\tfrac{1}{2})$$

and if $u = 2x$. [illegible]

point that $-\pi <$ arg [illegible]

tion $x = \frac{1}{2}(x' + \ldots)$ [illegible]

integral of (8) becomes

(9) $$\frac{1}{2\sqrt{\pi}}\left(\frac{u}{2}\right)^{1-2p}\int_{-2l+2p-2}^{\infty}\frac{F(x)\,u^{x}\,dx}{\Gamma(x+1)}$$

where

$$F(x)=\frac{\Gamma\left(\frac{x+1}{2}\right)\Gamma\left(\frac{x+2}{2}\right)}{\Gamma\left(\frac{x+2k_1-2p+1}{2}\right)\Gamma\left(\frac{x+2k_2-2p+1}{2}\right)}.$$

Up to this point the constant p has been arbitrary. If p be assigned the value,

(10) $$p=\frac{2k_1+2k_2-1}{4}$$

then it follows that

(11) $$F(x)=1+\sum_{n=1}^{s}\frac{c_n}{(x+1)(x+2)\cdots(x+n)}+\frac{\delta(x,s)}{(x+1)(x+2)\cdots(x+n)},$$

where $\lim_{x\to\infty}\delta(x,s)=0.$[3]

Therefore, the integral of (9) immediately takes the form,

(12) $$\frac{1}{2\sqrt{\pi}}\left(\frac{u}{2}\right)^{1-2p}\left\{\sum_{n=0}^{s}c_n\int_{-2l+2p-2}^{\infty}\frac{u^{x}\,dx}{\Gamma(x+n+1)}\right.$$

$$\left.+\int_{-2l+2p-2}^{\infty}\frac{\delta(x,s)\,u^{x}\,dx}{\Gamma(x+s+1)}\right\},$$

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3 Note Henry Van Engen, "Concerning Gamma Function Expansions," Tohoku Mathematical Journal, Vol. 45, Part I (September, 1938), pp. 124- 129.

where $c_0 = 1$, $\delta(x,s)$ has the property given in (11), and $-\pi < \arg u < 0$.

If a similar study be made upon the second integral of (7), the integral becomes

$$(13)\qquad \frac{1}{2\sqrt{\pi}}\left(\frac{u'}{2}\right)^{1-2p}\left\{\sum_{n=0}^{s} c_n \int_{-2l+2p-2}^{\infty}\frac{u'^{x}\,dx}{\Gamma(x+n+1)} + \int_{-2l+2p-2}^{\infty}\frac{\delta(x,s)\,u'^{x}\,dx}{\Gamma(x+s+1)}\right\},$$

where $c_0 = 1$, $\delta(x,s)$ has the property given in (11), and $\arg u' = \arg u + \pi$.

After making obvious transformations upon the integrals of (12) and (13), we may summarize the results by rewriting (7) as

$$(14)\qquad \varphi(u) = \frac{1}{2\sqrt{\pi}}\left(\frac{u}{2}\right)^{1-2p}\left\{\sum_{n=0}^{s}\frac{c_n}{u^n}\int_{-2l+2p+n-2}^{\infty}\frac{u^{x}\,dx}{\Gamma(x+1)} + \frac{1}{u^s}\int_{-2l+2p+s-2}^{\infty}\frac{\delta(x-s,s)\,u^{x}\,dx}{\Gamma(x+1)}\right\} + \frac{1}{2\sqrt{\pi}}\left(\frac{u'}{2}\right)^{1-2p}\left\{\sum_{n=0}^{s}\frac{c_n}{u'^n}\int_{-2l+2p+n-2}^{\infty}\frac{u'^{x}\,dx}{\Gamma(x+1)} + \frac{1}{u'^s}\int_{-2l+2p+s-2}^{\infty}\frac{\delta(x-s,s)\,u'^{x}\,dx}{\Gamma(x+1)}\right\} - \sum_{m=-l}^{-1} g(m)\left(\frac{u}{2}\right)^{2m} + \xi_1(l,u);$$

wherein

$$(15)\qquad \lim_{|u|\to\infty} u^{l}\,\xi_1(l,u) = 0\,, \quad \lim_{x\to\infty}\delta(x-s,s) = 0,$$

$-\pi < \arg u < 0$, and $\arg u' = \arg u + \pi$.

Employing an argument[4] similar to that of Ford, let L represent the largest integer in $2l-2R(p)-n+2$, wherein has been taken so large that $2l-2R(p)-n+2 > s$; $n = 0, 1, 2, \ldots s$. Since

$$2l-2R(p)-n+2 \geq 2l-2R(p)-s+2$$

the desired condition will be realized if

(16) $$l > R(p)+s-1 .$$

Under such a condition upon l as given above, and with the arguments of u and u` restricted as heretofore specified, it follows that

$$\int_{-2l+2p+n-2}^{\infty} \frac{u^x\,dx}{\Gamma(x+1)} = e^u - \Delta_n(u,L)$$

and

$$\int_{-2l+2p+n-2}^{\infty} \frac{u'^x\,dx}{\Gamma(x+1)} = e^{u'} - \Delta_n(u',L)$$

where

(17) $$\lim_{|u|\to\infty} u^n \Delta_n(u,L) = 0 \quad \text{and} \quad \lim_{|u'|\to\infty} u'^n \Delta_n(u',L) = 0;$$

$n = 0, 1, 2, \ldots s$. It is to be understood that the path of integration consists of the line drawn from the

4 Ford, op. cit., p. 65.

point $-2l+2p+n-2$ to infinity in the direction of the positive x-axis. We now have therefore,

$$(18)\quad \phi(u) = \frac{e^{u}}{2\sqrt{\pi}}\left(\frac{u}{2}\right)^{1-2p}\sum_{n=0}^{s}\frac{c_n}{u^n} - \frac{1}{2\sqrt{\pi}}\left(\frac{u}{2}\right)^{1-2p}\sum_{n=0}^{s}\frac{c_n D_n(u,L)}{u^n}$$

$$+\frac{1}{2\sqrt{\pi}}\left(\frac{u}{2}\right)^{1-2p}\frac{1}{u^s}\int_{-2l+2p+s-2}^{\infty}\frac{\delta(x-s,s)u^x dx}{\Gamma(x+1)} + \frac{e^{u'}}{2\sqrt{\pi}}\left(\frac{u'}{2}\right)^{1-2p}\sum_{n=0}^{s}\frac{c_n}{u'^n}$$

$$-\frac{1}{2\sqrt{\pi}}\left(\frac{u'}{2}\right)^{1-2p}\sum_{n=0}^{s}\frac{c_n D_n(u',L)}{u'^n} + \frac{1}{2\sqrt{\pi}}\left(\frac{u'}{2}\right)^{1-2p}\frac{1}{u'^s}\int_{-2l+2p+s-2}^{\infty}\frac{\delta(x-s,s)u'^x dx}{\Gamma(x+1)}$$

$$-\sum_{m=-l}^{-1} g(m)\left(\frac{u}{2}\right)^{2m} + \xi_1(l,u).$$

In studying this result it should be recalled that the properties (15) and (17) are applicable.

At this point in the analysis it is desirable to consider two sectors within the range of u, namely,

(Case I) $-\pi < \arg u < -\frac{\pi}{2}$ and (Case II) $-\frac{\pi}{2}$ arg $u < 0$;

the situation where arg $u = -\frac{\pi}{2}$ is excluded in this study.

Case I

The immediate problem in the consideration of this case pertains to the first integral of (18). This integral, ignoring the factor in front, may be readily changed to

$$(19) \qquad \frac{1}{u}\int_{-2l+2p+s-1}^{\infty}\frac{\delta(x-s-1,s)\,u^{x}\,dx}{\Gamma(x)}$$

In view of the fact that $u = u'(\text{Cos}-\pi + i\ \text{Sin}-\pi)$, which implies that $0 < \arg u' < \frac{\pi}{2}$, the integral factor of (19) may be written as

$$(20) \qquad I(u') = \int_{-2l+2p+s-1}^{\infty}\frac{P_1(x)\,u'^{x}\,dx}{\Gamma(x)} - i\int_{-2l+2p+s-1}^{\infty}\frac{P_2(x)\,u'^{x}\,dx}{\Gamma(x)}$$

where

$$P_1(x) = [\delta(x-s-1,s)][\text{Cos}\,\pi x]$$

and

$$P_2(x) = [\delta(x-s-1,s)][\text{Sin}\,\pi x].$$

Recalling relation (11), it is observed that

$$\delta(x,s) = A_1(x) + A_2(x)[(x+1)(x+2)\cdots(x+s)]\,\Gamma\left(\frac{x+1}{2}\right)\Gamma\left(\frac{x+2}{2}\right),$$

where $A_1(x)$ and $A_2(x)$ represent functions which are analytic in the finite portion of the plane. Thus

$$\delta(x,s) = A_1(x) + \frac{\sqrt{\pi}A_2(x)\,\Gamma(x+s+1)}{2^{x}},$$

since

$$\Gamma(2y) = \frac{2^{2y-1}}{\sqrt{\pi}}\Gamma(y)\Gamma\left(y+\frac{1}{2}\right),$$

and

$$\delta(x-s-1,s) = A_1(x-s-1) + \frac{\sqrt{\pi}\,A_2(x-s-1)\,\Gamma(x)}{2^{x-s-1}}$$

Hence it is readily observed that the integrands of

(20) are bounded for any particular finite portion of the plane. Therefore

(21) $$\lim_{|u'|\to\infty} e^{-u'} \int_{\alpha}^{\beta} \frac{P_1(x)\, u'^{x}\, dx}{T(x)} = 0,$$

and

$$\lim_{|u'|\to\infty} e^{-u'} \int_{\alpha}^{\beta} \frac{P_2(x)\, u'^{x}\, dx}{T(x)} = 0,$$

if α and β are chosen as any finite constants.

Also the integrals,

$$\int_0^{\infty} \frac{P_1(x)\, u'^{x}\, dx}{T(x)} \quad \text{and} \quad \int_0^{\infty} \frac{P_2(x)\, u'^{x}\, dx}{T(x)}$$

satisfy the conditions of a theorem due to Ford[5] if $P_1(x)$ and $P_2(x)$ are properly defined in the vicinity of the origin; thus,

$$\lim_{|u'|\to\infty} e^{-u'} \int_0^{\infty} \frac{P_1(x)\, u'^{x}\, dx}{T(x)} = 0,$$

and

$$\lim_{|u'|\to\infty} e^{-u'} \int_0^{\infty} \frac{P_2(x)\, u'^{x}\, dx}{T(x)} = 0,$$

since $\lim_{x\to\infty} P_1(x) = 0$ and $\lim_{x\to\infty} P_2(x) = 0$.

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5 Ford, op. cit., p. 38.

So, in summary,

(22) $$\lim_{|u'|\to\infty} e^{-u'} I(u') = 0.$$

It is now possible to simplify relation (18) under the conditions of Case I as follows:

(23) $$\varphi(u) = \frac{e^{u}}{2\sqrt{\pi}}\left(\frac{u}{2}\right)^{1-2p}\sum_{n=0}^{s}\frac{c_n}{u^n}$$

$$+\frac{e^{u'}}{2\sqrt{\pi}}\left(\frac{u'}{2}\right)^{1-2p}\left[\sum_{n=0}^{s}\frac{c_n}{u'^n}+\frac{R(u',s)}{u'^s}\right],$$

wherein $\lim_{|u'|=|u|\to\infty} R(u',s) = 0$, in view of the fact that

$$R(u',s) = e^{-u'}\Bigg[\left(\frac{u'}{u}\right)^{2p+s}\frac{I(u')}{u'}+\frac{1}{u'}\int_{-2l+2p+s-1}^{\infty}\frac{\delta(x-s-1,s)u'^x\,dx}{\Gamma(x)}$$

$$-\left(\frac{u'}{u}\right)^{2p-1}u'^s\sum_{n=0}^{s}\frac{c_n\Delta_n(u,L)}{u^n} - u'^s\sum_{n=0}^{s}\frac{c_n\Delta_n(u',L)}{u'^n}$$

$$-2\sqrt{\pi}\left(\frac{u'}{2}\right)^{2p-1}u'^s\sum_{m=-l}^{-1}g(m)\left(\frac{u}{2}\right)^{2m}+2\sqrt{\pi}\left(\frac{u'}{2}\right)^{2p-1}u'^s\xi_1(l,u)\Bigg],$$

where s is understood to be any arbitrarily large positive integer.

Using the symbol of asymptotic representation,

relation (23) may be written in the form,

$$(24)\quad \phi(u) \sim \frac{e^{u}}{2\sqrt{\pi}}\left(\frac{u}{2}\right)^{1-2\rho}\left[1+\frac{c_1}{u}+\frac{c_2}{u^2}+\cdots+\frac{c_n}{u^n}+\cdots\right]$$

$$+\frac{e^{u'}}{2\sqrt{\pi}}\left(\frac{u'}{2}\right)^{1-2\rho}\left[1+\frac{c_1}{u'}+\frac{c_2}{u'^2}+\cdots+\frac{c_n}{u'^n}+\cdots\right].$$

For Case II, where $-\frac{\pi}{2} < \arg u < 0$, the analysis is virtually identical with that employed in Case I. However, in considering Case II, the portion of (24) involving u becomes dominant instead of that involving u' as in the previous study. In truth, the result obtained in the second case is the same as (24).

Therefore, recalling that $u = 2z$ and that $u' = 2z' = -2z$, it follows that

$$(25)\quad f(z) \sim \frac{e^{2z} z^{3/2-(k_1+k_2)}}{2\sqrt{\pi}}\left[1+\frac{c_1}{2z}+\frac{c_2}{(2z)^2}+\cdots+\frac{c_n}{(2z)^n}+\cdots\right]$$

$$+\frac{e^{-2z}(-z)^{3/2-(k_1+k_2)}}{2\sqrt{\pi}}\left[1+\frac{c_1}{(-2z)}+\frac{c_2}{(-2z)^2}+\cdots+\frac{c_n}{(-2z)^n}+\cdots\right]$$

for the sectors, $-\pi < \arg z < -\frac{\pi}{2}$ and $-\frac{\pi}{2} < \arg z < 0$.

It is also a matter of interest to observe that the development (25) is also obtained for (Case I) $0 < \arg z < \frac{\pi}{2}$ and (Case II) $\frac{\pi}{2} < \arg z < \pi$ if in equation

(5) -1 be replaced by $\cos -\pi + i \sin -\pi$. The possibility of doing this follows as an immediate consequence of the fact that

$$\sum_{n=0}^{\infty} g(n) z^{2n} = \sum_{n=0}^{\infty} g(n)\left[z(\cos -\pi + i \sin -\pi)\right]^{2n}.$$

So in conclusion, we have obtained by quite a direct process the asymptotic development of f(z) as defined by

$$f(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{\Gamma(n+k_1)\Gamma(n+k_2)} .$$

Moreover, the form (25) furnishes a satisfactory generalization of the classical representation of the Bessel function obtained by Hankel.

CHAPTER II

FUNCTIONS OF THE DOUBLE BESSEL TYPE

In the first chapter, our study was concerned with an f(z) in which the g(n) was the reciprocal of the product of two gamma functions. This chapter purports to generalize the previous study to an extent.

Suppose that

$$(1) \qquad f(z) = \sum_{n=0}^{\infty} g(n)\, z^{4n}, \quad \text{rad. of conv.} = \infty;$$

where
$$g(n) = \frac{1}{\Gamma(n+k_1)\Gamma(n+k_2)\Gamma(n+k_3)\Gamma(n+k_4)}$$
and z^4 plays the same role as z in the theorem of Newsom.

It may readily be shown that g(n) satisfies the conditions set up in the aforementioned theorem when $k = 3$.[6]

Thus, it follows that

$$(2) \qquad f(z) = \int_{-l-\frac{1}{2}}^{\infty} \left\{ g(x)\, z^{4x} \frac{\sin 3\pi x}{\sin \pi x} \right\} dx - \sum_{m=-l}^{-1} g(m)\, z^{4m} + \xi(l,z);$$

where $\xi(l,z)$ has the property (4), (Chap. I), and $-\frac{\pi}{4} < \arg z < \frac{\pi}{4}$.

Due to a well known trigonometric formula, the

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6 Note Ford, op. cit., p. 61.

integral of (1) may be written as

$$(3)\qquad \int_{-\ell-\frac{1}{2}}^{\infty} \left\{ g(x)\, z^{4x} (1 + 2\cos 2\pi x) \right\} dx$$

$$= \int_{-\ell-\frac{1}{2}}^{\infty} \left\{ g(x) \left[z^{4x} + z'^{4x} + z''^{4x} \right] \right\} dx ;$$

where $\arg z' = \arg z - \frac{\pi}{2}$, and $\arg z'' = \arg z + \frac{\pi}{2}$.

In view of the nature of g(x), the existence of the integrals,

$$\int_{-\ell-\frac{1}{2}}^{\infty} g(x) z^{4x} dx\ , \int_{-\ell-\frac{1}{2}}^{\infty} g(x) z'^{4x} dx, \text{ and } \int_{-\ell-\frac{1}{2}}^{\infty} g(x) z''^{4x} dx$$

is readily demonstrable for any finite z.

Thus,

$$(4)\qquad f(z) = \int_{-\ell-\frac{1}{2}}^{\infty} \frac{z^{4x}}{\Gamma(x+k_1)\Gamma(x+k_2)\Gamma(x+k_3)\Gamma(x+k_4)} + \int_{-\ell-\frac{1}{2}}^{\infty} \frac{z'^{4x}\, dx}{\Gamma(x+k_1)\Gamma(x+k_2)\Gamma(x+k_3)\Gamma(x+k_4)}$$

$$+ \int_{-\ell-\frac{1}{2}}^{\infty} \frac{z''^{4x}\, dx}{\Gamma(x+k_1)\Gamma(x+k_2)\Gamma(x+k_3)\Gamma(x+k_4)} - \sum_{m=-\ell}^{-1} g(m) z^{4m} + \xi(\ell, z);$$

where $\xi(\ell,z)$ has the property (4), (Chap. I), and $-\frac{\pi}{4} < \arg z < \frac{\pi}{4}$, $\arg z' = \arg z - \frac{\pi}{2}$, $\arg z'' = \arg z + \frac{\pi}{2}$.

Employing a procedure similar to the paragraph following (8), (Chap. I), the first integral of (4) may be re-expressed as

$$(5)\qquad \frac{2^{2\rho_1+2\rho_2-2}}{\pi} \int_{-\ell-\frac{1}{2}}^{\infty} \frac{F_1(x)\cdot F_2(x)\cdot (2z)^{4x}\, dx}{\Gamma(2x+2\rho_1)\Gamma(2x+2\rho_2)}$$

where
$$F_1(x) = \frac{\Gamma(x+p_1)\Gamma(x+p_1+\frac{1}{2})}{\Gamma(x+k_1)\Gamma(x+k_2)}$$
$$F_2(x) = \frac{\Gamma(x+p_2)\Gamma(x+p_2+\frac{1}{2})}{\Gamma(x+k_3)\Gamma(x+k_4)}.$$

Using the identity, $\Gamma(2y) = \frac{2^{2y-1}}{\sqrt{\pi}}\Gamma(y)\Gamma(y+\frac{1}{2})$, it is readily seen that (5) becomes

(6)
$$\frac{2^{2p_1+2p_2+4p_3-3}}{\pi^{3/2}}\int_{-l-\frac{1}{2}}^{\infty}\frac{F_1(x)\cdot F_2(x)\cdot F_3(x)\,(4z)^{4x}\,dx}{\Gamma(4x+4p_3)},$$

where
$$F_3(x) = \frac{\Gamma(2x+2p_3)\Gamma(2x+2p_3+\frac{1}{2})}{\Gamma(2x+2p_1)\Gamma(2x+2p_2)}.$$

Upon making the transformations, $4x = x' - 4p_3 + 1$ and $z = u/4$, and dropping primes (6) becomes

(7)
$$\frac{2^{2p_1+2p_2}}{2^4\pi^{3/2}}\left(\frac{u}{2}\right)^{1-4p_3}\int_{-4l+4p_3-3}^{\infty}\frac{F_1'(x)\cdot F_2'(x)\cdot F_3'(x)\,u^x\,dx}{\Gamma(x+1)}$$

where $-\frac{\pi}{4} < \arg u < \frac{\pi}{4}$,
$$F_1'(x) = \frac{\Gamma(\frac{x}{4}+p_1-p_3+\frac{1}{4})\Gamma(\frac{x}{4}+p_1-p_3+\frac{3}{4})}{\Gamma(\frac{x}{4}-p_3+k_1+\frac{1}{4})\Gamma(\frac{x}{4}-p_3+k_2+\frac{1}{4})}$$
$$F_2'(x) = \frac{\Gamma(\frac{x}{4}+p_2-p_3+\frac{1}{4})\Gamma(\frac{x}{4}+p_2-p_3+\frac{3}{4})}{\Gamma(\frac{x}{4}-p_3+k_3+\frac{1}{4})\Gamma(\frac{x}{4}-p_3+k_4+\frac{1}{4})}$$

and
$$F_3'(x) = \frac{\Gamma(\frac{x+1}{2})\Gamma(\frac{x+2}{2})}{\Gamma(\frac{x}{2}+2p_1-2p_3+\frac{1}{2})\Gamma(\frac{x}{2}+2p_2-2p_3+\frac{1}{2})}.$$

Now it is possible to expand $F_1'(x)$ and $F_2'(x)$ in asymptotic series of the form

$$F_1'(x) \sim 1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n} + \cdots,$$

$$F_2'(x) \sim 1 + \frac{b_1}{x} + \frac{b_2}{x^2} + \cdots + \frac{b_n}{x^n} + \cdots,$$

if

$$p_1 = \frac{2k_1 + 2k_2 - 1}{4}$$

and $$p_2 = \frac{2k_3 + 2k_4 - 1}{4}.$$ [7]

But due to the well known properties of asymptotic series, it follows that

(8) $$F_1'(x) \cdot F_2'(x) = F_4(x) \sim 1 + \frac{d_1}{x} + \frac{d_2}{x^2} + \cdots + \frac{d_n}{x^n} + \cdots,$$

where $d_n = b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n$.[8]

By similar analysis, it follows that

(9) $$F_3'(x) \sim 1 + \frac{f_1}{x} + \frac{f_2}{x^2} + \cdots + \frac{f_n}{x^n} + \cdots$$

if $$p_3 = \frac{4p_1 + 4p_2 - 1}{4} = \frac{2k_1 + 2k_2 + 2k_3 + 2k_4 - 3}{8}.$$

Thus, multiplying series (8)and (9), we find that

(10) $$F_3'(x) \cdot F_4(x) = F(x) \sim 1 + \frac{g_1}{x} + \frac{g_2}{x^2} + \cdots + \frac{g_n}{x^n} + \cdots,$$

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7 Van Engen, _op. cit._, p. 125.

8 Walter B. Ford, _Studies on Divergent Series and Summability_, (The MacMillan Company, New York, 1916) p. 24.

where $g_n = d_n + f_1 d_{n-1} + \cdots + f_n$.

Since it is well known that a series of the form (10) may be written as a faculty series,[9]

$$(11) \qquad F(x) = 1 + \frac{e_1}{x+1} + \frac{e_2}{(x+1)(x+2)} + \cdots + \frac{c_n + \delta(x,s)}{(x+1)(x+2)\cdots(x+s)},$$

where $\lim_{x\to\infty} \delta(x,s) = 0$.

Hence, it is now possible to re-express (7) in the form

$$(12) \qquad \frac{2^{2p_1+2p_2}}{2^4 \pi^{3/2}} \left(\frac{u}{2}\right)^{1-4p_3} \int_{-4l+4p_3-3}^{\infty} \frac{F(x)\, u^x\, dx}{\Gamma(x+1)}$$

where $-\frac{\pi}{4} < \arg u < \frac{\pi}{4}$, and $F(x)$ is given by (11).

It follows immediately that (12) takes the form,

$$(13) \qquad \frac{2^{2p_1+2p_2}}{2^4 \pi^{3/2}} \left(\frac{u}{2}\right)^{1-4p_3} \left\{ \sum_{n=0}^{s} c_n \int_{-4l+4p_3-3}^{\infty} \frac{u^x\, dx}{\Gamma(x+n+1)} + \int_{-4l+4p_3-3}^{\infty} \frac{\delta(x,s)\, u^x\, dx}{\Gamma(x+s+1)} \right\}$$

where $c_0 = 1$, $\delta(x,s)$ has the property (11), and $-\frac{\pi}{4} < \arg u < \frac{\pi}{4}$.

Similar studies upon the other two integrals of (4) give analogous representations. Hence, after making

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9 Note Van Engen, op. cit., p. 125.

obvious transformations, we may write (4) as

$$(14)\quad \varphi(u) = \frac{2^{2p_1+2p_2}}{2^4\pi^{3/2}}\Big[\Big(\frac{u}{2}\Big)^{1-4p_3}\Big\{\sum_{n=0}^{s}\frac{c_n}{u^n}\int_{-4\ell+4p_3+n-3}^{\infty}\frac{u^x\,dx}{\Gamma(x+1)}$$

$$+\frac{1}{u^s}\int_{-4\ell+4p_3+s-3}^{\infty}\frac{\delta(x-s,s)u^x\,dx}{\Gamma(x+1)}\Big\}+\Big(\frac{u'}{2}\Big)^{1-4p_3}\Big\{\sum_{n=0}^{s}\frac{c_n}{u'^n}\int_{-4\ell+4p_3+n-3}^{\infty}\frac{u'^x\,dx}{\Gamma(x+1)}$$

$$+\frac{1}{u'^s}\int_{-4\ell+4p_3+s-3}^{\infty}\frac{\delta(x-s,s)u'^x\,dx}{\Gamma(x+1)}\Big\}+\Big(\frac{u''}{2}\Big)^{1-4p_3}\Big\{\sum_{n=0}^{s}\frac{c_n}{u''^n}\int_{-4\ell+4p_3+n-3}^{\infty}\frac{u''^x\,dx}{\Gamma(x+1)}$$

$$+\frac{1}{u''^s}\int_{-4\ell+4p_3+s-3}^{\infty}\frac{\delta(x-s,s)u''^x\,dx}{\Gamma(x+1)}\Big\}\Big]-\sum_{m=-\ell}^{-1}g(m)\Big(\frac{u}{4}\Big)^{4m}+\xi(\ell,u);$$

wherein it is recalled that $-\frac{\pi}{4}<\arg u<\frac{\pi}{4}$, $\arg u' = \arg u-\frac{\pi}{2}$, and $\arg u'' = \arg u+\frac{\pi}{2}$.

Now let L be the largest integer in $4\ell-4R(p)-n+3$, where ℓ has been taken so large that $4\ell-4R(p)-n+3>s$, $n = 0, 1, 2, \ldots s$. Since $4\ell-4R(p)-n+3 \geq 4\ell-4R(p)-s+3$, the desired condition is realized if

$$(15)\qquad \ell > R(p)+\frac{s}{2}-\frac{3}{2}.$$

Under these conditions and those on u, u', and u'', it follows that[10]

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[10] See the analysis in Chapter I following statement (15).

$$(16)\quad \phi(u) = \frac{2^{2p_1+2p_2}}{2^4\pi^{3/2}}\Bigg[e^{u}\left(\frac{u}{2}\right)^{1-4p_3}\sum_{n=0}^{s}\frac{c_n}{u^n} - \left(\frac{u}{2}\right)^{1-4p_3}\sum_{n=0}^{s}\frac{c_n D_n(u,L)}{u^n}$$

$$+\left(\frac{u}{2}\right)^{1-4p_3}\frac{1}{u^s}\int_{-4l+4p_3+s-3}^{\infty}\frac{\sigma(x-s,s)\,u^x\,dx}{\Gamma(x+1)} + e^{u'}\left(\frac{u'}{2}\right)^{1-4p_3}\sum_{n=0}^{s}\frac{c_n}{u'^n}$$

$$-\left(\frac{u'}{2}\right)^{1-4p_3}\sum_{n=0}^{s}\frac{c_n D_n(u',L)}{u'^n} + \left(\frac{u'}{2}\right)^{1-4p_3}\frac{1}{u'^s}\int_{-4l+4p_3+s-3}^{\infty}\frac{\sigma(x-s,s)\,u'^x\,dx}{\Gamma(x+1)}$$

$$+e^{u''}\left(\frac{u''}{2}\right)^{1-4p_3}\sum_{n=0}^{s}\frac{c_n}{u''^n} - \left(\frac{u''}{2}\right)^{1-4p_3}\sum_{n=0}^{s}\frac{c_n D_n(u'',L)}{u''^n}$$

$$+\left(\frac{u''}{2}\right)^{1-4p_3}\frac{1}{u''^s}\int_{-4l+4p_3+s-3}^{\infty}\frac{\sigma(x-s,s)\,u''^x\,dx}{\Gamma(x+1)}\Bigg] - \sum_{m=-l}^{-1} g(m)\left(\frac{u}{4}\right)^{4m} + f(l,w).$$

At this point it is desirable to divide the analysis into two cases, (Case I) $-\frac{\pi}{4} < \arg u < 0$ and (Case II) $0 < \arg u < \frac{\pi}{4}$. Employing an argument similar to that immediately following statement (19) in Chapter I, it is readily found that

$$(17)\quad \phi(u) = \frac{2^{2p_1+2p_2}}{2^4\pi^{3/2}}\Bigg[e^{u}\left(\frac{u}{2}\right)^{1-4p_3}\left\{\sum_{n=0}^{s}\frac{c_n}{u^n} + \frac{R(u,s)}{u^s}\right\}$$

$$+e^{u'}\left(\frac{u'}{2}\right)^{1-4p_3}\sum_{n=0}^{s}\frac{c_n}{u'^n} + e^{u''}\left(\frac{u''}{2}\right)^{1-4p_3}\left\{\sum_{n=0}^{s}\frac{c_n}{u''^n} + \frac{R(u'',s)}{u''^s}\right\}\Bigg],$$

where

$$R(u,s) = e^{-u}\left[-u^{s}\sum_{n=0}^{s}\frac{c_n\Delta_n(u,L)}{u^{n}} + \frac{1}{u}\int_{-4l+4p_3+s-2}^{\infty}\frac{\delta(x-s-1,s)u^{x}dx}{\Gamma(x)}\right.$$

$$-\left(\frac{u'}{u}\right)^{1-4p_3}u^{s}\sum_{n=0}^{s}\frac{c_n\Delta_n(u',L)}{u'^{n}} + \left(\frac{u'}{u}\right)^{1-4p_3-s}\frac{I(u)}{u'}$$

$$\left.+\frac{2^4\pi^{3/2}}{2^{2p_1+2p_2}}\left(\frac{u}{2}\right)^{4p_3-1}u^{s}\left\{-\sum_{m=-l}^{-1}g(m)\left(\frac{u}{4}\right)^{4m} + \xi(l,u)\right\}\right],$$

$$R(u'',s) = e^{-u''}\left[-u''^{s}\sum_{n=0}^{s}\frac{c_n\Delta_n(u'',L)}{u''^{n}} + \frac{1}{u''}\int_{-4l+4p_3+s-2}^{\infty}\frac{\delta(x-s-1,s)u''^{x}dx}{\Gamma(x)}\right],$$

$$I(u) = \int_{-4l+4p_3+s-2}^{\infty}\frac{\delta(x-s-1,s)u'^{x}dx}{\Gamma(x)},$$

$\lim_{|u|\to\infty} R(u,s) = 0$, and $\lim_{|u''|=|u|\to\infty} R(u'',s) = 0$.

It is to be noted at this point that $\delta(x,s)$ and hence $\delta(x-s-1,s)$ can be treated as in the paragraphs following statement (20), (Chapter I). Of course, in this case $\delta(x,s)$ is found to be related to

a number of similar functions[11] which are all bounded for any finite portion of the plane. Thus, it follows that $\delta(x-s-1,s)$ must be bounded in any finite portion of the plane.

Thus, using the symbol of asymptotic representation, relation (15) may be written in the form,

$$(18)\quad \phi(u) \sim \frac{2^{2p_1+2p_2}}{2^4\pi^{3/2}} e^{u}\left(\frac{u}{2}\right)^{1-4p_3}\left[1+\frac{c_1}{u}+\frac{c_2}{u^2}+\cdots+\frac{c_n}{u^n}+\cdots\right]$$

$$+\frac{2^{2p_1+2p_2}}{2^4\pi^{3/2}} e^{u'}\left(\frac{u'}{2}\right)^{1-4p_3}\left[1+\frac{c_1}{u'}+\frac{c_2}{u'^2}+\cdots+\frac{c_n}{u'^n}+\cdots\right]$$

$$+\frac{2^{2p_1+2p_2}}{2^4\pi^{3/2}} e^{u''}\left(\frac{u''}{2}\right)^{1-4p_3}\left[1+\frac{c_1}{u''}+\frac{c_2}{u''^2}+\cdots+\frac{c_n}{u''^n}+\cdots\right].$$

Of course, it is to be observed that in this case, the portions involving u and u'' are dominant. If the same analysis is repeated for Case I[1], an identical representation is obtained, but the portions involving u and u' are now dominant.

Therefore, recalling that $u = 4z$, $u' = 4z' = -4iz$, $u'' = 4z'' = 4iz$, it follows that

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11 See the expansions found in the paragraphs containing relations (7)-(10).

$$(19)\quad f(z) \sim \frac{e^{4z} z^{5/2-(k_1+k_2+k_3+k_4)}}{2^{5/2}\pi^{3/2}}\left[1+\frac{c_1}{4z}+\frac{c_2}{(4z)^2}+\cdots+\frac{c_n}{(4z)^n}+\cdots\right]$$

$$+\frac{e^{-4iz}(-iz)^{5/2-(k_1+k_2+k_3+k_4)}}{2^{5/2}\pi^{3/2}}\left[1+\frac{c_1}{-4iz}+\frac{c_2}{(-4iz)^2}+\cdots+\frac{c_n}{(-4iz)^n}+\cdots\right]$$

$$+\frac{e^{4iz}(iz)^{5/2-(k_1+k_2+k_3+k_4)}}{2^{5/2}\pi^{3/2}}\left[1+\frac{c_1}{4iz}+\frac{c_2}{(4iz)^2}+\cdots+\frac{c_n}{(4iz)^n}+\cdots\right]$$

for the sectors $-\frac{\pi}{4} < \arg z < 0$ and $0 < \arg z < \frac{\pi}{4}$.

It is again a matter of interest to observe that the development (19) is also obtained for the sector $-\frac{3\pi}{4} < \arg z < -\frac{\pi}{4}$, if one considers the two distinct cases, (Case I) $-\frac{3\pi}{4} < \arg z < -\frac{\pi}{2}$, and (Case II) $-\frac{\pi}{2} < \arg z < -\frac{\pi}{4}$. The possibility of this new analysis follows from the fact that in relation (2) the factor of unity can be replaced by $\operatorname{Cos} 2\pi + i \operatorname{Sin} 2\pi$, inasmuch as

$$\sum_{n=0}^{\infty} g(n) z^{4n} = \sum_{n=0}^{\infty} g(n)\left[z\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\right]^{4n}.$$

A similar analysis will show that (19) holds over the sector $\frac{\pi}{4} < \arg z < \frac{3\pi}{4}$ if one considers (Case I) $\frac{\pi}{4} < \arg z < \frac{\pi}{2}$ and (Case II) $\frac{\pi}{2} < \arg z < \frac{3\pi}{4}$. For this study, the factor of unity is replaced by $\operatorname{Cos}(-2\pi) + i \operatorname{Sin}(-2\pi)$ in relation (4).

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