Upon the Asymptotic Representation of Entire Functions Where the General Coefficient is the Product of Two Gamma Functions

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UPON THE ASYMPTOTIC REPRESENTATION OF ENTIRE FUNCTIONS WHERE THE GENERAL COEFFICIENT IS THE PRODUCT OF TWO GAMMA FUNCTIONS

By

James R. Ellis

A Thesis

Submitted in partial fulfillment of the

Requirements for the Degree of

Master of Science in Mathematics

University of New Mexico

1939

This thesis, directed and approved by the candidate's committee, has been accepted by the Graduate Committee of the University of New Mexico in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

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ACKNOWLEDGMENT

I desire to express my appreciation to Dr. C. V. Newsom whose helpful suggestions and constructive criticism have been invaluable in the development of this work, To Dr. H. D. Larsen for his aid in some of the more intricate operations and for his helpful suggestions pertaining to the final form of the work, and to Mr. LeRoy Gibson for criticism of the final manuscript.

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CHAPTER I

PRELIMINARY CONSIDERATION OF THE THEOREM OF NEWSOM

The essential purpose of this paper is to obtain further information in regard to the asymptotic representation of the power series,

$$(1)\qquad \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+k_1)\Gamma(n+k_2)\cdots\cdots\Gamma(n+k_n)},$$

where z is a complex variable and k_1, k_2, ----k_3 are constants, real or complex. This series is a special case of the summation,

$$(2)\qquad \sum_{n=0}^{\infty} g(n)z^n, \text{ radius of convergence} = \infty.$$

The greater part of this work will, therefore, necessarily be concerned with the general series as given in (2). The treatment is simply a continuation of the studies that have been made by Barnes, Ford, Newsom, Harp, Van Engen, and others.[1]

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[1] For a rather complete summary of the work which been done see Walter B. Ford, The Asymptotic Developments of Functions Defined by McLaurin Series, (Ann Arbor, Michigan, University of Michigan Press, 1936).

Van Engen[2] was especially concerned with the consideration of series of the type (1). He was, however, handicapped by lack of an adequate theorem covering the general case, (2). In this study, the advantages of a very recent theorem established by Newsom[3] will be utilized.

The theorem of Newsom is as follows:

Theorem: Let it be assumed that the coefficient g(n) occurring in the general term of the power series (2) may be regarded as a function g(w) of the complex variable $w = x+iy$ and as such satisfies the following two conditions:

(a) is single valued and analytic throughout the finite w-plane.

(b) is such that for all values of x and y one may write

$$(3) \qquad \left| g(x+iy) \right| < K e^{k\pi|y|}$$

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[2] Henry Van Engen, "On Asymptotic Behavior of Analytic Functions," (unpublished Doctor's dissertation, University of Michigan, Ann Arbor, Michigan, 1934. 57 pp.)

[3] Newsom, C. V., "On the Character of Certain Entire Functions in Distant Portions of the Plane," American Journal of Mathematics, Vol. LX, No. 3, (July, 1938), pp. 561-572.

where K is a constant and independent of x and y, and k is a positive integer depending upon the $|g(x+iy)|$.

Then the functions $f(z)$ defined by the series (2) when considered for all values of z satisfying the conditions, $-\pi < \arg \pm z < \pi$ may be expressed in the form

$$(4) \quad f(z) = \int_{-l-\frac{1}{2}}^{\infty} \left\{ g(x) [\pm z] \frac{\sin k\pi x}{\sin \pi x} \right\} dx - \sum_{m=-l}^{-1} g(m) z^m + \xi_k(l, z);$$

wherein l is any arbitrary positive integer and the upper or lower of the signs $\pm$ is taken according as k is odd or even. Moreover, the expression $\xi_k(l, z)$, is such that

$$(5) \quad \lim_{|z| \to \infty} z^l \xi_k(l, z) = 0$$

irrespective of the value chosen for l.

The difficulty encountered in the use of this theorem is that the integral appearing in (4) has not been well understood. In truth, further studies upon the integral must precede any important uses of the theorem. Accordingly, our attention for the time being will be confined to this integral.

It is desirable first that we should establish the following relation concerning the trigonometric factor involved in the integrand of (4):

$$(6)\quad \frac{\sin k\pi x}{\sin \pi x} = 2\cos \pi x + 2\cos 3\pi x + \cdots + 2\cos (k-1)\pi x,$$

where k is any even positive integer.

For this proof it is found advantageous to write the relation in the form:

$$\frac{\sin 2n\pi x}{\sin \pi x} = 2\cos \pi x + 2\cos 3\pi x + \cdots + 2\cos (2n-1)\pi x.$$

Using the method of mathematical induction, the observation is first made that when $n=1$, the following relation is known to be true:

$$\frac{\sin 2\pi x}{\sin \pi x} = \frac{2\sin \pi x \cos \pi x}{\sin \pi x} = 2\cos \pi x.$$

Thus the formula to be established is correct for the case, $n=1$. Let us then assume the truth of the general statement,

$$\frac{\sin 2n\pi x}{\sin \pi x} = 2\cos \pi x + 2\cos 3\pi x + \cdots + 2\cos (2n-1)\pi x.$$

It follows that, adding $2\cos(2n+1)$ to both members,

$$\frac{\sin 2n\pi x + 2\cos(2n+1)\pi x \sin \pi x}{\sin \pi x} =$$

$$2\cos \pi x + 2\cos 3\pi x + - - + 2\cos(n-1)\pi x$$
$$+ 2\cos(n+1)\pi x.$$

To complete the induction it is necessary to show that

$$\sin 2n\pi x + 2\cos(2n+1)\pi x \sin \pi x = \sin(2n+2)\pi x.$$

To demonstrate this let us rewrite the left hand member as,

$$\sin(\overline{2n+2}-2)\pi x + 2\cos(\overline{2n+2}-1)\pi x \sin \pi x.$$

Employing the trigonometric identities for the sine and cosine of the difference of two angles the expression becomes,

$$\sin(2n+2)\pi x \cos 2\pi x - \cos(2n+2)\pi x \sin 2\pi x$$
$$+ 2\sin \pi x\{\cos(2n+2)\pi x \cos \pi x + \sin(2n+2)\pi x \sin \pi x\}$$

Substituting the values of $\cos 2\pi x$ and $\sin 2\pi x$ in the first and second terms we may write,

$$\sin(2n+2)\pi x\{1 - 2\sin^2 \pi x\}$$
$$- \cos(2n+2)\pi x \cdot 2\sin \pi x \cos \pi x$$
$$+ 2\sin \pi x\{\cos(2n+2)\pi x \cos \pi x$$
$$+ \sin(2n+2)\pi x \cdot \sin \pi x.$$

After performing the indicated multiplications this expression reduces to,

$$\sin(2n+2)\pi x.$$

Hence the induction is complete and the relation (6) has been established.

By employing the same procedure it may be established that,

$$\frac{\sin k\pi x}{\sin \pi x} = 1 + 2\cos 2\pi x + 2\cos 4\pi x + \cdots + 2\cos(k-1)\pi x, \tag{7}$$

where k is odd.

After applying the facts established in relation (6) and (7) to the integrand of (4), two possibilities appear; namely, where k is larger than the least value of k for which the relation (4) holds true, and (2) where k is the least value for which the conditions are satisfied.

The ensuing analysis will therefore be divided into two parts,--the first, where k is larger than the least value, and the second, where k is the least value which k may assume.

After performing the indicated multiplication in the last expression it becomes

$$\sin(km+k)\pi x.$$

Hence the induction is complete and the relation (6) has been established.

By employing the same procedure it may be established that,

$$\frac{\sin k\pi x}{\sin \pi x} = 1 + 2\cos 2\pi x + 2\cos 4\pi x + \cdots + 2\cos(k-1)\pi x, \tag{7}$$

where k is odd.

After applying the Fourier analysis to relations (6) and (7) to the integrand of (2), we [illegible] appear[illegible], where k [illegible] the least value of k for which the conditions (ii), [illegible] and (i) [illegible] where k is the least value for which the conditions are satisfied.

The ensuing analysis will therefore be divided into two parts,--the first, where k is larger than the least value, and the second, where k is the least value which k may assume.

Part I

(k is greater than the least value for which the relation is true)

Inasmuch as k can be any positive integer there are two separate studies involved in part I, depending upon whether k is odd or even.

The first study undertaken will involve the case in which k assumes odd values. As a result of this assumption, the expansion of the trigonometric factor within the integral of (4) will take the form shown in (7), the integral then becoming,

$$(8) \int_{-e-1/2}^{\infty} \{g(x) Z^{x}\}\{1 + 2\cos 2\pi x + 2\cos 4\pi x + --- + 2\cos(k-1)\pi x\} dx.$$

If c represents the least value which k can assume, then k may be represented by $k=(c+2n)$ when c is odd, and by $k=(c+2n+1)$ when c is even.

Using the notation for the case where k and c are both odd and $n=0$, the integral (8) becomes,

$$\int_{-e-1/2}^{\infty} \{g(x) Z^{x}\}\{1 + 2\cos 2\pi x + 2\cos 4\pi x + -- + 2\cos(c-1)\pi x\} dx.$$

Replacing the integral appearing in (4) by the expression just developed, $f(z)$ is expressible as,

(9) $$f(z) = \int_{-k-\frac{1}{2}}^{\infty} \{g(x) z^x\} \{1 + 2\cos 2\pi x + 2\cos 4\pi x + \cdots + 2\cos (c-1)\pi x\} dx - \sum_{m=-l}^{-1} g(m) z^m + \xi_c(l, z).$$

Now $f(z)$ is known to exist. Since the right hand member is equal to $f(z)$ it must exist also. Due to condition (5), $\xi_c(l, z)$ exists and approaches zero as a limit as z approaches infinity. In the summation, the summation is finite and involves z to negative exponents, hence it will approach zero as z approaches infinity. Therefore it follows that the integral must exist.

When $n = 1$, (k and c both odd) the integral (8) becomes,

$$\int_{-k-\frac{1}{2}}^{\infty} \{g(x) z^x\} \{1 + 2\cos 2\pi x + 2\cos 4\pi x + \cdots + 2\cos (c-1)\pi x + 2\cos (c+1)\pi x\} dx.$$

Using the foregoing integral to replace its equivalent in (4), $f(z)$ is also expressible as,

(10) $$f(z) = \int_{-k-\frac{1}{2}}^{\infty} \{g(x) z^x\} \{1 + 2\cos 2\pi x + \cdots + 2\cos (c-1)\pi x + 2\cos (c+1)\pi x\} dx - \sum_{m=-l}^{-1} g(m) z^m + \xi_{c+2}(l, z).$$

By the process used to establish the existence of the integral in (9) it follows that the integral of (10) exists also.

Knowing the characteristics of each of the expressions appearing in (9) and (10) they may be combined in the following manner in view of the fact they are each equal to $f(Z)$.

$$\int_{-l-1/2}^{\infty}\{g(x)Z^x\}\{1+2\cos 2\pi x+2\cos 4\pi x+--$$
$$+2\cos(c-1)\pi x\}dx-\sum_{m=-l}^{-1} g(m)Z^m+\mathcal{G}_c(l,Z)=$$
$$\int_{-l-1/2}^{\infty}\{g(x)Z^x\}\{1+2\cos 2\pi x+--+2\cos(c-1)\pi x$$
$$+2\cos(c+1)\pi x\}dx-\sum_{m=-l}^{-1} g(m)Z^m+\mathcal{G}_{c+2}(l,Z).$$

After collecting the integrals on the left and the functions on the right, this equality reduces to,

$$\int_{-l-1/2}^{\infty}\{g(x)Z^x\}\{1+2\cos 2\pi x+2\cos 4\pi x+--$$
$$+2\cos(c-1)\pi x\}dx-\int_{-l-1/2}\{g(x)Z^x\}\{1+2\cos 2\pi x+-$$
$$--+2\cos(c-1)\pi x+2\cos(c+1)\pi x\}dx$$
$$=\mathcal{G}_{c+2}(l,Z)-\mathcal{G}_{c}(l,Z).$$

Since the two integrals exist they may be combined and the resulting integral is,

$$\int_{-\ell-1/2}^{\infty}\{g(x)Z^{x}\}\{[1+2\cos 2\pi x+2\cos 4\pi x+\cdots+2\cos(c-1)\pi x]$$

$$-[1+2\cos 2\pi x+\cdots+2\cos(c-1)\pi x+2\cos(c+1)\pi x]\}dx.$$

It is apparent that after simple algebraic cancellation, we obtain,

$$\int_{-\ell-1/2}^{\infty}\{g(x)Z^{x}\}\{-2\cos(c+1)\pi x\}dx=\xi_{c+2}(\ell,Z)-\xi_{c}(\ell,Z).$$

Referring to (5) we know that both $\xi_{c+2}(\ell,Z)$ and $\xi_{c}(\ell,Z)$, when multiplied by Z^{ℓ}, approach zero as a limit as Z approaches infinity; therefore,

$$\lim_{|Z|\to\infty} Z^{\ell}\int_{-\ell-1/2}^{\infty}\{g(x)Z^{x}\}\cos(c+1)\pi x\}dx=0.$$

By a similar procedure involving the cases where $n=1$ and $n=2$, one can establish the fact that,

$$\lim_{|Z|\to\infty} Z^{\ell}\int_{-\ell-1/2}^{\infty}\{g(x)Z^{x}\}\{\cos(c+3)\pi x\}dx=0.$$

or in general,

$$(11)\quad \lim_{|z|\to\infty} z^{c} \int_{-c-\frac{1}{2}}^{\infty} \{g(x) z^{x}\}\{\cos(c+2n+1)\pi x\}\, dx = 0,$$

where $n = 0, 1, 2, 3, \cdots,$ and c is odd.

Continuing the discussion to the situation where k is odd and c is even, an analysis of the preceding type shows that,

$$(12)\quad \lim_{|z|\to\infty} z^{c} \int_{-c-\frac{1}{2}}^{\infty} \{g(x) z^{x}\}\{\cos(c+2n)\pi x\}\, dx = 0,$$

where $n = 1, 2, 3, \cdots,$ and c is even.

Next it will be necessary to study the case where k is even. It will be recalled that the expansion of the trigometric factor of the integral of (4) will take the form shown in (6), and the integral will then become,

$$(13)\quad \int_{-c-\frac{1}{2}}^{\infty} \{g(x)(-z)^{x}\}\{2\cos\pi x + 2\cos 3\pi x + \cdots + 2\cos(k-1)\pi x\}\, dx.$$

A treatment of this form for the integral of (4) similar to the analysis where k is odd, will show for cases where k

is even,

$$(14)\quad \lim_{|Z|\to\infty} Z^{l}\int_{-l-1/2}^{\infty}\{g(X)(-Z)^{X}\}\{\cos(c+2n+1)\pi X\}\,dX=0$$

where $n = 0, 1, 2, 3, \cdot\cdot\cdot$, and c is even; and

$$(15)\quad \lim_{|Z|\to\infty} Z^{l}\int_{-l-1/2}^{\infty}\{g(X)(-Z)^{X}\}\{\cos(c+2n)\pi X\}\,dX=0,$$

where $n = 1, 2, 3, \cdot\cdot\cdot\cdot\cdot$, and c is odd.

It has thus been demonstrated that in any problem, if the integer used to represent k is larger than the least value for which the conditions are satisfied, the result obtained will be the same in the limit as that obtained if the minimum integer had been used.

The analysis thus far has been made under the assumption that the restrictions on the argument of Z as set forth in the theorem of Newsom, namely, $(-\pi < \arg \pm Z < \pi)$ are valid. However, more careful attention to the argument will be necessary in the ensuing analysis. Therefore, if θ is the argument of Z, the following observations are made;

(16) when k is odd,

$$-\pi < \theta < \pi;$$

(17) and, when k is even;

$$-2\pi < \theta < 0$$

Part II

(k is the least value for which the conditions are satisfied).

In this treatment let us consider the possibilities, $k = c = 1, 2, 3, \cdots$

For the case where $c = 1$, the expression (4) immediately becomes,

$$f(Z) = \int_{-k-\frac{1}{2}}^{\infty} g(x) Z^x dx - \sum_{m=-k}^{-1} g(m) Z^m + \xi_1(k, Z);$$

where $-\pi < \theta < \pi$.

For the case where $c = 2$, it is evident that the trigonometric factor within the integral of (4) will be of the form exhibited in (6). In addition, since c is even, $(-Z)^x$ appears in the integrand and the argument of Z will be restricted to the region defined in (17). Therefore, $f(Z)$ is expressible as,

$$(18) \quad f(Z) = \int_{-k-\frac{1}{2}}^{\infty} \{ g(x)(-Z)^x (2 \cos 2\pi x) \} dx$$

$$- \sum_{m=-k}^{-1} g(m) Z^m + \xi_2(k, Z);$$

where $-2\pi < \theta < 0.$

The integral appearing above will now be the subject of our consideration.

If $Z = \rho(\cos Q + i \sin Q)$

then,

$$(-Z)^x = [\rho\{\cos(Q+\pi) + i \sin(Q+\pi)\}]^x.$$

Applying the DeMoivre's theorem to the right hand member of this equation, the expression may be written,

$$(-Z)^x = \rho^x\,[\cos(QX+\pi X) + i \sin(QX+\pi X)].$$

Recalling that when complex numbers are multiplied the arguments are added and the absolute values multiplied, the right hand member is expressible as,

$$\rho^x(\cos QX + i \sin QX)(\cos \pi X + i \sin \pi X).$$

But, the product of the first two members of this expression may be equated to Z^x ; therefore,

(19) $$(-Z)^x = Z^x(\cos \pi X + i \sin \pi X).$$

Replacing $(-Z)^x$ in the integral of (18) by the equivalent expressed in (20), the integral becomes,

$$\int_{-e-\frac{1}{2}}^{\infty}\{g(x)(\cos \pi X + i \sin \pi X)(Z \cos \pi X)Z^x\}dx.$$

This integral may be changed at once to the form,

$$\int \{ g(x)\, 2\cos^2 \pi x + 2 i \sin \pi x \cos \pi x)\, Z^x \} \, dx.$$

Replacing $2\cos^2 \pi x$ by $1 + \cos 2\pi x$ and $2 \sin \pi x \cos \pi x$ by $\sin 2\pi x$, and then grouping terms, we have,

$$\int_{-l-1/2}^{\infty} g(x)\, Z^x [1 + (\cos 2\pi x + i \sin 2\pi x)]\, dx;$$

or,

$$\int_{-l-1/2}^{\infty} g(x)\, [Z^x + (\cos 2\pi x + i \sin 2\pi x)\, Z^x]\, dx.$$

The second member of the binomial within the integrand may be written,

$$\rho^x (\cos 2\pi x + i \sin 2\pi x)(\cos Qx + i \sin Qx).$$

Recalling again, that in multiplying complex numbers we add their arguments, the last expression becomes,

$$\rho^x [\cos (Q + 2\pi) x + i \sin (Q + 2\pi) x],$$

which in turn, is

$$\rho^x [\cos (Q + 2\pi) + i \sin (Q + 2\pi)]^x$$

or,

$$\{ \rho [\cos (Q + 2\pi) + i \sin (Q + 2\pi)] \}^x.$$

We shall now define

$$\rho\,[\cos(\theta+2\pi)+i\sin(\theta+2\pi)] \text{ as } Z';$$

where

$$|Z'| = \rho = |Z|,$$

and

$$\arg Z' = \theta + 2\pi.$$

Therefore the integral may now be written,

$$\int_{-\ell-1/2}^{\infty} g(x)\,[Z^x + Z'^x]\,dx;$$

where,

$$-2\pi < \theta < 0,$$

and,

$$\arg Z' = \theta + 2\pi.$$

Before separating this integral into two integrals it is necessary that the resulting integrals, namely,

$$\int_{-\ell-1/2}^{\infty} g(x)\,Z^x\,dx \text{ and } \int_{-\ell-1/2}^{\infty} g(x)\,Z'^x\,dx,$$

should exist.

Now,

$$\left|\int_{-\ell-1/2}^{\infty} g(x)\,Z^x\,dx\right| \leq \int_{-\ell-1/2}^{\infty} |g(x)|\,\rho^x\,dx,$$

and,

$$\left|\int_{-\ell-1/2}^{\infty} g(x)\,Z'^x\,dx\right| \leq \int_{-\ell-1/2}^{\infty} |g(x)|\,\rho^x\,dx.$$

Thus, if the integral appearing in each case upon the right exists, each of the original integrals exists.

The immediate applications of this work are to those series wherein $g(n)$ is the reciprocal of the product of gamma functions as displayed in (1).

It is known that

$$\log \Gamma(x) = \log \sqrt{2\pi} + (x - \tfrac{1}{2}) \log x - x + \omega(x),$$

where $\lim_{x \to \infty} \omega(x) = 0$.[4] Thus the existence of the integral under consideration depends upon the behavior of

$$\frac{\rho^x}{x^{x-\frac{1}{2}} e^{-x}}$$

as x approaches infinity. But, this fraction approaches zero to a very high order for any particular finite value of ρ, as x approaches infinity. Thus the integral under consideration exists for any particular value of z.

Since the integrals exist, then,

$$\int_{-e-\frac{1}{2}}^{\infty} g(x)(z^x + z'^x)\,dx = \int_{-e-\frac{1}{2}}^{\infty} g(x)\, z^x\,dx + \int_{-e-\frac{1}{2}}^{\infty} g(x)\, z'^x\,dx.$$

Replacing the integral in (18) by this equivalent

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[4] Ford, op. cit., p. 74, (6).

Thus, if the integral converging in each of the [illegible] exists, each of the original integrals exists.

The immediate applications of this work are to those series wherein $g(n)$ is the [illegible] of the product of gamma functions discussed in (1).

It is known that

$$\log \Gamma(x) = \log \sqrt{2\pi} + (x-\tfrac{1}{2}) \log x - x + \omega(x),$$

where $\lim_{x \to \infty} \omega(x) = 0$. Thus the existence of the integral under consideration depends upon the behavior of

$$\frac{\rho^{x}}{x^{x-1/2}\, e^{-x}}$$

as x approaches infinity. Now, this fraction approaches zero to a very high order for any arbitrary finite value of ρ, as x approaches infinity, [illegible] integral under consideration exists for any particular value of z.

Since the integrals exist, then,

$$\int_{-z-1/2}^{\infty} g(x)(z^{x}+z^{-x})\,dx = \int_{-z-1/2}^{\infty} g(x) z^{x}\,dx + \int_{-z-1/2}^{\infty} g(x) z^{-x}\,dx$$

Replacing the integral in (1) by this equivalent

Pearl, op. cit., p. [illegible]

expression, $f(Z)$ for the case where $k=2$, may be written as,

$$(20)\quad f(Z)=\int_{-l-\frac{1}{2}}^{\infty} g(x)Z^{x}\,dx+\int_{-l-\frac{1}{2}}^{\infty} g(x)Z'^{x}\,dx-\sum_{m=-l}^{-1} g(m)Z^{m} +\xi_2(l,Z);$$

where, $-2\pi<Q<0$,

and $\arg Z' = Q+2\pi$.

Proceeding to the case where $C=3$, it will be noticed that in the integrand of (4), the trigonometric factor may be replaced by the expansion (7), while Z will assume the upper sign within the brackets and the range of its argument will be that expressed in (16). Whereupon $f(Z)$ is developable as,

$$(21)\quad f(Z)=\int_{-l-\frac{1}{2}}^{\infty}\{g(x)Z^{x}(1+2\cos 2\pi x)\}\,dx-\sum_{m=-l}^{-1} g(m)Z^{m} +\xi_3(l,Z);$$

where, $-\pi<Q<\pi$

Directing our attention to the integral in (21), we may write immediately,

$$\int_{-l-\frac{1}{2}}^{\infty}\{g(x)(Z^{x}+2\cos 2\pi x\, Z'^{x})\}\,dx.$$

Here again, the existence of

$$\int_{-e-1/2}^{\infty} g(x)\,Z^x\,dx \quad \text{and} \quad \int_{-e-1/2}^{\infty} g(x)(2\cos 2\pi x)\,Z^x\,dx$$

must be established before the integral can be separated into these two parts.

The existence of $\int_{-e-1/2}^{\infty} g(x)\,Z^x\,dx$ may be established by a process similar to that employed in the previous case. Due to the peculiar property of the function, $\cos \pi x$, its contribution to the integral is bounded for all values of x however large. Therefore the absolute value of the integral is less than

$$M \left| \int_{-e-1/2}^{\infty} g(x)\,Z^x\,dx \right| ,$$

which can be shown to exist.

In view of these facts we may write,

$$(22) \quad \int_{-e-1/2}^{\infty} g(x)\{Z^x + 2\cos 2\pi x)\,Z^x\}\,dx =$$

$$\int_{-e-1/2}^{\infty} g(x)\,Z^x\,dx + \int_{-e-1/2}^{\infty} g(x)(2\cos 2\pi x)\,Z^x\,dx.$$

Our concern now is for the second integral of the right hand member of the above equality. Letting

$$z' = \rho\left[\cos(\theta - 2\pi) + i\sin(\theta - 2\pi)\right],$$

replacing z^x by $\left[(\cos 2\pi + i\sin 2\pi)z'\right]^x$
the integral becomes,

$$\int_{-c-1/2}^{\infty}\{g(x)(\cos 2\pi x + i\sin 2\pi x)(2\cos 2\pi x)z'^x\}dx.$$

This integral may be changed at once to the form,

$$\int_{-c-1/2}^{\infty}\{g(x)(2\cos^2 2\pi x + 2i\sin 2\pi x\cos 2\pi x)z'^x\}dx.$$

Replacing the two factors within the integrand by their equivalent trigonometric expressions and grouping terms, the integral becomes,

$$\int_{-c-1/2}^{\infty}\{g(x)\left[z'^x + (\cos 4\pi x + i\sin 4\pi x)z'^x\right]\}dx.$$

Considering the factor $(\cos 4\pi x + i\sin 4\pi x)z'^x$, one sees that it may be written as,

$$\left[(\cos 4\pi + i\sin 4\pi)z'\right]^x$$

or the argument of z' is increased by 4π.

Let us define

$$(\cos 4\pi + i \sin 4\pi) Z' \text{ as } Z'',$$

where $|Z''| = \rho = |Z'| = |Z|$, and, $\arg Z'' = \arg Z' + 4\pi = Q + 2\pi$, and $-\pi < Q < \pi$.

Thus the integral under consideration may be written,

$$\int_{-e-\frac{1}{2}}^{\infty} g(x)(Z'^{x} + Z''^{x})\,dx.$$

By a proof analogous to that on pages 17-18, the integrals,

$$\int_{-e-\frac{1}{2}}^{\infty} g(x) Z'^{x}\,dx \text{ and } \int_{-e-\frac{1}{2}}^{\infty} g(x) Z''^{x}\,dx,$$

may be shown to exist and one may write,

$$\int_{-e-\frac{1}{2}}^{\infty} g(x)(Z'^{x} + Z''^{x})\,dx = \int_{-e-\frac{1}{2}}^{\infty} g(x) Z'^{x}\,dx + \int_{-e-\frac{1}{2}}^{\infty} g(x) Z''^{x}\,dx.$$

Replacing the second integral of (22) by its equivalent expressed above, we then have

$$\int_{-e-\frac{1}{2}}^{\infty} g(x)(1 + 2\cos 2\pi x) Z^{x}\,dx = \int_{-e-\frac{1}{2}} g(x) Z^{x}\,dx$$

$$+ \int_{-e-\frac{1}{2}}^{\infty} g(x) Z'^{x}\,dx + \int_{-e-\frac{1}{2}}^{\infty} g(x) Z''^{x}\,dx,$$

and (20) is expressible as,

$$f(z)=\int_{-l-1/2}^{\infty} g(x)\, z^{x}\, dx+\int_{-l-1/2}^{\infty} g(x)\, z'\, dx+\int_{-l-1/2}^{\infty} g(x)\, z''^{x}\, dx$$

$$-\sum_{m=-l}^{-1} g(m)\, z^{m}+\xi_{3}(l,z);$$

where, $-\pi<\theta<\pi$,

$\arg z=\theta-2\pi$,

and, $\arg z'=\theta+2\pi$.

For the case where $l=4$, the integral of (4) employs $(-z)^{x}$, the trigonometric factor of the integrand is replaced by (6), and the argument of z is as set forth in (17). The relation (4) then becomes,

$$(23)\quad f(z)=\int_{-l-1/2}^{\infty} g(x)(-z)^{x}(2\cos\pi x+2\cos 3\pi x)\, dx$$

$$-\sum_{m=-l}^{-1} g(m)\, z^{m}+\xi_{4}(l,z);$$

where $-2\pi<\theta<0$.

For the study of the integral of the above expression it is advisable to write it as,

$$(24)\quad \int_{-l-1/2}^{\infty}\{g(x)(2\cos\pi x)(-z)^{x}+(2\cos 3\pi x)(-z)^{x}\}\, dx.$$

In the previous case it was shown that such an integral as

$$\int_{-e-\frac{1}{2}}^{\infty}\{g(x)(\cos\pi x)Z^{x}\}dx$$

exists. Inasmuch as the analysis was based upon the absolute value of Z , the integrals

$$\int_{-e-\frac{1}{2}}^{\infty}\{g(x)(\cos\pi x)(-Z)^{x}\}dx \text{ and } \int_{-e-\frac{1}{2}}^{\infty}\{g(x)(2\cos 3\pi x)(-Z)^{x}\}dx$$

also exist. Therefore the integral (23) may be separated into the two parts; namely,

$$\int_{-e-\frac{1}{2}}^{\infty}\{g(x)(-Z)^{x}(2\cos\pi x+2\cos 3\pi x)\}dx =$$

$$\int_{-e-\frac{1}{2}}^{\infty}\{g(x)(2\cos\pi x)(-Z)^{x}\}dx+\int_{-e-\frac{1}{2}}^{\infty}\{g(x)(2\cos 3\pi x)(-Z)^{x}\}dx.$$

Using the results of the previous study of $(-Z)^{x}$ as expressed in (21), the two integrals of the right hand members become,

$$(25)\quad\int_{-e-\frac{1}{2}}^{\infty}\{g(x)(\cos\pi x+i\sin\pi x)(2\cos\pi x)Z^{x}\}dx$$

$$+\int_{-e-\frac{1}{2}}^{\infty}\{g(x)(\cos\pi x+i\sin\pi x)(2\cos 3\pi x)Z^{x}\}dx.$$

Replacing Z^x in the second integral by $(\cos 2\pi x + i \sin 2\pi x) Z'^x$, the integral becomes,

$$\int_{-e-\frac{1}{2}}^{\infty} \{ g(x)(\cos 3\pi x + i \sin 3\pi x)(2\cos 3\pi x) Z'^x \} dx;$$

where $\arg Z' = Q - 2\pi$.

These integrals of (25) may now be changed to the form,

$$\int_{-e-\frac{1}{2}}^{\infty} \{ g(x)\, 2(\cos^2 \pi x + i \sin \pi x \cos \pi x) Z^x \} dx$$

$$+\int_{-e-\frac{1}{2}}^{\infty} \{ g(x)\, 2(\cos^2 3\pi x + i \sin 3\pi x \cos 3\pi x) Z'^x \} dx;$$

where $-2\pi < Q < 0$,

and $\arg Z' = Q - 2\pi$

If the equivalent trigonometric expressions are used to replace the factors within the parentheses of each integrand, the integrals become,

$$\int_{-e-\frac{1}{2}}^{\infty} g(x) \{ Z^x + (\cos 2\pi x + i \sin 2\pi x) Z^x \} dx$$

$$+\int_{-e-\frac{1}{2}}^{\infty} g(x) \{ Z'^x + (\cos 6\pi x + i \sin 6\pi x) Z'^x \} dx;$$

From previous discussions it is known that the following equalities are permissable:

$$(26)\ \int_{-l-\frac{1}{2}}^{\infty} g(x)\{z^x+(\cos 2\pi x+i\sin 2\pi x)z^x\}dx$$

$$=\int_{-l-\frac{1}{2}}^{\infty} g(x)z^x dx+\int_{-l-\frac{1}{2}}^{\infty} g(x)(\cos 2\pi x+i\sin 2\pi x)z^x dx;$$

and,

$$(27)\ \int_{-l-\frac{1}{2}}^{\infty} g(x)\{z'^x+(\cos 3\pi x+i\sin 3\pi x)z'^x\}dx$$

$$=\int_{-l-\frac{1}{2}}^{\infty} g(x)z'^x dx+\int_{-l-\frac{1}{2}}^{\infty} g(x)(\cos 6\pi x+i\sin 6\pi x)z'^x dx;$$

where $-2\pi<\theta<0$,

and, $\arg z' = \theta - 2\pi$

If we define z'' as $(\cos 2\pi + i\sin 2\pi)z$, where $|z''| = \rho = |z|$ and,

$$\arg z'' = \arg z + 2\pi = \theta + 2\pi;\ -2\pi<\theta<0;$$

then the second integral of (26) is expressible as,

$$\int_{-l-\frac{1}{2}}^{\infty} g(x)z''^x dx.$$

Defining z''' as $(\cos 6\pi + i\sin 6\pi)z'$, where $|z'''| = |z'|$, and

$$\arg z''' = \arg z' + 6\pi = \theta + 4\pi;\ -2\pi<\theta<0;$$

then one may write the second integral of (27) as,

$$\int_{-\ell-1/2}^{\infty} g(x)\, Z'''^{\,x}\, dx.$$

Collecting the four integrals involving Z, Z', Z'' and Z''' which have resulted from the analysis of (24), the expression (23) may now be written as,

$$f(Z) = \int_{-\ell-1/2}^{\infty} g(x) Z^{x} dx + \int_{-\ell-1/2}^{\infty} g(x) Z'^{\,x} dx + \int_{-\ell-1/2}^{\infty} g(x) Z''^{\,x} dx$$

$$+ \int_{-\ell-1/2}^{\infty} g(x) Z'''^{\,x} dx - \sum_{m=-\ell}^{-1} g(m) Z^{m} + \xi_4(\ell, Z);$$

where $-2\pi < \theta < 0$,

$\arg Z' = \theta - 2\pi$,

$\arg Z'' = \theta + 2\pi$,

and $\arg Z''' = \theta + 4\pi$.

When $c = 5$, equation (4) may be written as

(28) $$f(Z) = \int_{-\ell-1/2}^{\infty} g(x) Z^{x} \{1 + 2\cos 2\pi x + 2\cos 4\pi x\} dx$$

$$- \sum_{m=-\ell}^{-1} g(m) Z^{m} + \xi_5(\ell, Z);$$

where $-\pi < \theta < \pi$.

As a result of the previous analysis it is known that the following separation of the integral of (28) may be made.

$$\int_{-e-\frac{1}{2}}^{\infty} g(x)\, Z^{x}(1+2\cos 2\pi x + 2\cos 4\pi x)\,dx$$

$$= \int_{-e-\frac{1}{2}}^{\infty} g(x)\, Z^{x}\,dx + \int_{-e-\frac{1}{2}}^{\infty} g(x)\, Z^{x}(2\cos 2\pi x)\,dx + \int_{-e-\frac{1}{2}}^{\infty} g(x)\, Z^{x}(2\cos 4\pi x)\,dx,$$

where $-\pi < Q < \pi$.

The last two integrals of the latter part of this expression may be recast as shown, thus,

$$\int_{-e-\frac{1}{2}} \{g(x)(\cos 2\pi x + i\sin 2\pi x)(2\cos 2\pi x)\, Z'^{x}\}\,dx$$

$$+ \int_{-e-\frac{1}{2}}^{\infty} \{g(x)(\cos 4\pi x + i\sin 4\pi x)(2\cos 4\pi x)\, Z''^{x}\}\,dx,$$

where $\arg Z' = Q - 2\pi$,

$\arg Z'' = Q - 4\pi$,

and $-\pi < Q < \pi$.

These integrals may be written immediately as,

$$\int_{-e-\frac{1}{2}}^{\infty} \{g(x)\, 2(\cos^2 2\pi x + i\sin 2\pi x \cos 2\pi x)\, Z'^{x}\}\,dx$$

$$+ \int_{-e-\frac{1}{2}} \{g(x)\, 2\cos^2 4\pi x + i\sin 4\pi x \cos 4\pi x)\, Z''^{x}\}\,dx.$$

Replacing the factors within the parentheses by the equivalent trigonometric expressions we have

$$\int_{-l-\frac{1}{2}}^{\infty}\{g(x)[Z'^{x}+(\cos 4\pi x+i\sin 4\pi x)Z'^{x}]\}dx$$

$$+\int^{\infty}\{g(x)[Z''^{x}+(\cos 8\pi x+i\sin 8\pi x)Z''^{x}]\}dx;$$

where $\arg Z' = Q - 2\pi$,

$\arg Z'' = Q - 4\pi$,

and $-\pi < Q < \pi$.

Designating, $(\cos 4\pi x + i\sin 4\pi x)Z'^{x}$ as Z'''^{x} and $(\cos 8\pi x + i\sin 8\pi x)Z''^{x}$ as $Z^{IV x}$ (28) becomes,

$$f(z) = \int_{-l-\frac{1}{2}}^{\infty} g(x)Z^{x}dx + \int_{-l-\frac{1}{2}}^{\infty} g(x)Z'^{x}dx + \int_{-l-\frac{1}{2}}^{\infty} g(x)Z''^{x}dx$$

$$+\int_{-l-\frac{1}{2}}^{\infty} g(x)Z'''^{x}dx + \int_{-l-\frac{1}{2}}^{\infty} g(x)Z^{IV x}dx - \sum_{m=-l}^{-1} g(m)Z^{m} + j_{5}(l,z);$$

where $-\pi < Q < \pi$,

$\arg Z' = Q - 2\pi$,

$\arg Z'' = Q - 4\pi$,

$\arg Z''' = Q + 2\pi$,

and $\arg Z^{IV} = Q + 4\pi$.

One observes also that,

$$|z| = \rho = |z'| = |z''| = |z'''| = |z^{iv}|.$$

In general, one concludes that,

$$(29)\quad \int_{-l-1/2}^{\infty} g(x)[\pm z]^x \frac{\sin k\pi x}{\sin \pi x}\,dx - \sum_{m=-l}^{-1} g(m) z^m + \xi_k(l,z)$$

$$= \int_{-l-1/2}^{\infty} g(x) z^x\,dx + \int_{-l-1/2}^{\infty} g(x) z'^x\,dx + \cdots + \int_{-l-1/2}^{\infty} g(x) z^{(k-1)^x}\,dx$$

$$- \sum_{m=-l}^{-1} g(m) z^m + \xi_k(l,z);$$

where $|z| = \rho = |z'| = |z''| = \cdots = |z^{(k-1)}|$;

$$\lim_{|z|\to\infty} z^l \xi_k(l,z) = 0;$$

and $$\lim_{|z|\to\infty} \sum_{m=-l}^{-1} g(m) z^m = 0.$$

It suffices then to say that the expression (4) is developable into a series of k integrals as exhibited in the right hand member of (29), wherein the argument of z has the following value:

(30) if k is odd;

$$-\pi < \theta < \pi,$$

$$\arg z' = \theta - 2\pi, \qquad \arg z^{\left(\frac{k+1}{2}\right)} = \theta + 2\pi,$$

$$\arg z'' = \theta - 4\pi, \qquad \arg z^{\left(\frac{k+3}{2}\right)} = \theta + 4\pi,$$

$$\text{------} = \text{------}, \qquad \text{------} = \text{------},$$

$$\arg z^{\left(\frac{k-1}{2}\right)} = \theta - (k-1)\pi, \qquad \arg z^{(k-1)} = \theta - (k-1)\pi;$$

(31) and if k is even

$$-2\pi < \theta < 0, \qquad \arg z^{(k/2)} = \theta + 2\pi,$$
$$\arg z' = \theta - 2\pi, \qquad \arg z^{(k/2+1)} = \theta + 4\pi,$$
$$\arg z'' = \theta - 4\pi, \qquad - - - - = - - - -,$$
$$- - - - = - - - -, \qquad \arg z^{(k-1)} = \theta + k\pi.$$
$$\arg z^{(k/2-1)} = \theta - (k-2)\pi$$

It will now be our purpose to observe the application of the above theory to a particular type of series. This discussion will be reserved for Chapter 2.

Chapter II

APPLICATION TO ENTIRE FUNCTIONS WHERE THE GENERAL COEFFICIENT IS THE RECIPROCAL OF TWO GAMMA FUNCTIONS

Before considering series of the type (1), certain general observations should be made upon coefficients of the kind appearing therein.

For this purpose the following theorem, recently demonstrated by Ford, will be found indispensable in the analysis which follows.

Theorem:[1]

(32) $$\left|\frac{1}{\Gamma(x+iy)}\right| < K e^{(\pi/2+\epsilon)|y|},$$

where K is any positive integer independent of x and y, and ϵ is arbitrarily small.

The general coefficient of the series to be considered in this particular work will be limited to the case where $g(n)$ is the reciprocal of the product of two gamma functions. However, the methods and ideas exhibited

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[1] Ford, op. cit., p. 61.

CHAPTER [illegible]

[illegible] OF [illegible] METHOD [illegible]

[illegible]

[illegible]

Before calculating further [illegible] (see [illegible]), certain general observations should be [illegible] with the first appearing [illegible].

For this purpose the following [illegible], [illegible] will [illegible] analysis which follows.

Theorem

$$\left|\frac{1}{\Gamma(x+iy)}\right| < K e^{(\frac{\pi}{2}-\epsilon)|y|} \tag{3a}$$

where K is any positive integer independent of x and y, and ϵ is arbitrarily small.

The general conditions of the [illegible] situated in this [illegible] the simplest possible case where $g(n)$ is the product of [illegible] gamma functions. However, [illegible]

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[3] [illegible]

will lend themselves immediately to extensions to cases involving more than two gamma functions.

The general coefficient,

$$g(x+iy) = \frac{1}{\Gamma(x+iy+k_1)} \cdot \frac{1}{\Gamma(x+iy+k_2)},$$

when considered in light of (32), introduces the following inequalities;

$$\left| \frac{1}{\Gamma(x+iy+k_1)} \right| < K_1 e^{(\pi/2+\epsilon_1)|y|},$$

and

$$\left| \frac{1}{\Gamma(x+iy+k_2)} \right| < K_2 e^{(\pi/2+\epsilon_2)|y|}.$$

Inasmuch as K_1 and K_2 are positive integers independent of x and y, the two inequalities may be combined and written,

$$(33) \quad \left| \frac{1}{\Gamma(x+iy+k_1)} \right| \cdot \left| \frac{1}{\Gamma(x+iy+k_2)} \right| < K e^{(\pi+\epsilon')|y|};$$

where, $K_1 \cdot K_2 = K$ and ϵ' is the sum of $\epsilon_1 + \epsilon_2$.

Since ϵ' must be arbitrarily small it follows that,

(34) $$\left|\frac{1}{\Gamma(x+iy+k_1)}\right| \cdot \left|\frac{1}{\Gamma(x+iy+k_2)}\right| < K e^{k\pi|y|},\ k \geq 2$$

or, $$\left|\frac{1}{\Gamma(x+iy+k_1)} \cdot \frac{1}{\Gamma(x+iy+k_2)}\right| < K e^{k\pi|y|}.$$

Inasmuch as the reciprocal of the gamma function possesses only one singularity which is at infinity, condition (a) of the theorem of Newsom is satisfied. From the discussion already presented in this chapter, section (b) is satisfied. Therefore,

(35) $$f(z) = \int_{-l-1/2}^{\infty} \left[\frac{[-z]^x}{\Gamma(x+k_1)\Gamma(x+k_2)} \cdot \frac{\sin 2\pi x}{\sin \pi x} \right] dy$$
$$- \sum_{m=-l}^{-1} g(m) z^m + \xi_2(l, z);$$

where $\lim_{|z| \to \infty} z^l \xi_2(l, z) = 0$,

and, $-2\pi < \theta < 0$.

Our immediate problem is the study of the integral involved in this equation. As a result of the previous study of the integral where $k=2$, (see 20), the integral of (35) may be expressed as

$$\int_{-l-1/2}^{\infty}\frac{z^x\,dx}{\Gamma(x+k_1)\Gamma(x+k_2)}+\int_{-l-1/2}^{\infty}\frac{z'^x\,dx}{\Gamma(x+k_1)\Gamma(x+k_2)},$$

where $-2\pi < \theta < 0$,

and $\arg z' = \theta + 2\pi$.

To simplify the ensuing analysis it is desirable to separate each of the integrals into two parts. To accomplish this, let us direct our attention to the first integral. It is certainly true that

$$\int_{-l-1/2}^{\infty}\frac{z^x\,dx}{\Gamma(x+k_1)\Gamma(x+k_2)}=\int_{0}^{\infty}\frac{z^x\,dx}{\Gamma(x+k_1)\Gamma(x+k_2)}+\int_{-l-1/2}^{0}\frac{z^x\,dx}{\Gamma(x+k_1)\Gamma(x+k_2)},$$

since both integrals exist. Moreover the second integral on the right exhibits the following property,

$$\text{(36)}\quad \lim_{|z|\to\infty}\int_{-l-1/2}^{0}\frac{z^x\,dx}{\Gamma(x+k_1)\Gamma(x+k_2)}=0.$$

The proof of this fact follows.

Proof: By a well known theorem,

$$\left|\int_{-l-1/2}^{0}\frac{z^x\,dx}{\Gamma(x+k_1)\Gamma(x+k_2)}\right|\le\int_{-l-1/2}^{\infty}\left|\frac{z^x\,dx}{\Gamma(x+k_1)\Gamma(x+k_2)}\right|.$$

But,

$$\int_{-l-1/2}^{0}\left|\frac{z^x dx}{\Gamma(x+k_1)\Gamma(x+k_2)}\right| \leq \int_{-l-1/2}^{0}\left|\frac{1}{\Gamma(x+k_1)\Gamma(x+k_2)}\right|\cdot\left|z^x dx\right|.$$

Due to relation (33) the absolute value of the reciprocal of the gamma functions is bounded; therefore,

$$\int_{-l-1/2}^{0}\left|\frac{1}{\Gamma(x+k_1)\Gamma(x+k_2)}\right|\cdot\left|z^x dx\right| \leq M\int_{0}^{l+1/2}\rho^{-x}dx = -M\left[\frac{\rho^{-x}}{\log\rho}\right]_0^{l+1/2};$$

thus completing the proof since this last quantity approaches zero as z approaches infinity.

By a similar method one may show that,

$$\lim_{|z|\to\infty}\int_{-l-1/2}^{0}\frac{z'^x dx}{\Gamma(x+k_1)\Gamma(x+k_2)} = 0.$$

Inasmuch as the contribution of these integrals with limits from $(-l-1/2)$ to 0 are known to become small for large values of z, in the future they will be denoted by $Q(z, l)$ and $Q(z', l)$, respectively,

(37) where, $\lim_{|z|\to\infty} Q(z, l) = 0,$

and $\lim_{|z|\to\infty} Q(z', l) = 0.$

Hence, $f(z)$ may now be studied in the form,

$$(38)\quad f(z)=\int_0^\infty \frac{z^x\,dx}{\Gamma(x+k_1)\Gamma(x+k_2)}+\int_0^\infty \frac{z'^x\,dx}{\Gamma(x+k_1)\Gamma(x+k_2)}$$

$$-\sum_{m=-l}^{-1} g(m)z^m+f_2(l,z)+Q(z,l)+Q(z';l);$$

where $-2\pi<\theta<0$,

and $\arg z'=\theta+2\pi$.

The immediate problem, then, is the analysis of the first integral of this expression.

Following the analysis of Harp[2], we write the integral in the form,

$$\frac{1}{\sqrt{\pi}}\int_0^\infty\left\{\frac{\sqrt{\pi}}{2^{2x+k_1-k_2-3/2}}\cdot\frac{\Gamma(2x+k_1+k_2-\frac{1}{2})}{\Gamma(x+k_1)\Gamma(x+k_2)}\cdot\right.$$

$$\left.\cdot\frac{2^{2x+k_1-k_2-3/2}}{\Gamma(2x+k_1+k_2-\frac{1}{2})}\right\}z^x\,dx.$$

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[2] E. L. Harp, "Upon the Asymptotic Representation of the Generalized Bessel's Function," (unpublished Master's thesis, University of New Mexico, Albuquerque, New Mexico, 1933), pp. 16-18.

Then upon letting $2x+k_1+k_2-\frac{1}{2}=2y$,
replacing $\Gamma(2y)$ by $\frac{2^{2y-1}}{\sqrt{\pi}}\Gamma(y)(y+\frac{1}{2})$, and
following this by the transformation $z^{1/2}=t$ and
$y=x'/2$, the integral becomes,

$$\frac{t^{-(k_1+k_2-\frac{1}{2})}}{4\sqrt{\pi}}\int_0^\infty\left\{\frac{\Gamma\left(\frac{x'}{2}\right)}{\Gamma\left(\frac{x'+k_1-k_2+\frac{1}{2}}{2}\right)}\cdot\frac{\Gamma\left(\frac{x'+1}{2}\right)}{\Gamma\left(\frac{x'+k_2-k_1+\frac{1}{2}}{2}\right)}\cdot\frac{(2t)^{x'}}{\Gamma(x')}\right\}dx;$$

due to the transformation $z^{1/2}=t$, the argument of
t is one half the argument of z, or $-\pi<\arg t<0$

The successive transformations, $t=u/2$,
$x'=x''+k_1+k_2-\frac{1}{2}$ and $P=k_1+k_2-\frac{1}{2}$, allow the
integral to be written,

$$(39)\qquad \frac{1}{2\sqrt{\pi}}\cdot\frac{1}{2^{1-P}}\int_0^\infty\frac{F(x)}{\Gamma(x+P)}u^x\,dx;$$

where

$$F(x)=\frac{\Gamma\left(\frac{x+k_1+k_2-\frac{1}{2}}{2}\right)\Gamma\left(\frac{x+k_1+k_2+\frac{1}{2}}{2}\right)}{\Gamma\left(\frac{x+2k_1}{2}\right)\quad\Gamma\left(\frac{x+2k_2}{2}\right)},$$

and $-\pi<\arg u<0$.

Recalling the transformation, $P = k_1 + k_2 - 1/2$ one may write

$$F(X) = \frac{\Gamma\left(\frac{X+k_1+k_2-1/2}{2}\right)\Gamma\left(\frac{X+k_1+k_2+1/2}{2}\right)}{\Gamma\left(\frac{X+2k_1}{2}\right)\Gamma\left(\frac{X+2k_2}{2}\right)} = \frac{\Gamma\left(\frac{[X+P-1]+1}{2}\right)\Gamma\left(\frac{[X+P-1]+2}{2}\right)}{\Gamma\left(\frac{[X+P-1]+k_1-k_2+3/2}{2}\right)\Gamma\left(\frac{[X+P-1]+k_2-k_1+3/2}{2}\right)}.$$

Letting $X + P - 1 = V$, the expression on the right becomes,

$$F(V-P+1) = \frac{\Gamma\left(\frac{V+1}{2}\right)\Gamma\left(\frac{V+2}{2}\right)}{\Gamma\left(\frac{V+k_1-k_2+3/2}{2}\right)\Gamma\left(\frac{V+k_2-k_1+3/2}{2}\right)}.$$

This may be expanded in the form,

$$F(V-P+1) = 1 + \frac{c_1}{(V+1)} + \frac{c_2}{(V+1)(V+2)} + \cdots + \frac{c_n + \delta(V,S)}{(V+1)\cdots(V+S)};$$ [3]

where, $\lim_{V\to\infty} \delta(V,S) = 0.$

However, $V = X + P - 1$, and upon making this

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[3] Van Engen, op. cit., pp. 8-11, (7).

replacement,

$$(40)\quad F(x) = 1 + \frac{c_1}{(x+p)} + \frac{c_2}{(x+p)(x+p+1)} + \cdots + \frac{c_n + \delta(x+p-1, s}{(x+p)(x+p+1)\cdots(x+p+s-1)},$$

where $\lim_{x \to \infty} \delta(x+p-1, s) = 0.$

Replacing $F(x)$ by its equivalent given above, the integral of (39) is expressible as,

$$\frac{1}{2\sqrt{\pi}} \cdot \frac{1}{2^{1-p}} \left\{ \int_0^\infty \frac{u^x\,dx}{\Gamma(x+p)} + c_1 \int_0^\infty \frac{u^x\,dx}{(x+p)\Gamma(x+p)} \right.$$

$$+ c_2 \int_0^\infty \frac{u^x\,dx}{(x+p)(x+p+1)\Gamma(x+p)} + \cdots + c_s \int_0^\infty \frac{u^x\,dx}{(x+p)(x+p+1)\cdots(x+p+n-1)\Gamma(x+p)}$$

$$\left. + \int_0^\infty \frac{\delta(x+p-1, s)\, u^x\,dx}{(x+p)(x+p+1)\cdots(x+p+s-1)\Gamma(x+p)} \right\};$$

where, $-\pi < \arg u < 0,$

and, $\lim_{x \to \infty} \delta(x+p-1, n) = 0.$

An analysis of the same type upon the second integral of (38) will produce exactly the same results, whereupon if we allow $A(U)$ to be $f(x)$ after X is transformed in terms of U, relation (38) may be written,

$$(41)\quad A(U)=\frac{1}{2\sqrt{\pi}}\frac{1}{2^{1-p}}\Big\{\int_0^\infty \frac{U^x dx}{\Gamma(X+P)}+C_1\int_0^\infty \frac{U^x dx}{(X+P)\Gamma(X+P)}$$

$$+C_2\int_0^\infty \frac{U^x dx}{(X+P)(X+P+1)\Gamma(X+P)}+\cdots+C_s\int_0^\infty \frac{U^x dx}{(X+P)\cdots(X+P+s-1)\Gamma(X+P)}$$

$$+\int_0^\infty \frac{\delta(X+P-1,s)\,U^x dx}{(X+P)(X+P+1)\cdots(X+P+s-1)\Gamma(X+P)}+\int_0^\infty \frac{U'^x dx}{\Gamma(X+P)}$$

$$+C_1\int_0^\infty \frac{U'^x dx}{(X+P)\Gamma(X+P)}+\cdots+C_s\int_0^\infty \frac{U'^x dx}{(X+P)\cdots(X+P+s-1)\Gamma(X+P)}$$

$$+\int_0^\infty \frac{\delta(X+P-1,s)\,U'^x dx}{(X+P)(X+P+1)\cdots(X+P+s-1)\Gamma(X+P)}-\sum_{m=-l}^{-1} g(m)\,Z^m$$

$$+\xi_2(l,U)+\phi_1(U,l)+\phi_0(U',l);$$

where

$$-\pi < \arg U < 0,$$
$$\arg U' = \arg U + \pi,$$
$$\lim_{X \to \infty} \delta(X+P-1, n) = 0$$

and where, $\phi_1(U, l)$, $\phi_1(U', l)$ and $\xi_2(l, U)$ have the properties of $\phi(Z, l)$, $\phi(Z', l)$ and $\xi_2(l, Z)$ respectively.

The expression within the brackets of (41) may be rewritten as,

$$\int_0^\infty \frac{U^x dx}{\Gamma(X+P)} + c_1 \int_0^\infty \frac{U^x dx}{\Gamma(X+P+1)} + c_2 \int_0^\infty \frac{U^x dx}{\Gamma(X+P+2)} + \cdots$$

$$+ c_s \int_0^\infty \frac{U^x dx}{\Gamma(X+P+S)} + \int_0^\infty \frac{\delta(X+P-1, S)\, U^x dx}{\Gamma(X+P+S)} + \int_0^\infty \frac{U'^x dx}{\Gamma(X+P)}$$

$$+ c_1 \int_0^\infty \frac{U'^x dx}{\Gamma(X+P+1)} + c_2 \int_0^\infty \frac{U'^x dx}{\Gamma(X+P+2)} + \cdots + c_s \int_0^\infty \frac{U'^x dx}{\Gamma(X+P+S)}$$

$$+ \int_0^\infty \frac{\delta(X+P-1, S)\, U'^x dx}{\Gamma(X+P+S)};$$

or,

$$\sum_{n=0}^{s} c_n \int_0^\infty \frac{U^x\,dx}{\Gamma(x+p+n)} + \int_0^\infty \frac{\delta(x+p-1,s)\,U^x\,dx}{\Gamma(x+p+s)}$$

$$+\sum_{n=0}^{s} c_n \int_0^\infty \frac{U'^x\,dx}{\Gamma(x+p+n)} + \int_0^\infty \frac{\delta(x+p-1,s)\,U'^x\,dx}{\Gamma(x+p+s)};$$

$c_0 = 1.$

Hence,

$$A(U) = \frac{1}{2\sqrt{\pi}}\cdot\frac{1}{2^{1-p}}\left[\sum_{n=0}^{s} c_n \int_0^\infty \frac{U^x\,dx}{\Gamma(x+p+n)}\right.$$

$$+\int_0^\infty \frac{\delta(x+p-1,s)\,U^x\,dx}{\Gamma(x+p+s)} + \sum_{n=0}^{s} c_n \int_0^\infty \frac{U'^x\,dx}{\Gamma(x+p+n)}$$

$$\left.+\int_0^\infty \frac{\delta(x+p-1,s)\,U'^x\,dx}{\Gamma(x+p+s)}\right] - \sum_{m=-l}^{-1} g(m)\left(\frac{U}{2}\right)^{2m}$$

$$+\xi_2(l,U) + \phi_1(U,l) + \phi_1(U',l);\quad c_0 = 1,$$

and where $-\pi < \arg U < 0,$
$\arg U' = \arg U + \pi,$

and, $\lim_{x\to\infty} \delta(x+p-1,s) = 0,$
$\lim_{|U|\to\infty} \xi_2(U,l) = 0,$

$$\lim_{|u|\to\infty} \phi_1(u, l) = 0,$$
$$\lim_{|u|\to\infty} \phi_1(u', l) = 0.$$

By an analysis precisely similar to that of Ford[4], we know that

$$(43) \quad \int_0^\infty \frac{u^x dx}{\Gamma(x+p+n)} = u^{1-p-n}\int_{p+n-1}^\infty \frac{u^w dw}{\Gamma(w+1)}$$
$$= u^{1-p-n} e^u - u^{1-p-n}\xi(u),$$

where $\lim_{|u|\to\infty} u^n \xi(u) = 0;\ n = 0, 1, 2, \cdots S;$
$|\arg u| < \pi.$

Moreover, exactly the same statement may be made in regard to

$$\int_0^\infty \frac{u'^x dx}{\Gamma(x+p+n)};\ |\arg u'| < \pi.$$

It follows at once, therefore that

$$(44) \quad A(u) = \frac{1}{2^r \pi}\cdot\frac{1}{2^{1-p}}\left\{u^{1-p} e^u\left[1 + \frac{c_1}{u} + \frac{c_2}{u^2} + \cdots\right.\right.$$

[4] Ford, op. cit., p. 65; (9),(11).

$$\cdots + \frac{C_s}{U^s}\Big] - U^{1-P}\sum_{m=0}^{s}\frac{C_m\xi(U)}{U^m} + \int_0^\infty \frac{\delta(X+P-1,S)\,U^X dx}{\Gamma(X+P+S)}$$

$$-U^{1-P}e^{U'}\Big[1+\frac{C_1}{U'}+\frac{C_2}{U'^2}+\cdots+\frac{C_s}{U^s}\Big] - U'^{1-P}\sum_{m=0}^{s}\frac{C_m\xi(U')}{U'^m}$$

$$+\int_0^\infty \frac{\delta(X+P-1,S)\,U'^X dx}{\Gamma(X+P+S)}\Big\} - \sum_{m=-l}^{-1} g(m)\left(\tfrac{U}{2}\right)^{2m}$$

$$\xi_2(l,U) + \phi_1(U,l) + \phi_1(U',l); \quad C_0 = 1,$$

and where $-\pi < \arg U < 0,$

$$\arg U' = \arg U + \pi;$$

and, $\lim_{X\to\infty} \delta(X+P-1,S) = 0$

$$\lim_{|U|\to\infty} \xi_2(l,U) = 0,$$

$$\lim_{|U|\to\infty} \phi_1(U,l) = 0,$$

$$\lim_{|U|\to\infty} \phi_1(U',l) = 0.$$

At this point it is desirable to consider separately the behavior of this result for the two cases $-\pi < \arg U < 0$, and $-\pi < \arg U < -\pi/2$. It is apparent at once that it suffices to analyze only the portion within the bracket.

Let us suppose, then, that $-\pi/2 < \arg U < 0$. Since $|e^U| = e^{|U|\cos(\arg U)}$, it follows e^U becomes large exponentially as $|U| \longrightarrow \infty$; also

$$\Delta |U| = e^{-U} \sum_{n=0}^{s} \frac{C_n \xi(U)}{U^n}$$ vanishes in such a manner that

$$\lim_{|U| \to \infty} U^n \Delta(U) = 0;\ n = 1, 2, 3, \cdots s.$$

(45) Likewise

$$\lim_{|U| \to \infty} e^{-U} U^{P-1+S} \int_0^\infty \frac{\delta(X+P-1, S)\, U^X dX}{\Gamma(X+P+S)} = 0,$$ [5]

Also, recalling that $|e^U| = e^{|U| \cos[\arg(U+\pi)]}$,

$$\frac{U'^{1-P} e^{U'}}{U'^{1-P} e^U} \left[1 + \frac{C_1}{U'} + \frac{C_2}{U'^2} + \cdots + \frac{C_s}{U'^s}\right]$$

approaches zero exponentially as $|U| \to \infty$. It is apparent that

$$\frac{U'^{1-P}}{U'^{1-P} e^U} \sum_{n=0}^{s} \frac{C_n \xi(U')}{U'^n}$$

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[5] Ibid., Note (18), p. 66.

also approaches zero exponentially as

We are now ready for a consideration of

$$(46)\qquad \frac{1}{u^{1-p-s}e^{u}}\int_0^{\infty}\frac{\delta(x+p-1,s)\,u'^{x}\,dx}{\Gamma(x+p+s)} =$$

$$\frac{1}{ue^{u}}\int_{p+s}^{\infty}\frac{\alpha(w,s)\,u'^{w}\,dw}{\Gamma(w)},$$

where

$$\lim_{w\to\infty}\alpha(w,s)=0$$

It is readily noted that $\pi/2<\arg u'<\pi$ when $-\pi/2 <\arg u<0$. Hence, the integral of (46) can not be considered in precisely the same manner as that of (45), for the theorem employed in the study of (45) requires that $|\arg u|<\pi/2$. This restriction upon the argument was introduced in most artificial fashion in the theorem due to Ford and it appears that the limitation is unduly restrictive. The consideration of the question, however, represents a serious problem and will not be undertaken in this paper. We shall merely add to our assumptions that $\alpha(w,s)$ is such a function that

$$\lim_{|u|\to\infty}\frac{1}{ue^{u}}\int_{p+s}^{\infty}\frac{\alpha(w,s)\,u'^{w}\,dw}{\Gamma(w)}=0,$$

when $\pi/2<\arg u'<\pi$.

Summarizing the above results, we have

$$(47)\quad A(u) = \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{2^{1-p}} e^{u} u^{1-p} \left[\sum_{n=0}^{s} \frac{c_n}{u^n} + \frac{\omega(u,s)}{u^s} \right];$$

where $\lim\limits_{|u| \to \infty} \omega(u,s) = 0,$

when $-\pi/2 < \arg u < 0.$

When the argument of u is restricted to the region $-\pi < \arg u < -\pi/2$, a result resembling (47) is obtained except that the terms in u' have become dominant. In fact, the analysis is of the same type as that which leads to (47). Thus,

$$(48)\quad A(u) = \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{2^{1-p}} e^{u} u'^{1-p} \left[\sum_{n=0}^{s} \frac{c_n}{u'^n} + \frac{\omega'(u',s)}{u'^s} \right];$$

wherein $\lim\limits_{|u|=|u'| \to \infty} \omega'(u',s) = 0,$

when $-\pi < \arg u < -\pi/2.$

The case where $\arg u = -\pi/2$ will not be considered in this work.

It is at once possible to summarize the results of (47) and (48) in the single statement,

$$A(U) = \frac{1}{2^{v}\pi} \cdot \frac{1}{2^{1-p}} \left\{ e^{U} U^{1-p} \left[\sum_{m=0}^{s} \frac{c_m}{U^m} + \frac{\omega(U,s)}{U^s} \right] \right.$$

$$\left. + e^{U'} U'^{1-p} \left[\sum_{m=0}^{s} \frac{c_m}{U'^m} + \frac{\omega(U',s)}{U'^s} \right] \right\};$$

where $\lim\limits_{|U| \to \infty} \omega(U,s) = 0,$

$\lim\limits_{|U'| \to \infty} \omega(U,s) = 0.$

$-\pi < \arg U < 0$, (with the exception of $\arg U = -\pi/2$),

and $\arg U' = \arg U + \pi.$

However it is desirable to have our final representations in functions of Z. To accomplish this let us first rewrite this expression as,

$$\frac{e^{U}}{2^{v}\pi} \cdot \left(\frac{U}{Z}\right)^{1-p} \left[\sum_{m=0}^{s} \frac{c_m}{U'^m} + \frac{\omega(U,s)}{U^s} \right]$$

$$+ \frac{e^{U}}{2^{v}\pi} \left(\frac{U'}{Z}\right)^{1-p} \left[\sum_{m=0}^{s} \frac{c_m}{U'^m} + \frac{\omega'(U,s)}{U'^s} \right];$$

where $-\pi < \arg U < 0$, (with the exception of $\arg U = -\pi/2$),

and $\arg U' = \arg U + \pi.$

In view of the relation between the arguments of U and U' it follows that $U' = -U$. Whereupon $A(U)$ may be written,

$$A(U) = \frac{e^{U}}{2\sqrt{\pi}}\left(\frac{U}{2}\right)^{1-p}\left[\sum_{m=0}^{s}\frac{c_m}{U^m} + \frac{\omega(U,s)}{U^s}\right]$$

$$+\frac{e^{-U}}{2\sqrt{\pi}}\left(\frac{-U}{2}\right)^{1-p}\left[\sum_{m=0}^{s}\frac{c_m}{(-U)^m} + \frac{\omega'(-U,s)}{(-U)^s}\right];$$

where $-\pi < \arg U < 0$, (with the exception of $\arg U = -\pi/2$).

Letting $U/2 = z^{1/2}$ and $p = k_1 + k_2 - 1/2$ this expression becomes,

$$\frac{e^{2z^{1/2}}(z^{1/2})^{-(k_1+k_2-3/2)}}{2\sqrt{\pi}}\left[\sum_{m=0}^{s}\frac{c_m}{(2z^{1/2})^m} + \frac{\omega(2z^{1/2},s)}{(2z^{1/2})}\right]$$

$$+\frac{e^{-2z^{1/2}}(-z^{1/2})^{-(k_1+k_2-3/2)}}{2\sqrt{\pi}}\left[\sum_{m=0}^{s}\frac{c_m}{(-2z^{1/2})^m} + \frac{\omega'(-2z^{1/2},s)}{(-2z^{1/2})^s}\right];$$

where $-2\pi < \theta < 0$, (with the exception of $\arg z = -\pi$),

and $\lim\limits_{|z|\to\infty} \omega(2z^{1/2}, S) = 0,$

$$\lim_{|z|\to\infty} \omega'(-2z^{1/2}, S) = 0.$$

Hence for values of z of large modulus we may write,

$$A(z) \sim \frac{(z^{1/2})^{-(k_1+k_2-3/2)}}{2\sqrt{\pi}} e^{2z^{1/2}} \sum_{m=0}^{\infty} \frac{c_m}{(2z^{1/2})^m} \tag{49}$$

$$+ \frac{(-2z^{1/2})^{-(k_1+k_2-3/2)}}{2\sqrt{\pi}} e^{-2z^{1/2}} \sum_{m=0}^{\infty} \frac{c_m}{(-2z^{1/2})^m};$$

where $-2\pi < \varphi < 0$, (with the exception of $\arg z = -\pi$).

It might be interesting to note the application of this result to the classical Bessel's Function of the order ν. The Bessel's Function of the order ν is defined as,[6]

$$J_\nu(w) = \left(\frac{w}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{w}{2}\right)^{2m}}{\Gamma(m+1)\Gamma(m+\nu+1)}. \tag{50}$$

6 Ford, op. cit., p. 73, (1).

Inspection shows this to be of the same general form as the series being considered and it is identical if the transformation, $k_1 = 1$, $k_2 = \nu + 1$ and $z^{1/2} = iw/2$ are made. Applying these transformations (49) becomes

$$A(w) \sim \left[\left(\frac{iw}{2}\right)^{-(1+\nu+1-3/2)} e^{\frac{2iw}{2}} \sum_{n=0}^{\infty} \frac{c_n}{\left(\frac{2iw}{2}\right)^n} \right.$$

$$\left. + \left(\frac{-iw}{2}\right)^{-(1+\nu+1-3/2)} e^{\frac{2(-iw)}{2}} \sum_{n=0}^{\infty} \frac{c_n}{\left(\frac{2(-iw)}{2}\right)^n} \right] \left(\frac{w}{2}\right)^{\nu};$$

where $-\pi < \arg w < 0$, (with the exception of $\arg w = -\pi/2$;

or

$$A(w) \sim \frac{w^{\nu}}{2} \left(\frac{iw}{2}\right)^{-\nu-1/2} \frac{e^{iw}}{2^{\nu}\pi} \sum_{n=0}^{\infty} \frac{c_n}{(iw)^n}$$

$$+ \frac{w^{\nu}}{2} \left(\frac{-iw}{2}\right)^{-\nu-1/2} \frac{e^{-iw}}{2^{\nu}\pi} \sum_{n=0}^{\infty} \frac{c_n}{(-iw)^n};$$

where $-\pi < \arg w < 0$, (with the exception of $\arg w = -\pi/2$).

This conclusion is standard in mathematical literature and the fact that the result was obtained serves as a valuable check upon the more general results of the paper.

The considerations of this work must be regarded as preliminary. It appears that the field opened up by this analysis is very broad, and it may be true that the asymptotic representations of many functions here-to-fore unknown can now be obtained.

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