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# Tracking Control of Uncertain Systems

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## Abstract

This paper deals with the problem of designing output tracking controllers for uncertain systems. The systems we consider may be non-minimum phase but are restricted to be linear. The problem is motivated by control applications where a desired output trajectory is specified, and the corresponding input to the system is to be found.

## 1 Introduction

Mathematically, the output-tracking problem considered may be studied as an inverse problem. The output tracking has a long history but more recently, Devasia et. al. [1] initiated a new line of research by studying stable inversion as a viable approach to exact-output tracking. This research has also benefited from contributions by Hunt et. al. [2, 3]. At this stage, it is understood that the stable inversion of non-minimum phase systems (linear or nonlinear, time invariant or time varying) is possible at least for a large class of trajectories. The basic idea is that rather than finding an inverse for all trajectories, one only need be concerned with the inverse of the particular trajectory to be tracked. The resulting input then has a noncausal component and is of the feedforward variety.

This paper uses a different approach to study a similar but more general problem in the linear system case. The previously mentioned results assume given the system dynamics even though some robustness issues have been studied in [4, 5]. In many cases, the system under study is highly uncertain and what we are given are samples of the desired output trajectory, rather than a closed-form time-dependent trajectory. Our approach relies on concepts from statistical learning theory and on the results of a recent paper by Efromovich and

Koltchinski [6]. As such, our approach has much in common with the recent book by Vidyasagar [7], and other recent papers which attempt to answer some control questions statistically.

## 2 Mathematical Set-Up

The authors in [6] were interested in estimating a function  $f \in L_2([0, 1])$  which is observed in Gaussian white noise modeled by the stochastic differential equation  $dY^\epsilon(t) = (Hf)(t)dt + \epsilon dW(t)$  where  $H$  is a linear operator,  $\epsilon > 0$  and  $W$  is a standard Brownian motion. This is again an inverse problem and applications of the approach to signal processing and communication theory is described by the authors of [6]. The novelty of their results lie in the facts that  $H$  may be unknown and that the problem may be ill-posed due to the non-invertibility of  $H$ . In the setting of tracking control, the output is the desired trajectory to be tracked,  $H$  is an unknown system and  $f$  corresponds to the input to be found. The non-invertibility of  $H$  may be due to it being non-minimum phase. In order to use the setup of [6], we first assume that samples of the desired output are given even though they may contain random errors, and that the system is unknown. We then use statistical learning theory [7] to generate training input functions along with the corresponding (possibly noisy) outputs. Based on this data, a data-driven learning machine is obtained to estimate the desired input for the original tracking problem.

More specifically, we review the mathematical setup of [6]: Let  $f \in \mathcal{F} \subseteq \mathcal{H}$  where  $\mathcal{H}$  is a Hilbert space, and let  $H$  be a linear operator from  $\mathcal{H}$  into another Hilbert space  $\mathcal{R}$  such that our unknown system satisfies the linear operator equation  $Hf = g$ . We do not assume that the system  $H$  is known but rather that the function class  $\mathcal{F}$  is given. The problem we study is then to find the input  $f$  such that the output  $g$  tracks a given trajectory, when we only have available noisy samples of the desired trajectory.

Suppose an orthonormal basis is given in  $\mathcal{H}$  by  $\{e_i; i \geq 1\}$  and an orthonormal basis of  $\mathcal{R}$  is  $\{\psi_j; j \geq 1\}$ .

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Both bases will be assumed known. We denote inner products over a Hilbert space by  $\langle \cdot, \cdot \rangle$ . The operator  $H$  defines the infinite matrix  $(h_{ij})_{i,j=1}^\infty$ , where  $h_{ij} = \langle H e_j, \psi_i \rangle$ . Note that one can approximate the possibly infinite-dimensional  $H$  by the  $n \times n$  matrixes  $H_n = (h_{ij})_{i,j=1}^n$ . Suppose that we are given a noisy sequence of the desired outputs  $Y_j = \langle H f, \psi_j \rangle + \epsilon \eta_j$ ;  $j = 1, 2, \dots$  and that we can generate a training set of data  $X_{(ji)} = h_{ji} + \sigma \zeta_{ji}$ ;  $i, j = 1, 2, \dots$ , where the random variables  $\{\eta_j, \zeta_{ji}; i, j = 1, 2, \dots\}$  are i.i.d and Normal, and where  $\epsilon, \sigma$  are nonnegative real numbers. The results of [6] then place conditions on the different mathematical objects so that we can obtain an asymptotic or a minimax estimate of the desired tracking input as  $\epsilon$  and  $\sigma$  tend to zero. The studied risk is the minimax integrated squared error

$$R(\hat{f}, f) = \inf_{\hat{f}} \sup_{f \in \mathcal{F}} \mathbb{E} \|\hat{f} - f\|^2 \quad (1)$$

We now define a few more terms to facilitate the statement of the main results: Let  $\mathcal{H}_n$  denote the subspace of  $\mathcal{H}$  spanned by  $\{e_1, \dots, e_n\}$  and  $\mathcal{R}_n$  be the subspace of  $\mathcal{R}$  spanned by  $\{\psi_1, \dots, \psi_n\}$ . Let  $P_n$  and  $\Pi_n$  denote the orthogonal projections on  $\mathcal{H}_n$  and  $\mathcal{R}_n$ , and  $f_n = P_n f$ . Let  $\|A\|$  be the operator norm of  $A$  and  $\|A\|_H = \sqrt{\text{Tr}(A^T A)}$  be the Hilbert norm of  $A$ . We let  $H_n = \Pi_n H P_n$  be an operator from  $\mathcal{H}$  into  $\mathcal{R}$ . If we restrict  $H_n$  to be an operator from  $\mathcal{H}_n$  to  $\mathcal{R}_n$ , we obtain a matrix  $[H]_n$  which is the  $n \times n$  principal submatrix of  $H_n$ . We assume throughout that for a large enough  $n$ ,  $[H]_n$  is invertible. Then, we let  $T_n = H_n^{-1} H$  be a projection operator from  $\mathcal{H}$  onto  $\mathcal{H}_n$ . Next, let

$$X_n = (X_{(ji)})_{j,i=1}^n; \quad \Xi_n = (\zeta_{ji})_{j,i=1}^n \quad (2)$$

so that  $X_n = [H]_n + \sigma \Xi_n$ . The projection estimate is then given by  $\hat{f}_n = \hat{H}_{n,\gamma}^{-1} Y^{(n)}$  where

$$\begin{aligned} Y^{(n)} &= \sum_{j=1}^n Y_j \psi_j; \quad = \Pi_n H f + \epsilon \eta^{(n)} \\ &= \Pi_n H f + \epsilon \sum_{j=1}^n \eta_j \psi_j \end{aligned} \quad (3)$$

and

$$\hat{H}_{n,\gamma}^{-1} = \begin{cases} X_n^{-1} & \text{if } X_n \text{ is nonsingular and } \|X_n^{-1}\| \leq \gamma \\ 0 & \text{otherwise} \end{cases}$$

In particular, we consider the following form for  $\gamma$ ,  $\gamma = 2 \frac{\alpha(\epsilon)}{\epsilon \sqrt{n}}$  where  $\alpha(\epsilon)$  is a nondecreasing function of  $\epsilon > 0$  such that  $\alpha(\epsilon) = o(\log^{-1/2}(\epsilon^{-1}))$ . Finally, we denote

$$A_n = \|H_n^{-1}\|; \quad B_n = \|H_n^{-1}\|_H; \quad \delta(\sigma, n) = \sigma \sqrt{n} A_n \quad (4)$$

The following theorem is then obtained:

**Theorem 1** [6] Suppose that

$$\begin{aligned} \sup_{\delta(\sigma, n) \leq \delta} \log(\sigma^{-1} B_n^{-1}) &= o(\delta^{-2}) \\ \gamma &\geq 2 A_n \\ \sup_{\delta(\sigma, n) \leq \delta} \log \gamma &= o(\delta^{-2}) \end{aligned}$$

Then, for some  $\beta > 0$  and for any  $C > 1$ , there exists  $\delta_0 > 0$  such that for all  $\sigma, \epsilon$ , and  $n$  satisfying

$$\delta(\sigma, n) \leq \frac{\delta_0}{\sqrt{\log(2 + \|H f\|)}}; \quad n \leq \frac{1}{\epsilon^2}$$

the following holds

$$\mathbb{E} \|\hat{f}_n - T_n f\| \leq C B_n^2 (\sigma^2 \|T_n f\|^2 + \epsilon^2) + e^{-\beta \delta^2(\sigma, n)}$$

Moreover, if  $\sigma \leq c\epsilon$ , the following holds

$$\mathbb{E} \|\hat{f}_n - T_n f\| \leq C B_n^2 (\sigma^2 \|T_n f\|^2 + \epsilon^2)$$

The above theorem illustrates the fact that as  $n$  increases, one is able to obtain a more accurate estimate of the desired tracking input  $\hat{f}$ . In fact, one can find the minimum number of samples needed to achieve a certain accuracy in the estimate of  $\hat{f}$ . Note also that if  $\epsilon$  and  $\sigma$  are zero (i.e. deterministic case), one can guarantee the convergence of the estimate to zero as  $n$  increases. In the full version of the paper, we illustrate the main theorem using numerical examples and present other relevant results from paper [6] which may be applied to control problems.

## References

- [1] S. Devasia, D. Chen, and B. Paden, "Nonlinear inversion-based output tracking," *IEEE Trans Auto. Control*, vol. AC-41, pp. 930-943, 1996.
- [2] L. Hunt, G. Meyer, and R. Su, "Noncausal inverses for linear systems," *IEEE Trans Auto. Control*, vol. AC-41, pp. 608-611, 1996.
- [3] L. Hunt, G. Meyer, and R. Su, "Driven dynamics for time-varying linear systems," *IEEE Trans Auto. Control*, vol. AC-42, pp. 1313-1317, 1997.
- [4] J. Huang and W. Spong, "On a nonlinear multivariable servomechanism problem," *Automatica*, vol. 26, pp. 963-972, 1990.
- [5] J. Huang and C. Lin, "Internal model principle and robust control of nonlinear systems," in *Proceedings 32nd Conf. Decision Contr.*, (San Antonio, TX), pp. 1501-1506, 1993.
- [6] S. Efremovich and V. Koltichinskii, "Projection estimation for inverse problems with unknown operator," *Preprint*, 1997.
- [7] M. Vidyasagar, *A Theory of Learning and Generalization*. Berlin: Springer-Verlag, 1996.