The Use of Kamke's Transformation in Approximating the Zeros of Orthogonal Polynomials

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THE USE OF KAMKE'S TRANSFORMATION - DANIELS

THE USE OF KAMKE'S TRANSFORMATION IN APPROXIMATING THE ZEROS OF ORTHOGONAL POLYNOMIALS

by

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Introduction

The importance of the classical orthogonal polynomials has long been acknowledged. It has not been possible, however, to represent them in such a way that all of their important properties are immediately evident. In particular, the location of the zeros of these polynomials is of considerable interest. A brief description of the polynomials which are considered in this thesis is presented in Chapter 1.

Repeated efforts have been directed toward approximating the values of the independent variable for which the polynomials are equal to zero. The traditional approach has been to use Sturm's theorems. A general description of this method is included in Chapter 2.

This thesis is primarily concerned with a different technique in which Kamke's transformation is applied to the differential equations frequently used to define these polynomials. The resulting trigonometric differential equations cannot be explicitly solved either, but certain characteristics of these solutions facilitate the derivation of approximations to the zeros of the solutions. The transformation is presented in Chapter 3.

Introduction

The importance of the classical orthogonal polynomials has long been appreciated. It has not been possible, however, to represent them in such a way that all of their important properties are immediately evident. In particular, the location of the zeros of these polynomials is of considerable interest. A brief description of the polynomials which are considered in this thesis is presented in Chapter 1.

Repeated efforts have been directed toward approximating the values of the independent variable for which the polynomials are equal to zero. The traditional approach has been to use Sturm's theorems. A general description of this method is included in Chapter 2.

This thesis is primarily concerned with a different technique in which the Liouville transformation is applied to the differential equations frequently used to define these polynomials. The resulting transformed differential equations can then be [illegible] solved exactly, with certain characteristics of these solutions [illegible] the derivation of [illegible] to the zeros of [illegible] solutions. The procedures [illegible] presented in Chapter 3.

The methods of approximation and calculation, together with a concluding comparison of the resulting estimates with others in current use, comprise the remaining chapters.

I. Orthogonal Polynomials

The Jacobi polynomials are frequently described as the polynomial solutions of the differential equation

$$(1.1) \qquad (1 - x^2)y'' + \left[b - a - (a + b + 2)x\right]y' + n(n + a + b + 1)y = 0$$

where $y = P_n^{(a,b)}(x)$ represents the Jacobi polynomial of degree n (an integer) in the independent variable x with parameters a and b. These polynomials are an orthogonal family in the range $-1 < x < 1$ in the sense that

$$\int_{-1}^{1} P_n^{(a,b)}(x)\ P_m^{(a,b)}(x)\ dx = \delta_{nm}$$

$$\text{where } \delta_{nm} = 0 \text{ if } n \neq m$$

$$\delta_{nm} = 1 \text{ if } n = m.$$

When $a > -1$ and $b > -1$, the roots of the polynomial are all real and lie in the interval $-1 < x < 1$. The following two values of the polynomial are known[1]:

$$(1.2) \qquad P_n^{(a,b)}(1) = \binom{n + a}{n}$$

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[1] G. Szegö, <u>Orthogonal Polynomials</u>, Amer. Math. Soc. Coll. Series, vol. 23, 1939, pp. 57-60.

$$P_n^{(a,b)}(-1) = (-1)^n \binom{n+b}{n}. \tag{1.3}$$

The lack of symmetry of the polynomial can be observed from the identity[1]

$$P_n^{(a,b)}(x) = (-1)^n P_n^{(b,a)}(-x). \tag{1.4}$$

The ultraspherical polynomials are obtained from Jacobi polynomials when the constants a and b are equal. In this case a new constant λ, introduced for the purpose of simplification, is defined by the relation $a = b = \lambda - \frac{1}{2}$. The resulting equation is

$$(1 - x^2)y'' - (2\lambda + 1)y' + n(n + 2\lambda)y = 0 \tag{1.5}$$

with polynomial solution $y = P_n^{(\lambda)}(x)$. These polynomials are also orthogonal in the range $-1 < x < 1$ and have real zeros when $\lambda > -\frac{1}{2}$. The symmetry of the ultraspherical polynomials is demonstrated by the identity

$$P_n^{(\lambda)}(-x) = (-1)^n P_n^{(\lambda)}(x) \tag{1.6}$$

which follows immediately from (1.4). The zeros are therefore distributed symmetrically about the point $x = 0$.

A special case occurs when $a = b = 0$ or when $\lambda = \frac{1}{2}$. In this case, the Jacobi and ultraspherical polynomials are Legendre polynomials and may be described by the equation

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[1] Ibid, 58.

$$(1.7) \qquad (1 - x^2)y'' - 2xy' + n(n + 1)y = 0$$

with polynomial solution $y = P_n(x)$.

Other special cases occur when $\lambda = 0$ and when $\lambda = 1$. In these cases, the polynomial solutions of (1.1) are the Tchebichef polynomials of the first and second[1] kind,

$$y = T_n(x) = T_n(\cos \theta) = \cos (n\theta)$$

$$y = U_n(\cos \theta) = \sin (n - 1)\theta \,/\, \sin \theta$$

respectively.

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[1] Ibid, 28.

II. Sturm's Method

One method used in approximating the zero points of orthogonal polynomials is based upon Sturm's three oscillation theorems. They describe several of the relationships between the zero points of the solutions of similar differential equations and are usually stated in the following manner[1].

Theorem 1. Let $f(x)$ and $F(x)$ be continuous functions in $x_0 < x < X_0$ with $f(x) \leq F(x)$. Let the functions $y(x)$ and $Y(x)$, both not identically zero, satisfy the differential equations

$$y'' + f(x)y = 0$$

$$Y'' + F(x)Y = 0$$

respectively. Let x' and x'', $x' < x''$, be two consecutive zeros of $y(x)$. Then the function $Y(x)$ has at least one variation of sign in the interval $x' < x < x''$ provided that $f(x) \not\equiv F(x)$ in $[x', x'']$.

Theorem 2. Let $\phi(x)$ be continuous and decreasing in $x_0 < x < X_0$, and let y be a solution of

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[1] Ibid., 19-20.

$$y'' + \phi(x)y = 0$$

which is not identically zero. Then $x' < x'' < x'''$ being three consecutive zeros of $y(x)$, we have $x'' + x' < x''' + x''$: that is, the sequence of the zeros of $y(x)$ is convex.

The last inequality holds also under the following more general condition:

$$\phi(x) < \phi(x'') < \phi(y)$$

$$\text{for } x < x'' < y < x'''.$$

Theorem 3. Let $f(x)$ be continuous and negative in $x_0 < x < X_0$. Then an arbitrary solution y of $y'' + f(x)y = 0$ for which $y \to 0$ if $x \to X_0$ cannot vanish in $x_0 < x < X_0$.

The technique of applying Sturm's theorem consists of comparing the coefficients of the differential equation defining an orthogonal polynomial with the corresponding coefficients of a differential equation defining a function with zeros that are known.

It is possible to use any second order equation, usually containing only y'' and y terms, for the comparison. Bessel equations are the most frequently used for this purpose.

A relatively simple example of this method is cited by Szegö for the case of Laguerre polynomials. These polynomials may be described as the polynomial solutions

of the differential equation

$$(2.1) \qquad u'' + \left\{\left[n + (a + 1)/2\right]/x + (1 - a^2)/(4x^2) - 1/4\right\}u = 0$$

where $u = e^{-x/2}\, x^{(a + 1)/2}\, L_n^{(a)}(x)$

is the solution involving the Laguerre polynomials $L_n^{(a)}(x)$. This can be readily compared with the Bessel equation[1]

$$U'' + \left\{\left[n + (a + 1)/2\right]/x + (1 - a^2)/(4x^2)\right\}U = 0$$

which has the solution

$$U = x^{\frac{1}{2}}\, J_a\left\{2x^{\frac{1}{2}}\left[n + (a + 1)/2\right]^{\frac{1}{2}}\right\}$$

where $J_a(z)$ is the Bessel function of the first kind of order a. It immediately follows that

$$x_v > \frac{(j_v/2)^2}{n + (a + 1)/2}$$

where j_v represents the vth positive zero of Bessel's function $J_a(z)$ and x_v is the vth zero of the Laguerre polynomial $L_n^{(a)}(x)$.

The upper bound of x_v can be obtained by letting B be such a constant that $B < 4n + 2(a + 1)$. In this case, however, the Laguerre equation (2.1) is compared with

$$V'' + \left\{\left[n + (a + 1)/2 - B/4\right]/x + (1 - a^2)/(4x^2)\right\}V = 0$$

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[1] Ibid., 123.

when $0 < x < B$. The solution of this differential equation is also a Bessel function of order a, namely

$$V = x^{\frac{1}{2}} J_a\left\{2x^{\frac{1}{2}}\left[n + (a + 1)/2\right]^{\frac{1}{2}}\right\}$$

where $J_a(z)$ is again the function described previously. Then, by the same reasoning that was used to determine the lower estimate,

$$x_v < \frac{(j_v/2)^2}{n + (a + 1)/2 - B/4} ,$$

providing that the right hand estimate is not greater than the value of B.

III. Kamke's Transformation

In contrast to the theorems of Sturm, E. Kamke has used a transformation of the dependent variables of a second order differential equation such that the zeros of the solution of the original differential equation become evident if one is able to solve the transformed differential equation. Although the exact solution of the transformed differential equation is equivalent to solving the original differential equation, the techniques for constructing an approximation to its solution are quite different from those used to apply Sturm's theorems to approximate the zeros of the solutions of the original differential equation.

The transformation[1] consists of introducing a second variable into the second order linear differential equation

$$y'' + g(x)y' + h(x)y = 0 \tag{3.1}$$

In accordance with the relations

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[1] E. Kamke, Differential Gleichungen, vol. 1, 1943, pp. 121ff.

E. Kamke, A New Proof of Sturm's Comparison Theorems, Amer. Math. Month., vol. 46, 1939, p. 417.

$$y' = P(x)y + Q(x)z \tag{3.2}$$

$$z' = R(x)y + S(x)z. \tag{3.3}$$

The solutions of (3.2) and (3.3) may be represented by the use of polar coordinates[1]. Let

$$W^2 = y^2 + z^2 \tag{3.4}$$

$$\theta = \arcsin y(y^2 + z^2)^{-\frac{1}{2}} = \arccos z(y^2 + z^2)^{-\frac{1}{2}}. \tag{3.5}$$

The functions y, y', z, and z' in (3.2) and (3.3) may be replaced by θ and W using (3.4), (3.5), and their first derivatives. The resulting differential equations for θ and W are

$$\theta' = Q \cos^2 \theta + (P - S) \sin \theta \cos \theta - R \sin^2 \theta \tag{3.6}$$

$$W' = \left[P \sin^2 \theta + (Q + R) \sin \theta \cos \theta + S \cos^2 \theta\right]W. \tag{3.7}$$

Then

$$y = CW(x) \sin \theta(x) \tag{3.8}$$

$$z = CW(x) \cos \theta(x) \tag{3.9}$$

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[1] Dr. Kamke also points out in the cited references that the transformation $y' = hz$, $z' = -gy$ was originally used by H. Prūfer in Mathematische Annalen, vol. 95, 1926, pp. 499ff. He also employed polar coordinates to obtain solutions $y = CW(x) \sin \theta$, $z = CW(x) \cos \theta$ where $W = \exp \frac{1}{2} \cdot \int (h - g) \sin 2\theta(x)dx$. J. Sturdivant also worked with these forms in Second Order Linear Systems with Summable Coefficients, Trans. Amer. Math. Soc., vol. 45, 1939, pp. 164-171. The present transformation is a logical extension of these early methods.

where C is a constant, represent the solutions of (3.2) and (3.3), respectively.

To avoid the trivial solution, the constant C is considered as being different from zero. The solution of (3.1) is consequently given by (3.8).

For the purposes of this work, solution (3.9) is relatively unimportant. The concern is directed wholly toward determining the values of x for which the value of y is equal to zero. If the value of W as obtained from (3.7) does not become zero in the region being investigated, the zero points of y will be located at those values of x which make θ equal to integral multiples of π.

The four coefficients P, Q, R, and S can be determined by differentiating the relations (3.2) and (3.3), eliminating the term z' and comparing terms with the equation resulting from the substitution of (3.2) into (3.1). The result is the following pair of equations:

$$P^2 + P' + gP + QR + h = 0 \tag{3.10}$$

$$P + S + Q'/Q + g = 0. \tag{3.11}$$

Any values of these Kamke coefficients[1] that satisfy (3.10) and (3.11) may be used in the transforma-

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[1] For the sake of simplicity, the term "Kamke coefficients" is used to mean the coefficients P, Q, R, and S as they appear in (3.2) and (3.3).

tion. If R is set equal to zero, however, (3.10) becomes

$$P^2 + P' + gP + h = 0$$

which is a standard form of the Riccati equation. The coefficients $g(x)$ and $h(x)$ are determined by the original differential equation (3.1). The result is a standard transformation of a second order differential equation to a Riccati equation. This also occurs when $P = kQ$ or $S = kQ$, k being a constant other than zero.

Other limitations imposed by the subject of this thesis will be considered later since they are not a part of the transformation itself.

IV. Methods of Approximation

The methods of approximation used are based upon the following ideas. If $\theta(x_0) = 0$ and $\theta'(x_0) > 0$, and if both $y(x_0) = 0$ and $y'(x) \geq \theta'(x)$ for $x_0 < x < b$, then $y(x) \geq \theta(x)$. This relation is then inverted to obtain the values of x for which $\theta(x)$ takes on given values. Let $\theta(x_1) = y(x_1') = \theta_0$, then $x_1' < x_1$, so that x_1' is a lower estimate of the value $x = x_1$ at which $\theta = \theta_0$.

Under similar conditions , if $\theta(x_0) = 0$ and $\theta'(x_0) > 0$, and if $y(x_0) = 0$ and $y'(x) \leq \theta'(x)$ for $x_0 < x <$ b, and if also $\theta(x_1) = y(x_1'') = \theta_0$, then $x_1'' > x_1$ so that x_1'', when it exists, is an upper estimate of the value $x = x_1$ at which $\theta = \theta_0$.

The following form of (3.6) was used for convenience:

$$(4.1) \qquad \theta' = \tfrac{1}{2}(Q + R)\cos 2\theta + \tfrac{1}{2}(P - S)\sin 2\theta + \tfrac{1}{2}(Q - R).$$

Because of the conditions stated above, it was necessary to restrict these terms to positive values. If this is not the case, the relative position of the limits may be reversed.

The fundamental idea behind the approximations is to construct a differential equation based on (4.1) with

a known solution such that its solution curve lies above or below the solution curve of (4.1). The values of the independent variable at which the solution of this constructed differential equation takes on the same value as the solution of (4.1) will form upper and lower estimates of the values of the independent variable for which the solutions of (4.1) take on presceribed values. Since the zeros of the function $y(x)$, as defined by (3.1), are the points of interest, (3.8) shows that these will be the values for which $\theta(x)$ equals $k\pi$, where k is an integer.

Two methods of attack are available which may be used separately or together. In the one, the trigonometric functions $\sin 2\theta$ and $\cos 2\theta$ may be replaced by other functions. In the other, the coefficients $\frac{1}{2}(Q - R)$ and $\frac{1}{2}(P - S)$ may be selected to satisfy (3.10) and (3.11) and then replaced by approximations that lead to a tractable differential equation. Both of the procedures may be used simultaneously.

Exploring further the first method of attack, its most elementary application is to consider the upper and lower bounds of the functions $\sin 2\theta$ and $\cos 2\theta$ to be plus one and minus one, respectively. Inserting these values into (4.1) gives the following upper and lower bounds of θ[1]

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[1] The upper and lower bounds of θ are represented in this thesis by the symbols θ_U and θ_L, respectively.

in the form of separable differential equations

(4.2) $$\theta_U' = Q + \tfrac{1}{2}(P - S)$$

(4.3) $$\theta_L' = -R + \tfrac{1}{2}(S - P).$$

The solution of these equations is obtained by quadrature. Because of its simplicity, this method can be used with any set of Kamke coefficients having positive values.

A more detailed approximation of the functions sin 2θ and cos 2θ may be obtained by considering the function in increments of length $\pi/4$. The straight line approximations given in Table 1 and Table 2 are then used. The notation for upper and lower bounds follows that described on page 15.

TABLE 1

Incremental Approximations of Sin 2θ

Range of θ	Sin $2\theta_U$	Sin $2\theta_L$
$(n - 1)\pi < \theta < (n - 3/4)\pi$	$2\theta - 2(n - 1)\pi$	$4\theta/\pi - 4n + 4$
$(n - 3/4)\pi < \theta < (n - 1/2)\pi$	$-2\theta + (2n - 1)\pi$	$-4\theta/\pi + 4n - 2$
$(n - 1/2)\pi < \theta < (n - 1/4)\pi$	$-4\theta/\pi + 4n - 2$	$(2n - 1)\pi - 2\theta$
$(n - 1/4)\pi < \theta < (n)\pi$	$4\theta/\pi - 4n$	$2\theta - 2n\pi$

The resulting differential equations are linear and may be integrated using the integrating factor

in the form of separate differential equations

$$\theta_1' = Q + \tfrac{1}{2}(P - R) \qquad (4.2)$$

$$\theta_2' = -R + \tfrac{1}{2}(Q - P). \qquad (4.3)$$

The solution of these equations is obtained by quadrature. Because of its simplicity, this method can be used with any set of Kasse coefficients having [illegible] values.

A more detailed approximation of the functions $\sin 2\theta$ and $\cos 2\theta$ may be obtained by considering the function in increments of length $\pi/4$. The straight line approximations given in Table 1 and Table 2 [illegible] used. The notation for upper and lower bounds follows that described on page 15.

TABLE 1

Incremental Approximations of sin 2θ

Range of θ	$\sin 2\theta_1$	$\cos 2\theta_1$
$(n - 1)\pi < \theta < (n - 3/4)\pi$	$2\theta - 2(n - 1)\pi$	$16\theta/\pi - 4n + 4$
$(n - 3/4)\pi < \theta < (n - 1/2)\pi$	$-2\theta + (2n - 1)\pi$	$-16\theta/\pi + 4n - 2$
$(n - 1/2)\pi < \theta < (n - 1/4)\pi$	$-16\theta/\pi + 4n - 2$	$2\theta - (2n - 1)\pi$
$(n - 1/4)\pi < \theta < (n)\pi$	$4\theta/\pi - 4n$	$-2\theta + 2n\pi$

The resulting differential equations are linear and may be integrated using the integrating factor

$\exp \int \phi \, dx$ where ϕ is a function of the Kamke coefficients. The integrals involved, however, are usually such that the double integration required may be difficult or impossible to carry out explicitly. In those cases used here, these difficulties did not permit satisfactory explicit integration.

TABLE 2

Incremental Approximations of Cos 2θ

Range of θ	Cos $2\theta_U$	Cos $2\theta_L$
$(n - 1)\pi < \theta < (n - 3/4)\pi$	$\frac{1}{2}(4n - 3)\pi - 2\theta$	$4n - 3 - 4\theta/\pi$
$(n - 3/4)\pi < \theta < (n - 1/2)\pi$	$4n - 3 - 4\theta/\pi$	$\frac{1}{2}(4n - 3)\pi - 2\theta$
$(n - 1/2)\pi < \theta < (n - 1/4)\pi$	$4\theta/\pi - 4n + 1$	$2\theta - 2n\pi + \frac{1}{2}\pi$
$(n - 1/4)\pi < \theta < (n)\pi$	$2\theta - 2n\pi + \frac{1}{2}\pi$	$4\theta/\pi - 4n + 1$

The second method of attack mentioned concerns the combinations of Kamke coefficients in (3.6) and may be approached in either of one or both of two ways. The first involves choosing the Kamke coefficients as defined by (3.10) and (3.11) for the express purpose of facilitating the solution of (4.1), (4.2), or (4.3). The most obvious choices are setting $P = S$ and $Q = -R$.

Not all the differential equations resulting from use of either of the two approaches thus far described

exp $\int \phi \, dx$ where ϕ is a function of the [illegible]. The integral is involved, however, [illegible] that the double integration required [illegible] to carry out [illegible] difficulties did not [illegible] integration.

Table 2

Intermediate [illegible] in terms of the [illegible]

Range of θ		
$(n - 1)\pi < \theta < (n - 3/4)\pi$	[illegible]	[illegible]
$(n - 3/4)\pi < \theta < (n - 1/2)\pi$	[illegible]	[illegible]
$(n - 1/2)\pi < \theta < (n - 1/4)\pi$	[illegible]	[illegible]
$(n - 1/4)\pi < \theta < (n)\pi$	[illegible]	[illegible]

The second method [illegible] combinations of [illegible] (3.5) and may be approached in [illegible] of these cases. The first involves [illegible] as defined by (3.10) and (3.11) [illegible] ting the solution of [illegible] obvious choices are [illegible]

Not all the [illegible] use of either of the two approaches [illegible]

can be explicitly integrated. Under such circumstances the second approach may sometimes be used, namely to employ a second time the approximation technique outlined at the beginning of the chapter to the equation and obtain an integrable form. This can be done when the unintegrable function of (4.1), (4.2), or (4.3) possesses a maximum and/or minimum value in the region of the independent variable under consideration. This value is inserted into (4.1), (4.2), or (4.3) which may then be solved to obtain an upper and/or lower bound of the dependent variable.

V. Estimate Calculations

It is desired to know those values of x for which y is equal to zero in (3.8),

(3.8) $$y = CW(x) \sin \theta.$$

The value of z, as given by (3.9), is of no importance here. It has already been stated that the value of C cannot be zero. The solutions to (3.7) indicate that W(x) is never equal to zero, except possibly in the cases where x is equal to plus or minus one. The values of x which are to be determined are those which set the estimates of θ equal to multiples of π.

Although the purpose of this thesis was to investigate the application of Kamke's transformation to both the Jacobi and ultraspherical polynomials, considerable difficulty was experienced in attempting to evaluate the function $\theta(x)$ which is obtained from (3.6). The values of these polynomials at plus one and minus one are known. Unfortunately, both the terms g(x) and h(x) of (3.1), and consequently some of the Kamke coefficients too, contain the term $(1 - x^2)^{-1}$. This term renders the solution of (3.6) and (3.7) indeterminant at those values.

The only other known points of the Jacobi polynomial are for specific values of the coefficients a and b. When $a = b = \frac{1}{2}$ or $a = b = -\frac{1}{2}$, the Tchebichef polynomials result. If both a and b are equal to zero, the Legendre polynomials result.

Only the ultraspherical polynomials with odd degree have a point within the range $-1 < x < 1$ which is known. In these cases, the middle zero is at the point $x = 0$. The estimates of the zeros which are derived in the following pages are confined to this particular case.

The sets of Kamke coefficients which were used in this study are given in Table 3.

The upper and lower bounds of $\theta(x)$ may now be obtained by most of the techniques described with no general difficulty.

Use of the incremental approximations of $\sin 2\theta$ and $\cos 2\theta$, however, is complicated by the presence of the $(1 - x^2)$ term in the denominators of both $g(x)$ and $h(x)$ of (3.1). Application of the approximations given in Tables 1 and 2 to (4.1) results in equations of the form

$$(5.1) \qquad \theta' + \left[G(x)/(1 - x^2)\right]\theta = H(x)/(1 - x^2)$$

which have the solutions

The only other known points of the Jacobi polynomial are for specific values of the coefficients a and b. When a = b = $\frac{1}{2}$ or a = b = $-\frac{1}{2}$, the Tchebichef polynomials result. If both a and b are equal to zero, the Legendre polynomials result.

Only the ultraspherical polynomials with odd degree have a point within the range $-1 < x < 1$ which is known. In these cases, the middle zero is at the point $x = 0$. The estimates of the zeros which are derived in the following pages are confined to this particular case.

The sets of Kamke coefficients which were used in this study are given in Table 5.

The upper and lower bounds of $\theta(x)$ may now be obtained by most of the techniques described with no general difficulty.

Use of the incremental approximations of $\sin 2\theta$ and $\cos 2\theta$, however, is complicated by the presence of the $(1 - x^2)$ term in the denominators of both $g(x)$ and $h(x)$ of (5.1). Application of the approximations given in Tables 1 and 2 to (4.1) results in equations of the form

$$(5.1) \qquad \theta' + \left[g(x)/(1 - x^2)\right]\theta = h(x)/(1 - x^2)$$

which have the solutions

TABLE 3

Sets of Kamke Coefficients

No.	P(x)	R(x)
1.	0	$n(n + 2\lambda)(1 - x^2)^{\lambda + \frac{1}{2}}$
2.	$(2\lambda + 1)x(1 - x^2)^{-1}$	$e^x\left[(2\lambda + 1 - n^2 - 2n)(1 - x^2)^{-1} + (4\lambda + 2)x^2(1 - x^2)^{-1}\right]$
3.	$- x^{-1}$	$(2\lambda + 1 - n^2 - 2n)(x - x^3)^{-1} - 2(x^{-1})$
4.	x	$-e^{\frac{1}{2}x^2}\left[1 - (2\lambda + 1)x(1 - x^2)^{-1} + x^2 + n(n + 2\lambda)(1 - x^2)^{-1}\right]$
5.	0	$n(n + 2\lambda)(1 - x^2)^{-1}$
6.	$(2\lambda + 1)x(1 - x^2)^{-1}$	$(4\lambda + 2)x^2(1 - x^2) - (2\lambda + 1 - n^2 - 2n)x(1 - x^2)^{-1}$
7.	x^{-1}	$(2\lambda + 1 - n^2 - 2n)x(1 - x^2)^{-1}$
8.	$(2\lambda + 1)x(1 - x^2)^{-1}$	$e^{-x}\left[(2\lambda + 1 - n^2 - 2n)(1 - x^2)^{-1} + (4\lambda + 2)x^2(1 - x^2)^{-1}\right]$

9.	0	$-n(n+2\lambda)e^{x}(1-x^2)^{-1}$
10.	0	$-n(n+2\lambda)$
11.	$(1-x^2)^{-1}$	$2\lambda - 2x - n(n+2\lambda)(1-x^2)$
12.	0	-1
13.	0	$(1-x^2)^{-1}$
14.	0	$n(n+2\lambda)(2\lambda x - x)^{-1}$
15.	0	$-\left[n(n+2\lambda)(1-x^2)^{-1}\right]^{\frac{1}{2}}$

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TABLE 3

Sets of Kamke Coefficients - continued

No.	Q(x)	S(x)
1.	0	$(1 - x^2)^{-\lambda - \frac{1}{2}}$
2.	e^{-x}	1
3.	x	$(2\lambda + 1)x(1 - x^2)^{-1}$
4.	$e^{-\frac{1}{2}x^2}$	$(2\lambda + 1)x(1 - x^2)^{-1}$
5.	$- 1$	$(2\lambda + 1)x(1 - x^2)^{-1}$
6.	x^{-1}	x^{-1}
7.	x^{-1}	$(2\lambda + 1)x(1 - x^2)^{-1}$
8.	e^x	$- 1$

9.	e^{x}	$-1 + (2\lambda + 1)x(1 - x^2)^{-1}$
10.	$(1 - x^2)^{-1}$	$(2\lambda + 1)x(1 - x^2)^{-1}$
11.	$(1 - x^2)^{-1}$	$-\left[1 + x(2\lambda + 3)\right](1 - x^2)^{-1}$
12.	$n(n + 2\lambda)(1 - x^2)^{-1}$	$(2\lambda + 1)x(1 - x^2)^{-1}$
13.	$-n(n + 2\lambda)$	$(2\lambda + 1)x(1 - x^2)^{-1}$
14.	$(2\lambda + 1)x(1 - x^2)^{-1}$	$-x^{-1} + (2\lambda + 1)x(1 - x^2)^{-1}$
15.	$\left[n(n + 2\lambda)(1 - x^2)^{-1}\right]^{\frac{1}{2}}$	$(2\lambda + 1)x(1 - x^2)^{-1} - x(1 - x^2)^{-\frac{1}{2}}$

$$(5.2) \qquad \theta \exp\int\left[G(x)/(1 - x^2)\right] dx = \int\left[H(x)/(1 - x^2)\right] \cdot$$

$$\exp\left[\int G(x)/(1 - x^2)dx\right]dx + c.$$

In its simplest form, the integrating factor is in the form $(1 - x)^c(1 - x)^d$ where c and d are positive constants. The right hand part of (5.2) can be integrated in the range $-1 < x < 1$ by making the substitution $x = 2z - 1$ and converting it into an incomplete beta function since

$$(5.3) \qquad \int_{-1}^{x}(1 - x^2)\, dx = (2)^{(2\lambda - 1)} \cdot \frac{(\lambda - \frac{1}{2})!}{(2\lambda)!} \cdot$$

$$\frac{(2\lambda)!}{(\lambda - \frac{1}{2})!} \cdot \int_{0}^{z}(z)^{\lambda}(1 - z)^{\lambda} dz .$$

when c or d is less than minus one, (5.3) has no solution because it cannot be explicitly integrated. An estimate obtained by this method might therefore be restricted to incremental values of θ.

The adaptability of the sets of coefficients listed in Table 3 to these various methods may now be evaluated. All except number four may be directly integrated in (4.2) and (4.3). In those cases where $\frac{1}{2}(P - S) <$ 0 and $\frac{1}{2}(Q - R) < 0$, the roles of the respective bounds are reversed. A trial integration of number four by series revealed that it was not as desirable as some of the other coefficient sets.

All sets in Table 3 except number fifteen could be

disregarded immediately by noting that their lower estimates did not contain n. This indicated that the distances between the upper and lower bounds of θ were too large for use.

None of the sets, therefore, necessitated the employment of the maximum or minimum value of unintegrable functions. In experimenting with Jacobi polynomials, however, the more extensive construction of g(x), (3.1), makes this method very practical on occasion.

Attempts to use the incremental approximations listed in Tables 1 and 2 consistently failed because of the fact that the integrating factor contained terms which could not be converted into the form $(1 - x)^c \cdot (1 - x)^d$. The factor could not therefore be converted into a form which could be explicitly integrated, and no solution could be obtained.

The equations for the upper and lower bounds of θ which resulted from the use of coefficient set number fifteen, Table 3, in (4.2) and (4.3) are

$$(5.4) \qquad \theta_U = (n^2 + 2n\lambda)^{\frac{1}{2}} \sin^{-1}x - \tfrac{1}{4}(2\lambda + 1)\log(1 - x^2) - \tfrac{1}{2}(1 - x^2)^{\frac{1}{2}} + \tfrac{1}{2}$$

$$(5.5) \qquad \theta_L = (n^2 + 2n\lambda)^{\frac{1}{2}} \sin^{-1}x + \tfrac{1}{4}(2\lambda + 1)\log(1 - x^2) + \tfrac{1}{2}(1 - x^2)^{\frac{1}{2}} - \tfrac{1}{2}$$

disregarded immediately, by noting that the [illegible] esti-
mates did not contain. With increasing n the dis-
tances between the upper and lower [illegible]
large for yes.

None of this [illegible] the
employment of the maximum [illegible] of [illegible]-
graphic functions. In experimenting with Jacobi poly-
nomials, however, the [illegible] of α [illegible]
(3.1), makes this method [illegible].

Attempts to use the [illegible]
listed in tables 1 and 2 [illegible] because of
the fact that the [illegible]
which could not be converted into the form $(1 - x)^a$
$(1 - x)^b$, the [illegible] could not therefore be converted
into a form which could be [illegible], and no
solution could be obtained.

The equations for the upper and lower [illegible]
which resulted in [illegible]
fifteen, Table 3, in [illegible] are

$$(5.4) \qquad e_u = (\alpha + \beta n)[illegible]$$

$$[illegible]$$

$$(5.5) \qquad e_l = (\alpha + \beta n)[illegible]$$

$$[illegible]$$

where n is odd and the notation is in accordance with the footnote on page 15.

VI. Evaluation of Result

Estimates of x which do not involve Bessel functions are[1]

$$(6.1) \qquad \theta_v > \frac{v - (1 - \lambda)/2}{n + \lambda}\pi$$

$$(6.2) \qquad \theta_v < \frac{v}{n + 1}\pi \qquad \text{if } \lambda > \tfrac{1}{2}$$

$$(6.3) \qquad \theta_v < \frac{v + \lambda - \frac{1}{2}}{n + 2\lambda}\pi \qquad \text{if } \lambda < \tfrac{1}{2}$$

where $v = 1, 2, \ldots n/2$, and $x_v = \cos \theta_v$. These relations are valid for all values of n. The subscript 'v' denotes the number of the zero, defining $v = 1$ for the zero nearest $+ 1$ with the numbers proceeding consecutively to the zero nearest $- 1$ whose v is n.

When $\lambda = 0$, (6.1) and (6.3) become equal. This is because the ultraspherical polynomial has degenerated into a Tchebichef polynomial of the first kind. Table 4 is a comparison of those known values of the polynomial when the particular situation $\lambda = 0$ exists and the

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[1] G. Szegö, loc. cit., pp. 134-136.

G. Szegö, Inequalities for the Zeros of Legendre Polynomials and Related Functions, Trans. Amer. Math. Soc. vol. 39, 1936, pp. 1-17.

(5.4)-(5.5) estimate for the same condition. The value of n is taken as nine.

TABLE 4

Comparison of the Estimates of Equations (5.4) and (5.5) with Known x_v's

Eqns. (5.4) & (5.5)	x_v	v
0.00	0.000	5
.335 - .345	.343	4
.62 - .66	.640	3
.84 - .91	.865	2
.96 - 1.00	.984	1

Since the solution of (5.4) and (5.5) for values of x for which θ_U and θ_L are multiples of π cannot be carried out explicitly in terms of the elementary functions, graphical solutions are used. Figure 1 illustrates how the estimates listed in Table 4 were obtained. Since the ultraspherical polynomials are symmetric about the origin, only positive x's are indicated.

With these points in mind, it is interesting to choose a comparison which is not concerned with a special case of either method of estimate. Table 5 lists the estimates for the zeros of the ultraspherical polynomial when n = 9 and λ = 6. This condition is also illustrated in Figure 2.

Tables 4 and 5 suggest two generalizations of the

(5.4)-(5.5) [illegible] when [illegible] of n is taken as [illegible].

[illegible]

[illegible]

Eqns. (5.4) - (5.5)		
[illegible]	.[illegible]	
[illegible]	.[illegible]	.
[illegible]	.[illegible]	.[illegible]
[illegible]	.[illegible]	.[illegible] - .[illegible]
[illegible]	.[illegible]	.95 - 1.00

Since the solution of (5.4) and (5.5) for values of x for which H_n and H_{n+1} are multiples of [illegible] cannot be carried out explicitly in terms of the elementary functions, graphical solutions are used. Figure [illegible] illustrates how the estimates listed in Table 4 were obtained. Since the [illegible] polynomials are symmetric about the origin, only positive x's are indicated.

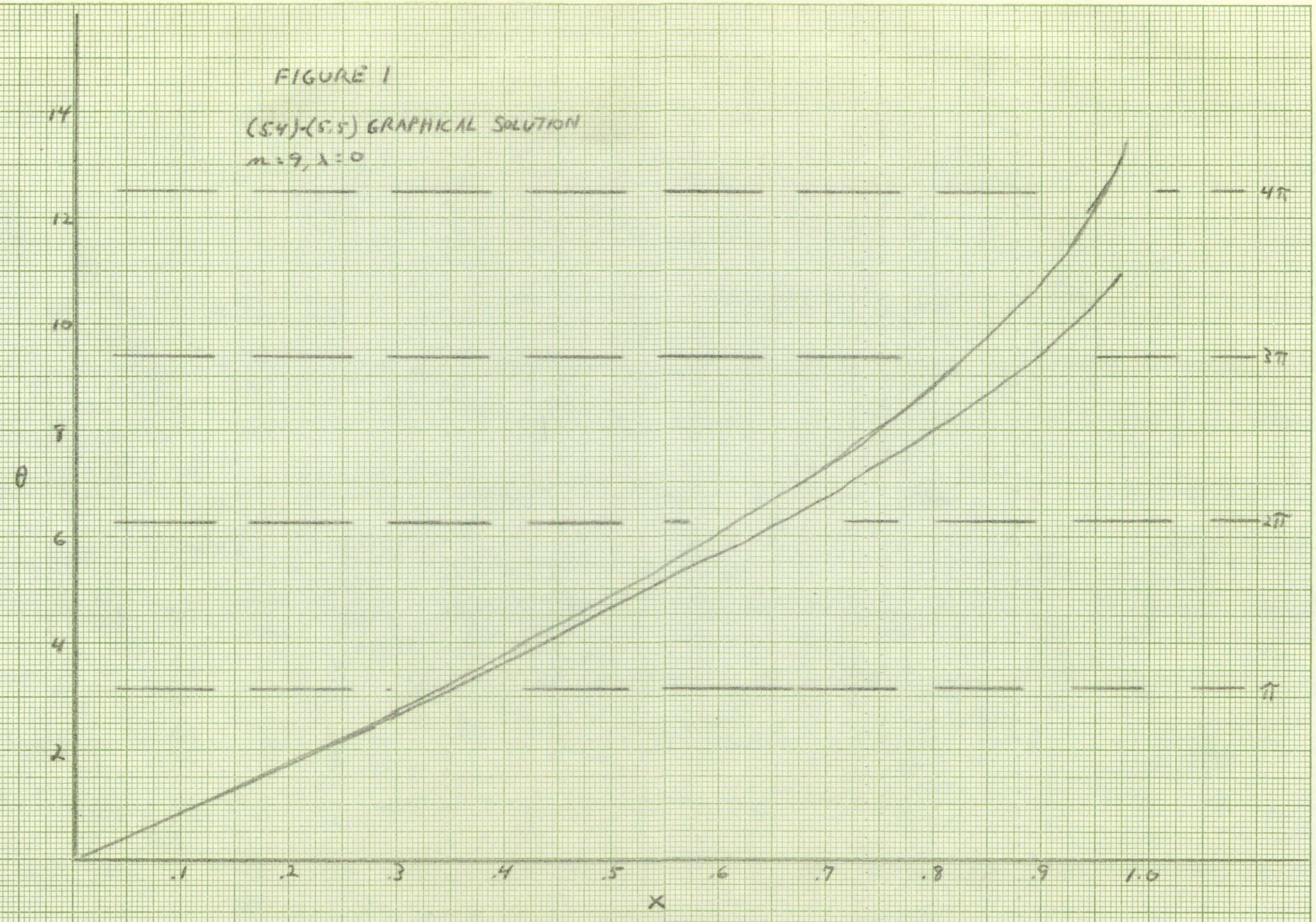
With these points in mind, it is interesting to choose a description [illegible] a special case of [illegible] method [illegible] the estimates for the [illegible] of the [illegible] polynomial when n = [illegible] and A = [illegible]. This [illegible] is also illustrated in Figure [illegible].

Tables 4 and 5 [illegible]

FIGURE 1

(5.4)-(5.5) GRAPHICAL SOLUTION

$m=9, \lambda=0$



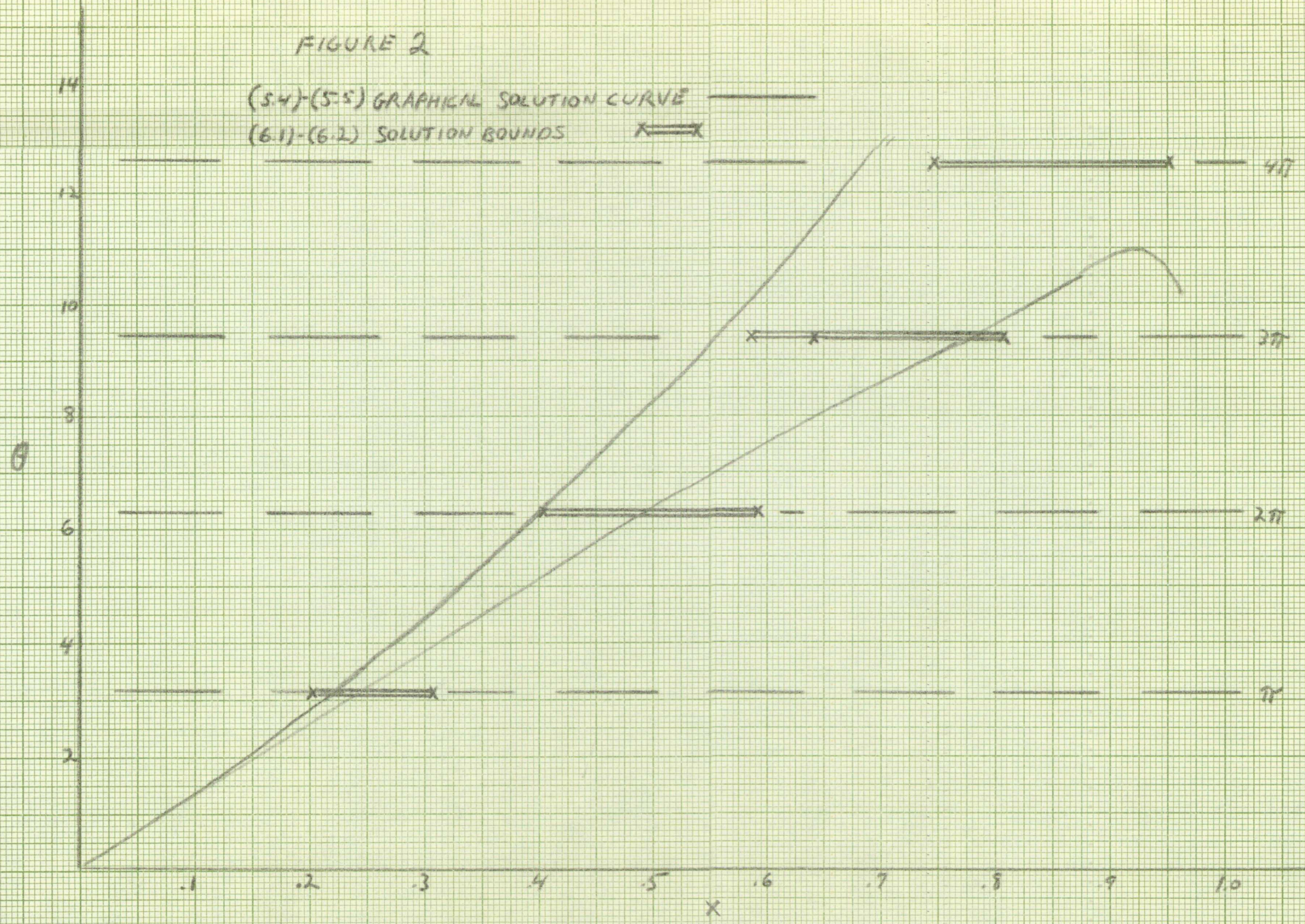

(5.4)-(5.5) GRAPHICAL SOLUTION CURVE
(6.1)-(6.2) SOLUTION BOUNDS
4π
3π
2π
π
θ
14
12
10
8
6
4
2
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
X

FIGURE 2

behavior of the (5.4)-(5.5) estimates with different values of n and λ. First, the estimate interval increases with the distance from zero. Second, the estimate interval increases as the values of λ increase and n decreases.

TABLE 5

Comparison of the Estimates of Equations (5.4) & (5.5) and (6.1) & (6.2)

Eqns. (5.4) & (5.5)	Eqns. (6.1) & (6.2)	v
0.00	0.00	5
.22 - .24	.207 - .311	4
.40 - .49	.406 - .588	3
.56 - .78	.587 - .808	2
.69 - 1.00	.745 - .951	1

These deductions can be verified by considering the extreme cases. In $n = 3$ and $\lambda = 0$, the exact value of x for which the polynomial equals zero is given by (6.1) and (6.3) to be 0.878. The (5.4)-(5.5) estimate is that it is greater than 0.78, less than the upper limit of one. This effect is amplified even further in the case where $n = 3$ and $\lambda = 6$. The (6.1)-(6.2) estimate places the zero point between 0.72 and 0.77. The (5.4)-(5.5) estimate is merely that it is between 0.38 and 1.00.

In summary, therefore, it may be stated that the estimates derived by use of the Kamke transformation

possess two properties: (1) for reasonably low values of λ the estimate interval may be less than that of some of the present methods, and (2) the (6.1)-(6.2 or 6.3) and (5.4)-(5.5) methods of estimate do not usually have the same medial value, so combined use may result in even shorter estimate intervals.

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