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The Motion of a Charged Particle in Homogenous Time-Varying Magnetic Field

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A CHARGED PARTICLE · BATES

NAME AND ADDRESS	DATE

THE MOTION OF A CHARGED PARTICLE

IN A HOMOGENEOUS TIME-VARYING MAGNETIC FIELD

By

Thomas R. Bates

A Thesis

Submitted in Partial Fulfillment of the

Requirements for the Degree of

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1956

This thesis, directed and approved by the candidate's committee, has been accepted by the Graduate Committee of the University of New Mexico in partial fulfillment of the requirements for the degree of

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John R Green

CHAIRMAN

Roy Thomas.

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CHAPTER I

INTRODUCTION

I. AN INTRODUCTORY DISCUSSION OF THE PROBLEM

This paper will discuss the motion of a non-relativistic, non-radiating charged particle in an infinite homogeneous magnetic field which varies linearly with time. We shall assume the motion to be confined to a plane perpendicular to the magnetic field. The force exerted on a charged particle by an electromagnetic field is given by

$$1\text{-}1 \qquad \vec{F} = q\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)$$

in mixed c.g.s. units where

F is the force on the particle in dynes;

q is the charge in statcoulcombs;

E is the electric field in statvolts/cm;

V is the velocity in centimeters per second;

B is the magnetic induction in gauss; and

c is equal to 3.10^{10} cm/sec.

The rate of change of kinetic energy of the particle is

$$1\text{-}2 \qquad \vec{F}\cdot\vec{v} = q\vec{E}\cdot\vec{v}$$

since

$$(\vec{v}\times\vec{B})\cdot\vec{v} = 0$$

As shown above the force on the particle due to the magnetic field alone is always at right angles to the direction of motion; and, therefore, cannot cause any change in the speed of the particle.

When the particle is in a constant homogeneous magnetic field, the force, $\vec{F}$, is

$$\vec{F} = \frac{q}{c} \vec{V} \times \vec{B}$$

and is of constant magnitude. As a result the particle travels in a circle with constant speed. One can analyze the motion by setting the centripetal force on the particle equal to the force exerted by the magnetic field.

$$\frac{m v^2}{R} = \frac{q}{c} v B \tag{1-3}$$

In vector notation one can write

$$\vec{\rho} = \frac{mc}{qB^2} \vec{V} \times \vec{B} \tag{1-4}$$

where $\vec{\rho}$ is a vector pointing from the particle to its center of motion. We can describe in more detail some of the characteristics of particle motion in a uniform homogeneous field.[1] By rewriting equation 1-3 we note that the frequency of rotation of the particle,

$$\omega = \frac{qB}{mc} \tag{1-5}$$

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[1] H. Alfeven, Cosmical Electrodynamics, pp. 16

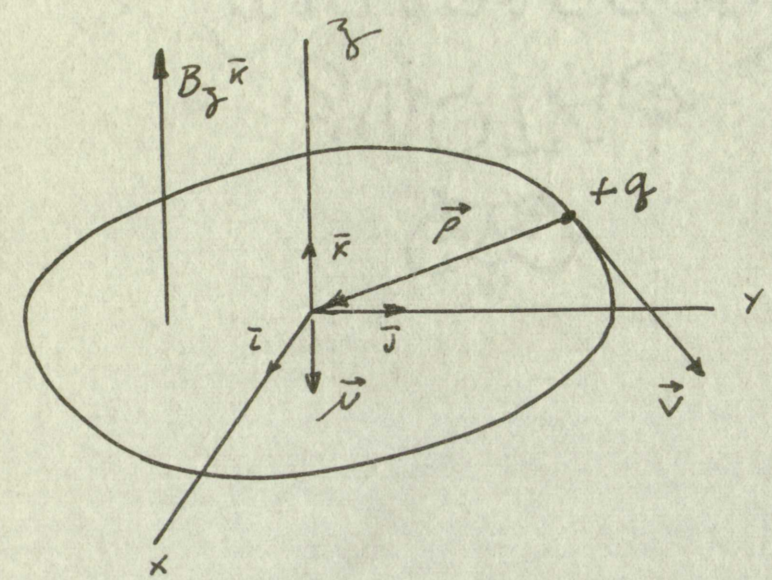

Fig I

is independent of the speed of the particle, and the period of rotation is given by

1-6 $$\tau = \frac{2\pi mc}{qB}$$

The average current at any point in the trajectory is the charge flowing past that point per unit of time or

1-7 $$I = \frac{q}{c}\,\frac{1}{\tau}$$

where I is the current in abamps. The effective magnetic dipole moment of the particle is then given by

1-8 $$\mu = \pi \rho^2 I$$

If substitutions are made for ρ and I, this expression reduces to

$$1\text{-}9 \qquad \mu = \frac{1}{2} m v^2 \frac{1}{B} = \frac{W}{B}$$

where W is the kinetic energy of the particle. It can be seen from Fig. I that this magnetic dipole is oriented antiparallel to the magnetic field. The magnetic flux (in maxwells) through the circular path due to the external magnetic field is

$$1\text{-}10 \qquad \phi = \pi \rho^2 B = \frac{2\pi m c^2}{q^2} \mu$$

Should the magnetic field now vary in time, an electric field will be induced according to Faraday's law of induction. The motion of the particle must then be described in terms of the varying magnetic field and the induced electric field. As one goes from a stationary field to a changing magnetic field, one can generally expect changes in the speed of the particle, the radius of curvature and the shape of its trajectory.

II. THE LITERATURE

The literature on the motion of charged particles in changing magnetic fields is very sketchy in both the periodicals and textbooks. Most textbooks limit their discussion of this topic to betatron acceleration.

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in a homogeneous time-varying magnetic field where the change in field strength during one rotation of the particle is very small compared to the field strength itself. He assumes then that each circulation of the particle forms a circular trajectory. We can then apply Faraday's law of induction

1-11 $$\oint \vec{E} \cdot d\vec{s} = -\frac{1}{c} \frac{d\phi}{dt}$$

where the flux linkage ϕ is

1-12 $$\phi = \pi R^2 B$$

The gain in kinetic energy, ΔW, in one rotation is

1-13 $$\Delta W = q \oint \vec{E} \cdot d\vec{s} = \frac{q}{c} \pi R^2 \frac{dB}{dt}$$

This gain in kinetic energy divided by the period is the rate of increase of kinetic energy

$$\frac{dW}{dt} = \frac{\Delta W}{\tau}$$

Alfven then substitutes equation (1-6), that gives the period of rotation in a constant field, into the above expression to obtain

$$\frac{dW}{dt} = \frac{q^2}{2mc^2} B R^2 \frac{dB}{dt}$$

Then rearranging equation (1-3) as follows

$$R = \frac{c}{qB} \sqrt{2mW}$$

and substituting this in the above expression for dW/dt

$$\frac{dW}{dt} = \frac{W}{B} \frac{dB}{dt}$$

This differential equation can be solved by multiplying by dt/W

$$\frac{dW}{W} = \frac{dB}{B}$$

and integrating

$$\ln W = \ln B + C$$

or

1-14 $$\frac{W}{B} = \text{Constant}$$

This shows that the magnetic moment μ remains constant. This in turn requires that the flux linkage remain constant; consequently, R must change even though we assumed it to be constant during one revolution. Applying equation (1-3) again

1-15 $$\mu = \frac{1}{2} m v^2 \frac{1}{B} = \frac{\varepsilon V R}{2c}$$

or

1-16 $$VR = \text{Constant}$$

Let us examine more closely what Alfven has done. For simplicity we shall assume dB/dt is constant. Under the assumption that dB/dt is small, equations (1-11, 1-12) are valid. However, in calculating $\frac{d\phi}{dt}$ in equation (1-13) he neglects the change in ϕ caused by the fact that the radius is actually changing. Now equation (1-13) together with the assumption that R is constant says that the energy gain

per cycle is constant. He then tacitly drops the assumption that R is constant and fits the condition that Δ W/cycle is constant into equation (1-3) which we know to be approximately correct since the predominating force on the particle is due to an approximately constant magnetic field. We note that he must drop the condition that R is constant since the three conditions, R being constant, Δ W/cycle being constant and equation (1-3) cannot all be satisfied simultaneously.

III. THE BETATRON

A comparison of our problem with betatron acceleration will yield some qualitative information on the nature of our problem. In the betatron it is desired to increase the kinetic energy of the particle by accelerating it in an induced electric field. It is desired, furthermore, to maintain the particle in a circular trajectory with constant radius. We can derive the betatron equation as follows:

Let the magnetic induction through which the particle travels be B_r and the average magnetic induction inside the closed trajectory be B_i (both normal to the plane of motion). If one then assumes that the only radial force on the particle is that due to the field B_r then by equations 1-1 and A-8

1-17 $$v = \frac{\varepsilon}{mc} R B_r$$

The tangential electric field can be derived from Faraday's law of induction.

1-18 $$e.m.f. = \oint \vec{E}\cdot d\vec{l} = -\frac{1}{c}\frac{\partial \phi}{\partial t}$$

e.m.f. is in statvolts

dl is a differential line element tangential to the trajectory.

ϕ is the magnetic flux linkage of the trajectory in maxwells.

Now the electric field along this circular trajectory does not vary from point to point (at a given instant) because of the symetrical nature of the betatron. Therefore, we may write for the second term

$$\oint \vec{E}\cdot d\vec{l} = \oint E_t\, dl = 2\pi R E_t$$

where E_t is the θ, or tangential component of the electric field and for the third term

$$-\frac{1}{c}\frac{\partial \phi}{\partial t} = -\frac{1}{c}\pi R^2 \frac{\partial B_i}{\partial t}$$

By equating the last two equations we can solve for E_t

1-19 $$E_t = -\frac{R}{2c}\frac{\partial B_i}{\partial t}$$

The rate of change of momentum of the particle is given by

$$\frac{d(mv)}{dt} = qE_T = -\frac{qR}{2c}\frac{\partial B_i}{\partial t}$$

Integrating, one obtains

1-20 $$v = -\frac{qRB_i}{2mc}$$

if V and B_i are both zero at time zero.

The centripetal force is given by

$$\frac{mv^2}{R} = -\frac{q}{c} v B_r$$

or

$$B_r = -\frac{mcv}{qR} \qquad (1\text{-}21)$$

Eliminating the velocity between equation (1-20) and (1-21) we obtain for the relation between B_r and B_i

$$B_r = \frac{B_i}{2} \qquad (1\text{-}22)$$

Thus the deflecting magnetic induction must be one-half the magnetic induction inside the closed path in order that the particle maintain a circular trajectory.

It is evident that in betatron acceleration the induced electric field is circular and is in the direction of motion of the particle. If the magnetic field in the orbital region were increased slightly, we see that the particle would spiral inwards because of the increase of the magnetic force over that force necessary to provide just the centripetal force required for circular motion. Therefore, it is reasonable to assume that if a particle were circulating in a homogeneous, changing magnetic field and the field change were in such a direction as to increase the kinetic energy of the particle, the particle would spiral inwards towards its center of motion.

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CHAPTER II

THE FORMAL METHOD OF ATTACKING THIS PROBLEM

I. GENERAL CONSIDERATIONS

In a sense our problem can be separated into two parts. The first is, with a given changing magnetic field, to find the effective electric field which acts on the particle. The second, once both fields are known, is to calculate the trajectory of the particle or at least some of the properties of its motion. Faraday's law of induction tells us that the e.m.f. generated about any closed path is proportional to the rate of decrease of magnetic flux through that closed path. However, in our problem we cannot expect to have a closed path as does a particle in the betatron; and it is difficult to see how one can apply the law of induction. Let us first consider what can be accomplished by applying the formalism of Lagrangian mechanics to the problem.

II. THE EQUATIONS OF MOTION IN THE LAGRANGIAN FORM

A typical derivation for the Lagrangian for a charged particle is as follows:

1-1 $$\vec{F} = q\left(\vec{E} + \frac{1}{c}\,\vec{v} \times \vec{B}\right)$$

The magnetic induction is related to the vector potential $\vec{A}$, as follows

$$\vec{B} = \text{curl}\,\vec{A} \tag{2-1}$$

and, if there is no electrostatic field,

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} \tag{2-2}$$

We may then write equation (1-1) as

$$\vec{F} = \frac{q}{c}\left(-\frac{\partial \vec{A}}{\partial t} + \vec{V} \times (\nabla \times \vec{A})\right)$$

We may substitute in the above equation the vector identity

$$\vec{V} \times \nabla \times \vec{A} = \nabla(\vec{V}\cdot\vec{A}) - (\vec{V}\cdot\nabla)\vec{A} \tag{2-3}$$

and the following expression which depends on the assumption that $\vec{A}$ is a function of the coordinates and time only.

$$\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\vec{V}\cdot\nabla)\vec{A} \tag{2-4}$$

After performing these substitutions $\vec{F}$ may be written as

$$\vec{F} = \frac{q}{c}\left[-\nabla(\vec{V}\cdot\vec{A}) - \frac{d\vec{A}}{dt}\right]$$

Since $\vec{A}$ is a function of the coordinates and time, but not of the velocity

$$\nabla_{\sim}\vec{A} = 0 \tag{2-5}$$

where $\nabla_{\sim}$ represents the "velocity gradient". Therefore, we may finally write $\vec{F}$ as

$$\vec{F} = \frac{q}{c}\left[-\nabla_{r}(\vec{V}\cdot\vec{A}) - \frac{d}{dt}\nabla_{\sim}(\vec{V}\cdot\vec{A})\right] \tag{2-6}$$

It is now apparent that $\frac{q}{c}\vec{V}\cdot\vec{A}$ can be chosen as the velocity-dependent potential. The Lagrangian for a charged particle in an electromagnetic field is therefore

2-7 $$L = W - \frac{e}{c}\vec{V}\cdot\vec{A}$$

where W is the kinetic energy of the particle.

The above derivation was presented in order to show that the vector potential and hence the electric field are assumed to be functions only of the coordinates and of the time in Lagrangian theory and are assumed not to be functions of the velocity of the particle. Later on we shall examine these assumptions in view of other requirements for the fields. Let us for the present find the equations of motion of the particle by use of this Lagrangian.

We can derive a general form for the vector potential for our particular problem. Since $\vec{B} = B\vec{k}$, we have

2-8 $$B\vec{k} = \operatorname{curl}\vec{A}$$

Thus

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0$$

$$\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0$$

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B$$

Because of the two-dimensional nature of the problem, neither A_y nor A_x can be functions of z. Thus

$$\frac{\partial A_y}{\partial z} = \frac{\partial A_x}{\partial z} = 0$$

Consequently

$$\frac{\partial A_z}{\partial y} = \frac{\partial A_z}{\partial x} = 0$$

Furthermore, since the electric field will by symmetry be restricted to the x,y plane,

$$\frac{\partial A_z}{\partial t} = 0$$

Thus it appears that A_z can at most be only a constant. This leaves

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B$$

If we assume A_x to be an arbitrary function of x,y, then

$$A_y = \int \left(B + \frac{\partial A_x}{\partial y} \right) dx + f(y)$$

or

2-9 $$A_y = Bx + \int \frac{\partial A_x}{\partial x} dx + f(y)$$

since B is not a function of x. Similarly

$$A_x = -By + \int \frac{\partial A_y}{\partial x} dy + g(x)$$

We see that there is an infinite variety of possible vector potentials.

We might arbitrarily pick the "vortex" type of vector potential for further exploration:

2-10 $$\vec{A} = \frac{B}{2}\left(-y\vec{i} + x\vec{j} \right)$$

Then

2-11 $$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 \right) + \frac{qB}{2c}\left(-y\dot{x} + x\dot{y} \right)$$

$$\frac{\partial L}{\partial x} = \frac{qB\dot{y}}{2c}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} - \frac{q}{2c} By$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} - \frac{q}{2c}(B\dot{y} + \dot{B}y)$$

and for y

$$\frac{\partial L}{\partial y} = -\frac{qB\dot{x}}{2c}$$

$$\frac{\partial L}{\partial \dot{y}} = m\dot{y} + \frac{qBx}{2c}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = m\ddot{y} + \frac{q}{2c}(B\dot{x} + \dot{B}x)$$

Lagrange's equations of motion are then:

2-12 $$m\ddot{x} - \frac{q}{2c}(2B\dot{y} + \dot{B}y) = 0$$

$$m\ddot{y} + \frac{q}{2c}(2B\dot{x} + \dot{B}x) = 0$$

We can combine these equations into one equation expressing the motion in the complex plane by multiplying the second equation by $i(\sqrt{-1})$, and adding it to the first.

2-13 If $\gamma = x + iy$

2-14 $$\ddot{\gamma} = \frac{-iq}{2mc}(2B\dot{\gamma} + \dot{B}\gamma)$$

Since B is a function of time, this equation would not have a simple solution.

If $\dot{B} = 0$

$$\ddot{\gamma} = -i\frac{qB_0}{mc}\dot{\gamma}$$

and

$$\dot{\gamma} = \dot{\gamma}_0 e^{-\frac{iqB_0}{mc}t}$$

which is recognized as the solution for motion in a constant field.

III. THE LAGRANGIAN IN CYLINDRICAL COORDINATES

The "vortex" vector potential

$$2\text{-}10 \qquad \vec{A} = \frac{B}{2}\left(-y\,\vec{i} + x\,\vec{j}\right)$$

can be written in cylindrical coordinates as follows:

$$\vec{A} = A_\theta\,\vec{\theta}_1 = \frac{r}{2}\,B\,\vec{\theta}_1$$

The velocity can be written

$$\vec{V} = \dot{r}\,\vec{r}_1 + r\,\dot{\theta}\,\vec{\theta}_1$$

The Lagrangian in cylindrical coordinates is then

$$2\text{-}15 \qquad L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{q}{c}\,\frac{r^2}{2}\,\dot{\theta}\,B$$

The canonical momentum associated with r is

$$2\text{-}16 \qquad P_r = \frac{\partial L}{\partial \dot{r}} = m\,\dot{r}$$

The canonical momentum associated with θ is

$$2\text{-}17 \qquad P_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\,r^2\dot{\theta} + \frac{q}{c}\,\frac{r^2}{2}\,B$$

We note that P_r is the ordinary radial component of momentum for a particle. The ordinary θ component of momentum is $mr^2\dot{\theta}$. However, we see here that P_θ contains an additional term. Hence, the appropriate conservation law to

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We note that P_r is the ordinary radial component of momentum for a particle. The ordinary θ component of momentum is $m r^2\dot{\theta}$. However, we see here that P_θ contains an additional term. Hence, the appropriate conservation law to

apply here is that P_θ is constant when the associated generalized force is zero.

IV. A CRITICISM OF THE PROBLEM AS SET UP IN SECTION II

Fig. II is a diagram showing some of the features of the problem as we have set it up using the vortex $\vec{A}$, which yields a vortex electric field.

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\frac{\dot{B}}{2c}(-y\vec{i} + x\vec{j}) \tag{2-18}$$

Here E has only a θ component.

There is a disturbing feature to this description of the motion. At time zero when the field change is begun the electric field is tangential to the trajectory. As time moves on, however, there appears a finite angle between the electric field and the trajectory. This is because the electric field is concentric with the origin of coordinates; whereas, the instantaneous center of motion of the particle has moved to a different position.

There is nothing unique about the origin of coordinates in so far as the fields are concerned except that it represents the original center of motion of the particle. Since the center of motion changes with time, it would seem more appropriate in this case to describe the electric field in terms of the instantaneous center of curvature.

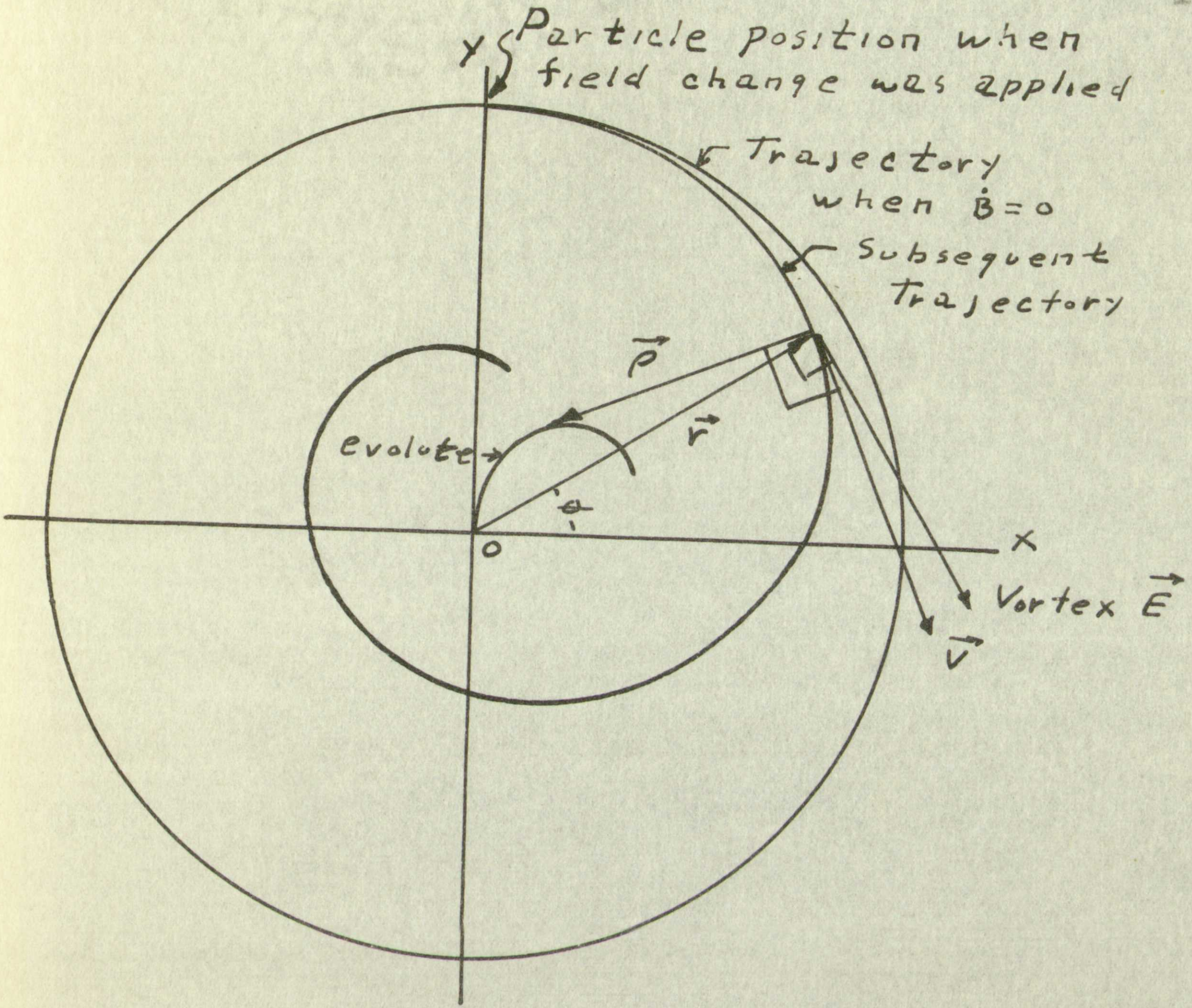
Y
Particle position when field change was applied
Trajectory when $\dot{B}=0$
Subsequent trajectory
$\vec{\rho}$
$\vec{r}$
evolute
θ
O
X
Vortex $\vec{E}$
$\vec{v}$

Fig II

CHAPTER III

A TREATMENT OF THE PROBLEM BASED ON THE LAW OF INDUCTION

I. THE REASONS FOR THIS TREATMENT

The electric field acting on the particle may be resolved into components in the direction of and perpendicular to the trajectory.

3-1 $$\vec{E} = E_n \vec{n} + E_t \vec{t}$$

where $\vec{n}$ and $\vec{t}$ are the unit normal and unit tangent respectively. (ref. to App. A)

The remainder of this thesis is based upon the assumption that $E_n = 0$. This was originally accepted by the author as a working hypothesis on the basis of rather intuitive considerations which will be outlined below. It was hoped during the course of the work that either a proof (or disproof) of this hypothesis would come to light or at least a theory would be evolved which would be consistent with known physical phenomena. The subsequent investigation has failed to supply a proof or disproof of our assumption. The evolved theory seems to be free of any glaring discrepancies in the light of a rather limited investigation.

The initial considerations which led to the working hypothesis that $E_n = 0$ were as follows.

Assume a particle is circulating in a homogeneous

CHAPTER III

A TREATMENT OF THE [illegible]

1. THE [illegible]

The electric field acting [illegible] may be resolved into components in the direction of [illegible] and perpendicular to the trajectory.

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where $\vec{n}$ and $\vec{t}$ are the unit normal and unit tangent vectors respectively. (ref. to App. A)

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Assume a particle is circulating [illegible]

magnetic field. A simple vector potential which describes the field is the "vortex" potential.

$$2\text{-}10 \qquad \vec{A} = \frac{B}{2}\left(-y\,\vec{i} + x\,\vec{j}\right)$$

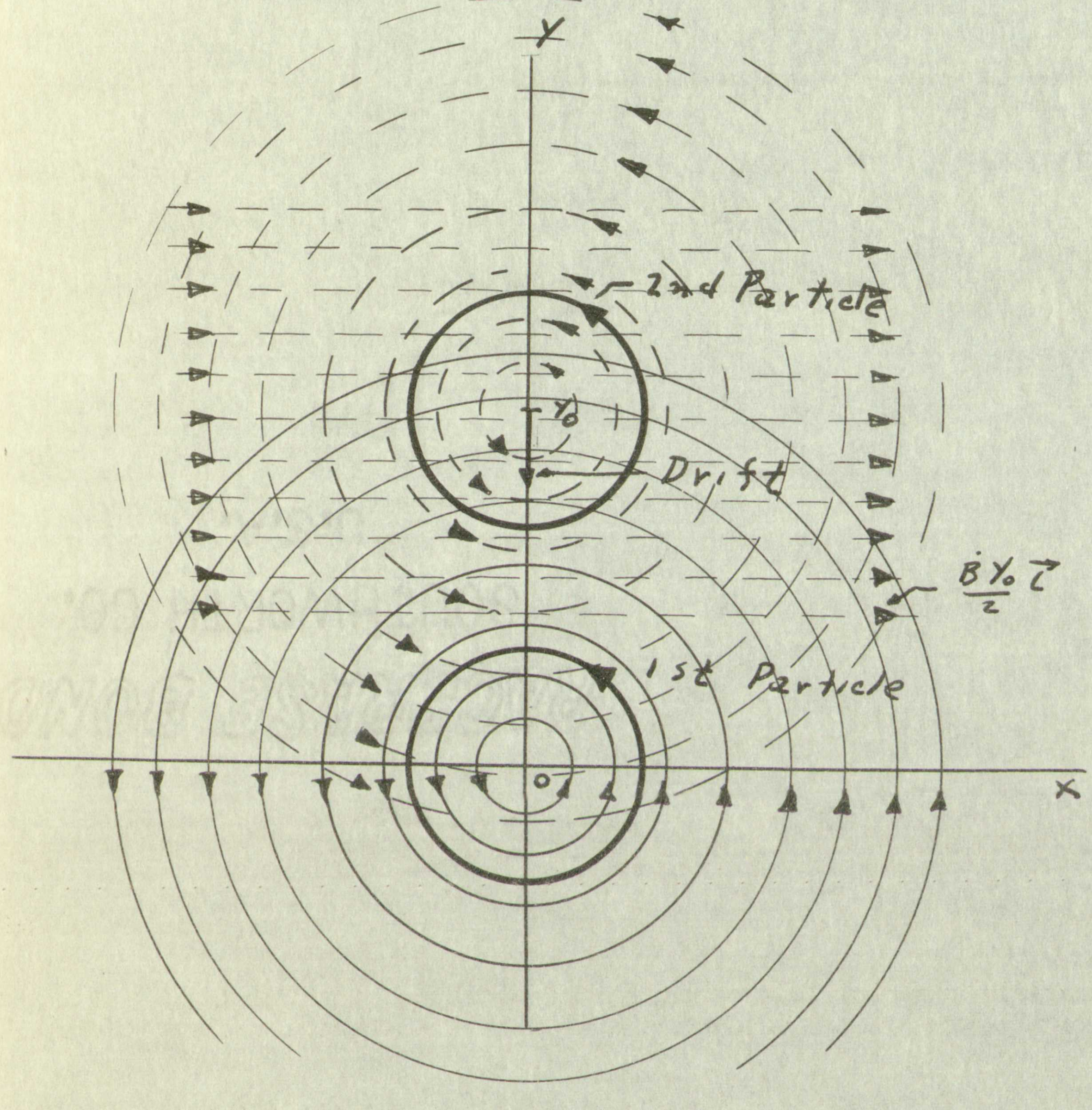
If the magnetic field were changing at a very low rate, the radius of curvature of the particle would remain approximately constant, and there would be a small induced electric field which would act on the particle. This electric field, as calculated from the vector potential would also be of the vortex type and, therefore, parallel to the direction of motion. Now assume we had another particle in the same field with a different center of motion (Fig. III). For simplicity we shall assume the second particle's center of motion to be at y_o on the x axis. If the electric field acting on the first particle is

$$2\text{-}18 \qquad \vec{E} = \frac{\dot{B}}{2}\left(-y\,\vec{i} + x\,\vec{j}\right)$$

this same electric field acts on the second particle. We may rewrite this electric field as

$$3\text{-}2 \qquad \vec{E} = \frac{\dot{B}}{2}\left[-(y-y_o)\,\vec{i} + x\,\vec{j}\right] - \frac{\dot{B}y_o}{2}\,\vec{i}$$

We see that the first term on the right represents a vortex electric field about the point $x=0,\ y=y_o$ of the same intensity as the vortex field about the origin. The two particles then both experience equal vortex fields about their respective centers, but the second particle, in addition,



$\vec{B}$, $\dot{\vec{B}}$ are into paper

$\vec{E}$ (as expressed by eqn. 2-18)

$\vec{E}$ (as expressed by eqn. 3-2)

Fig III

is acted on by the field $\frac{-\dot{B}y_0 i}{2}$. By comparing this situation to the well-known particle motion in crossed electric and magnetic fields one can see that the field is such as to cause the second particle's center to drift towards the origin. It, therefore, seems that the nature of the motion is different for each particle.

Now if our rate of change of magnetic field is large enough to cause the radius of curvature to change appreciably, the center of curvature must change. It would seem then that if at some future time we wish to describe the forces acting on the particle, it would be more appropriate to use as our origin the new center of motion. (However, we are still viewing the motion from a stationary coordinate frame.) Therefore, when we try to set up the electric field using the law of induction, we take as our origin the center of motion.

In our discussion of the betatron it was pointed out that there was no radial component of electric field. If the same thing is to be true for our charged particle motion, then $E_\eta = 0$ since the radius of curvature vector is always perpendicular to the trajectory of the particle, and the electric field would be a vortex about the center of motion.

II. AN ASSUMED ELECTRIC FIELD

Our problem in this section is to find a way of applying the induction law to a particle traveling in an unclosed path.

If a particle were constrained to travel in a circle of radius R_o in a homogeneous, time-varying magnetic field, we would apply the law of induction as follows:

3-3 $$\mathcal{E}.m.f. = \oint \vec{E} \cdot d\vec{l} = \oint E_t \, dl = -\frac{\pi R_o^2 \dot{B}}{c}$$

3-4 $$\oint E_t \, dl = -2\pi R_o E_t$$

3-5 $$E_t = -\frac{R_o \dot{B}}{2c}$$

where E_t is the tangential component of the induced electric field. Now the electric field at any point may be resolved into components in the direction of, and normal to, the trajectory.

3-1 $$\vec{E} = E_n \vec{n} + E_t \vec{t}$$

where $\vec{t}$ and $\vec{n}$ are the unit tangent and unit normal respectively. If we select E_t on the basis of the induction law, we might select

3-6 $$E_t = -\frac{R \dot{B}}{2c}$$

where R is the instantaneous radius of curvature.

At present we shall assume E_t to be referred to the stationary frame. We shall also assume that $E_n = 0$. Then the complete electric field we have assumed is given by

3-7 $$\vec{E} = \frac{R \dot{B}}{2c} \vec{t}$$

in the stationary frame. The - sign has been replaced by making E point in the direction of motion.

The unit vector in the direction of motion can be written as

$$\vec{t} = \frac{\vec{v}}{v}$$

Substituting this into equation (3-7) we obtain

3-8 $$\vec{E} = \frac{R\dot{B}\vec{v}}{2cv}$$

Since $E_n = 0$, the only force normal to the trajectory is that due to the magnetic field

$$F_n \vec{n} = \frac{q}{c} \vec{v} \times \vec{B}$$

or

$$F_n = \frac{q}{c} vB$$

This is equal to the instantaneous centripetal force acting on the particle

1-3 $$\frac{mv^2}{R} = \frac{q}{c} vB$$

This is the same relationship as that describing the motion in a constant, homogeneous field except that V, B and R are now variables. We can rewrite this as

$$\frac{R}{v} = \frac{mc}{qB}$$

and substitute this in equation (3-8) to obtain

3-9 $$\vec{E} = \frac{m\dot{B}\vec{v}}{2qB}$$

This, then, is the final version of our assumed electric

in the stationary frame. The - sign has been replaced by making E point in the direction of motion.

The unit vector in the direction of motion can be written as

$$\hat{\tau} = \frac{\vec{V}}{V}$$

Substituting this into equation (3-7) we obtain

$$3-8 \qquad \vec{E} = \frac{R\dot{B}}{2cV}\vec{V}$$

Since $F_\tau = 0$, the only force normal to the trajectory is that due to the magnetic field

$$\vec{F}_B = \frac{q}{c}\vec{V}\times\vec{B}$$

or

$$F_B = \frac{q}{c}VB$$

This is equal to the instantaneous centripetal force acting on the particle

$$1-3 \qquad \frac{mV^2}{R} = \frac{q}{c}VB$$

This is the same relationship as that describing the motion in a constant, homogeneous field except that V, B and R are now variables. We can rewrite this as

$$\frac{R}{V} = \frac{mc}{qB}$$

and substitute this in equation (3-8) to obtain

$$3-9 \qquad \vec{E} = \frac{m\dot{B}}{2qB}\vec{V}$$

This, then, is the final version of our induced electric

field.

III. THE RESULTING PARTICLE MOTION

If we assume that the electric field is given by equation (3-9), the total force on the particle becomes

$$\vec{F} = m\dot{\vec{V}} = q\left(\frac{m\dot{B}\vec{V}}{2qB} + \frac{1}{c}\vec{V}\times\vec{B}\right) \tag{3-10}$$

The two component equations are

$$\ddot{x} = \frac{q}{m}\left(\frac{m\dot{B}\dot{x}}{2qB} + \frac{B}{c}\dot{y}\right) \tag{3-11}$$

$$\ddot{y} = \frac{q}{m}\left(\frac{m\dot{B}\dot{y}}{2qB} - \frac{B}{c}\dot{x}\right)$$

We can express the motion in terms of the complex variable $\gamma = x + iy$ by multiplying the second equation by $i(\sqrt{-1})$, and adding it to the first equation.

$$\ddot{\gamma} = \frac{d\dot{\gamma}}{dt} = \frac{q}{m}\left(\frac{m\dot{B}}{2qB} - i\frac{B}{c}\right)\dot{\gamma} \tag{3-12}$$

$$\frac{d\dot{\gamma}}{\dot{\gamma}} = \frac{\dot{B}}{2B}\,dt - \frac{iqB}{mc}\,dt$$

$$\ln\dot{\gamma} = \frac{\dot{B}}{2}\int\frac{dt}{B} - \frac{iq}{mc}\int B\,dt$$

Now we assume a linearly rising magnetic field

$$B = B_0 + \dot{B}t \tag{3-13}$$

Equation (3-12) becomes

$$\ln\dot{\gamma} = \frac{\dot{B}}{2}\int\frac{dt}{B_0+\dot{B}t} - \frac{iq}{mc}\int(B_0+\dot{B}t)\,dt$$

$$\ln\dot{\gamma} = \frac{1}{2}\ln\frac{B}{\dot{B}} - \frac{iq}{mc}\left(B_0 + \frac{\dot{B}t}{2}\right)t + C$$

$$\dot{\gamma} = C_1\sqrt{\frac{B}{\dot{B}}}\;e^{-\frac{iq}{mc}\left(B_0+\frac{\dot{B}t}{2}\right)t}$$

when $t=0$, $B=B_0$, $\dot{r}=\dot{r}_0$ so that

$$C_1 = \dot{r}_0 \sqrt{\frac{B}{B_0}}$$

The velocity of the particle in the complex plane is then

3-14 $$\dot{r} = \dot{r}_0 \sqrt{\frac{B}{B_0}}\, e^{-\frac{iq}{mc}\left(B_0 + \frac{\dot{B}t}{2}\right)t}$$

This equation cannot be integrated to find the particle's trajectory without the use of infinite series.

We can, however, find some interesting properties of the motion from this equation. We first note that if $\dot{B}$ is zero, the solution reduces to

$$\dot{r} = \dot{r}_0\, e^{-i\frac{qB_0 t}{mc}}$$

which is correct for a stationary field.

It is interesting to note that the quantity $B_0 + \frac{\dot{B}t}{2}$ in the exponent is the average value of the magnetic field from $t=0$ to $t=t$.

It should be pointed out that the frequency of rotation of the rotating vector $\dot{r}$ is not necessarily the same as the frequency of rotation of the position vector r.

In order to find the magnitude of the velocity, we use the relation

3-15 $$\dot{r}\,\dot{r}^* = V^2$$

Hence from equation (3-14)

3-16 $$\frac{V^2}{B} = \frac{V_0^2}{B_0}$$

This can be rewritten as

3-17 $$\frac{W}{B} = \frac{W_0}{B_0} = \mu = \text{Constant}$$

where μ is the magnetic moment defined in Chapter I, Section I. This is in agreement with Alfven's derivation. As shown below, this is equivalent to saying that the flux linkage through the instantaneous circle of curvature is constant.

In this section we have used the relation

1-3 $$\frac{mv^2}{R} = \frac{q}{c} vB$$

or

$$R = \frac{mcv}{qB}$$

The flux through the instantaneous circle of curvature is

3-18 $$\phi = \pi R^2 B$$

or

$$\phi = \pi \left(\frac{mcv}{qB} \right)^2 B$$

Thus

3-19 $$\phi = \frac{2\pi mc^2}{q^2} \frac{W}{B} = \text{Constant}$$

We also see that from equations (1-3) and (3-16) that

3-20 $$mRv = \frac{mcv^2}{qB} = \text{Constant}$$

At first glance the result that mRv is constant would seem to violate Newton's second law of motion inasmuch as

the quantity qER represents a torque on the particle about its center of motion and should change the angular momentum of the particle. However, this is not a real discrepancy. The canonical momentum of the particle is the quantity which must be conserved in the absence of the generalized force associated with that momentum. The form of the canonical variables associated with the radius of curvature is not known.

IV. BOUNDARY CONDITIONS

A particle is rotating in a constant magnetic field. At time t_0 the magnetic field starts to change. The transition period between the constant field and the linearly changing field we consider to be quite small, and we here neglect the effects of the higher time derivatives of B existing during the transition period. We wish to compare the various properties of the motion at a time $t_1 = t_0 - \Delta t$ just before the field change to their values at a time $t_2 = t_0 + \Delta t$ just after the field change is applied.

We assume that $E_n = 0$. Then the equation

1-4 $$\vec{\rho} = \frac{mc}{qB} \vec{v} \times \vec{k}$$

applies both before and after t_0. Neglecting the differential differences the following identities are apparent.

3-21 $$\vec{r_1} = \vec{r_2}$$

where $\vec{r}$ is the position vector of the particle

$$\vec{V}_1 = \vec{V}_2$$

Since the forces remain finite, the impulse $qEdt \to 0$

$$\vec{\rho}_1 = \vec{\rho}_2 \qquad (3\text{-}24)$$

and

$$\vec{\rho}_1 + \vec{r}_1 = \vec{\rho}_2 + \vec{r}_2 \qquad (3\text{-}25)$$

The last equation shows that the center of curvature does not change discontinuously.

The following argument is offered to show that our solution given by equation (3-14) applies as soon as the magnetic field is changing at a constant rate. Assume that at some time t, a particle has a given velocity V, in a given field, B, with a given rate of change $\dot{B}$. E_n is zero, and the radius of curvature which determines E_t is in turn determined by V and B. The subsequent trajectory is determined by V at t and the forces which then act on the particle. Higher time derivatives of the trajectory (r) than the second have no forces associated with them. Hence, even though the nature of the trajectory changes at time t_o, equation (3-14) should apply as soon as the field is changing at a constant rate.

If

$$\dot{B} = 0 \quad \text{for } t < t_0$$
$$\dot{B} = \text{Constant} \quad \text{for } t_1 > t > t_0$$
$$\dot{B} = 0 \quad \text{for } t > t_1$$

then the initial and final centers of rotation are end points on the evolute describing that trajectory which would be obtained by integrating equation (3-14).

V. THE CENTER OF MOTION

Fig. IV shows the position of the particle at a time t and a time $t+dt$ with the corresponding positions of the center of curvature.

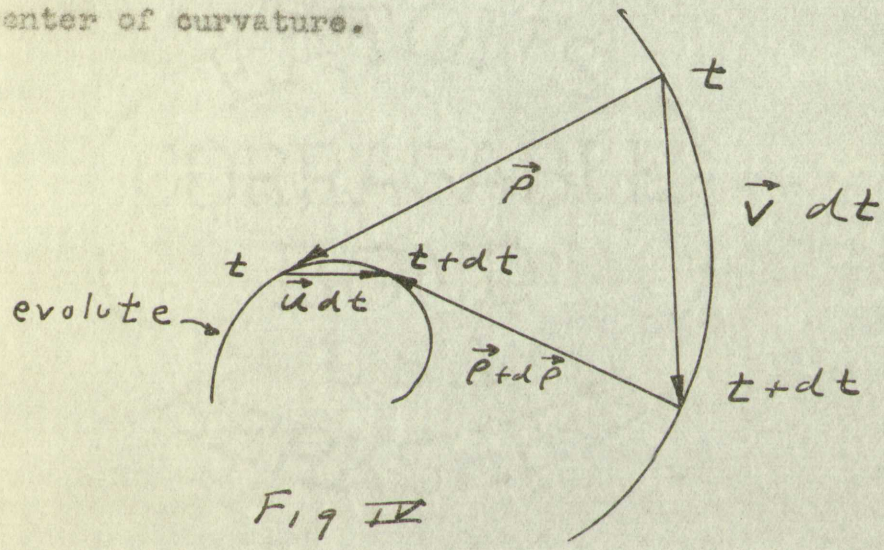


Fig IV

$\vec{U}$ is the vector velocity of the center of curvature.

From Fig. IV we see that

$$\vec{\rho}+\vec{u}\,dt=\vec{V}\,dt+\vec{\rho}+d\vec{\rho}$$

or

3-26 $$\frac{d\vec{\rho}}{dt}=\vec{u}-\vec{V}$$

Let us find out what we can about the center of motion assuming for the present only that $E_\eta = 0$. Since the only force normal to the trajectory is that due to the magnetic field

1-4 $$\vec{\rho} = \frac{mc}{qB}\,\vec{V}\times\vec{K}$$

Differentiating with respect to time

$$\frac{d\vec{\rho}}{dt} = \frac{mc}{q}\left(-\frac{\dot{B}V}{B^2} + \frac{\dot{\vec{V}}}{B}\right)\times\vec{K}$$

but

1-1 $$\dot{\vec{V}} = \frac{q}{m}\left(\vec{E} + \frac{1}{c}\vec{V}\times\vec{B}\right)$$

Then

$$\frac{d\vec{\rho}}{dt} = \frac{mc}{qB}\left[-\frac{\dot{B}}{B}\vec{V}\times\vec{K} + \frac{q}{m}\vec{E}\times\vec{K} + \frac{qB}{mc}(\vec{V}\times\vec{K})\times\vec{K}\right]$$

Now since

$$(\vec{V}\times\vec{K})\times\vec{K} = -\vec{V}$$

3-27 $$\frac{d\vec{\rho}}{dt} = \frac{mc}{qB}\left(-\frac{\dot{B}\vec{V}}{B} + \frac{q}{m}\vec{E}\right)\times\vec{K} - \vec{V}$$

Hence from equation (3-26),

3-28 $$\vec{u} = \frac{mc}{qB}\left(-\frac{\dot{B}\vec{V}}{B} + \frac{q}{m}\vec{E}\right)\times\vec{K}$$

Now then let us also assume as before that

3-9 $$\vec{E} = \frac{m\dot{B}\vec{V}}{2qB}$$

Then substituting this in equation (3-28)

3-29 $$\vec{u} = \frac{mc}{qB}\left(-\frac{\dot{B}\vec{V}}{B} + \frac{\dot{B}\vec{V}}{2B}\right) = -\frac{mc\dot{B}}{2qB^2}\vec{V}\times\vec{K}$$

or in terms of $\vec{E}$,

$$\vec{u} = -c\,\frac{\vec{E}}{B} \times \vec{\kappa} \tag{3-30}$$

Now let us assume that field transformation equation which transforms the electric field to a moving coordinate system is valid instantaneously for the reference frame (non-rotating) which travels with the center of curvature.

$$\vec{E}' = \vec{E} + \frac{1}{c}\,\vec{u} \times \vec{B} \tag{3-31}$$

Then

$$\vec{E}' = \vec{E} + \frac{1}{c}\left(-c\,\frac{B}{B}(\vec{E} \times \vec{\kappa}) \times \vec{\kappa}\right)$$

$$\vec{E}' = 2\,\vec{E} \tag{3-32}$$

or

$$\vec{E}' = \frac{m\dot{B}\vec{V}}{qB} \tag{3-33}$$

It was not at all necessary to assume a specific form for $\vec{E}$ other than the general condition that $E_\eta = 0$ to find $\vec{E}'$. Equation (3-28) was derived only on the assumption that $E_\eta = 0$. If we substitute the value of $\vec{U}$ given by equation (3-28) into (3-31) it is readily seen that we get the above value of $\vec{E}'$ as a result.

Now let us attempt to view the motion from the moving coordinate frame assuming equation (3-9) represents the electric field in the stationary frame and assuming $\dot{\vec{U}}$ is small enough for equation (3-31) to be true. From equation (3-30) we can calculate $\dot{\vec{U}}$

3-34 $$\dot{\vec{u}} = -c\left(-\frac{\dot{B}}{B^2}\vec{E} + \frac{\dot{\vec{E}}}{B}\right)\times\vec{k}$$

From equation (3-9) we can calculate $\dot{\vec{E}}$

3-35 $$\dot{\vec{E}} = \frac{m\dot{B}}{2q}\left(\frac{\dot{\vec{V}}}{B} - \frac{\dot{B}}{B^2}\vec{V}\right)$$

Then eliminating $\dot{\vec{E}}$ in equation (3-34) and reducing

3-36 $$\dot{\vec{u}} = \frac{3mc\dot{B}^2}{4qB^3}\vec{V}\times\vec{k} + \frac{\dot{B}\vec{V}}{2B}$$

Now $\vec{V}'$ the velocity of the particle in the moving frame is

3-37 $$\vec{V}' = \vec{V} - \vec{u}$$

By using equations (3-30) and (3-9) this reduces to

3-38 $$\vec{V}' = \vec{V}\left(1 + \frac{mc\dot{B}}{2qB^2}\right)$$

By using the above equation we can express $\vec{E}'$ in terms of $\vec{V}'$.

3-39 $$\vec{E}' = \frac{\vec{V}'}{\left(\frac{2qB}{m\dot{B}} + \frac{c}{B}\right)}$$

The force on the particle in the moving frame is

3-40 $$m\dot{\vec{V}}' = q\left(\vec{E}' + \frac{1}{c}\vec{V}'\times\vec{B}'\right) - m\dot{\vec{u}}$$

where the last term is an inertial force since the primed system is accelerating.

It is shown in Appendix B we can set $\vec{B}' = \vec{B}$.

By substituting the value of $\vec{E}'$ from equation (3-39) and $\dot{\vec{U}}$ from equation (3-36) and $\vec{V}$ from equation (3-38), equation (3-40) reduces to

3-41 $$\dot{\vec{V}}' = \left[\frac{qB}{mc} - \frac{3\dot{B}}{2B\left(\frac{2qB^2}{mc\dot{B}} + 1\right)}\right]\vec{V}'\times\vec{k}$$

This equation can be solved in the complex plane by the means used in Chapter III, Section III.

Setting

$$r' = x + iy$$

a first integral of the motion can be readily obtained in the form

$$3\text{-}42 \qquad \dot{r}' = \dot{r}'_0 \, e^{-i\int\left[\frac{qB}{mc} - \frac{3\dot{B}}{2B\left(\frac{2qB^2}{mc\dot{B}}+1\right)}\right]dt}$$

Since the square of the velocity is given by $v^2 = \dot{r}'\dot{r}'^*$, it is obvious that the energy of the particle in the moving frame is constant.

VI. A DISCUSSION OF OUR ASSUMED FIELD

In Section II an attempt was made to relate the electric

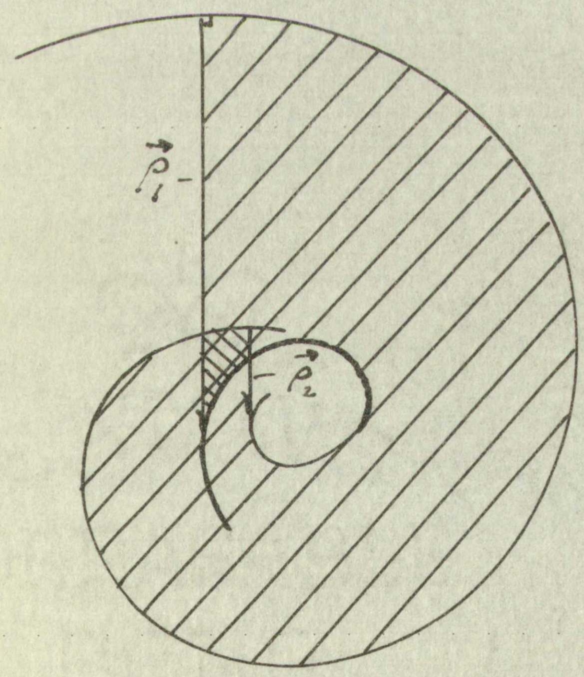


Fig V

field to the trajectory - through the particle's "rate of generation of area" using its center of curvature as an apex. However, if one considers the motion through one entire rotation of its position vector, one has the situation shown in Fig. V. The shaded area represents the area "inside" our unclosed path through which we considered the effect of the rate of change of flux. Let us try to compare this to closed circuit induction.

If we consider the trajectory and that portion of $\vec{P}_1$ which connects the two points of the spiral as the closed path about which we are developing our e.m.f., then since we have assumed $E_n = 0$ the e.m.f. along the actual particle trajectory should equal the rate of decrease of flux inside the closed path. We note, however, that the area under the evolute is left out and that a portion of the area is included twice. We also notice that $\vec{P}_1$ will not be perpendicular to the inner leg of the spiral unless the successive legs are parallel (such as they would be if the evolute itself were a closed curve, in which case the overlap area would disappear).

Hence, the application of the law of induction to charged particle motion should be analogous but not identical to its application to a closed circuit.

VII. AN ATTEMPT TO VIEW OUR ASSUMED ELECTRIC FIELD FROM THE STANDPOINT OF FIELD THEORY

If a vector potential, $\vec{A}$, could be found which satisfied

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = \frac{m\dot{B}\vec{V}}{2qB} \qquad (3\text{-}43)$$

and

$$\operatorname{curl}\vec{A} = B\vec{K} \qquad (3\text{-}44)$$

we would have a tool which would help us investigate the consistency of our assumptions with Maxwell's equations.

Now when one derives equations of motion by use of the Lagrangian, one assumes that the velocities are not explicit functions of the coordinates

$$\frac{\partial \dot{x}}{\partial x} = 0 \qquad \frac{\partial \dot{x}}{\partial y} = 0 \qquad (3\text{-}45)$$

If, however, we try to satisfy Maxwell's equations

$$\operatorname{curl}\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\operatorname{curl}\frac{m\dot{B}\vec{V}}{2qB} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad (3\text{-}46)$$

$\vec{V}$ must be considered a function of the coordinates.

Our difficulty here may be that we have assumed a problem free of radiation effects and that this is inconsistent with the above equations. It is also clear that the

finite velocity of propagation of electric and magnetic fields would make it impossible to have a strictly homogeneous magnetic field that is simultaneously changing in time.

CHAPTER IV.

CONCLUSION

Our most basic problem in this discussion has been concerned with the determination of the electric field which acts on the particle. For a physically realizable case the electric field is regarded as a function of the coordinates. As described in Chapter III, Section I, the induced vortex electric field acting on a particle will have two tendencies; one is to accelerate the particle about its center of motion; the other is to cause the particle to drift towards the origin of the vortex. As shown in Chapter II, Section II, the equation of motion of such a particle (equation 2-14) is quite difficult to solve. If a solution were obtained, the resulting trajectory would contain both the aspects of induction acceleration and drift.

The problem as we have treated it in Chapter III involves only the induction acceleration aspect of the motion. This interpretation of what we have done is consistant with the basic assumption we have made which is that the forces on the particle should depend only on the velocity, the magnetic field and its rate of change and should be independent of the particle's position.

Our treatment here might be regarded as a counterpart of charged particle motion in constant crossed electric and

magnetic fields which is also an idealized treatment utilizing homogeneous fields.

A further investigation which the author thinks would be of interest would be to examine the motion in a more realistic field and determine to what extent the induction acceleration aspect of the motion can be separated from the particle's drift.

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APPENDIX

APPENDIX A. KINEMATICS OF A PARTICLE IN A PLANE[2] AND RADIUS OF CURVATURE

If $\vec{r}$ is the position vector of a moving particle, the unit vector in the direction of motion is given by

$$\text{A-1} \qquad \vec{t} = \frac{d\vec{r}}{ds}$$

where $\vec{t}$ is the unit tangent, and S is the measure of arc length along the trajectory.

The curvature at any point of the trajectory is defined as

$$\text{A-2} \qquad K = \frac{d\theta}{ds}$$

where $\Delta\theta$ is the angle between the two unit tangents of two points on the curve separated by the arc length ΔS. $\Delta\vec{t}$ is the difference between these two unit tangents. In the limit (as the two points approach each other) the vector $\Delta\vec{t} \to d\vec{t}$ where $d\vec{t}$ is perpendicular to the curve but lies in the plane of the curve.

Now

$$\left|\frac{d\vec{t}}{ds}\right| = \frac{d\theta}{ds}$$

since $\vec{t}$ has a constant magnitude and

$$\text{A-3} \qquad \frac{d\vec{t}}{ds} = K\vec{n}$$

---

[2] C. E. Weatherburn, Elementary Vector Analysis, Chapter VI.

where $\vec{n}$ (the unit normal) is the unit vector perpendicular to the curve.

The inverse of the curvature is the radius of curvature R.

$$\text{A-4} \qquad R\vec{n} = \frac{1}{\kappa}\vec{n}$$

The vector $R\vec{n}$ is a vector from the position of the particle to the center of curvature of its trajectory.

$$\text{A-5} \qquad R\vec{n} = \vec{\rho}$$

As the vector moves along its trajectory (described by the vector $\vec{r}$), the vector $\vec{r}+\vec{\rho}$ describes another curve known as the evolute of the trajectory. The evolute has the geometrical property[3] that if one considered it to be a raised cylinder (from the plane of the trajectory) and if inextensible string were wrapped around the cylinder at its base (the string being kept taut and there being no sliding of the string on the evolute) then some constant point on the string would describe the trajectory.

The components of velocity and acceleration in the normal and tangential directions may be found as follows: The velocity of the particle is

$$\text{A-6} \qquad \vec{V} = \frac{d\vec{r}}{dt}$$

---

[3] Granville, Smith & Longley, Elements of the Differential and Integral Calculus, Chapter X.

which may be written

A-7 $$\vec{V} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = v\vec{t}$$

The acceleration is

$$\dot{\vec{V}} = \frac{d(v\vec{t})}{dt} = v\frac{d\vec{t}}{dt} + \frac{dv}{dt}\vec{t}$$

since

$$\frac{d\vec{t}}{dt} = \frac{d\vec{t}}{ds}\frac{ds}{dt}$$

We can write

$$\frac{d\vec{t}}{dt} = v\frac{d\vec{t}}{ds} = v\kappa\vec{n}$$

and the acceleration is given by

$$\dot{\vec{V}} = v^2\kappa\vec{n} + \dot{v}\vec{t}$$

or

A-8 $$\dot{\vec{V}} = \frac{v^2}{R}\vec{n} + \dot{v}\vec{t}$$

APPENDIX B. THE FIELD TRANSFORMATION EQUATIONS

The field transformation equations for the field components perpendicular to the direction of motion are

B-1 $$\vec{E}' = \vec{E} + \frac{1}{c}\vec{u} \times \vec{B}$$

B-2 $$\vec{B}' = \vec{B} - \frac{1}{c}\vec{u} \times \vec{E}$$

where $\vec{E}$ and $\vec{B}$ are the field values as measured in the stationary reference frame, and $\vec{E}'$ and $\vec{B}'$ are the field values as measured in a reference frame moving with a velocity $\vec{U}$ with respect to the stationary frame. The field components in the direction of motion do not change.

We wish to examine the relative applicability of these equations in describing the forces on a charged particle as observed from the coordinate frame moving with the velocity $\vec{U}$.

The force on the particle on the stationary frame is

1-1 $$\vec{F} = q\left(\vec{E} + \frac{1}{c}\vec{V} \times \vec{B}\right)$$

For simplicity, let us assume the forces due to the electric and magnetic fields are equal; and $\vec{E}$, $\vec{B}$ and $\vec{V}$ are mutually perpendicular. Then

B-3 $$\vec{E} = \frac{1}{c}\vec{V} \times \vec{B}$$

We can write for $\vec{E}'$

$$\vec{E}' = \vec{E} + \frac{1}{c}\vec{u} \times \vec{B} = \vec{E} + \vec{E}_1$$

Forming the ratio

$$B-4 \qquad \frac{E_1}{E} = \frac{u}{v}$$

shows that the change in electric field in going from the stationary frame to the moving frame is a gross effect. We can write for $\vec{B}'$

$$\vec{B}' = \vec{B} - \frac{1}{c}\,\vec{u} \times \vec{E} = \vec{B} - \vec{B}_1$$

Forming the ratio

$$B-5 \qquad \frac{B_1}{B} = -\frac{vu}{c^2}$$

shows that for small velocities the change in magnetic induction in going from the stationary frame to the moving frame is negligible.

Hence, we may consider the following as the transformation equations.

$$B-1 \qquad \vec{E}' = \vec{E} + \frac{1}{c}\,\vec{u} \times \vec{B}$$

$$\vec{B}' = \vec{B}$$