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The Diophantine Equation

Frederic C. Barnett

BARNETT — THE DIOPHANTINE EQUATION $x^n + y^n = z^n$

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THE DIOPHANTINE EQUATION $x^n + y^n = z^n$

By

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INTRODUCTION

In any list of famous unsolved problems of mathematics, one must surely include Fermat's last theorem. It deserves this acclaim not only for the long list of great mathematicians who have been concerned with its proof over the past three centuries, which is indicative of its charm; but also for the importance of the mathematical developments adduced in the process of its attempted proof, which is indicative of its subtlety. For example, Dickson[1] states that Kummer held "always before himself as goal the complete proof of Fermat's last theorem and the general reciprocity law", and declares that "he would probably have abandoned his theory of ideal numbers, in spite of its success, if he had known another method of proving that theorem and law".

Perhaps the most fascinating aspect of this theorem is the ease with which it can be stated and grasped. It is a familiar fact that there exist triplets of integers such that the squares of two of them add up to the square of the third. It is then simplicity itself to grasp the meaning of the problem to find a triplet of integers such that the sum of the nth powers of two of them is equal to the nth power of the third. Fermat stated definitely that he had

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[1] L. E. Dickson, Annals of Mathematics, Series 2, 18:161.

found a "truly wonderful" proof of this proposition, which has become known as "Fermat's Last Theorem". This, then, is the problem with which we are concerned in this paper.

Propositions of such simplicity are often easily discovered by induction, and yet are often of so recondite a character that a satisfactory proof remains hidden for many years. Accordingly, it might be suspected that Fermat had found, but not proved, his last theorem. But as we have shown in the first chapter, Fermat was not only a mathematician of supreme ability in number theory, but was also a mathematician of peerless accuracy - he never claimed to have a proof where later the conclusion reached was shown to be false.

Though Fermat's last theorem has been proved for many particular values of n, the proofs in question have not been generalized to arbitrary n. The next step taken by mathematicians has then been to try to find implications in the field of algebra of the hypothesis that Fermat's relation has solutions. By this means, certain tests have been obtained which for particular values of n provide a ready verification, and others have been obtained that are of theoretical interest only. But it is important to note that even in this direction no published proof has been given of Fermat's "conjecture" for arbitrary n.

The indirect algebraic attack, as we may term the method referred to in the last paragraph, suggested the main problem attacked in this thesis - that of initiating an indirect geometric attack on the last theorem of Fermat. This is accomplished by noting that each triplet of non-zero integers satisfying the Fermat relation, if such triplets exist, is included among the set of all real number triplets which for integral n greater than 2 satisfy the equation $x^n + y^n = z^n$. But the latter equation defines for each value of n a surface F_n which can be investigated in various ways in the attempt to find implications either of the assumption that integral triads are not present or of the assumption that integral triads do occur on the surfaces in question.

In the first part of Chapter II of this thesis a study is made of these surfaces F_n. Such surfaces have many interesting analytical geometric properties, among which is that as n increases without bound through even and odd integral values, respectively, they approach definite limiting surfaces, S_E and S_O.

If a rational point (x,y,z) did exist on F_n, $xyz \neq 0$, $n > 2$, we would have a contradiction of Fermat's last theorem, as is shown in the second chapter. On the other hand, it is shown there that rational points exist, and in fact, form an everywhere dense set on the limiting surfaces.

Yet an attempt to use this fact to demonstrate the existence of rational points of F_n is shown to require a solution of the Fermat relation itself!

In the second part of Chapter II, several differential geometric properties of F_n are adduced, and again, an attack based on certain of these is shown to lead back to the very theorem we seek to prove.

Finally, we obtain three geometrical implications of the assumption that an integral triad exists on F_n suggesting new lines of attack on the theorem of Fermat. These implications must be evaluated as important only in so far as further development in this direction utilizes them in obtaining proofs of Fermat's theorem for particular values of n, or perhaps, if fortune prevails, in obtaining a proof for arbitrary n.

CHAPTER I

1. RELEVANT HISTORY

1.1 The man, Fermat. Having received a classical education, Pierre de Fermat (1608 - 1665) became at the age of 30 commissioner of requests at Toulouse and five years later, king's councillor in the parliament of Toulouse. It is thus at least unusual that he should have completed works in mathematics that have earned him the highest praise. M. Cantor[1] ranks Fermat and Descartes as the greatest mathematicians of the seventeenth century and implies that Fermat's was the greater fame. He concludes, however, that such considerations "hat für die Gesammtwurdigung beider Geisteshelden keine Bedeutung".

1.2 Definition of Diophantine problems. The Arithmetics of Diophantos (A. D. 250) require the reader to find rational numbers satisfying prescribed conditions. For example, problem 8 of book II requests that a given square number be decomposed into the sum of two square numbers. Thus all problems for which one is required to find rational solutions are called Diophantine problems.

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[1] M. Cantor, Vorlesungen über Geschichte Der Mathematik, 2:876.

The particular Diophantine problem alluded to above inspired an attempted generalization by Fermat.

1.3 Fermat's statement. Fermat's copy of Bachet's translation of Diophantos (1621) contains a marginal comment opposite problem 8 of book II: "On the other hand it is impossible to separate a cube into two cubes, or a biquadrate into two biquadrates, or generally any power except a square into two powers with the same exponent. I have discovered a truly remarkable proof of this, which, however, the margin is not large enough to contain." Since the time of Fermat, mathematicians have been divided on the question of whether he really possessed a proof, since no proof has been found, though the greatest of them have used subtle tools of great power unknown to Fermat, in a futile attempt to find one. On the other hand, Fermat often made such claims, and with this one exception his proofs have since been rediscovered by other investigators. For example, in a letter to the French mathematician Roberval, he states that he had found by use of his favorite method of infinite descent a proof that positive integers can be written as the sum of at most four squares. "I confess openly that in the theory of numbers I have found nothing which I have enjoyed more than the proof of this theorem, and I should be pleased if you would attempt to find it. . ." Although Euler himself

attacked this problem for many years, a proof was not found until J. L. Lagrange obtained one in 1770. Similarly, as is stated above, although Fermat was concerned with deep-lying properties of numbers, proofs have been found for each of his theorems but the one considered in this paper.

1.4 Scope of investigations devoted to Fermat's Theorem. Some idea can be obtained of the diversity of the mathematical investigations devoted to Fermat's last theorem from a comment made by Vandiver.[2] "Efforts on my part to clear up the question have led me into the following topics: Bernoulli numbers and polynomials and generalizations; Euler and Genocchi numbers; Euler and Mirimanoff polynomials; partitions modulo m; finite fields and rings, including a great many types of congruences; the Dirichlet Zeta function and the related Dedekind function; the Lagrange resolvent and Jacobi ϕ-numbers and various generalizations including generalized Gauss sums; theory of Kummer fields, class numbers, class fields, power characters and laws of reciprocity in the theory of algebraic fields; transformations of Elliptic functions and complex multiplication; Fermat's quotient and other arithmetic quotient forms; congruence

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[2] H. S. Vandiver, American Mathematical Monthly, 53:555-78, 1946

theories as applied to power series; abstract algebra including, particularly, group theory and semi-groups; and many types of Diophantine equations aside from the Fermat relation itself."

1.5 Paul Wolfskehl's prize. Adding not an iota to the mathematical interest of Fermat's Theorem, but a great deal to public curiosity concerning it has been the prize of 100,000 marks offered by the German mathematician Paul Wolfskehl for a proof of it, or for a complete determination of the values of n for which it is true, in case it is not universally true. The conditions were first outlined in the Jahresbericht der Deutschen Mathematiker-Vereinigung, June 1908, p. 111. Some of these are[3] that "no manuscripts will be considered, but only printed articles or monographs. These articles or monographs must have appeared in regular periodicals or they must have been for sale in the open market. The prize will not be awarded for articles or monographs which have been before the public less than two years: the latest date for awarding the prize being September 13, 2007."

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[3] G. A. Miller, Historical Introduction to Mathematical Literature, p. 156.

2. THE DIRECT ATTACK

2.1 Fermat's method of "infinite descent". To students of beginning algebra is presented the method of proof by induction which is used to establish the truth of an infinite sequence of propositions, $P_1, P_2, \ldots, P_n, \ldots$ This method is based on an assumption that if for every positive integer k, P_k implies P_{k+1}, and P_1 is known to be true, then all of the propositions P_k, $k = 1, 2, 3, \ldots$ are true. The closely analogous principle of infinite descent may be phrased as follows: if the assumption that a problem can be solved in positive integers (proposition P_{a_k}) can be shown to imply there is a solution to that problem in smaller positive integers (proposition $P_{a_{k+1}}$, $a_{k+1} < a_k$), then an infinity of propositions P_{a_k} are true, each successive proposition referring to smaller positive integers than does its predecessor. But in this case a contradiction is obtained which denies the truth of P_{a_k} for any set of integers; for although the set of positive integers is not bounded above, it is certainly bounded below. Just as the difficulty in any proof by induction usually lies in the proof that P_k implies P_{k+1}, so the difficulty in the infinite descent proof lies in the demonstration that P_{a_k} implies $P_{a_{k+1}}$. We may conjecture that Fermat was actually able to show this for the general Fermat

Theorem. After obtaining the general solution of $x^2 + y^2 = z^2$ in positive integers we shall indicate how he actually did carry through such a demonstration for the equation $x^4 - y^4 = z^2$, and how this result immediately implies the impossibility of $x^4 + y^4 = z^4$.[4]

2.2 General solution of $x^2 + y^2 = z^2$. The following chain of reasoning[5] gives us the general solution of $x^2 + y^2 = z^2$ in positive integers. We first observe that $(x,y) = (x,z) = (y,z)$, and that this implies that all solutions can be expressed as multiples of the solutions of the primitive equation $x^2 + y^2 = z^2$, where $(x,y) = 1$. Accordingly, we assume that $(x,y) = 1$, from which it follows that z is odd. (For, if z were even, both of x and y would have to be odd. But if they were both odd, z^2 would be divisible by 2 but not by 4, which is impossible.) Thus, in the primitive equation, one of x and y is odd, the other even. Since x and y enter symmetrically, there is no loss in generality in taking x even. We may now state and prove: "The necessary and sufficient condition that

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[4]Fermat's copy of Bachet, Diophantos, problem 26, book VI.

[5]Edmund Landau, Vorlesungen über Zahlentheorie, 3:204-5.

$x^2 + y^2 = z^2$, $(x,y) = 1$, $x \equiv 0 \pmod 2$ is that $x = 2uv$, $y = u^2 - v^2$, $z = u^2 + v^2$, where u and v have relatively prime integral values such that $u > v > 0$ and $u + v \not\equiv 0 \pmod 2$."

For, from $(y,z) = 1$, $\frac{z+y}{2} + \frac{z-y}{2} = z$, and $\frac{z+y}{2} - \frac{z-y}{2} = y$, it is clear that $\left(\frac{z+y}{2}, \frac{z-y}{2}\right) = 1$. Since $z > y$, it follows from $(\frac{1}{2}x)^2 = \frac{z+y}{2} \cdot \frac{z-y}{2}$ that: $\frac{z+y}{2} = u^2$, $u > 0$, $\frac{z-y}{2} = v^2$, $v > 0$, where u and v are integers such that $u > v$, $(u,v) = 1$ and $(u+v)^2 \equiv u^2 + v^2 \not\equiv 0 \pmod 2$, so that $u + v \not\equiv 0 \pmod 2$. Furthermore, $(\frac{1}{2}x)^2 = (uv)^2$ implies that $x = 2uv$.

Conversely, given $x = 2uv$, $y = u^2 - v^2$, $z = u^2 + v^2$, where u and v are integers such that $u > v > 0$, $(u,v) = 1$, $(u+v) \not\equiv 0 \pmod 2$, we have $x^2 + y^2 = (2uv)^2 + (u^2 - v^2)^2 = z^2$, where $xyz \neq 0$, $x \equiv 0 \pmod 2$, $(x,y) = 1$, and x, y, z are all integers.

2.3 Impossibility of $x^4 + y^4 = z^4$. We are now prepared to follow an argument by infinite descent proving that both $x^4 - y^4 = z^2$ and $x^4 + y^4 = z^4$ are impossible in positive integers. To this end we define a primitive right triangle to be one whose sides have relatively prime integral lengths, and prove that the area of a primitive right triangle is never a square.

Since Pythagoras' Theorem assures us that the sides of a right triangle have the relation $x^2 + y^2 = z^2$, and in a primitive right triangle $(x,y) = 1$, the foregoing theorem gives us its area as $uv(u + v)(u - v)$, which is to be a perfect square. If the factors of this expression for the area were relatively prime, we would know thereby each of them to be a square. But they <u>are</u> relatively prime, the only pair for which this is not obvious being $u + v$, $u - v$. These are relatively prime, for any common factor would divide their sum <u>and</u> difference, $2u$ and $2v$. By the foregoing theorem u and v are relatively prime, and since $u + v$ and $u - v$ are odd, 2 could not be a common factor. Thus, $u = a^2$, $v = b^2$, $u - v = c^2$, $u + v = d^2$, where a, b, c, d are relatively prime in pairs.

By algebra,

$$(2.3.1) \qquad 2a^2 = c^2 + d^2 \quad \text{and}$$

$$(2.3.2) \qquad 2b^2 = (d + c)(d - c)$$

Since one of u and v is even and the other odd, a and b must have the same property. For the same reason c and d are both odd, so that both $d + c$ and $d - c$ are even. Hence, from (2.3.2), b is even. Let $b = 2b_1$ and therefore

$$(2.3.3) \qquad 2b_1^2 = \frac{d - c}{2} \cdot \frac{d + c}{2}, \text{ where } \left(\frac{d - c}{2}, \frac{d + c}{2}\right) = 1,$$

their sum and difference being relatively prime. From (2.3.3) one of $\frac{d - c}{2}$, $\frac{d + c}{2}$ is even. Suppose in particular that $\frac{d - c}{2} = 2k^2$, $\frac{d + c}{2} = j^2$. This implies that $b_1^2 = k^2j^2$. If on the other hand $\frac{d + c}{2}$ is even, say $\frac{d + c}{2} = 2k^2$, and $\frac{d - c}{2} = j^2$, we still have $b_1^2 = k^2j^2$. Thus, from (2.3.1)

(2.3.4) $\quad a^2 = (j^2)^2 + (2k^2)^2$

We have so far proved that the assumption that there exists a primitive right triangle with area $uv(u^2 - v^2) > \frac{V}{4}$ implies the existence of another with area $b_1^2 = \frac{V}{4}$. This solution in turn assures us of another yet smaller, etc. So that we have proved there is an infinity of primitive right triangles with areas perfect squares, each less than that of the preceding. But this is impossible, and we know thereby that no primitive right triangle has an area equal to a perfect square.

We can conclude from this almost immediately that there exist no integers p, q, r, with $pqr \neq 0$ such that $p^4 - q^4 = r^2$. For, this equation implies that $(p^4 - q^4)p^2q^2 = r^2p^2q^2$, where $(2p^2q^2)^2 + (p^4 - q^4)^2 = (p^4 + q^4)^2$: that is, that a primitive right triangle does

exist whose area is a square. Since no true proposition can imply a false one, we have proved that $p^4 - q^4 = r^2$ is impossible with p,q,r integers such that $pqr \neq 0$.

Finally, there exists no integers x,y,z with $xyz \neq 0$ such that $x^4 + y^4 = z^4$, for this is an obvious special case of the preceding result that none exist such that $p^4 - q^4 = r^2$, where $p = z$, $q = x$ and $r = y^2$.

2.4 The case $(xyz, n) = 1$. Restricting the integers that enter into the Fermat equation so that $(xyz, n) = 1$, we are able to obtain[6] a particularly elegant proof that $x^n + y^n + z^n = 0$ is impossible[7] for $n = 3$ and $n = 5$.

Let,

$$L = x + y + z$$

$$S_i = x^i + y^i + z^i .$$

Then, identically,

$$L^3 - S_3 = 3(x + y)(y + z)(z + x)$$

$$L^5 - S_5 = 5(x + y)(y + z)(z + x)\frac{L^2 + x^2 + y^2 + z^2}{2} \qquad (2.4.1)$$

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[6] P. Bachman, Niedere Zahlentheorie, 2:461-63.

[7] Section 3.2 shows how $x^n + y^n = z^n$ may be reduced to $x^n + y^n + z^n = 0$.

Fermat's well-known "little theorem" implies that

$$S_p \equiv x + y + z \pmod{p}$$

for every prime p. Therefore, if x,y,z are solutions of (3.2.1),

$$x + y + z \equiv 0 \pmod{p}.$$

In particular, if

$$x^3 + y^3 + z^3 = 0$$

$$x^5 + y^5 + z^5 = 0$$

then,

$$x + y + z \equiv 0 \pmod{3}$$

$$x + y + z \equiv 0 \pmod{5}$$

and relations (2.4.1) yield

$$3(x + y)(y + z)(z + x) \equiv 0 \pmod{3^3}$$

(2.4.2)

$$5(x + y)(y + z)(z + x)\cdot\frac{L^2 + x^2 + y^2 + z^2}{2} = 0 \pmod{5^5}.$$

From the first of congruences (2.4.2) it follows that at least one of the factors $(x + y),(y + z),(z + x)$ is divisible by 3, and therefore, from $x + y + z \equiv 0 \pmod{3}$, that one of the numbers x,y,z is divisible by 3.

From the second congruence we infer either the divisibility of one of the three factors $(x + y), (y + z), (z + x)$, or of the expression $L^2 + x^2 + y^2 + z^2$, by 5.

As before, we infer that if one of the factors $(x + y), (y + z), (z + x)$ is divisible by 5, so is one of the numbers x, y, z, which is a contradiction of $(xyz, p) = 1$. If on the other hand it is $L^2 + x^2 + y^2 + z^2$ that is divisible by 5, then

$$x^2 + y^2 + z^2 \equiv 0 \pmod 5,$$

for $L \equiv 0 \pmod 5$. But this too is impossible unless one of x, y, z is divisible by 5.[8] Bachman concludes, however, that other methods must be developed to handle $x^n + y^n + z^n = 0$ for larger values of p.

2.5 Conclusion. Existing elementary direct proofs become more difficult with increasing prime n. Already with $n = 3$, Uspensky[9] requires thirteen pages of preparatory development and four[10] more of specific treatment to obtain such a proof. More discouraging, these ponderous proofs do

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[8] If no one of x, y, z is divisible by 5, then x, y, z are congruent to ± 1 or $\pm 2 \pmod 5$ in some order; whence, $x^2 + y^2 + z^2 \equiv (\pm 1) + (\pm 1) + (\pm 1) \not\equiv 0 \pmod 5$.

[9] J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, p. 484.

[10] Ibid., pp. 408-11.

not seem to suggest a method which can be generalized to all values of n. In fact, as we shall discover in the next section, the indirect attack seems to be more productive of important results.

3. INDIRECT ALGEBRAIC ATTACKS

3.1 Purpose of this section. As a preface to the second and geometrical portion of this thesis we shall set forth in this section certain known implications of one or the other of the alternative assumptions,

(a) that solutions do exist for the Fermat problem,

(b) that such solutions do not exist.

Such implications are many and varied, and it is clearly impossible in the present brief review either to deal comprehensively with them or to go into complete detail in connection with the relatively few which we choose to present as examples. Our real purpose is to make clear that when the direct attack on the Fermat problem seemed unproductive, a variety of indirect algebraic attacks were discovered. This procedure motivated the present thesis which in the second chapter seeks to discover certain geometrical implications of the basic assumptions (a) and (b) to which we have alluded above.

3.2 Reduction of the problem. In discussing Fermat's equation $x^n + y^n = z^n$ it is sufficient to consider a solution in which $(x,y) = 1$. For if there exists a solution in integers (x,y,z) where $(x,y) = k > 1$, then $z = kZ$ and we may set $x = kX$, $y = kY$. On cancelling k^n we have the Fermat equation $X^n + Y^n = Z^n$ in which $(X,Y) = 1$. Furthermore, if Fermat's theorem is true for some n, it is true for any multiple m of n, for if $(x^m)^n + (y^m)^n = (z^m)^n$ were possible, so would be $x^n + y^n = z^n$. Thus, since every integer greater than 2 is divisible by 4 or by an odd prime, it suffices to prove the Fermat conjecture for $n = 4$ and for $n = p$, p being an odd prime. Having already proved it for $n = 4$, the only exponents for which we must prove it are odd primes. Since in the algebraic treatment negative integers are admissible values of (x,y,z), we may clearly write the Fermat equation in the form:

$$(3.2.1) \quad x^p + y^p + z^p = 0, \quad p \text{ an odd prime.}$$

3.3 Implications of the existence of solutions. In 1850, E. E. Kummer[11] proved that if (3.2.1) holds, then n divides at least one of the numerators of the Bernoulli numbers $B_1, B_2, \ldots, B_{\frac{p-3}{2}}$.

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[11] L. E. Dickson, History of the Theory of Numbers, 2:742. For the definition of B_1, see Niels Nielsen, Traite Elementaire Des Nombres De Bernoulli, p. 14.

A. Wieferich[12] proved that if equation (3.2.1) is possible in integers prime to p, where p is an odd prime, then $2^{p-1} - 1$ is divisible by p^2. This criterion is satisfied only for p = 1093 and p = 3511 for p < 16000.

Under the same conditions, D. Mirimanoff[13] showed that $3^{p-1} \equiv 1 \pmod{p^2}$. By combining criteria of this nature, the Lehmers[14] established the case (xyz, p) = 1 of Fermat's Theorem for all p less then 253,747,889.

Let $L = x + y + z$, $M = xy + yz + zx$, $N = xyz$. Then (3.2.1) can be written in the form:

$$G_p(L, M, N) = 0 \tag{3.3.1}$$[15]

where x, y, z are roots of

$$t^3 - Lt^2 + Mt - N = 0. \tag{3.3.2}$$

(3.2.1) can have a rational solution only when all roots of (3.3.2) are rational, its coefficients being subject to criterion (3.3.1). This result is due to Mirimanoff[16].

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[12] Dickson, op. cit., p. 764.

[13] Ibid., p. 768.

[14] D. H. & Emma Lehmer, "On the First Case of Fermat's Last Theorem", Bulletin American Mathematical Society, 47:139-42.

[15] $G_p(L, M, N)$ is the expression obtained on substituting the solution for x, y, z in terms of L, M, N into (3.2.1).

[16] Dickson, op. cit., p. 766.

3.4 Implications if no solutions exist. We now attempt to find solvable Diophantine implications of (3.2.1) with the hope of finding a contradiction to the conjecture that (3.2.1) is impossible.

V. A. Lebesque[17] stated in 1840, that if $x^n + y^n = z^n$ is impossible in integers, so is $x^{2n} + y^{2n} = z^2$. This result follows readily from the general solution of $x^2 + y^2 = z^2$. For if $(x^n)^2 + (y^n)^2 = z^2$ has solutions they are given by $x^n = u^2 - v^2$, $y^n = 2uv$ where $(u,v) = 1$. Thus,

$$u - v = a^n$$
$$u + v = b^n$$
$$u = 2^{n-1} c^n$$
$$v = d^n$$

in some order. Hence,

$$a^n \pm b^n = (2c)^n ;$$

i.e., $x^n + y^n = z^n$ is possible.

J. Liouville[18] noted a similar result in the same year. If $x^n + y^n = z^n$ is impossible in integers not zero, then so is $x^{2n} - y^{2n} = 2z^n$.

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[17] Dickson, op. cit., p. 737.

[18] Ibid., p. 738.

Finally, A. Hurwitz[19] proved that if $x^k + y^k = z^k$ is impossible for every integer K, $K > 2$, then

$$u^m v^n + v^m w^n + w^m u^n = 0$$

is impossible for every pair of integral m and n except where (u, v, w) = (1,0,0), (0,1,0), or (0,0,1).

3.5 Absence of indirect geometric attacks. Although indirect algebraic attacks abound in the literature, I have been unable to locate any investigation giving geometric implications of the (a) existence, or (b) non-existence of solutions of the Fermat problem. If such could be found, the proof of Fermat's last theorem would become a problem of geometry.

It is not inconceivable that the assumed existence of a solution for the Fermat problem would imply a geometrical impossibility. It is in the desire to throw the problem into a geometrical one that we have developed in the next chapter the geometry of Fermat surfaces,

$$F_n : \quad x^n + y^n = z^n$$

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[19] Ibid., p. 764.

CHAPTER II

1. ANALYTIC GEOMETRY OF F_n

1.1 F_n a conical surface. The intersections of the surfaces F_n and $y = ax$ have the equations

$$(1.1.1)\quad \frac{x}{(1+a^{-n})^{1/n}} = \frac{y}{(1+a^{n})^{1/n}} = (\pm 1)^{n+1}\frac{z}{(a^n + a^{-n} + 2)^{1/n}}$$

and hence comprise a family of straight lines passing through (0,0,0). For $x^n + y^n = z^n$ and $y^n = a^n x^n$ imply that $x = (\pm 1)^{n+1}\frac{z}{(1+a^n)^{1/n}}$. But $x = \frac{y}{a} = \frac{x}{1}$

Hence $\frac{x}{1} = \frac{y}{a} = (\pm 1)^{n+1}\frac{z}{(1+a^n)^{1/n}}$ Multiplication by $\frac{1}{(1+a^{-n})^{1/n}}$ gives (1.1.1). Thus, the Fermat surfaces are cones with vertices at the origin.

As is true with the familiar F_2, all the surfaces F_n are symmetric with respect to the origin, which can be seen from equations (1.1.1). Hence, in the sequel we shall limit ourselves without loss of generality to the portions of these conical surfaces for which $z \geq 0$.

1.2 Contour lines of F_n. The contour lines of the Fermat cones fall into categories corresponding, respectively, to even or odd values of n.

Case I, $n = 2m$.

In case n is even, the contour lines are curves parallel to the x,y-plane whose projections on that plane are given by the equations $x^n + y^n = z_0{}^n$, where z_0 is their distance above that plane. We infer from the equations that x is defined only for $|y| \leq |z_0|$ and y for $|x| \leq |z_0|$, and hence, that no asymptote is possible.

From the fact that n is even we know that these contours are symmetric with respect to both the x- and y-axes. The intercepts are at $(0, \pm z_0)$ and $(\pm z_0, 0)$.

Case II, $n = 2m + 1$.

In case n is odd, the contour lines are again curves parallel to the x,y-plane whose projections on that plane are given by the equations $x^n + y^n = z_0{}^n$, where z_0 is their distance above that plane. These equations are unchanged when x is replaced by y and y by x, so that $y = x$ is a line of symmetry. The points $(z_0, 0)$ and $(0, z_0)$ are the only intercepts. We infer from the equations that points exist on these contours at every real value of x, so that we may profitably test for asymptotes.

We have from $x^n + y^n = z_0^n$ that $y' = -x^{n-1}y^{1-n}$ and hence, the equation of the tangent at (x,y) is given by $y^{n-1}Y + x^{n-1}X = z_0^n$, where (X,Y) are the running coordinates. The intercepts of the tangent line are therefore, $\left(\frac{z_0^n}{x^{n-1}}, 0\right)$ and $\left(0, \frac{z_0^n}{y^{n-1}}\right)$. As (x,y) moves off to infinity these intercepts approach $(0,0)$ so that any asymptote of these contours will pass through the origin. We can find the slope of the asymptotes by considering $\lim_{x\to\infty} y'$. But $y' = -\left(\frac{z_0^n}{x^n} - 1\right)^{\frac{1-n}{n}}$ and $\lim_{x\to\infty} y' = -1$. Thus, there is a single asymptote whose equation is $y = -x$.

The following two pages contain sketches of the contour lines in the plane $z = z_0 = 1$ for even and odd n, respectively, and for several values of n in each case to show the manner in which for fixed z these contours approach limiting positions shown by the heavy lines.

To derive equations of limiting positions of $x^n + y^n = z^n$ for increasing even and odd values of n, respectively, we shall first consider two lemmas and a definition.

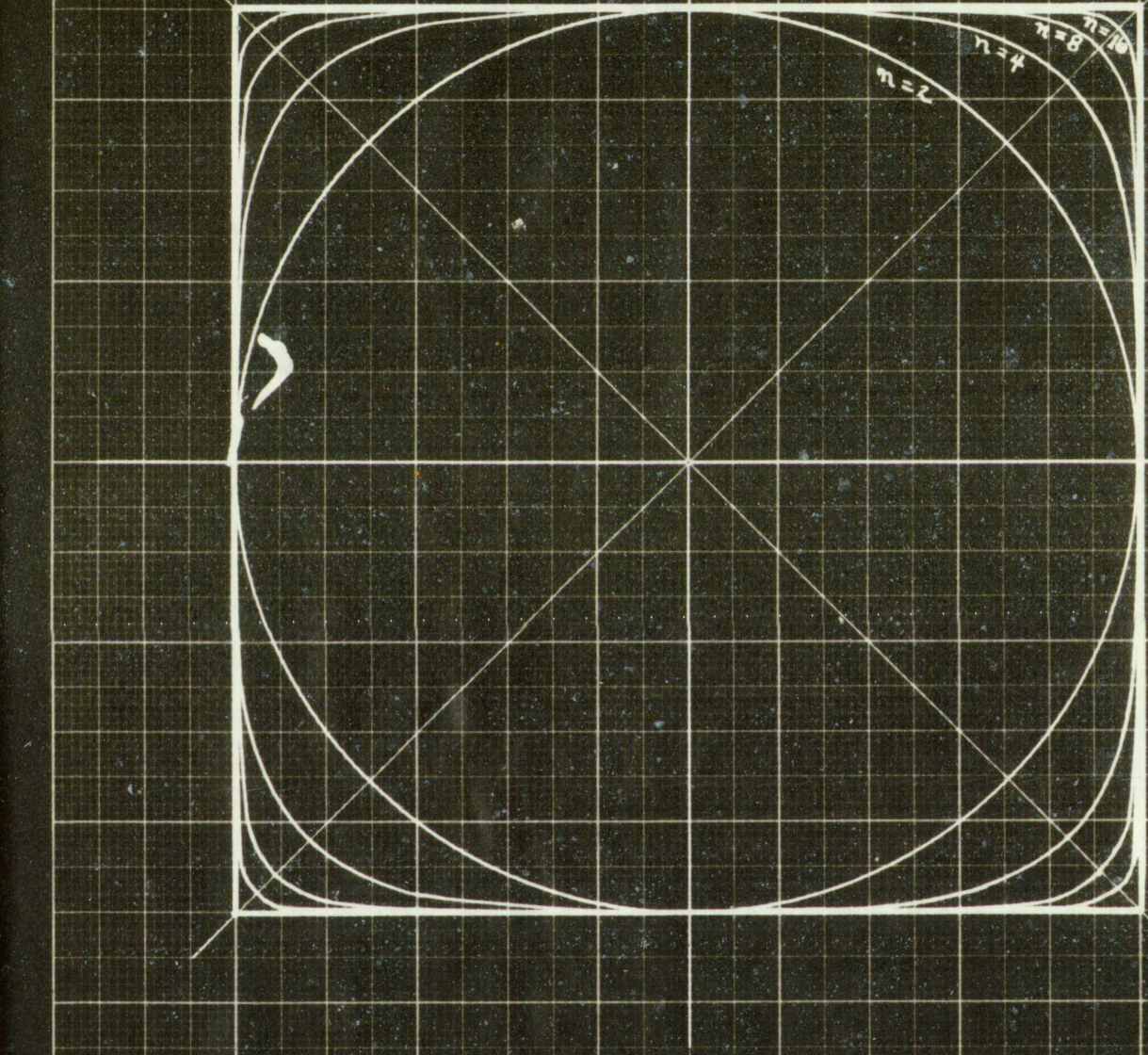


FIGURE 1

CONTOUR LINES OF F_n IN THE PLANE $z = 1$: $n = 2m$

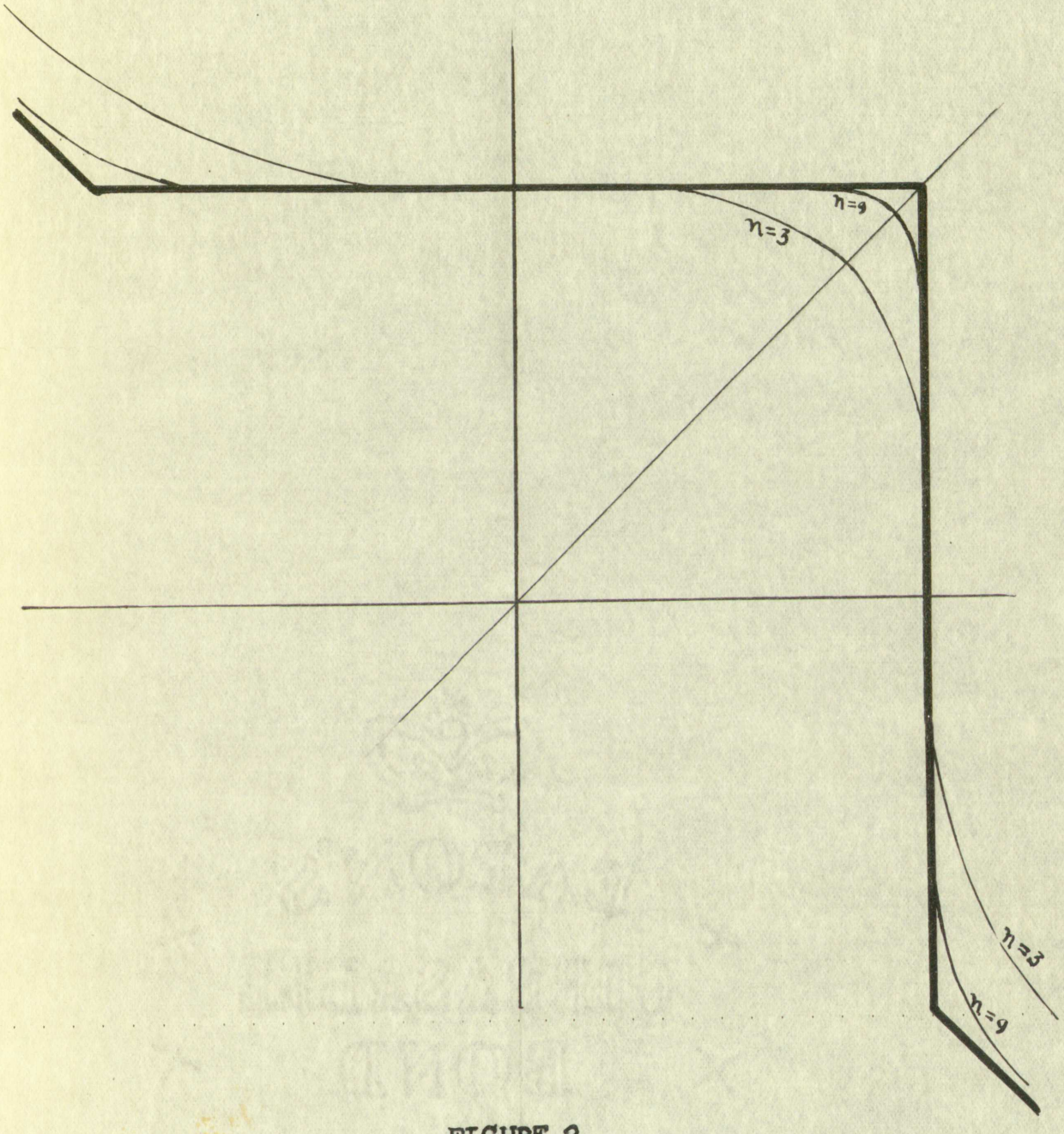
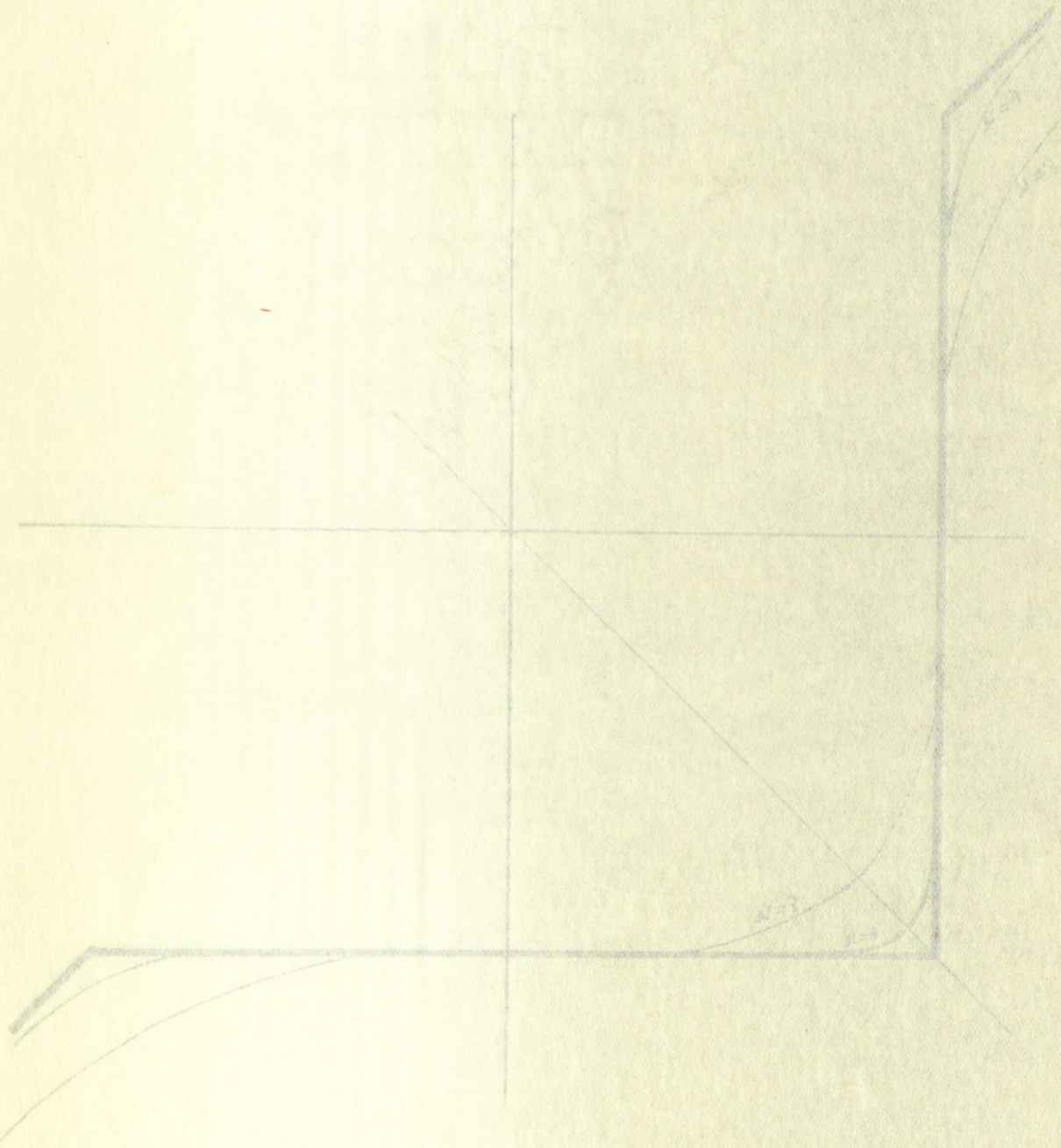


FIGURE 2

CONTOUR LINES OF F_n IN THE PLANE $z = 1$: $n = 2m + 1$



1.3 Lemmas from the theory of real variables.

Let a and b be any real numbers such that $a \geq b \geq 0$. Then $a = tb \geq 0$, where t is a real number greater than or equal to 1. Whence, if we restrict attention to positive nth roots only,

$$\lim_{n\to\infty} (a^n + b^n)^{1/n} = \lim_{n\to\infty} (t^n b^n + b^n)^{1/n} = \lim_{n\to\infty} b\,(t^n + 1)^{1/n}.$$

But,

$$\lim_{n\to\infty} b\,(t^n)^{1/n} \leq \lim_{n\to\infty} b\,(t^n + 1)^{1/n} \leq \lim_{n\to\infty} b\,(t^n + t^n)^{1/n}.$$

So that,

$$bt \leq \lim_{n\to\infty} b\,(t^n + 1)^{1/n} \leq bt \lim_{n\to\infty} 2^{1/n} = bt.$$

Whence, $$\lim_{n\to\infty} (a^n + b^n)^{1/n} = a.$$

Similarly, if $0 \leq a \leq b$, $\lim_{n\to\infty} (a^n + b^n)^{1/n} = b$.

Consequently, if we define

$$M(a,b) = a \quad \text{if } |a| > |b|$$
$$M(a,b) = b \quad \text{if } |b| > |a|$$
$$M(a,b) = \frac{a+b}{2} \quad \text{if } |a| = |b|$$

we may state:

Lemma I. $\lim_{n\to\infty} (a^n + b^n)^{1/n} = M(a,b)$.

If a and b are any two real numbers such that $|b| < |a|$ and $n = 2m + 1$, then $a = tb$ where t is a real number such that $|t| > 1$. If $1 \leq t$, the theory of Lemma I carries over to this case. If, however, $t \leq -1$, consider

$$\lim_{n\to\infty} (a^n + b^n)^{1/n} = \lim_{n\to\infty} b(1 + t^n)^{1/n}.$$

Now,

$$\lim_{n\to\infty} b(t^n + t^n)^{1/n} \leq \lim_{n\to\infty} b(1 + t^n)^{1/n} \leq \lim_{n\to\infty} b(t^n)^{1/n} = bt.$$

so that,

$$bt = bt \lim_{n\to\infty} 2^{1/n} \leq \lim_{n\to\infty} b(1 + t^n)^{1/n} \leq bt.$$

whence, $\lim_{n\to\infty} (a^n + b^n)^{1/n} = a.$

Similarly, if

$|a| < |b|$, $\lim_{n\to\infty} (a^n + b^n)^{1/n} = b.$

If $a = -b$, $\lim_{n\to\infty} (a^n + b^n)^{1/n} = 0 = \frac{a+b}{2}.$

Consequently, we may state:

Lemma II. $\lim_{n\to\infty} (a^n + b^n)^{1/n} = M(a, b).$

1.4 Limiting Fermat surfaces. Divide the family of surfaces F_n: $x^n + y^n = z^n$ where x, y, z are real, $z > 0$, $n > 0$, into the two subfamilies,

(1.4.1) $x^{2m} + y^{2m} = z^{2m}$

(1.4.2) $x^{2m+1} + y^{2m+1} = z^{2m+1}.$

For any choice of x and y in (1.4.1) we have

$$\lim_{n\to\infty} z = \lim_{n\to\infty} (x^n + y^n)^{1/n}$$

$$= \lim_{n\to\infty} (|x|^{2m} + |y|^{2m})^{1/2m}$$

$$= M(|x|, |y|), \text{ by Lemma I.}$$

Define the surface $z = M(|x|, |y|)$ to be the limiting Fermat surface, S_E, in this case.

On the other hand, from (1.4.2) we have

$$z = (x^{2m+1} + y^{2m+1})^{1/2m+1}$$

$$y = (z^{2m+1} - x^{2m+1})^{1/2m+1}$$

$$x = (z^{2m+1} - y^{2m+1})^{1/2m+1}.$$

Whence by Lemma II

$$\lim_{m\to\infty} z = M(x,y)$$

$$\lim_{m\to\infty} y = M(z,-x)$$

$$\lim_{m\to\infty} x = M(z,-y)$$

We define the limiting surface S_0 to be the point set in space which consists of number triplets which satisfy one or more of the relations:

$$z = M(x,y)$$

$$y = M(z,-x)$$

$$x = M(z,-y)$$

While S_E is symmetric with respect to all three coordinate planes, S_O is symmetric with respect to the origin and the plane $y = x$. On the following two pages are three-dimensional sketches of S_E and S_O, respectively.

1.5 Existence of rational points on S_E. If there did exist a rational point on any Fermat surface F_n, $n > 2$, say $\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}\right)$, then Fermat's conjecture would be disproved, for we would then have

$$(a_1 b_2 b_3)^n + (b_1 a_2 b_3)^n = (b_1 b_2 a_3)^n$$

for some n greater than 2, and obviously the quantities in parentheses are integers. For this reason, we should be happy to discover such a point, or to be able to prove that none exists, but up to the present no proof or disproof of the existence of such points has been discovered.

On the other hand, we shall see that such points do exist on the limiting Fermat surfaces, and in abundance, and it might seem that this fact would lead directly to an answer to the question proposed in the last paragraph. This is not true as we shall show below.

Consider the intersections of the planes $x = \frac{a_3}{a_4} y$ with the contour of S_E in the plane $z = \frac{a_1}{a_2}$, a_i integers.

While S_n is symmetric with respect to all three coordinate planes, S_0 is symmetric with respect to the origin and the plane $y = x$. On the following two pages are three-dimensional sketches of S_n and S_0, respectively.

1.6 <u>Existence of rational points on S_n</u>. If there did exist a rational point on any Fermat surface F_n, $n > 2$, say $\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}\right)$, then Fermat's conjecture would be disproved, for we would then have

$$(a_1 b_2 b_3)^n + (b_1 a_2 b_3)^n = (b_1 b_2 a_3)^n$$

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Consider the intersections of the planes $x = \frac{a_3}{a_4} y$ with the contour of S_n in the plane $z = \frac{a_1}{a_2}$, a_i integers.

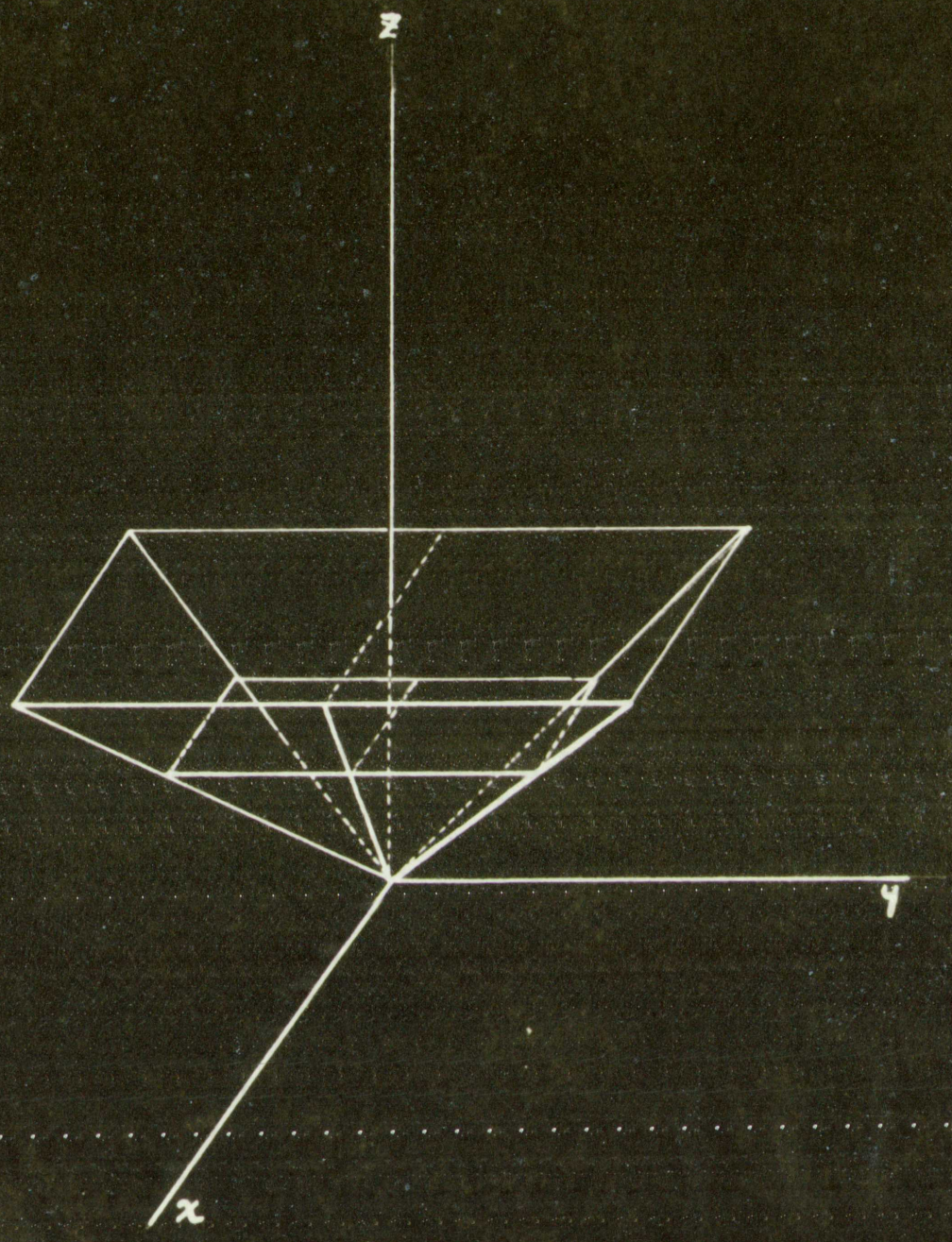

FIGURE 3

FERMAT SURFACE S_E: $z \geq 0$

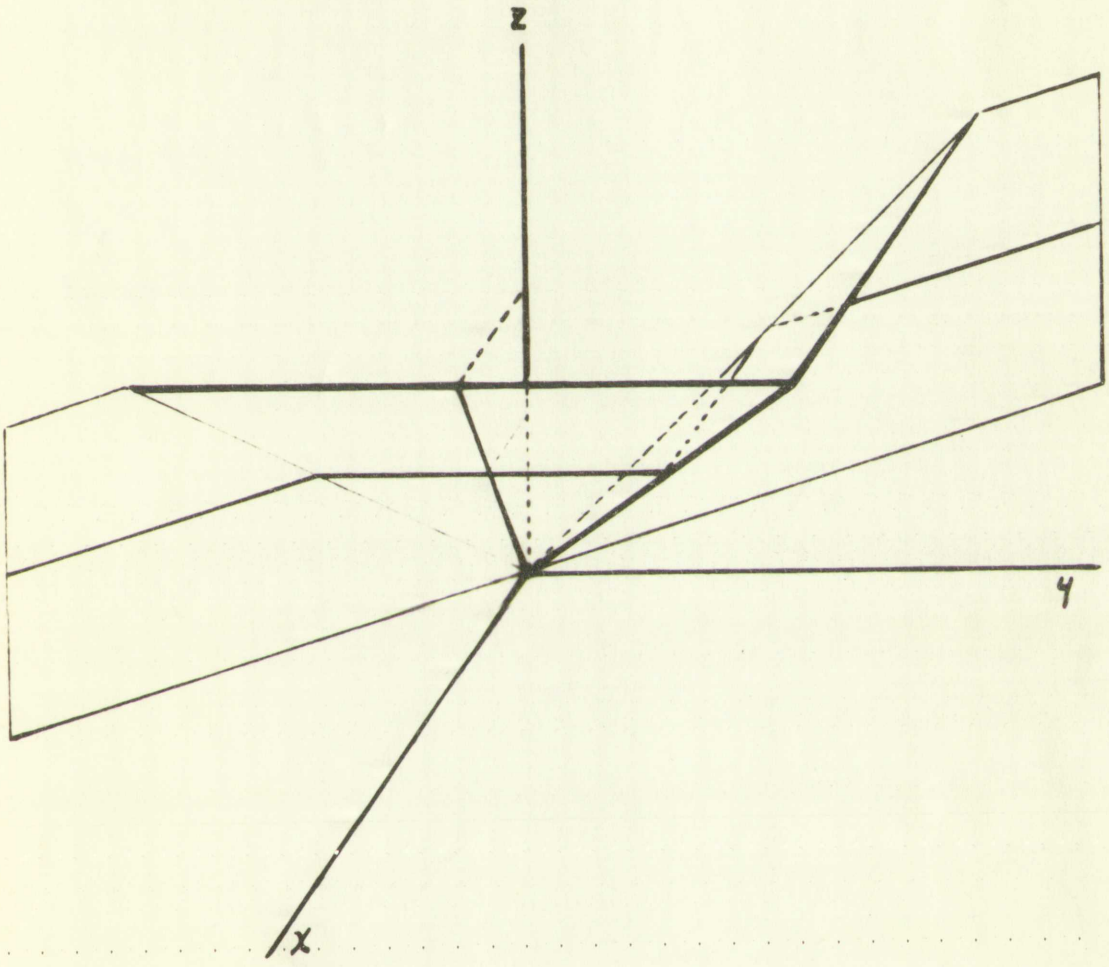

FIGURE 4

FERMAT SURFACE S_0: $z \geq 0$

The intersections are rational points, and in fact, all rational points on S_E may be obtained this way. The points in question constitute an everywhere dense set on F_n.

But consider the intersections with the surfaces F_n of the lines $x = \frac{a_3}{a_4} y$, $z = \frac{a_1}{a_2}$ used to determine the rational points on S_E. These intersections are

$$a_2^n y^n (a_3^n + a_4^n) = a_1^n a_4^n .$$

From this we see that y is rational if and only if $a_3^n + a_4^n$ is a perfect nth power, which is the very issue at question in our investigation.

2. DIFFERENTIAL GEOMETRY OF F_n

2.1 PARAMETERIZATIONS OF F_n

2.1.1 $\underline{Y(u,v_1) = uX(v_1),\ v_1}$ $\underline{\text{arc length}}$. The search for the most useful parameterization of F_n first yielded the following. We consider the curve C_n of intersection of F_n with the unit sphere whose center is at (0,0,0). Points P_{v_1} on this curve are uniquely determined in position by their arc length distance v_1 measured along C_n from an arbitrary point on C_n corresponding to arc length distance 0. Let $X(v_1)$ be the unit vector[1] emanating from the origin with terminal point at P_{v_1}. Let u be a scalar, so that for fixed v_1 and variable u, $uX(v_1)$ is the vector representation of the line through the origin and P_{v_1}. Using C as directrix and $uX(v_1)$ as generator, we have one simple representation of F_n.

2.1.2 $\underline{Y(u,v_2) = uX(v_2),\ v_2}$ $\underline{\text{not arc length}}$. One such representation may be obtained as follows. We again let $X(v_2)$ be the unit vector emanating from the origin and terminating on C_n, where this time v_2 is the tangent of the angle made with the positive x-axis by the projection of $X(v_2)$ on the x,y-plane. There results a representation of

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[1] Here and elsewhere in this section, the capital letters X, Y, N, will be used to denote three-dimensional vectors.

F_n as $uX(v_2)$, where $X(v_2)$ can be written explicitly:

$$(2.1.2.1) \quad X(v_2) : \left[\frac{1}{w^{1/2}} , \frac{v_2}{w^{1/2}} , \frac{(1+v_2^n)^{1/n}}{w^{1/2}} \right]$$

where $$w = 1 + v_2^2 + (1+v_2^n)^{2/n} .$$

2.1.3 Explicit expression for X in parameterization 2.1.1. While in parameterization (2.1.2) we have an explicit expression for $X(v_2)$, the vector $X(v_1)$ has not been given explicitly in (2.1.1); although such an explicit expression possibly would have been of great value in the sequel. In this section we justify our omission by carrying through the explicit determination of $X(v_1)$ in the special case n 2 and showing that an attempt to generalize this to arbitrary n would only lead us into hopeless complexities. Several indirect attempts to obtain $S(v_1)$ explicitly were also unsuccessful.

Using parameterization 2.1.2, arc length along C_n from $v_2 = 0$ to $v_2 = v_2$ is given by $v_1 = \int_0^{v_2} \sqrt{\dot{X}(v_2) \mid \dot{X}(v_2)} \, dv_2$, where

$$(2.1.3.1) \quad \dot{X}(v_2) \mid \dot{X}(v_2) = \frac{1 + (1+v_2^n)^{\frac{2-2n}{n}} + v_2^{2n-2}(1+v_2^n)^{\frac{2-2n}{n}}}{w^2} .$$

We interrupt our development to show on the following page how (2.1.3.1) was derived from (2.1.2.1).

Given The vector $X(v_2)$ as in (2.1.2.1), we find that $\dot{x}_1 = -\left[v_2 + v_2^{n-1}(1+v_2^n)^{\frac{2-n}{n}}\right] w^{-\frac{3}{2}}$

$\dot{x}_2$ and $\dot{x}_3$ can be given in terms of $\dot{x}_1$ as: $\dot{x}_2 = v_2\dot{x}_1 + x_1$, $\dot{x}_3 = (1+v_2^n)^{1/n}\dot{x}_1 + v_2^{n-1}(1+v_2^n)^{\frac{1-n}{n}}$

Hence, $\dot{x}|\dot{x} = \dot{x}_1^2 + v_2^2\dot{x}_1^2 + 2v_2\dot{x}_1x_1 + x_1^2 + (1+v_2^n)^{\frac{2-2n}{n}}v_2^{2n-2}\dot{x}_1^2 + 2(1+v_2^n)^{\frac{2-n}{n}}v_2^{n-1}x_1\dot{x}_1 + (1+v_2^n)^{2/n}\dot{x}_1$

and $w^2\dot{x}|\dot{x}$ is given by each of the following expressions:

$$\left[v_2 + (1+v_2^n)^{\frac{2-n}{n}}v_2^{n-1}\right]^2 - 2\left[v_2 + (1+v_2^n)^{\frac{2-n}{n}}v_2^{n-1}\right]^2 + \left[1 + v_2^{2n-2}(1+v_2^n)^{\frac{2-2n}{n}}\right] w$$

$$-v_2^2 - 2v_2^n(1+v_2^n)^{\frac{2-n}{n}} - v_2^{2n-2}(1+v_2^n)^{\frac{4-2n}{n}} + 1 + v_2^2 + (1+v_2^n)^{2/n} + v_2^{2n-2}(1+v_2^n)^{\frac{2-2n}{n}} + v_2^{2n}(1+v_2^n)^{\frac{2-2n}{n}} + v_2^{2n-2}(1+v_2^n)^{\frac{4-2n}{n}}$$

$$-2v_2^n(1+v_2^n)^{\frac{2-2n}{n}} - 2v_2^{2n}(1+v_2^n)^{\frac{2-2n}{n}} + 1 + (1+v_2^n)^2(1+v_2^n)^{\frac{2-2n}{n}} + v_2^{2n-2}(1+v_2^n)^{\frac{2-2n}{n}} + v_2^{2n}(1+v_2^n)^{\frac{2-2n}{n}}$$

$$-2v_2^n(1+v_2^n)^{\frac{2-2n}{n}} - 2v_2^{2n}(1+v_2^n)^{\frac{2-2n}{n}} + 1 + (1+v_2^n)^{\frac{2-2n}{n}} + 2v_2^n(1+v_2^n)^{\frac{2-2n}{n}} + v_2^{2n}(1+v_2^n)^{\frac{2-2n}{n}} + v_2^{2n-2}(1+v_2^n)^{\frac{2-2n}{n}} + v_2^{2n}(1+v_2^n)^{\frac{2-2n}{n}}$$

Hence, $$w^2\dot{x}|\dot{x} = 1 + (1+v_2^n)^{\frac{2-2n}{n}} + v_2^{2n-2}(1+v_2^n)^{\frac{2-2n}{n}}.$$

Divide both sides of the last equation by w^2 to obtain (2.1.3.1).

The right member of (2.1.3.1) is essentially bulky, for $\frac{2-2n}{n}$ is not an integer except for n = 1 or n = 2, as may be seen from the congruence $2 - 2n \equiv 0 \pmod{n}$, which is true if and only if $2(1 - n) \equiv 0 \pmod{n}$. Since 1 - n is never divisible by integral n except for n = 1, it follows that $2(1 - n) \equiv 0 \pmod{n}$ only for n = 1 or n = 2.

In case n = 2, the right member of (2.1.3.1) reduces to $\frac{1}{2(1+v_2^2)^2}$ and $v_1 = \frac{\sqrt{2}}{2}\int_0^{v_2} \frac{dv_2}{1+v_2^2} = \frac{\sqrt{2}}{2}\tan^{-1}v_2$.

Thus, when n 2, $X(v_1)$ is given explicitly by

$$(2.1.3.2)\quad \left[\frac{\cos(\sqrt{2}\,v_1)}{\sqrt{2}}, \frac{\sin(\sqrt{2}\,v_1)}{\sqrt{2}}, \frac{1}{2} \right].$$

However, inspection of the general formula (2.1.3.1) from whence this explicit result was derived by integration in the special case n = 2 will convince the reader that this direct procedure is not feasible for general n.

2.1.4 $\underline{Y(\bar{u}, v_3)}$: $\left[\bar{u}, v_3, (\bar{u}^n + v_3^n)^{1/n} \right]$. This vector representation of F_n is obtained immediately from $x^n + y^n = z^n$ on setting $x = \bar{u}$, $y = v_3$, $z = (\bar{u}^n + v_3^n)^{1/n}$.

2.2 THE FUNDAMENTAL SURFACE QUANTITIES

2.2.1 Parameterization 2.1.1. Since $X(v_1)$ is a unit vector defining C_n, along which v_1 is arc length, we have the relations $X|X = 1$, $X|\dot{X} = 0$, $\dot{X}|\dot{X} = 1$, and $\dot{X}|\ddot{X} = 0$. Hence, $Y_u = X$, $Y_{v_1} = u\dot{X}$, $Y_{uu} = 0$, $Y_{uv_1} = \dot{X}$, and $Y_{v_1v_1} = u\ddot{X}$.

It follows that

(2.2.1.1) $E = 1$, $F = 0$, $G = u^2$, $D^2 = u^2$ and

(2.2.1.2) $e = 0$, $f = 0$, $g = u(X\dot{X}\ddot{X})$, $d^2 = 0$.

Furthermore, if $\mathcal{f}$ is the unit normal at the point $Y(u, v_1)$ on F_n, $\mathcal{f} = X\hat{\dot{X}}$, $\mathcal{f}_u = 0$, $\mathcal{f}_{v_1} = X\hat{\ddot{X}}$, so that

(2.2.1.3) $\mathcal{E} = 0$, $\mathcal{F} = 0$, $\mathcal{G} = \ddot{X}|X - u^2$ and $\mathcal{D}^2 = 0$.

It is clearly remarkable that these comprehensive results could have been obtained without an explicit expression for $X(v_1)$.

2.2.2 Parameterization 2.1.2. From expression (2.1.2.1) for $X(v_2)$ and $Y = uX$, we have $Y_u = X$ and

$$Y_{v_2} : \left[u\frac{-v_2 - v_2^{n-1}(1+v_2^n)^{\frac{2-n}{n}}}{W^{3/2}},\ u\frac{1+(1+v_2^n)^{2/n} - v_2^n(1+v_2^n)^{\frac{2-n}{n}}}{W^{3/2}},\ u\frac{(1+v_2^n)^{1/n}\{-v_2+(1+v_2^2)(1+v_2^n)^{-1}v_2^{n-1}\}}{W^{3/2}} \right]$$

We interrupt 2.2.2 to show how the components of Y_{v_2} were derived from the results on page 32.

From P.32, $Y_{v_2} : \left[u \frac{-v_2 - v_2^{n-1}(1+v_2^n)^{\frac{2-n}{n}}}{W^{3/2}}, u(v_2\dot{x}_1 + x_1), u\left[(1+v_2^n)^{1/n}\dot{x} + v_2^{n-1}(1+v_2)^{\frac{1-n}{n}}x_1\right]\right]$

But, $v_2\dot{x}_1 + x_1 = \frac{-v_2\left[v_2 + v_2^{n-1}(1+v_2^n)^{\frac{2-n}{n}}\right]}{W^{3/2}} + \frac{W}{W^{3/2}}$

$$= \frac{-v_2^2 - v_2^n(1+v_2^n)^{\frac{2-n}{n}} + 1 + v_2^2 + (1+v_2^n)^{2/n}}{W^{3/2}}$$

$$= \frac{1}{u}\left[\text{component 2 of } Y_{v_2}\right].$$

$(1+v_2^n)^{1/n}\dot{x}_1 + v_2^{n-1}(1+v_2^n)^{\frac{1-n}{n}}x_1$ is given by each of :

$$\frac{-(1+v_2^n)^{1/n}\left[v_2 + v_2^{n-1}(1+v_2^n)^{\frac{2-n}{n}}\right]}{W^{3/2}} + \frac{v_2^{n-1}(1+v_2^n)^{\frac{1-n}{n}}W}{W^{3/2}}$$

$$\frac{-v_2(1+v_2^n)^{1/n} - v_2^{n-1}(1+v_2^n)^{\frac{3-n}{n}} + v_2^{n-1}(1+v_2^n)^{\frac{1-n}{n}} + v_2^{n+1}(1+v_2^n)^{\frac{1-n}{n}} + v_2^{n-1}(1+v_2^n)^{\frac{3-n}{n}}}{W^{3/2}}$$

$$= \frac{(1+v_2^n)^{1/n}\left[-v_2 + v_2^{n-1}(1+v_2^n)^{-1} + v_2^{n+1}(1+v_2^n)^{-1}\right]}{W^{3/2}}$$

$$= \frac{(1+v_2^n)^{1/n}\left[-v_2 + v_2^{n-1}(1+v_2^n)^{-1}(1+v_2^2)\right]}{W^{3/2}}$$

$$= \frac{1}{u}\left[\text{component 3 of } Y_{v_2}\right].$$

It follows that

$$(2.2.2.1)\quad E = 1,\ F = 0,\ G = u^2\left[\frac{1+(1+v_2^n)^{\frac{2-2n}{n}}+v_2^{2n-2}(1+v_2^n)^{\frac{2-2n}{n}}}{\{1+v_2^2+(1+v_2^n)^{2/n}\}^2}\right]$$

and

$$(2.2.2.2)\quad e = 0,\ f = 0,\quad g = \frac{(Y_{v_2v_2} \times Y_{v_2})}{\sqrt{G}}$$

In this case, an explicit expression could be obtained for g by dint of considerable effort. However, we have found in the sequel no use for this result, and consequently, have not obtained it.

2.2.3 Parameterization 2.1.3. We obtain from expression (2.1.4) for $Y(u, v_3)$ that

$$(2.2.3.1)\quad \begin{cases} E = 1 + x^{2n-2}(x^n + y^n)^{\frac{2-2n}{n}} \\ F = x^{n-1}y^{n-1}(x^n + y^n)^{\frac{2-2n}{n}} \\ G = 1 + y^{2n-2}(x^n + y^n)^{\frac{2-2n}{n}} \end{cases}$$

and

$$(2.2.3.2)\quad \begin{cases} e = \dfrac{(n-1)x^{n-2}y^n(x^n + y^n)^{\frac{1-2n}{n}}}{V} \\ f = \dfrac{(1-n)x^{n-1}y^{n-1}(x^n + y^n)^{\frac{1-2n}{n}}}{V} \\ g = \dfrac{(n-1)x^n y^{n-2}(x^n + y^n)^{\frac{1-2n}{n}}}{V} \end{cases}$$

where $V = \left[1 + x^{2n-2}(x^n+y^n)^{\frac{2-2n}{n}} + y^{2n-2}(x^n+y^n)^{\frac{2-2n}{n}}\right]^{1/2}$.

Furthermore,

$$(2.2.3.3)\quad f : \left[\frac{-x^{n-1}(x^n+y^n)^{\frac{1-n}{n}}}{V}, \frac{-y^{n-1}(x^n+y^n)^{\frac{1-n}{n}}}{V}, \frac{1}{V} \right].$$

2.3 IMPLIED DIFFERENTIAL GEOMETRIC PROPERTIES OF F_n

2.31 Parameterizations 2.11 and 2.12. We first obtain the differential equations of the geodesics on F_n. These equations have the form[2]

$$(2.3.1.1)\quad \begin{cases} \dfrac{d^2u}{ds^2} + C_{11}^{1}\left(\dfrac{du}{ds}\right)^2 + 2C_{12}^{1}\dfrac{du}{ds}\dfrac{dv}{ds} + C_{22}^{1}\left(\dfrac{dv}{ds}\right)^2 = 0 \\ \dfrac{d^2v}{ds^2} + C_{11}^{2}\left(\dfrac{du}{ds}\right)^2 + 2C_{12}^{2}\dfrac{du}{ds}\dfrac{dv}{ds} + C_{22}^{2}\left(\dfrac{dv}{ds}\right)^2 = 0. \end{cases}$$

But from parameterization 2.1.1, we have

$$C_{11}^{1} = 0, \quad C_{12}^{1} = 0, \quad C_{22}^{1} = u(\ddot{X}\,|\,X) = -u$$

$$C_{11}^{2} = 0, \quad C_{12}^{2} = \frac{1}{u}, \quad C_{22}^{2} = \dot{X}\,|\,\ddot{X} = 0$$

Hence, for F_n, (2.3.1.1) reduces to

$$(2.3.1.2)\quad \begin{cases} \dfrac{d^2u}{ds^2} - u\left(\dfrac{dv}{ds}\right)^2 = 0 \\ \dfrac{d^2v}{ds^2} + \dfrac{2}{u}\left(\dfrac{du}{ds}\right)\left(\dfrac{dv}{ds}\right) = 0 \end{cases}$$

We immediately verify that the u-curves are geodesics.

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[2] W. C. Graustein, Differential Geometry, p. 155.

To obtain other important systems of curves on F_n we turn to the equation of the lines of curvature[3]

$$(2.3.1.3)\quad (Ef - Fe)\,du^2 + (Eg - Ge)\,du\,dv + (Fg - Gf)\,dv^2 = 0$$

and that of the asymptotic lines[4]

$$(2.3.1.4)\quad e\,du^2 + 2f\,du\,dv + g\,dv^2 = 0.$$

Using parameterization 2.1.1,

$$e = f = F = 0,\quad E = 1,\quad G = u^2,\quad g = u(x\dot{x}\ddot{x})$$

so that (2.3.1.3) and (2.3.1.4) reduce to

$$(2.3.1.5)\quad u(x\dot{x}\ddot{x})\,du\,dv = 0$$

and

$$(2.3.1.6)\quad u(x\dot{x}\ddot{x})\,dv^2 = 0$$

respectively. Because $e = f = 0$, we could not have also $g = 0$, since this would imply that F_n is a plane. Hence, for parameterizations 2.1.1 and 2.1.2 the parametric curves are the lines of curvature, while the u-curves are the asymptotic lines.

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[3] Ibid., p. 111.

[4] Ibid., p. 127.

2.3.2 Parameterization 2.1.3. The angle between the directed u- and v-curves is given by

$$\cos\omega = \frac{x^{n-1}y^{n-1}(x^n+y^n)^{\frac{2-2n}{n}}}{\left\{\left[1+x^{2n-2}(x^n+y^n)^{\frac{2-2n}{n}}\right]\left[1+y^{2n-2}(x^n+y^n)^{\frac{2-2n}{n}}\right]\right\}^{1/2}},$$

since along the curves v = constant, the direction of the tangent is X_u; along u = constant, it is X_v. Thus,

$$\cos\omega = \frac{X_u \mid X_v}{\sqrt{E}\,\sqrt{G}},$$

which reduces to the value given above.

Let S: $x^2 + y^2 + z^2 - r^2 = 0$ intersect F_n at (x,y,z). The normal N to S at this point has direction components x,y,z. From (2.2.3.3), direction components of the normal N_n to F_n at this point are

$$\left[-x^{n-1}(x^n+y^n)^{\frac{1-n}{n}},\ -y^{n-1}(x^n+y^n)^{\frac{1-n}{n}},\ 1\right].$$

Hence, $N \mid N_n = 0$, and the normals intersect orthogonally at each point of their curve of intersection. This result verifies the famous theorem of Joachimsthal[5], that if a curve of intersection of two surfaces is a line of curvature on both, the surfaces intersect at the same angle along the curve. This is the situation obtaining in our case, since as is well known, every curve on a sphere is a line of

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[5] Graustein, op. cit., p. 121.

curvature, and we have found above that C_n is also a line of curvature on F_n.

Following an attack suggested by this result, let S_p denote the sphere passing through the point P on F_n. Direction numbers of the normal to the sphere are (x,y,z). If P is an integral point, these direction numbers (A, B, C) are integers. Let (a,b,c) be the set of direction numbers of the normal to F_n at P given by

$$a = -x^{n-1}(x^n+y^n)^{\frac{1-n}{n}}, \quad b = -y^{n-1}(x^n+y^n)^{\frac{1-n}{n}}, \quad c = 1.$$

The orthogonality property implies that $Aa + Bb = C$ or $[A^n + B^n]^{1/n} = C$. We do not attain a contradiction of our assumption that an integral point does exist on F_n unless it is impossible that $A^n + B^n = C^n$. Thus again, as in section 1.5, our indirect attack has led back to the very theorem we seek to prove.

Finally, from parameterization 2.1.4 we obtain three hueristic geometric implications of Fermat's last theorem:

Theorem I. If there exists an integral triad on F_n, where these surfaces are represented parametrically in terms of x and y, then the coefficients, E, F, G, of the first fundamental differential quadratic form of F_n are integers.

Theorem II. If when F_n is represented parametrically in terms of x and y there exists on the surface F an integral triad (x,y,z) such that $x^n + y^n = (z^{1/2})^n$, then the coefficients of the first fundamental differential quadratic form of F_n are integers.

Theorem III. If there exists an integral triad on F_n, where these surfaces are represented parametrically in terms of x and y, then the quotients, obtained by dividing the coefficients e, f, g, of the second fundamental differential quadratic form of F_n by the corresponding components of the unit normal, are rational numbers.

The first two theorems derive from inspection of the coefficients of the first fundamental form, while the divisions mentioned in the third theorem yield

$$\frac{e}{f_1} = \frac{(1-n)y^n}{x(x^n+y^n)} \quad , \quad \frac{f}{f_2} = \frac{(n-1)x^{n-1}}{x^n+y^n} \quad , \quad \frac{g}{f_3} = \frac{(n-1)x^n y^{n-2}(x^n+y^n)^{1/n}}{(x^n+y^n)^2}$$

which are seen to be rational numbers if $x^n + y^n = z^n$, x, y, z, integers.

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