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Some remarks on the delay stabilizing effect in SISO systems

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Abstract

This note addresses the stabilization problem of a class of SISO systems with a time delay in the input, and explore the stabilizing effect of time delay. More precisely, for a fixed feedback gain such that the closed loop system is unstable when the delay is set to zero, we shall present *necessary and sufficient* conditions for the delays such that the stability in closed-loop is achieved, and provide an explicit construction of the controllers. Next, we shall analyze conditions for preserving the closed-loop stability if parametric or time-varying delay uncertainties are present in the control law. Illustrative examples are also proposed.

1 Introduction

The existence of a time-delay at the actuating input in a feedback control system is ususally known to cause *instability* or poor performance for the closed-loop schemes [12, 13, 7] (and the references therein). This note addresses the opposite problem: characterizing the situations when a delay has a *stabilizing effect*. In other words, we consider the situation where the delay free feedback system is unstable, and it becomes asymptotic stable due to the presence of appropriate delay in the actuating input.

Consider the following class of *strictly proper* SISO open-loop systems:

$$H_{yu}(s) = c^T(sI_n - A)^{-1}b = \frac{P(s)}{Q(s)} \qquad (1)$$

where (A, b, c^T) is a state-space representation of the open-loop system, and $deg(Q(s)) > deg(P(s))$, with the controller

$$u(t) = -ky(t-\tau). \qquad (2)$$

The *stabilizing delay effect* problem mentioned above can be defined as follows:

Problem 1 (Delay stabilizing effect) *Find explicit conditions on the pair (k, τ), such that the controller (2) stabilizes (1), but with the closed-loop system would be unstable if the delay τ is set to zero $\tau = 0$.*

As we shall see below, the conditions derived will lead to an explicit construction of the controller. Furthermore, for each stabilizing pair, we may define a stabilizing delay interval, which can be seen as *robustness measure* of the corresponding control law if the delay is subject to parametric uncertainty.
The next step is to analyze the robustness with respect to uncertain time-varying delay. Roughly speaking, the corresponding robust stability problem of the closed-loop system can be formulated as follows:

Problem 2 (Time-varying delay uncertainty) *For a given stabilizing pair (k, τ) such that the closed-loop system is unstable if $\tau = 0$, find conditions on the time-varying delay uncertainty $\delta(t)$ satisfying $\dot{\delta}(t) \le \beta$ for some real $0 \le \beta < 1$, such that the control law:*

$$u(t) = -ky(t-\tau-\delta(t)), \qquad (3)$$

still stabilizes (1).

The interest of solving such problems is twofold: first, the resulting design is rather simple and delay is rather easy to implement; second, explore the potential of using such a controller (using delay as a *design parameter*) in situations where it is not easy to design or implement a controller without delay. Some discussions in this direction have been considered in [1, 3, 15], but without any attempt to treat the problem in the general setting. A Nyquist criterion was used in [1] to prove that a pair (gain,delay) may stabilize second-order oscillatory systems. A different approach was proposed in [3], where upper and lower bounds of the delay are given such that the closed-loop system is stable, under the assumption that the system is stable with some known nominal delay values. Finally, the paper [15] addresses the general static delayed output feedback problem, and some *existence results* (delay-independent, delay-dependent, instability persistence) are derived, but without any *explicit construction* of the controllers. More specifically, [15] compares the stability of the closed-loop schemes with or without delays in the corresponding control laws.

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Although only strictly proper SISO systems are considered above, the ideas still work for more general SISO systems, such as a restricted class of (not necessarily strictly) proper systems, or systems with internal delays in addition to the feedback input delay.

Both problems proposed here will be handled using frequency-domain methods. First, we shall analyze the sensitivity of the roots in terms of delays, and we shall derive *necessary and sufficient* conditions for the delay values in the control law such that a pair of unstable complex conjugate roots cross the imaginary axis to the left hand plane. An explicit construction of the controller will be given in the following form: for any gain satisfying some assumptions, a delay interval guaranteeing stability will be computed. The method is inspired by the developments in [4] (only second-order systems including discrete or pointwise delay) and extended in [5] (more general analytic functions), with further generalizations, comments, discussions and related references in [13]. Second, we shall use the integral quadratic constraint (IQCs) based approach for handling the time-varying delay uncertainty. Both conditions are easy to check.

2 Main results

In this section, we consider the first problem. In order to prove our *main* results, some prerequisites from the standard output feedback stabilization problem for SISO systems (free of delays) are needed.

2.1 Basic results in the case free of delays

The difficulty in designing static output feedback stabilization problem (see, for instance, [18] and the references therein) is well known. However, in the SISO system case, the problem is reduced to a one-parameter problem, which is relatively easy. Indeed, there exist several methods to solve it: This include (standard) graphical tests (root-locus, Nyquist), and computation of the real roots of an appropriate set of polynomials. In addition to these standard methods, we may cite two interesting approaches [2, 10] based on *generalized eigenvalues computation* of some appropriate matrix pencils defined by the corresponding Hurwitz [2], and Hermite [10] matrices. The approach below is inspired by Chen's characterization [2] for systems without delay.

As we shall see in the next paragraphs (see also the problem statement in the Introduction), we are interested in finding gains k for which the closed-loop system is *unstable*, but with an *appropriate number of roots* in $\mathbf{C}^+$. Later on, we will discuss the use of time delay to *"move"* them from $\mathbf{C}^+$ to $\mathbf{C}^-$. For these reasons, one needs to adapt the results cited above in order to handle this situation also.

Introduce the following Hurwitz matrix associated to the denominator polynomial $Q(s) = \sum_{i=0}^{n} q_i s^{n-i}$ of the transfer function:

$$H(Q) = \begin{bmatrix} q_1 & q_3 & q_5 & \cdots & q_{2n-1} \\ q_0 & q_2 & q_4 & \cdots & q_{2n-2} \\ 0 & q_1 & q_3 & \cdots & q_{2n-3} \\ 0 & q_0 & q_2 & \cdots & q_{2n-4} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & q_n \end{bmatrix} \in \mathbf{R}^{n \times n}, \tag{4}$$

where the coefficients $q_l = 0$, for all $l > n$. Next, we interpret the numerator polynomial $P(s)$ of the transfer function as a nth order polynomial: $P(s) = \sum_{i=0}^{n} \tilde{p}_i s^{n-i}$, where $\tilde{p}_i = 0$, for all $i = 0, 1, \ldots (n-m+1)$, and $\tilde{p}_i = p_{n-i}$, for all $i = (n-m), \ldots n$. Corresponding to this interpretation, we construct $H(P)$ as a $n \times n$ matrix by the same procedure as (4) with the understanding that $\tilde{p}_l = 0$ for all $l > m$.

The following result is a slight modification, and generalization of Theorem 2.1 by Chen [2]:

Lemma 3 *Let $\lambda_1 < \lambda_2 < \ldots \lambda_h$, with $h \leq n$ be the real eigenvalues of the matrix pencil $\Sigma(\lambda) = \mathbf{det}(\lambda H(P) + H(Q))$. Then $H_{yu}(s)$ cannot be stabilized by the controller $u(t) = ky(t)$ for any $k = \lambda_i$, $i = 1, 2, \ldots h$. Furthermore, if there are r unstable closed-loop roots ($0 \leq r \leq n$) for $k = k^*$, $k^* \in (\lambda_i, \lambda_{i+1})$, then, there are r unstable closed-loop roots for any gain $k \in (\lambda_i, \lambda_{i+1})$. In other words, the number of unstable closed-loop roots remains constant as k varies within each interval $(\lambda_i, \lambda_{i+1})$.*

Remark 4 *For $r = 0$, we recover to the class of stabilizing controllers defined by some appropriate intervals, if any (Theorem 2.1 in Chen [2]).*

2.2 Existence results

Define the polynomial F:

$$F(\omega) = \mid Q(j\omega) \mid^2 - k^2 \mid P(j\omega) \mid^2, \tag{5}$$

and denote by S_+ the set of positive roots of $F(\omega)$.

With these notations, definitions and prerequisites, we have the following result:

Theorem 5 (Existence results) *Assume $Q(s)$ unstable, and let k a real number such that the polynomial $Q(s) + kP(s)$ has a pair of strictly complex conjugate unstable roots with remaining roots stable, and such that all the roots of $F(\omega)$ are simple. Then the delay stabilizing problem has a solution if and only if $\mathbf{card}(S_+) \geq 2$, and the following inequality is satisfied:*

$$\tau_- < \tau_+, \tag{6}$$

where[1]*:*

$$\tau_- = \min_{l \in \mathbf{Z}} \min_{\left\{ \begin{array}{c} \omega \in S_+, \\ F'(\omega) < 0 \end{array} \right\}} \left\{ \frac{1}{\omega} Log\left(-\frac{Q(j\omega)}{kP(j\omega)} + 2\pi l \right) > 0 \right\}, \tag{7}$$

$$\tau_+ = \min_{l \in \mathbf{Z}} \min_{\left\{ \begin{array}{c} \omega \in S_+, \\ F'(\omega) > 0 \end{array} \right\}} \left\{ \frac{1}{\omega} Log\left(-\frac{Q(j\omega)}{kP(j\omega)} + 2\pi l \right) > 0 \right\}, \tag{8}$$

The complete proof can be found in [16] (full version of the paper).

[1] Here, "*Log*" denotes the principal value of the logarithm.

Remark 6 *The condition that there exists only one pair of strictly unstable roots is essential. Indeed, it is not difficult to see*[2] *that there exists chains of oscillators (with an appropriate distribution of the characteristic roots on the imaginary axis) which can be stabilized by an output feedback using a single delay.*

Remark 7 (Gain's choice) *Lemma 3 gives explicitly a way to define such gains, by computing the generalized eigenvalues of the matrix pencil* $\Sigma(\lambda)$, *and selecting only the real eigenvalues:* $\lambda_1 < \lambda_2 < \ldots \lambda_h$, *with* $h \leq n$. *Note however, that one needs to check the number of unstable roots for each* $(\lambda_i, \lambda_{i+1})$ *interval, which implies further computations,* etc *(see also the illustrative example in Section 4).*

Remark 8 (Delay-independent instability) *Based on the proof above, it follows that if* $\tau_+ < \tau_-$, *and* $\mathbf{card}(S_+) \geq 1$, *the first crossing will be towards instability, which is equivalent to say, that the system will* never *recover closed-loop* stability *for* any *positive* delay τ, *since at each crossing the number of roots with positive real part will be always strictly positive (stability/instability crossing alternates each-other,* etc*).*
If $\mathbf{card}(S_+) = 1$, *then the crossing direction will be towards instability, and the closed-loop system will become more and more unstable, when the delay is increased.*
Furthermore, if $\mathbf{card}(S_+) = 0$, *then we recover the* delay-independent hyperbolicity *property (two strictly unstable roots for all positive delays), as defined in [8] (see also [9]).*

Remark 9 (Neutral case) *If we assume that the transfer function* $H_{yu}(s)$ *is* not strictly *proper, that is there exists a direct link d between the input and the output:*

$$H_{yu}(s) = c^T(sI_n - A)^{-1}b + d = \frac{P(s)}{Q(s)} + d, \tag{9}$$

the result above still works if $\mid d \mid < 1$. *Note however that the argument in [5] is* not sufficient *for handling such a case.*

Remark 10 (Internal delay case) *As specified in the Introduction, the argument still works for transfer functions with internal delays, if one assumes that* $F(\omega) = 0$ *has a* finite *number of roots:*

$$H_{yu}(s) = c^T\left(sI_n - A - \sum_{i=1}^{r} A_i e^{-s\tau_i}\right)^{-1} b = \frac{P(s, e^{-s})}{Q(s, e^{-s})}. \tag{10}$$

Such an analysis can be found in [14] for a class of second-order systems including two delays, and encountered as congestion control algorithms. Note however, that the test on the existence of a gain k such that the closed-loop system with $u(t) = -ky(t)$ *has only two unstable roots becomes more difficult to be worked out.*

3 Robustness issues

In the sequel, we shall focus on the robustness of the control scheme with respect to *parametric* and *time-varying uncertainty* in the delay term.

[2] for the brevity of the paper, such a case study is omitted

3.1 Parametric uncertainty: defining delay intervals

Theorem 11 (Delay intervals) *Assume that the conditions in Theorem 5 are verified. Then a stabilizing controller (2) is defined by the gain k and any delay* $\tau \in (\underline{\tau}, \overline{\tau})$ *with:*

$$\underline{\tau} = \tau_-, \overline{\tau} = \tau_+, \tag{11}$$

$\tau_\pm$ *being defined by (7)-(8).*
Furthermore, for a given gain k, the number of stabilizing delay intervals is always finite.

The complete proof can be found in [16] (full version of the paper).

First, note that Theorem 11 is a straightforward consequence of the proof of Theorem 5, and it defines a class of stabilizing controllers: a delay interval $(\underline{\tau}, \overline{\tau})$ for each gain k satisfying the corresponding constraints. It is easy to see that this delay interval can be interpreted as a (parametric) *robustness measure.* Indeed, for a given delay $\tau_0 \in (\underline{\tau}, \overline{\tau})$, the controller:

$$u(t) = -ky(t - \tau_0 - \delta),$$

stabilizes $H_{yu}(s)$ for any real parametric uncertainty δ, with $\delta \in (\tau_0 - \underline{\tau}, \overline{\tau} - \tau_0)$, *etc.*

Remark 12 *The last statement of Theorem 11 represents the so-called* instability persistence property *discussed in [15] (see also [13] in a more general setting).*

A natural consequence of the result above is the following corollary:

Corollary 13 (all stabilizing delay controllers) *Assume that the conditions in Theorem 5 are satisfied for some positive real gain k, and assume further that* $\mathbf{card}(S_+) = 2$. *Then, all stabilizing delay controllers (2) defined by the gain k are characterized by* $\tau \in (\underline{\tau_l}, \overline{\tau_l})$, $l = 0, 1, 2, \ldots l_m$, *where:*

$$\underline{\tau_l} = \tau_- + \frac{2\pi l}{\omega_-}, \overline{\tau_l} = \tau_+ + \frac{2\pi l}{\omega_+},$$

and l_m *is the largest integer to satisfy* $\underline{\tau_l} < \overline{\tau_l}$, *which can be explicitly expressed as*

$$l_m = \max_{l \in \mathbf{Z}} \left\{ l \leq \frac{\omega_+ \omega_-}{2\pi} \cdot \frac{\tau_+ - \tau_-}{\omega_+ - \omega_-} \right\}. \tag{12}$$

Remark 14 *Corollary 13 is a natural extension of the stabilization problem for second-order* oscillatory *systems using a delay, and a positive gain (see, for instance, [1] or the illustrative example treated below). Note also the distinct approaches proposed in [3] (discrete counterpart of some derivative controllers) and [15] (crossing directions based argument, but without any deep analysis of the mechanism) for deriving the corresponding delay intervals.*

Remark 15 *The results above are still valid for the transfer function of the form:*

$$H_{yu}(s) = c^T(sI_n - A)^{-1}be^{-s\tau_1} = \frac{P(s)}{Q(s)}e^{-s\tau_1}, \tag{13}$$

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with the difference that we may have two distinct situations. First, for a given k, the input delay τ_1 belongs to the corresponding stabilizing delay interval $(\underline{\tau}, \overline{\tau})$, and then any control law: $u(t) = -ky(t-\tau_2)$, with $\tau_2 \in [0, \overline{\tau} - \tau_1)$ is a stabilizing control law.
Second, if $\tau_1 \notin (\underline{\tau}, \overline{\tau})$, one needs to capture one stabilizing delay interval including τ_1. Note that in this second case, if τ_1 is very large, such a delay interval may not exist.

3.2 Time-varying uncertainty

The next step is to assume that the delay in the input may be subject to *time-varying uncertainty*, and then to analyze the *stability robustness* of the corresponding scheme.
In conclusion, based on the results above, consider a stabilizing pair (k, τ) for the transfer $H_{yu}(s)$ satisfying the constraints in Theorem 5, and let $(\underline{\tau}, \overline{\tau})$ be the corresponding stabilizing delay interval.

Assume now the existence of a time-varying delay uncertainty in the control law, that is:

$$u(t) = -ky(t - \tau - \delta(t)), \tag{14}$$

where $\delta(\cdot)$ is a continuous time-varying bounded function, with bounded derivative:

$$0 \leq \delta(t) \leq \overline{\delta}, \dot{\delta}(t) \leq \rho < 1. \tag{15}$$

In the sequel, we are interested in deriving bounds on ε and β, such that the closed-loop system is uniformly asymptotically stable.

If (A, b, c^T) is a state-space representation of the transfer $H_{yu}(s)$, then the closed-loop system can be rewritten in time-domain as follows:

$$\dot{x}(t) = Ax(t) - kbc^T x(t - \tau - \delta(t)), \tag{16}$$

with $x \in \mathbf{R}^n$, $A \in \mathbf{R}^{n \times n}$, and $b, c \in \mathbf{R}^n$. Based on the remarks above, it follows that the system without uncertainty:

$$\dot{x}(t) = Ax(t) - kbc^T x(t - \tau), \tag{17}$$

is asymptotically stable for all $\tau \in (\underline{\tau}, \overline{\tau})$, where the bounds are given by Theorem 5.

The next step is the use of a classical *model transformation* (see, for instance, [13]) of the original system (16) by integrating over the delay interval $[t - \tau - \delta(t), t - \tau]$. Thus, (16) rewrites as follows:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + bkc^T x(t - \tau) \\ &\quad - bkc^T \int_0^{\delta(t)} \frac{\partial}{\partial t} x(t - \tau - \theta) d\theta \\ &= Ax(t) + bkc^T x(t - \tau) - bkc^T A \int_0^{\delta(t)} x(t - \theta) d\theta \\ &\quad - (bkc^T)^2 \int_0^{\delta(t)} x(t - \tau - \theta - \delta(t - \tau - \theta)) d\theta.\end{aligned} \tag{18}$$

The above system can be written as one with time-invariant delay subject to uncertain feedback [7]:

$$\dot{x}(t) = Ax(t) + bkc^T x(t - \tau) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \cdot \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix}, \tag{19}$$

where: $b_1 = bkc^T$, $b_2 = bkc^T$, and:

$$u_1(t) = -A \int_0^{\delta(t)} x(t - \theta) d\theta, \tag{20}$$

$$u_2(t) = -bkc^T \int_0^{\delta(t)} x(t - \tau - \theta - \delta(t - \tau - \theta)) d\theta. \tag{21}$$

The equation (19) represents the forward part of the system, and (20)-(21) can be seen as dynamic uncertain feedback. Let us estimate the gain of u with respect to x in the feedback (20)-(21). Using the Jensen inequality combined with the Hölder inequality, we get:

$$\begin{aligned}&\int_0^t u_1(\theta)^T u_1(\theta) d\theta \leq \int_0^t \delta(\theta) \cdot \\ &\quad \cdot \left[\int_0^{\delta(\theta)} x(\theta - \xi)^T (A^T A) x(\theta - \xi) d\xi \right] d\theta,\end{aligned} \tag{22}$$

$$\begin{aligned}&\int_0^t u_2(\theta)^T u_2(\theta) d\theta \leq \int_0^t \delta(\theta) \cdot \\ &\quad \cdot \left[\int_0^{\delta(\theta)} x(\theta - \tau - \xi - \delta(\theta - \tau - \xi))^T (bkc^T ckb^T) \cdot \right. \\ &\quad \left. \cdot x(\theta - \tau - \xi - \delta(\theta - \tau - \xi)) d\xi \right] d\theta.\end{aligned} \tag{23}$$

Simple computations prove that:

$$\int_0^t u_1(\theta)^T u_1(\theta) d\theta \leq \varepsilon^2 \|A\|^2 \int_0^t x(\theta)^T x(\theta) d\theta, \tag{24}$$

where we used Fubini theorem [17] (under zero initial conditions).
Define $\nu(\eta) = \eta - \delta(\eta)$. Since $\dot{\delta}(\cdot)$ is bounded by ρ, it follows that the inverse function $\eta = \eta(\nu)$ is uniquely defined, and:

$$\frac{\partial \eta}{\partial \nu} = \frac{1}{1 - \dot{\delta}(\eta)} \leq \frac{1}{1 - \rho}. \tag{25}$$

Also due to the range of δ, we can easily verify that

$$\nu \leq \eta(\nu) \leq \nu + \varepsilon \tag{26}$$

A change of integration variable from ξ to $\nu = \nu(\theta - \tau - \xi)$ yields

$$\begin{aligned}&\int_0^t u_2(\theta)^T u_2(\theta) d\theta \leq \int_0^t \delta(\theta) \cdot \\ &\quad \cdot \int_{\nu(\theta - \tau - \delta(\theta))}^{\nu(\theta - \tau)} \frac{1}{1 - \rho} x(\nu - h_k)^T (bkc^T ckb^T) x(\nu - \tau) d\nu d\theta \\ &\leq \frac{\varepsilon^2 k^2 (c^T b)^2}{1 - \rho} \int_0^t x(\theta)^T x(\theta) d\theta,\end{aligned} \tag{27}$$

where the last condition was derived using Fubini theorem (we assume zero initial conditions).
The system can be written in the following form:

$$\left\{ \begin{array}{l} \dot{x}(t) = Ax(t) + bkc^T x(t - \tau) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix} \\ y_i(t) = \varepsilon c_i x(t), \quad i = 1, 2, \end{array} \right. \tag{28}$$

under the feedback:

$$u_i = \Delta_i y_i, \quad 1 \leq i \leq 2 \tag{29}$$

where c_i, $i = 1, 2$ are given by:

$$c_1 = \|A\|, c_2 = \frac{kc^T b}{\sqrt{1 - \rho}}. \tag{30}$$

the gain of the "uncertainty" Δ_i is bounded by 1.

In conclusion, we have the following result:

Theorem 16 *The original closed-loop system (16) is uniformly asymptotically stable for all pairs (k,τ) satisfying Theorem 5, and for any time-varying delay uncertainty $\delta(t)$ satisfying (15), if there exists scalars α_i, $i = 1,2$ such that:*

$$\|\Lambda H(j\omega)\Lambda^{-1}\|_\infty < \frac{1}{\varepsilon}, \qquad (31)$$

where: $\Lambda = \mathbf{diag}(\alpha_1 I_n, \alpha_2 I)$, *and:*

$$H(s) = \begin{bmatrix} c_1 I_n \\ c_2 I_n \end{bmatrix} \left(sI - A - bkc^T e^{-s\tau}\right)^{-1} \begin{bmatrix} b_1 & b_2 \end{bmatrix}. \qquad (32)$$

Remark 17 (Lyapunov approach) *A different way to handle the time-varying delay uncertainty was proposed in [11] using a time-domain approach based on an appropriate Liapunov-Krasovskii functional construction.*

Remark 18 (ε bounds) *The condition (31) above can be also used to* estimate *a bound $\varepsilon_{max} > 0$ of the uncertain time-varying delay $\delta(t)$, such that the closed-loop stability is guaranteed. This bound on ε can be seen as a* measure *of the degree of robustness of the corresponding delay system,* etc.

4 Illustrative example

Consider the following second-order system:

$$H_{yu}(s) = \frac{1}{s^2 - \alpha s + 2}, \qquad (33)$$

with $\alpha \in [0, 2\sqrt{2})$ a real parameter. Simple computations prove that the polynomial $Q(s) = s^2 - \alpha s + 2$ is unstable, and for all $k \in \mathbf{R}$, the polynomial $Q(s) + kP(s)$ has at least one unstable root. Furthermore, if $\alpha = 0$, then $H_{yu}(s)$ in (33) corresponds to an oscillator (characteristic equation has two roots on the imaginary axis). The upper bound $\alpha = 2\sqrt{2}$ corresponds to a double positive root of $Q(s)$, and for all $\alpha \in (0, 2\sqrt{2})$, $Q(s)$ has two complex conjugate strictly unstable roots.
Choose the controller $u(t) = ky(t-\tau)$. The corresponding characteristic equation of the closed-loop system is:

$$s^2 - \alpha s + 2 - ke^{-s\tau} = 0. \qquad (34)$$

The polynomial $F(\omega)$ is:

$$\begin{aligned} F(\omega) &= |\, Q(j\omega)\,|^2 - |\, P(j\omega)\,|^2 = (2-\omega^2)^2 + \alpha^2\omega^2 - k^2 \\ &= \omega^4 - (4-\alpha^2)\omega^2 + (4-k^2). \end{aligned} \qquad (35)$$

It is clear that if $\alpha \in [2, 2\sqrt{2})$, then $\mathbf{card}(S_+) \leq 1$ for all real k, and, in conclusion, there does not exist any (gain,delay) pair which stabilizes $H_{yu}(s)$. Some straightforward computations prove that the condition $\mathbf{card}(S_+) = 2$ requires: $\alpha < 2$, and $k \in \left(2\sqrt{1-\left(1-\frac{\alpha^2}{4}\right)^2}, 2\right)$. Based on Corollary 13, the condition above is also sufficient.

Proposition 19 *For any pair (α, k) of positive numbers satisfying the constraints:*

$$\alpha \in [0,2), k \in \left(2\sqrt{1-\left(1-\frac{\alpha^2}{4}\right)^2}, 2\right), \qquad (36)$$

there exists a delay interval $(\underline{\tau}, \overline{\tau})$ including τ, such that the controller $u(t) = ky(t-\tau)$ stabilizes $H_{yu}(s)$. The bounds $\underline{\tau} = \tau_-$ and $\overline{\tau} = \tau_+$ are given by:

$$\tau_\pm = \frac{1}{\omega_\pm}\left[arctan\frac{\alpha\omega_\pm}{2-\omega_\pm^2} + \varepsilon_{\omega_\pm}\pi\right], \qquad (37)$$

where:

$$\omega_\pm = \sqrt{1-\frac{\alpha^2}{4}} \cdot \sqrt{2\left(1 \pm \sqrt{1 - \frac{1-\frac{k^2}{4}}{\left(1-\frac{\alpha^2}{4}\right)^2}}\right)}, \qquad (38)$$

and:

$$\varepsilon_{x\in\mathbf{R}} = \begin{cases} 0, & \text{if } x^2 \leq 2 \\ 1, & \text{otherwise.} \end{cases} \qquad (39)$$

Remark 20 *It is easy to see that for all $\alpha \in [0,2)$, $\omega_-^2 < 2$, but $\omega_+^2 < 2$ only for $\alpha \in [\sqrt{2}, 2)$. Thus, for all $\alpha \in [0, \sqrt{2})$, $\omega_+^2 \geq 2$, which explain the term $\varepsilon_{\omega_\pm}$ given by (39) in the definition of $\tau_\pm$.*

Remark 21 (Stabilizing oscillations) *If $\alpha = 0$, then $k \in (0,2)$, and we recover the results proposed in [1, 15, 3]:*

$$\tau_- = 0, \tau_+ = \frac{\pi}{\sqrt{2+k}}.$$

Furthermore, the number of delay intervals is given by:

$$\max_{l\in\mathbf{Z}}\left\{ l \leq \frac{1}{2} \cdot \frac{1}{\sqrt{\frac{2+k}{2-k}} - 1} \right\}.$$

Roughly speaking, the smaller the gain is, the smaller the number of stabilizing delay intervals is, property coherent with the graphical representation in [1], etc.

Remark 22 *Simple computation proves that choosing $\alpha = 1$, and $k = \sqrt{2}$, which belongs to the interval $\left(\frac{\sqrt{7}}{2}, 2\right)$ defined by Proposition 19, we get:*

$$\omega_- = 1, \omega_+ = \sqrt{2}.$$

Then, the first delay interval guaranteeing closed-loop asymptotic stability is:

$$\tau \in \left(\frac{\pi}{4}, \frac{\pi}{2\sqrt{2}}\right).$$

Furthermore, using Corollary 13, it follows that there does not *exist* other delay intervals *guaranteeing the closed-loop asymptotic stability.*

5 Concluding remarks

This note was devoted to the stabilization problem of a class of SISO systems subject to output delayed feedback. More precisely we considered the problem where the delay in the control law may induce a *stabilizing* effect, that is the closed-loop stability is guaranteed due to the delay existence. *Necessary and sufficient* conditions have been derived using a frequency-domain approach. Furthermore, we considered also the related *robustness* problem with respect to the delay terms (parametric and time-varying uncertainty). A simple illustrative example was also proposed.

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