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Delay Effects on Static Output Feedback Stabilization

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Abstract

This paper addresses conditions for characterizing static output feedback controllers including delays for some proper (finite-dimensional) transfer functions. The interest of such study is in controlling systems which can not be stabilized by the classical, non-delayed static output feedback, and its difficulty lies in computing delay intervals guaranteeing closed-loop stability, since stability switches/reversals may occur for the same (matrix) gain if the delay is seen as a 'free' (design) parameter. The derived conditions are expressed in terms of some appropriate matrix pencils or MIMO Nyquist tests. Illustrative examples are also presented.

Keywords: Stability, delay switches/reversals, matrix pencils, Nyquist.

1 Introduction and Problem Formulation

In this paper, we consider mainly the following:
Problem 1: Given a strictly proper transfer function
\[ H(s) \in \mathbb{C}^{p \times m}, (p, m \geq 1) \] with a state-space representation \((u \in \mathbb{R}^m, y \in \mathbb{R}^p, x \in \mathbb{R}^n)\):
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{align*}
\]

find all pairs \((K, \tau) \in \mathbb{R}^m \times \mathbb{R}^+\) such that the static delayed output feedback \(u(t) = Ky(t - \tau)\) stabilizes the system (1.1).

It is clear that for \(\tau = 0\), we have the 'classical' static output feedback problem, which has been thoroughly studied (see for example [31] and the references therein). We are interested in introducing a delay in the control law of the class of systems (1.1) for which the static output stabilization fails. It would seem that the class of the transfer functions which are closed-loop asymptotically stable may become larger if one uses infinite-dimensional controllers, for stabilizing finite-dimensional systems.

In practice, the delay effects on the system's phase may be sufficient in some cases to guarantee the stability of the control scheme in closed-loop (see, for example, in the stabilization of some oscillatory systems). Indeed, consider the following SISO system:
\[ y(t) = H(s) \cdot u(t), \quad H(s) = \frac{1}{s^2 + \omega_0^2}, \]
with \(\omega_0 > 0\). It is clear that any feedback \(u(t) = ky(t), k \in \mathbb{R}\) does not stabilize the system. However, the control \(u(t) = ky(t - \tau), k > 0\) may ensure the stability for sufficiently "small" delays [1, 24]. Even for such a simple system, the behavior with respect to the pair \((k, \tau)\) is very complicated: if \(\tau\) is thought of as a parameter, then for a fixed 'gain' \(k\) we may have a sequence of stability and instability regions in the parameter space \((k, \tau)\).

Although the destabilizing effect of a delay in a system model is well known in the control literature, see, e.g. the reference list in [24], the 'switch' from instability to stability (called also reversal, see, e.g. [8]) has not been sufficiently addressed. Note however that, to the best authors' knowledge, it was first discussed by Minorsky [23] in the 40s for a second-order (delayed) friction equation. Further comments and remarks on delayed oscillatory systems can also be found in [3, 12].

A different but related problem to problem 1 may be stated as follows:
Problem 2: Assume that (1.1) can be stabilized by a static output feedback, then we want to know how robust is the closed-loop stability with respect to the delay.

It seems natural that, for 'small gains', it is possible to ensure the closed-loop stability for any delay value, i.e. stability is a delay-independent property, and for 'large gains', stability may be guaranteed only in the first-delay interval \([0, \tau(K)]\), i.e. it is a delay-dependent property. The existence of other delay intervals guaranteeing stability in the closed-loop system is also analyzed in this paper, along with several robust-stability existence results.

The proposed approach is based on generalized eigenvalues distribution with respect to the unit circle of some appropriate matrix pencils associated to the system. The use of matrix pencils for characterizing the existence of static output feedback controllers for delay-free systems was already considered in the literature [18, 8]. Some constructive procedures were presented in [9]. Furthermore, matrix pencils techniques...
in the stability of time-delay systems is intimately related to the development of 2D stability analysis [6] in the commensurate delay case [22, 16] (variables on the imaginary axis, and on the unit circle in the complex plane), and to the linearization of some matrix polynomials [19, 20] after 'reducing' one of the variables [10, 24, 25, 26] (the variable on the imaginary axis). Note that such variable-reducing ideas were already encountered in the 1960s in [2] in a different framework.

Due to the difficulty of the general design problem, we shall analyze in this paper the analysis problem by considering delay-intervals ensuring closed-loop stability for a given, known $K$. Some remarks on the case $K = kI_m$ ($m = p$) with $k$ free are also included using the above approach, but also the MIMO Nyquist theorem. The notations are standard, except when otherwise noted.

2 Preliminary Results and Definitions

For a given $K \in \mathbb{R}^{m \times m}$, system (1.1), and the corresponding controller $u(t) = KY(t - \tau)$, define the following matrix pencils $\Sigma_1 \in \mathbb{C}^{p \times 2p}$, $\Sigma_2 \in \mathbb{C}^{n \times n}$:

$$\Sigma_1(z, K) = z \begin{bmatrix} I_p & 0 \\ 0 & \phi_\theta(BKC, I_n) \end{bmatrix} + \begin{bmatrix} \phi_\theta(I_n, (BKC)^T) & -I_p \\ \phi_\theta(A, AT)^T \end{bmatrix}, \quad (2.2)$$

$$\Sigma_2(z, K) = zBKC + A. \quad (2.3)$$

where $\phi_\theta, \phi_{\theta'} : \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \to \mathbb{R}^{p \times p}$ are defined as follows: For all $P, Q \in \mathbb{R}^{n \times n}$,

$$p = n^2: \quad \frac{\phi_\theta(P, Q)}{\phi_\theta(P, Q)} = P \circ Q, \quad (2.4)$$

$$p = \frac{n(n - 1)}{2}: \quad \frac{\phi_\theta(P, Q)}{\phi_\theta(P, Q)} = P \circ Q. \quad (2.5)$$

The symbols $\circ$ and $\circ'$ are the product and the sum of Kronecker respectively, and the symbols $\circ$ and $\circ'$ are defined as follows:

$$P \circ Q = [c_{ij}] \in \mathbb{R}^{p \times p}, \quad \text{where}$$

$$c_{ij} = \frac{1}{2} (p_{1i1}q_{1j1} + p_{1i2}q_{1j2} - p_{1i1}q_{1j2} - p_{1i2}q_{1j1}),$$

with $(i_1, i_2)$ the ith pair of the sequence $(1, 2), (1, 3), \ldots, (1, n), (2, 3), \ldots, (2, n), \ldots, (n, n)$ and $(j_1, j_2)$ is generated by duality. For $P \circ Q$, we use the classical definition of the Kronecker sum:

$$P \circ Q = P \circ I_n + I_n \circ Q.$$ 

Using the same arguments as in [24, 25], we may prove the following result:

Lemma 1 The following are true:

1) The complex number $z \in \mathbb{C}^*$, $|z| \neq 1$ is a generalized eigenvalue of the matrix pencil $\Sigma_1$ if and only if $z^{-1}$ is an eigenvalue of $\Sigma_2$.

2) All the generalized eigenvalues on the unit circle of the matrix pencil $\Sigma_2$ are also eigenvalues on the unit circle for $\Sigma_1$.

3 Existence Results

Based on the continuity property with respect to the delay of the roots of the closed-loop characteristic equation [14], we may prove several existence stability results.

Denote $\sigma_i(K) = \sigma(\Sigma_i(z, K))$ ($i = 1, 2$) the set of generalized eigenvalue of the matrix pencil $\Sigma_i(z, K)$, and let $\sigma(K) = \sigma_1(K) - \sigma_2(K)$ denote the set of points in $\sigma_1(K)$ but not in $\sigma_2(K)$. Assume also that $\Sigma_1$ are regular matrix pencils (see [28] for the exact meaning of this assumption).

Then we have the following results (see [28] for the complete proofs):

Proposition 1 (delay-independent) Consider a gain matrix $K$ such that $\Sigma_1$ is regular, and such that $\sigma = \emptyset$ on the unit circle of the complex plane. Then, the following assertions are equivalent:

i) The static output feedback $u(t) = KY(t)$ is a stabilizing law for (1.1);

ii) The static output feedback $u(t) = KY(t - \tau)$ is a stabilizing law for (1.1) for any delay value $\tau$.

Remark 1 (Generalized eigenvalue distribution) Note that if the matrix pencil $\Sigma_1$ has no eigenvalues on the unit circle, then it is dichotomically separable with respect to the unit circle [25, 24]. Such condition guarantees $\sigma = \emptyset$, but it is not sufficient to guarantee the delay-independent type property, as remarked in [25]. In fact, $\Sigma_1$ may have generalized eigenvalues on the unit circle, but these generalized eigenvalues should be identical to those of $\Sigma_2$.

Remark 2 (Strong/weak delay-independent) In terms of system's parameters, the common generalized eigenvalues of $\Sigma_1$ and $\Sigma_2$ ($\sigma(K)$ should be non-empty) generate hypersurfaces in the parameter-space, which have to be included in the corresponding stability regions, since they are related, in some sense, to the case $\tau \to +\infty$. Such aspect was pointed out in [7], and exploited in [24]. One may differentiate strong and weak delay-independent stability notions, by including or not the corresponding hypersurfaces in the stability regions [24] (and the references therein). In this framework, the results proposed in [22] are strong, and the results in [16] are weak delay-independent, etc. Note that the strong stability notion allows a complete decoupling of the complex variables between those on the imaginary axis, and on the unit circle, respectively. The weak stability notion allows that 0 is an accumulation point, in some sense, for generalized eigenvalues (continuously depending on the delay values in the Dukić's sense [14]), if the delay $\tau \to +\infty$. Such problems are better explained in a hyperbolicity framework (see also [25]).
Remark 3 In conclusion, if $\sigma(K) = \emptyset$ on the unit circle of the complex plane, the static delayed output feedback does not improve the closed-loop stability with respect to the delay-free case. The problem is reduced to a static output feedback problem [31].

Proposition 2 (first-delay interval) Consider a gain matrix $K$ such that $\Sigma_1$ is regular, and such that $\sigma \neq \emptyset$ on the unit circle of the complex plane. Then, the following assertions are equivalent:

i) The static output feedback $u(t) = Ky(t)$ is a stabilizing law for (1.1);

ii) The static output feedback $u(t) = Ky(t - \tau)$ is a stabilizing law for (1.1) for any delay value, $\tau \in (0, \pi(K))$, where:

$$\pi(K) = \min_{1 \leq \alpha \leq \pi(K)} \frac{\alpha k}{\omega_k} > r.$$ 

Consider the closed-loop system of the strictly proper transfer function $H(s) \in \mathbb{C}^{m \times m}$ with the feedback $u(t) = Ky(t - \tau)$ for some gain $K \in \mathbb{R}^{m \times p}$, and some positive $r > 0$. Introduce now the sets:

$$\Lambda_{r,+,K} = \left\{ (\tau_k, \alpha_k) : \tau_k - \frac{\alpha_k}{\omega_k > r} : e^{-j\alpha_k} \in \sigma(K), j\omega_k \in \sigma(A + e^{-j\alpha_k}(BKC)) \right\}$$

$$\Lambda_{r,-,K} = \left\{ (\tau_k, \alpha_k) : \tau_k - \frac{\alpha_k}{\omega_k < r} : e^{-j\alpha_k} \in \sigma(K), j\omega_k \in \sigma(A + e^{-j\alpha_k}(BKC)) - \{0\} \right\}$$

The main (existence) result may be written as follows:

Proposition 3 (general delay-intervals) The strictly proper transfer function $H(s)$ can be stabilized by delayed output feedback of the form $u(t) = Ky(t - \tau)$ on the delay interval $(\pi, \pi)$ if and only if:

i) it can be stabilized by the same law for some delay $\tau_0$ in the same interval, and

ii) the following inequalities hold simultaneously:

$$\sup \{ \tau : (\tau, \alpha) \in \Lambda_{r,-,K} \} \leq \tau_0$$

$$\inf \{ \tau : (\tau, \alpha) \in \Lambda_{r,+,K} \} \geq \pi.$$ 

Theorem 1 Let $H(s)$ be a square transfer matrix with $P_1$ unstable Smith-Macmillan poles. Let $\lambda_i(s) : i = 1, \ldots, m$ be the eigenvalues of $G(s)$. Then, the closed-loop system with feedback input $u = -Ky(t - \tau)$
with $\mathcal{K} = kI$ is stable if and only if the graphs of $e^{-\tau \rho} \lambda_i(s); i = 1, \ldots, n$ taken together encircle the -1 point, $\rho_0$ times in the counterclockwise direction.

5 Examples

We shall apply the previous results to three different examples.

5.1 Stabilizing oscillations using delays

Consider a simple second-order oscillatory system [1]:

$$ g(t) + \omega_0^2 y(t) = u(t), \quad (5.12) $$

with $\omega_0 \in \mathbb{R}^+$. As specified, it is not possible to stabilize it by static delayed output feedback of the form:

$$ u(t) = k_2 (t - \tau), \quad (5.13) $$

By taking

$$ A = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ +\omega_0 \end{bmatrix}, \quad (5.14) $$

we may apply directly the proposed approach, and thus, we have the following stabilization result:

**Proposition 5** The system (5.12) can be stabilized by delayed output feedback $u(t) = k_2 (t - \tau)$ for all $k(k, \tau)$ satisfying simultaneously:

i) the gain $k \in (0, \omega_0^2)$, and

ii) the delay $\tau \in (\tau_i(k), \tau_r(k))$ where:

$$
\begin{align*}
\tau_i(k) &= \frac{2\pi}{\sqrt{\omega_0^2 + k}} \\
\tau_r(k) &= \frac{(2i + 1)\pi}{\sqrt{\omega_0^2 + k}}
\end{align*}
$$

for $i = 0, 1, \ldots$. Furthermore, if $\tau = \tau_i(k)$ or $\tau = \tau_r(k)$, the corresponding characteristic equation in closed-loop has at least one eigenvalue on the imaginary axis.

The regions of stabilizing $k$ shrink as the delay $\tau$ gets larger, and furthermore for each $k$ there exists a value $\tau^*(k)$, such that for any $\tau > \tau^*(k)$ the closed-loop system is unstable.

So, if we take $i = 0$, we see that the first delay interval guaranteeing the closed-loop stability is given by:

$$ \tau \in \left(0, \frac{\pi}{\sqrt{\omega_0^2 + k}}\right). $$

Using a different argument to those already proposed in the paper, let us prove that for sufficiently small delays $\tau = \epsilon > 0$ the closed-loop system is stable for any $k \in (0, \omega_0^2)$. Consider the characteristic equation associated to the system:

$$ s^2 + \omega_0^2 - ke^{-\tau\epsilon} = 0. \quad (5.15) $$

If $\tau = 0$, the corresponding roots are on the imaginary axis $s = \pm j\omega_0$. Consider now a delay $\tau = \epsilon > 0$. Simple computations give:

$$ \frac{ds}{d\tau} = -\frac{s(s^2 + \omega_0^2)}{\epsilon(s^2 + \omega_0^2) + 2s}, \quad (5.16) $$

Since the roots on the imaginary axis of (5.15) are simple $\omega = \sqrt{\omega_0^2 \pm k}$, it follows that the crossing direction of the roots (from left to right, or from right to left) is given by the sign of $Re\left(\frac{ds}{d\tau}\right)$ when $s = j\omega$: "+1" from stability to instability, and "-1" from instability to stability (Note that the condition on the simplicity of the roots is necessary, since if not the corresponding derivative will be 0, etc.).

Simple computations in (5.16) lead to the following:

$$ Re\left(\frac{ds}{d\tau}\right) = -\frac{\omega^2(k^2 - \omega_0^2)}{4\omega^2 + \epsilon^2(\omega_0^2 - \omega^2)^2}, $$

which is always negative, for sufficiently small value $\epsilon > 0$, and for any $k \in (0, \omega_0^2)$, etc.

**Remark 5** The same results can be obtained using different approaches, as for example, the study of the corresponding characteristic equation [21], or using Nyquist criterion [1].

**Remark 6** A similar analysis can be done if we assume that $\epsilon < 0$. Note that for such situation the system is still unstable for sufficiently small delay values $\tau = \epsilon > 0$, and any $k < 0$, $|k| < \omega_0^2$. However, it will be stabilized on some delay intervals, etc.

5.2 Delay measurements in active displacement

In active displacement control (flexible structures), a time delay $\tau$ always exists between measuring the deflection and applying the active displacement feedback. Since the corresponding delay-free, closed-loop model is, in general, stable, the problem is to study the delay effects on the closed-loop stability, with respect to the two parameters: a) the (point or lumped) delay; b) the gain of the active displacement feedback.

Based on the study proposed in [30], the stability (instability) problem can be reduced to the analysis of the following transcendental equation:

$$ s^2 + \mu_n s + \lambda_n + ke^{-\tau\epsilon} = 0, \quad (5.17) $$

with $\mu_n, \lambda_n$ (associated eigenvalues corresponding to some orthogonal eigenvectors of some self-adjoint operators, etc.) and $k$ (displacement control feedback parameter) positive.

Some algebraic manipulations combined with the existence results given above lead to the following results:

**Proposition 6** (Delay-independent results) The following statements are equivalent:

i) The system (5.17) is delay-independent asymptotically stable.

ii) The parameters $(\mu_n, \lambda_n, k)$ satisfy the following con-
If the conditions above are not satisfied, the only possibility to have is a delay-dependent type result. We shall consider two cases: only one switch (with no reversal), and several switches and reversal. The first case is already encountered in the scalar case, and the second one appears firstly with the second order systems. The proposed results can be summarized as follows (the proofs are in the full version of the paper [28]):

**Proposition 7 (Only one switch)** The following statements are equivalent:
1) The system (5.17) is delay-dependent stable, and there exists only one switch from stability to instability without any reversal.
2) The parameters \((\mu_n, \lambda_n, k)\) satisfy the following constraints:
\[
\begin{align*}
\mu_n &\in \mathbb{R}^+ \\
\lambda_n &\in (2\mu_n^2, +\infty) \\
k &\in \left[-\lambda_n, 2\mu_n\sqrt{\lambda_n - \mu_n^2}\right) \\
&\cup (2\mu_n\sqrt{\lambda_n - \mu_n^2}, +\infty)
\end{align*}
\] (5.19)

**Proposition 8 (Delay bound)** Consider the second order system (5.17), and assume \(k > 0\). If the parameters \((k, \mu_n, \lambda_n)\) satisfy the constraints (5.19), then the system (5.17) is asymptotically stable for all delays \(\tau \in (0, \tau_{\text{switch}})\), and unstable for all delays \(\tau \in (\tau_{\text{switch}}, \tau_{\text{reversal}})\), where the delay switch \(\tau_{\text{switch}}\) is given by:
\[
\frac{1}{\omega_+} \arccos \left( \frac{\sqrt{k^2 + 4\mu_n^4 - 4\mu_n^2\lambda_n - 2\mu_n^2}}{k} \right)
\] (5.22)
and the delay reversal \(\tau_{\text{reversal}}\) is given by:
\[
\frac{1}{\omega_-} \arccos \left( \frac{-\sqrt{k^2 + 4\mu_n^4 - 4\mu_n^2\lambda_n - 2\mu_n^2}}{k} \right)
\] (5.23)
with:
\[
\omega_{\pm} = \sqrt{\lambda_n - 2\mu_n^2} \pm \sqrt{k^2 + 4\mu_n^4 - 4\mu_n^2\lambda_n}
\] (5.24)
At \(\tau \in \{\tau_{\text{switch}}, \tau_{\text{reversal}}\}\), the characteristic equation has two complex conjugate eigenvalues on the imaginary axis.

**Remark 7** If we take \(\mu_n = 0\), we recover the bounds proposed in §4.1 (stabilizing oscillations).

5.3 Integrodifferential models for commodity markets
In [4], the following distributed-discrete delay model \((R, Q, \tau > 0)\):
\[
\dot{z}(t) + \frac{Q}{R} \int_{-\infty}^{0} e^{R(t-\theta)} z(t+\theta) d\theta + z(t-\tau) = 0
\] (5.25)
has been used for describing interactions between consumer memory and price fluctuations on commodity markets. Simple computations prove that (5.25) has the same characteristic equation as the differential equation with discrete delays:
\[
\dot{z}(t) + \frac{1}{R} \dot{z}(t-\tau) + \frac{Q}{R} \tau(t) + \frac{1}{R} \tau(t-\tau) = 0
\] (5.26)
which is a second-order delay differential system. The analysis of the stability regions in the parameters space \((Q, R, \tau)\) may be easily transformed to an output static delay feedback problem:

Given the system \(y(s) = \frac{s + \frac{1}{R}}{u(s) = \frac{s^2 + \frac{1}{R}}{s^2 + \frac{1}{R}}},\) find all the delay-intervals \((\tau_i(Q, R), \tau_j(Q, R)), i = 0, \ldots,\) such that the closed-loop is stable via the output feedback \(u(t) = -y(t-\tau)\).

Using the results presented above the complete characterization of the stability regions becomes an easy task.
6 Conclusions

This paper presented a compilation of the effects of delay on the static output feedback problem. We have generalized our earlier research on the presence of stability "switches" using a matrix pencil approach and have laid the foundation for future work on multivariable static controllers. We have also provided numerous examples to illustrate the applicability of our results.

References


