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We introduce a general strategy for preparation of arbitrary quantum states via optimal control of repeated conditional measurements. The effectiveness of this strategy in generating finite Fock-state superpositions with a high level of confidence from experimentally accessible coherent states is demonstrated for the simple and well known Jaynes-Cummings model dynamics. [S1050-2947(96)07406-9]

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State preparation of quantum systems is a prerequisite for studying fundamental aspects of quantum measurement theory [1], as well as for encoding quantum information [2] and its processing (computing) [3]. Problems of state preparation have been dealt with most extensively in the realm of cavity quantum electrodynamics [4–8]. One proposal for state preparation of a cavity field mode [8] relies entirely on its *unitary evolution*, via coupling with a rather complex system, an atom having several Zeeman sublevels. This coupling results (under *perfectly adiabatic conditions*) in a one-to-one mapping of the initial sublevel superposition to a superposition of Fock (photon-number) states. Can one alternatively use a *simple* field-atom interaction, e.g., the resonant Jaynes-Cummings (JC) model [4] or the off-resonant Kerr-like interaction [5], followed by a measurement on the atom leaving the cavity, and repeat the process over and over again until the desired state of the field is attained?

In general, measurements of atomic observables after the interaction would yield *random* results for the prepared field state [4,5]. In order to prepare *predetermined* field states, the conditional measurement (CM) approach has been suggested [6]. In this approach, only those sequences of atoms in which each atom is found after the interaction to be in a chosen state are used to guide the field evolution to the desired state, whereas all other measurement sequences are discarded, at the price of atomic post selection probability, which is less than unity at each step of the sequence. The CM approach has been significantly enriched by a recipe for constructing an *arbitrary superposition* of Fock states [7]. It is based on a recurrence relation, which allows one to retract the desired superposition back to the starting vacuum state, by determining the possible initial atomic states and interaction times (in the JC model) at each step of the CM sequence. The practical restriction on this recipe is that the probability of the resulting CM sequences falls off rapidly with the maximal photon number in the superposition.

Our aim here is to address the basic questions of state preparation via quantum measurements: (a) Given a simple field-atom interaction, as in the JC model, and a choice of experimentally realizable initial field states (e.g., coherent states), can the field converge to any desired ''target'' state to within the required accuracy via a *finite* number of measurements? Hilbert space topology arguments supported by numerical calculations are given to show that such convergence is in general attained, provided that the number of control parameters per CM is comparable to the dimensionality of the target-state subspace. (b) How can one choose a CM sequence connecting the initial and ''target'' states, so as to maximize its success probability and minimize its length (the required number of CMs)? Although in principle it should be possible to choose the CM sequence with the highest ratio of probability to length, in practice such optimization amounts to the formidable task of a global search over a huge parameter space (whose dimensionality is the number of parameters per CM times the maximal admissible number of CMs). We demonstrate that there is a simple and computationally fast alternative, namely, stepwise optimization by search over the parameter space of one CM at a time, allowing one to choose a high-probability CM sequence from among those that converge to the target state monotonically.

Let us first formulate our strategy in general terms, suitable for any dynamical model. Suppose that we have started from the field state $|\psi_0\rangle=\Sigma_n C_{n,0}|n\rangle$. After $K-1$ CMs the field state is $|\psi_{K-1}\rangle=\Sigma_n C_{n,K-1}|n\rangle$ and we are trying to obtain the ''target'' state $|\psi_t\rangle=\Sigma_{n=n_{\min}}^{n\ \max} C_{n,t}|n\rangle$ via an optimal route in Hilbert space. Choosing the next atom to be in initial state $|\phi_K^{(i)}\rangle$, we unitarily evolve the initial field-atom product state $|\psi_{K-1}\rangle\otimes|\phi_K^{(i)}\rangle$ over time τ_K by the operator $U(\tau_K)$ and perform a CM by projecting the resulting entangled state onto a postselected final atomic state $|\phi_K^{(f)}\rangle$. This choice of initial and final atomic states and of τ_K corresponds to choosing a field-state *CM transformation*

$$|\psi_{K-1}\rangle\rightarrow|\psi_K\rangle=P_K^{-1/2}\langle\phi_K^{(f)}|U(\tau_K)[|\psi_{K-1}\rangle\otimes|\phi_K^{(i)}\rangle], \qquad (1)$$

where $P_K=|\langle\phi_K^{(f)}|U(\tau_K)[|\psi_{K-1}\rangle\otimes|\phi_K^{(i)}\rangle]|^2$ is the success probability of the corresponding CM.

The application of a sequence of CM transformations to an initial coherent state effects convergence to the target state via two processes taking place in parallel: (a) gradual elimination (*filtering out*) of all undesired Fock states $|n\rangle$ from the starting coherent state, while leaving the amplitudes

of the Fock states $n_{\min} \leqslant n \leqslant n_{\max}$ almost intact—this is achievable by adjusting the control parameters so as to leave the CM transformation *near unity* for $n_{\min} \leqslant n \leqslant n_{\max}$ and considerably less than unity for all other n (see below); (b) amplitude *fine tuning* in the Hilbert *subspace* spanned by Fock states $|n\rangle$, $n_{\min} \leqslant n \leqslant n_{\max}$, wherein the target state is contained—after the first few CMs the field state remains confined to this subspace by the filtering action of the first process.

The *physical states* within the above $(N = n_{\max} - n_{\min} + 1)$-dimensional Hilbert subspace are represented by unit norm vectors with global phase factors ignored. They constitute a manifold of $D_N = 2(N-1)$ real dimensions. An important part of the optimization is to fix a *distance* function on this manifold, in terms of which proximity to the target site is to be measured. A particularly useful family of such distances

$$d_{u,K} = \{u d_{\mathrm{mo},K}^2 + d_{\mathrm{ph},K}^2\}^{1/2} \tag{2}$$

is obtained by assigning an *adjustable* weight u to the *moduli distance* from the target

$$d_{\mathrm{mo},K} = \left\{ \sum_n \big| |C_{n,K}|^2 - |C_{n,t}|^2 \big|^2 \right\}^{1/2} \tag{3}$$

relative to the *phase-factor distance*

$$d_{\mathrm{ph},K} = \left\{ \sum_n \big| e^{i\chi_K} C_{n,K}/|C_{n,K}| - C_{n,t}/|C_{n,t}| \big|^2 \right\}^{1/2}. \tag{4}$$

The phase factor $e^{i\chi_K}$ is chosen to be such that $\chi_K + \arg(C_{n,K}) = \arg(C_{n,t})$ for a chosen n, thus eliminating the arbitrariness in the overall phases of $|\psi_K\rangle$ and $|\psi_t\rangle$. The current field state $|\psi_K\rangle$ can then be viewed as lying on the $(D_N - 1)$-dimensional hypersurface of states *equidistant* (in terms of d_u) from the target state [Fig. 1(a)]. The family of all possible CM transformations acting on the field state *includes* transformations close to the identity, whence it spans a *submanifold* containing $|\psi_K\rangle$, whose real dimensionality is, normally, the number of control parameters D_c [Fig. 1(a)].

Assuming that $D_c \leqslant D_N$, how can we achieve optimized convergence to the target? The answer is that, generically, the intersection of the spanned submanifold with the equidistant-state hypersurface, which occurs at $|\psi_K\rangle$, is *transversal* for any $D_c \geqslant 1$. Hence the spanned submanifold contains states *closer* to the target than $|\psi_K\rangle$, as well as states further away from it. Among those states *closer* to the target than $|\psi_K\rangle$, we may choose an *optimal* one, having high success probability P_{K+1} for the corresponding CM transformation. The chosen $|\psi_{K+1}\rangle$ minimizes a *cost function* G_{K+1}, which, albeit *arbitrary*, must increase with $d_{u,K+1}$ and be reduced with success probability P_{K+1}. The simplest choice, which has proven to be highly effective is

$$G_{K+1} = \frac{d_{u,K+1}^2}{P_{K+1}^r}. \tag{5}$$

Here the adjustable exponent $r > 0$ determines the sensitivity to success probability: The maximization of P_{K+1} is most drastic when r is large, but excessive r values render G_{K+1} insensitive to distance, thus hampering convergence.

Repeating this process, again and again, we obtain a sequence of field states with *monotonically* decreasing distances to the target along with maximized probabilities. This holds true for arbitrary D_N and $D_c \geqslant 1$, but as D_c increases, so does the choice of optimized states $|\psi_K\rangle$, allowing higher CM success probabilities. Moreover, when D_c approaches D_N we may expect the residual distance $\lim_{k\to\infty} d_{u,K}$ to be small, allowing good proximity to the target.

We shall now apply this general optimization strategy to experimental setups which realize the resonant JC model [4], where *convergence* is *successful* for $N \leqslant 4$ $(D_N \leqslant 6)$ since $D_c = 5$ (see below). For convenience, the quantum states of the cavity field and atomic systems are written in the *interaction picture*.

(1) The first stage of the considered setup is the atomic preparation region: each atom enters the region in its ground state $|g\rangle$ and with controllable velocity, then interacts with classical fields which transform $|g\rangle$ into a superposition of excited and ground states with two independent parameters

$$|\phi_K^{(i)}\rangle = \alpha_K^{(i)}|e\rangle + \beta_K^{(i)}|g\rangle, \tag{6}$$

the superscript signifying the initial atomic state.

(2) The second stage of the setup is the cavity, in which the atom interacts for a time τ_K (determined by its velocity) with the resonant field mode. The starting field state $|\psi_0\rangle = \Sigma_n C_{n,0}|n\rangle$ is a coherent (or squeezed) state prepared by means of a classical oscillator that is phase locked (synchronized) with the resonant classical field in stage 1. The second stage is described by the unitary evolution operator of the resonant JC model, $U(\tau_K)$, which entangles the field and atom states according to

$$\begin{aligned} |n\rangle|e\rangle &\to \cos\theta_{n,K}|n\rangle|e\rangle - i\sin\theta_{n,K}|n+1\rangle|g\rangle, \\ |n\rangle|g\rangle &\to \cos\theta_{n-1,K}|n\rangle|g\rangle - i\sin\theta_{n-1,K}|n-1\rangle|e\rangle, \end{aligned} \tag{7}$$

where $\theta_n = \lambda\tau_K\sqrt{n+1}$, 2λ being the vacuum Rabi frequency (and $|-1\rangle$ is formally defined to be zero).

(3) Upon exiting the cavity, the atom encounters the last stage, in which it interacts again with classical fields. This last stage is aimed at projecting the entangled field-atom state onto the atomic state

$$|\phi_K^{(f)}\rangle = \alpha_K^{(f)}|e\rangle + \beta_K^{(f)}|g\rangle. \tag{8}$$

Here $\alpha_K^{(f)}$ and $\beta_K^{(f)}$ are the final (postselected) counterparts of the initial atomic parameters in Eq. (6). The classical fields in this stage are so set as to unitarily transform the state $|\phi_K^{(f)}\rangle$ into $|g\rangle$, whence successful detection of the atoms in the state $|g\rangle$ then corresponds to projection onto $|\phi_K^{(f)}\rangle$, whereas detection of the $|e\rangle$ state corresponds to projection onto the state orthogonal to $|\phi_K^{(f)}\rangle$.

The three consecutive stages described above: preparation of $|\phi_K^{(i)}\rangle$, then resonant JC unitary evolution $U(\tau_K)$, followed by projection onto $|\phi_K^{(f)}\rangle$, effect the field-state transformation

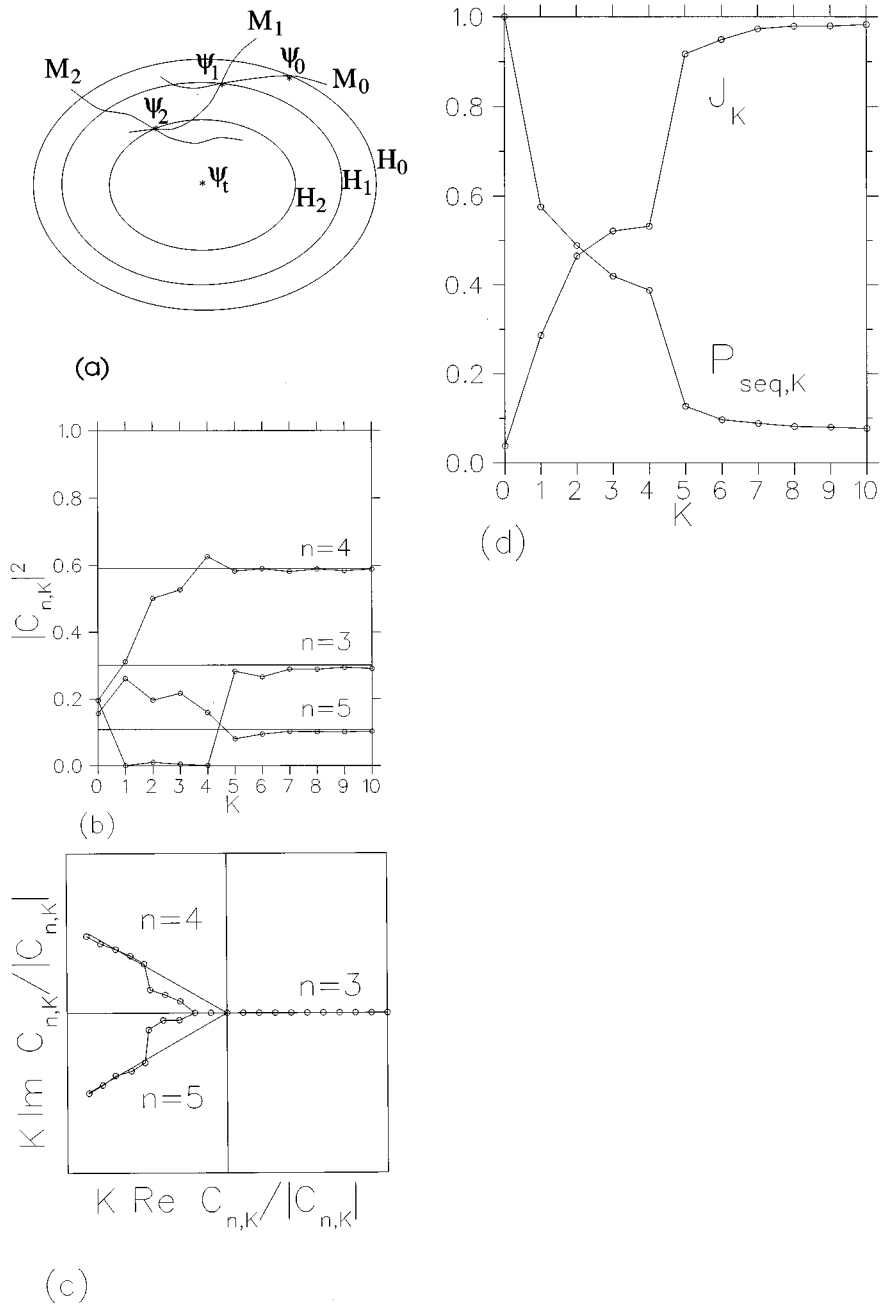

FIG. 1. (a) Schematic Hilbert-space map of the optimized convergence to the target state ψ_t. The successive field states $\psi_0,\ldots,\psi_2$, reached in the process of optimization, lie on the hypersurfaces $H_0,\ldots,H_2$ of equidistant points. $M_0,\ldots,M_2$ are the submanifolds spanned by CM transformations applied to $\psi_0,\ldots,\psi_2$. (b)–(d) Optimized convergence of field state as a function of the number of CMs, K (dots), to target state $|\psi_t\rangle=(5|3\rangle+7e^{(5/6)\pi i}|4\rangle+3e^{(7/6)\pi i}|5\rangle)/(5^2+7^2+3^2)^{(1/2)}$, in the resonant JC model, from coherent state $|\psi_0\rangle=\Sigma_{n=0}^{\infty}\sqrt{e^{-4}4^n/n!}|n\rangle$, using $r=1.5$ and $u=8$ in Eq. (5). (b) Convergence of moduli squared $|C_{n,K}|^2$ to target's squared moduli $|C_{n,t}|^2$ (horizontal lines). (c) Idem, for phase factors $C_{n,K}/|C_{n,K}|$ (scaled by K) in the complex plane to directions of target's phase factors $C_{n,t}/|C_{n,t}|$ (radial lines). (d) Sequence probability $P_{\mathrm{seq},K}=\Pi_{l=1}^{K}P_l$ and squared projection $J_K=|\langle\psi_K|\psi_t\rangle|^2$. Success probability of the total sequence is 8%, double the initial squared projection of 4%.

$$|\psi_{K-1}\rangle=\sum_n C_{n,K-1}|n\rangle\rightarrow|\psi_K\rangle=\sum_n C_{n,K}|n\rangle,$$

$$\begin{aligned}C_{n,K}=P_K^{-1/2}\{&(\alpha_K^{(f)*}\alpha_K^{(i)}\cos\theta_{n,K}\\&+\beta_K^{(f)*}\beta_K^{(i)}\cos\theta_{n-1,K})C_{n,K-1}\\&-i\beta_K^{(f)*}\alpha_K^{(i)}\sin\theta_{n-1,K}C_{n-1,K-1}\\&-i\alpha_K^{(f)*}\beta_K^{(i)}\sin\theta_{N,K}C_{n+1,K-1}\}\end{aligned}\qquad(9)$$

(where we define $C_{-1,K-1}=0$). There are five controllable parameters of this CM transformation that should be determined by minimizing G_K [Eq. (5)] for each K: the two complex pairs $\alpha_K^{(i)}$ and $\beta_K^{(i)}$, and $\alpha_K^{(f)}$ and $\beta_K^{(f)}$ (each pair having two independent parameters due to normalization and global-phase redundancy), as well as $\lambda\tau_K$. The accuracy to which the target state can be attained is (at best) that of controlling the above five parameters.

From numerous computations based on the present strategy we can infer that the success probability of the optimized sequence is close to, and often *higher* than the squared projection $|\langle\psi_0|\psi_t\rangle|^2$ of the starting coherent state on the target state. This implies that convergence to the target state is genuine optimization, and not merely the filtering-out (erasure) of unwanted parts of the initial Fock-state distribution. The *filtering out* is effected by the diagonal terms ($C_{n,K-1}$) in (9), for which $\lambda\tau_K$ is such that $|\cos\theta_{n,K}|\ll 1$, whereas for $n_{\min}\leqslant n\leqslant n_{\max}$, $|\cos\theta_{n,K}|\approx 1$. By contrast, the off-diagonal terms ($C_{n\pm1,K-1}$) in (9) are responsible for the *fine tuning* (reshaping) of the Fock-state distribution.

The results of the optimization strategy outlined in Eqs. (5)–(9) are illustrated in Figs. 1 and 2. In Fig. 1 we explore

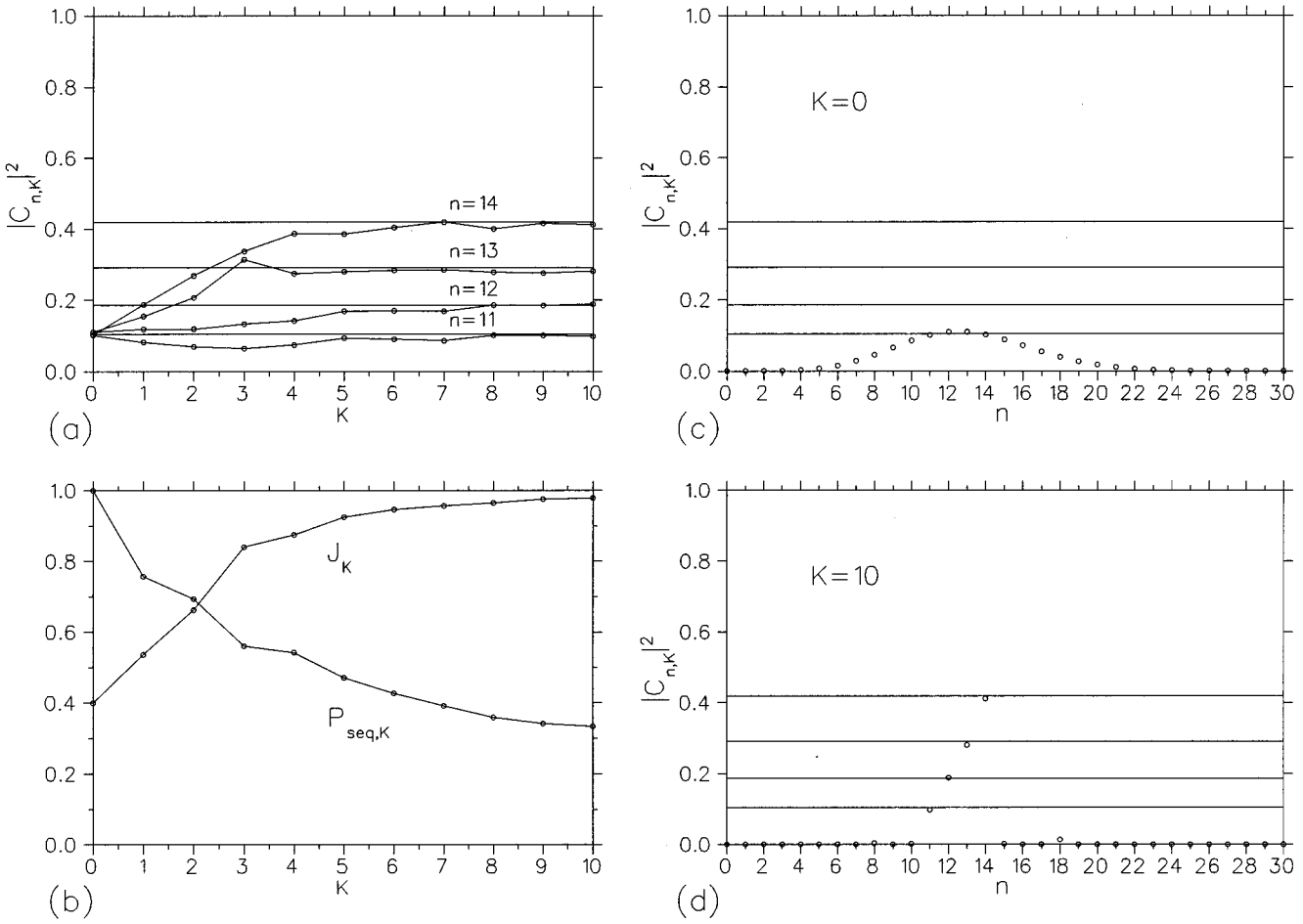

FIG. 2. (a), (b): Same as Fig. 1 for target $|\psi_t\rangle=(3|11\rangle+4|12\rangle+5|13\rangle+6|14\rangle)/(3^2+4^2+5^2+6^2)^{1/2}$, starting with coherent state $|\psi_0\rangle=\Sigma_{n=0}^{\infty}\sqrt{e^{-13}13^n/n!}|n\rangle$, using $r=3$ and $u=4$ in Eq. (5). Success probability of the total sequence is 33%, comparable with the initial squared projection of 40%. (c), (d): Initial ($K=0$) and final ($K=10$) Fock-state distributions with reference lines as in (a).

the difficult challenge of starting from a coherent state and attaining a superposition of three successive number (Fock) states whose amplitudes have prescribed *different phases, as well as, moduli*. Such a state can encode one quantum *ternary* digit. The optimization yields a CM sequence whose success probability $P_{\rm seq}$ is *twice* the squared projection of the initial state on the target state, $P_{\rm seq}\sim 2|\langle\psi_0|\psi_t\rangle|^2$ (8% compared to 4% initially). In Fig. 2 we optimize the generation of a superposition of four successive number states with different moduli but *equal phases* of the amplitudes. Such a state (with different phases allowed) can encode two quantum *binary* digits (qubits) [3]. By optimization we obtain for this state $P_{\rm seq}\sim|\langle\psi_0|\psi_t\rangle|^2$. Typically, five to ten CMs suffice to attain 95% and higher overlap with the target.

The outlined strategy for field-state preparation via optimal control of repeated CMs is universal, in that it is applicable to *any* field-atom interaction and starting-field state (as long as it is pure). The effectiveness of this strategy in generating *finite* Fock-state superpositions, which has been demonstrated in Figs. 1, and 2, is contingent on two conditions: (a) The lifetime (decoherence time) of the mode divided by the photon number must be much longer than the state preparation time. This is achievable at present in high-Q microcavities [4,6]. (b) High detection efficiency of the final atomic states (90–95 %, according to our numerical checks) is needed for high reliability of state preparation.

If these two conditions are satisfied, then the present CM strategy allows one to ''load'' the cavity field with quantum information (m qubits in superpositions of 2_m Fock states) at a rather high rate: M repetitions of the CM sequence, whose size $K_{\rm max}$ suffices for convergence to the target state (within the accuracy margin), can guarantee the state ''loading'' with confidence level $P_{\rm con}$ (desired success probability), if

$$M=\log(1-P_{\rm con})/\log(1-P_{\rm seq}), \tag{10}$$

where $P_{\rm seq}$ is the success probability of a single CM sequence. For $P_{\rm con}=0.95$ and $P_{\rm seq}=0.05$ this yields $M=58$, whence a rather modest number of trials; $MK_{\rm max}\sim 500$ is then required for $K_{\rm max}\sim 10$. Practically, the loading rate $1/M(\underline{K_{\rm max}}\overline{\tau+}\,\tau_{\rm reset})$ is determined by the average interaction time τ, the time to reset the field to its initial state after an unsuccessful measurement sequence, $\tau_{\rm reset}$, and the requirement that $P_{\rm con}$ be comparable with the atomic detection efficiency or the field-decoherence probability (during the state preparation). By comparison, $P_{\rm con}$ in unitary nearly adiabatic

schemes [8] is limited by the probability of intermediate-level decay or field decoherence, whereas the loading rate is determined by the duration of the miltilevel atomic-state preparation and the subsequent nearly adiabatic coherence transfer to the field. These estimates suggest that our optimized CM strategy, based on simple dynamics, such as the JC model, is a *viable alternative* to unitary evolution schemes involving multilevel-atom dynamics.

[1] *Quantum Theory and Measurement*, edited by J. A. Wheeler and W. H. Zurek (Princeton University Press, Princeton, NJ, 1983).

[2] C. M. Caves and P. D. Drummond, Rev. Mod. Phys. **66**, 481 (1994).

[3] A. Ekert and R. Jozsa, Rev. Mod. Phys. (to be published).

[4] Fock-state generation by measurements following the JC evolution has been studied by J. Krause, M. O. Scully, and H. Walther, Phys. Rev. A **36**, 4547 (1987); J. Krause, M. O. Scully, T. Walther, and H. Walther, *ibid.* **39**, 1915 (1989); P. Meystre, Opt. Lett. **12**, 669 (1987).

[5] Schrödinger-cat generation by measurements following off-resonant (Kerr-like) interaction has been proposed by M. Brune, S. Haroche, J. M. Raimond, L. Davidovich, and N. Zagury, Phys. Rev. A **45**, 5193 (1992).

[6] The CM approach to field-state preparation has been introduced by B. Sherman and G. Kurizki, Phys. Rev. A **45**, R7674 (1992); B. Sherman, G. Kurizki, and A. Kadyshevitch, Phys. Rev. Lett. **69**, 1927 (1992); B. M. Garraway, B. Sherman, H. Moya-Cessa, P. L. Knight, and G. Kurizki, Phys. Rev. A **49**, 535 (1994).

[7] K. Vogel, V. M. Akulin, and W. P. Schleich, Phys. Rev. Lett. **71**, 1816 (1993).

[8] A. S. Parkins, P. Marte, P. Zoller, and H. J. Kimble, Phys. Rev. Lett. **71**, 3095 (1993).