Existence and Characterizations of Weighted Chebyshev Polynomials

Notation and Definitions

Notation:

- $E \subset \mathbb{C}$ a compact set with infinitely many points.
- $w : E \to [0, \infty)$ an upper semi-continuous function.
- $\mathbb{P}_n = \{$monic polynomials of degree $n\}$
- $\mathcal{P}_n = \{$polynomials of degree $n\}$
- $||f||_E = \sup_{x \in E} |f(x)|$

Definitions:

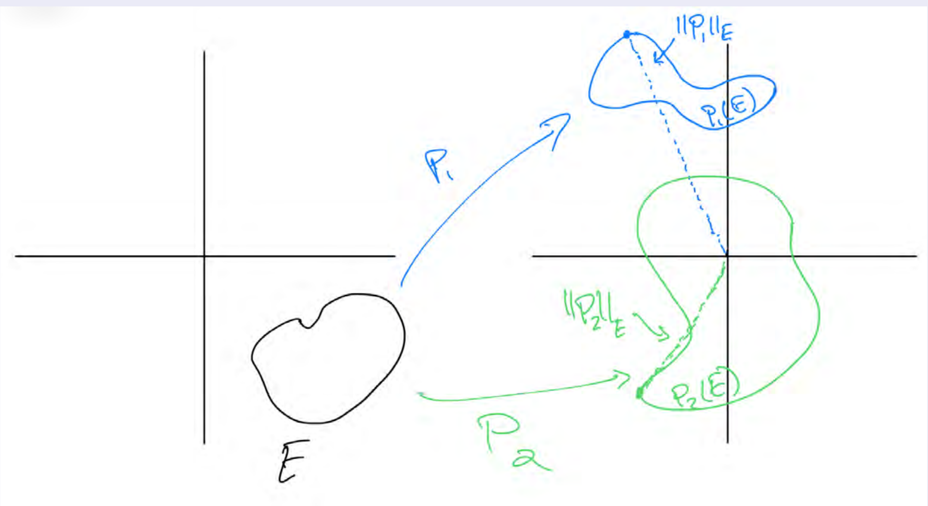
- A weighted polynomial of degree n (on E with weight w) is a function of the form wp with $p \in \mathcal{P}_n$.
- For a fixed w, E as above $T_n \in \mathbb{P}_n$ is called a weighted Chebychev polynomial (w.r.t w, E) if it minimizes $||wp_n||_E$ over all $p_n \in \mathbb{P}_n$. That is,

$$||wT_n||_E = \inf\{||wp_n||_E : p_n \in \mathbb{P}_n\} \qquad (*)$$

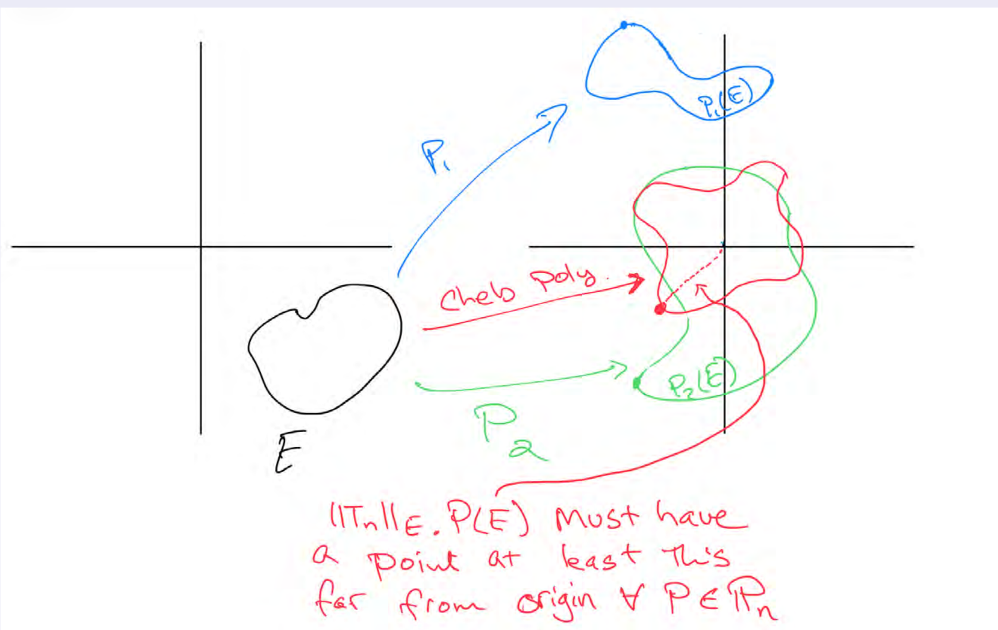
- A point $z_0 \in E$ is called an weighted extremal point of a function p_n (w.r.t w, E) when $|(wp)(z_0)| = ||wp||_E$.
- The set of all weighted extremal points of p_n (w.r.t w, E) will be denoted A_0.

Chebyshev Polynomials

It is hard to visualize graphs of complex valued functions (since the graph would have to be 4 dimensional). What we do instead is think of how the domain of a function changes.



$||p||_E$ is the maximum distance from the origin to a point on $p(E)$, the Chebyshev Polynomial minimizes $||p||_E$.



Upper Semi-Continuous Functions

Definition:

- A function $E \to [0, \infty)$ is said to be upper-semi-continuous iff $w(z) \geq \limsup_{E \ni \zeta \to z} w(\zeta)$ for each accumulation point $z \in E$.

Upper Semi-Continuous functions can have various types of discontinuities, the key point is that the function value is always $\geq$ any limit values.

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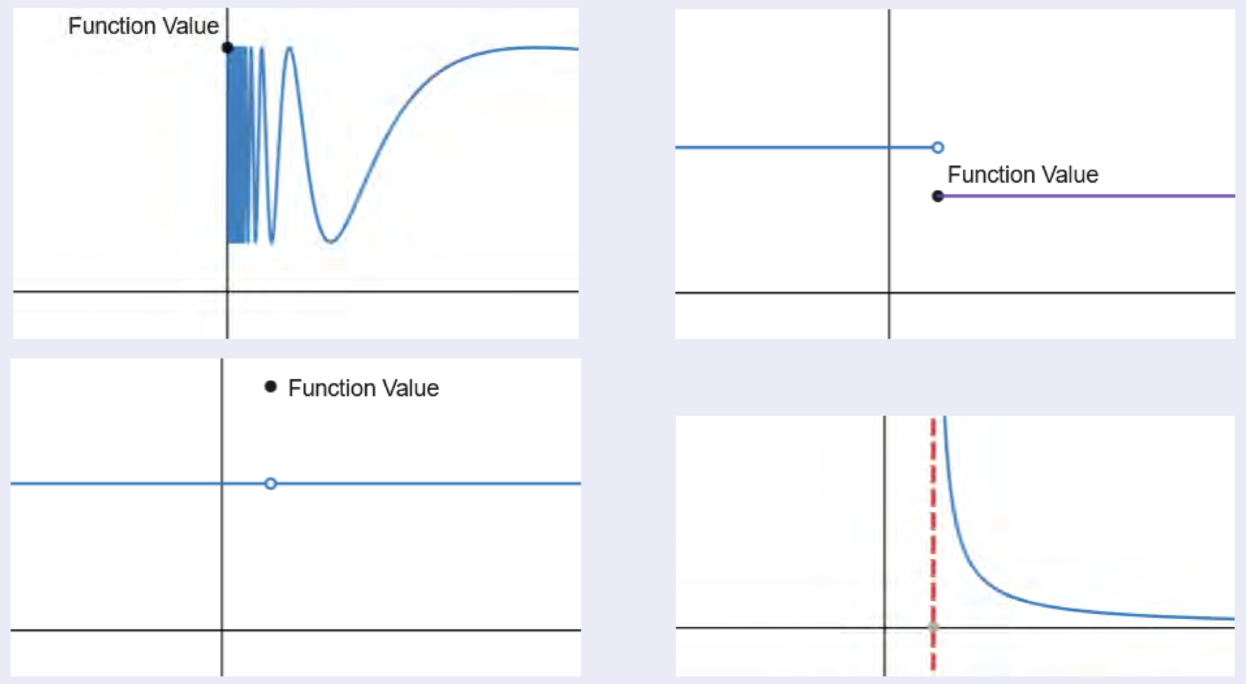
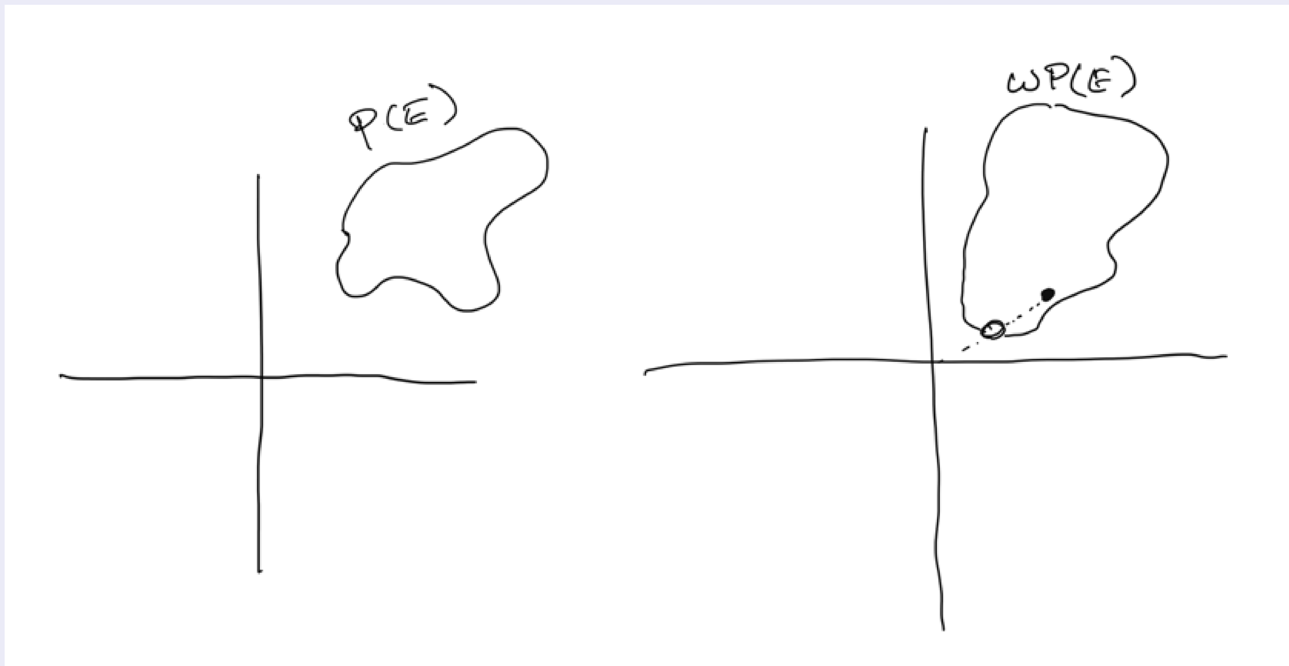


Weighted Polynomials

Because the weight functions are upper semi continuous, we have extremal points where we expect them. Intuitively, the discontinuities in the weight function always have to push the graph away from the origin.



This makes weighted polynomials well behaved with respect to extremal points. In particular our weighted polynomials always have extremal points where we expect to find them (and some places we might not).

Weighted Chebyshev Polynomials Exist

Theorem:

- For each fixed pair w, E as above there is some $T_n \in \mathbb{P}_n$ such that

$$||wT_n||_E = \inf\{||wp_n||_E : p_n \in \mathbb{P}_n\}$$

Weighted Chebyshev Polynomials are Unique

Theorem:

- T_n is unique if w is non-zero at least $n + 1$ points.

Weighted Chebyshev Polynomials Alternate

Theorem:

- If $E \subset \mathbb{R}$ then wp is a weighted Chebyshev polynomial of degree n iff it has extremal points $x_0 < \ldots < x_n$ with $wp(x_i) = -wp(x_{i+1})$ for all $i \in \{0, \ldots, n-1\}$.

Kolmogorov's Characterization

Definitions:

- A weighted continuous function on E is a function of the form wf where $w : E \to [0, \infty)$ is an upper semi continuous weight function and $f \in C(E)$.
- We say a weighted polynomial wp, $p \in \mathcal{P}_n$ is a best approximation to the weighted continuous function wf on the compact set $E \subset \mathbb{C}$ if $||wf - wp||_E = \inf_{q \in \mathcal{P}_n} ||wf - wq||_E$.

Theorem:

- A weighted polynomial wp, $p \in \mathcal{P}_n$, is a best approximation to the weighted continuous function wf iff for all $q \in \mathcal{P}_{n-1}$

$$\max_{z \in A_0} \Re\left((wf(z) - wp(z))\overline{wq(z)}\right) \geq 0$$

where

$A_0 = \{z \in E : |(wf)(z) - (wp)(z)| = ||wf - wp||_E\}$.

Rivlin's Characterization

Theorem:

- A monic polynomial $p \in \mathbb{P}_n$ is a weighted Chebyshev polynomial on E iff there exist $m \leq 2n + 1$ extremal points $z_1, \ldots, z_m$ and positive numbers $\alpha_1, \ldots, \alpha_m$ such that $\sum_{j=1}^{\infty} \alpha_j = 1$ and for each $q \in \mathbb{P}_{n-1}$,

$$\sum_{j=1}^{\infty} \alpha_j p(z_j)\overline{q(z_j)} = 0.$$