Neutrosophic Sets and Systems

Volume 73 *Neutrosophic Sets and Systems, Vol. 73, 2024*

Article 52

9-10-2024

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Tomasz Witczak

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Tomasz Witczak[1]

[1]Institute of Mathematics, Faculty of Technology, University of Silesia, Bankowa 14, 40-007 Katowice, Poland.

*Correspondence: tm.witczak@gmail.com

Abstract. In this paper we define the notion of *possibly paraconsistent sets*. We introduce algebraic operations on them and we analyze their properties. Moreover, it is shown that our class can be considered as isomorphic to the classes of intuitionistic and weak rough sets (with the assumption that intersection, union and complement are understood in an appropriate manner). Hence, the framework of paraconsistent sets can be treated as a new semantics for three-valued logic. Two less typical operations on possibly paraconsistent sets are studied too. They do not give us de Morgan algebra but rather a bisemilattice with only one absorption law. Finally, we pay attention to the fact that possibly paraconsistent sets can be treated as neutrosophic crisp sets of type 2. As for the exact isomorphism, it should be a matter of further research.

Keywords: possibly paraconsistent sets, neutrosophic sets, intuitionistic sets, weak rough sets

1. Introduction

Assume that our initial universe X is divided into three mutually disjoint parts. Then we can treat each such triple as a new non-classical set (that is, some kind of "data container"). Then we can define various operations on these sets and study their properties.

For example, we may assume that our triple consists of two distinguished sets A_T and A_F such that their intersection is empty while the third component is (obviously but also tacitly) $(A_T \cup A_F)^c$. These are *intuitionistic sets* in the sense of Çoker (see [3] and [4]). We may define their complement, union and intersection in a very natural way that leads to the structure of de Morgan algebra. In fact, intuitionistic sets have more to do with the three-valued logic of Łukasiewicz than with the intuitionistic logic of Brouwer and Heyting. Hence, some authors refuse to call them "intuitionistic". Instead of this, they use the notion of *orthopairs*. For example, in [2] Ciucci analyzed orthopairs in the context of granular computing.

Alternatively, we may assume that our universe consists of $A_1 \subseteq A_2$ and A_2^c (where the last component is not explicitly mentioned). These are *flou sets* of Gentilhomme (see e.g. [12]), known as *double sets* (see [16] for their soft version) or *weak rough sets* too (the last notion was used by Yong-jin in [18] together with the concept of weak rough numbers). Again, we may easily give them a structure of de Morgan algebra. Weak rough sets can be considered as isomorphic to intuitionistic sets.

Another approach is the one that is typical for *neutrosophic crisp sets of type 2*. This is a subclass of the wider algebra of *neutrosophic crisp sets* that has been introduced by Salama and Smarandache (see [13]). In this case we assume that we have three independent and mutually disjoint sets A_1, A_2 and A_3 and that their union is the whole X.

In the rest of the paper we shall go back to the concepts mentioned above. Now we can point out that all these ideas are crisp variants of some "fuzzy" solutions and due to this fact they are used to model the phenomenon of uncertainty and ambiguity. For example, Çoker sets refer to the notion of Atanassov's *intuitionistic fuzzy sets* (see [1]).

In this paper a new semantics is given. While intuitionistic sets sets are based on the supposition that our distinguished sets A_T and A_F have empty intersection, our idea is to consider A and $\sim A$ such that their intersection can be non-empty. Moreover, we assume that $A^c \subseteq \sim A$. In intuitionistic sets we assume that A_F is somewhat *weaker* than the classical complement of A_T. It gathers exactly those elements that are openly *rejected*. They do not belong to A_T. On the other hand, $(A_T \cup A_F)^c$ consists of those elements that are *neutral*. In our approach, $\sim A$ is *stronger* (at least in some sense) than the complement of A. Thus, in $A \cap \sim A$ we have those elements that are ambiguous.

The idea is to establish some appropriate operations in this framework. We prove that possibly paraconsistent sets equipped with these operations are isomorphic with other classes mentioned above. In this sense, our paper fills some gap in understanding of three-valued logics and tripartite division of space.

However, we also study some less typical operations that do not give us de Morgan algebra. This part of our study refers to the concept of bisemilattices. They are much weaker than de Morgan algebras and even weaker than lattices.

2. **On the algebra of possibly paraconsistent sets**

In this section we define our structures and we investigate their algebraic properties. In particular, we propose somewhat natural partial order together with the operations of union, intersection and complement.

2.1. *Basic notions*

We need to define basic components of our semantics.

Definition 2.1. Let $X \neq \emptyset$. Assume that $A, \sim A \subseteq X$ and $A^c \subseteq \sim A$. Then the ordered pair $\mathcal{A} = (A, \sim A)$ is called a *possibly paraconsistent set* on X.

Example 2.2. Let $X = \{a, b, c, d, e, f, g\}$, $A = \{a, b, g\}$ and $\sim A = \{b, c, d, e, f\}$. Then $\mathcal{A} = (A, \sim A)$ is a possibly paraconsistent set on X. In this case $A \cap \sim A = \{b\}$.

Analogously, if $B = \{d, e\}$ and $\sim B = \{a, b, c, f, g\}$, then $\mathcal{B} = (B, \sim B)$ is a possibly paraconsistent set. Note, however, that $\mathcal{B}$ is an intuitionistic set too (because in this case $\sim B = B^c$). Clearly, each intuitionistic set is a possibly paraconsistent set too. Obviously, the converse is not true.

Example 2.3. Let $X = \mathbb{R}_+ \cup \{0\}$ (that is, the set of positive real numbers with zero). Assume that $A = [0, 100)$ and $\sim A = [90, +\infty)$. Then $\mathcal{A} = (A, \sim A)$ is a possibly paraconsistent set.

Example 2.4. Let $X = \mathbb{N}$ (that is, the set of natural numbers with the assumption that zero is natural). Let $A = 2\mathbb{N}$ (even numbers) while $\sim A = (2\mathbb{N})^c \cup 8\mathbb{N}$ (odd numbers together with multiplicities of 8). Now $\mathcal{A} = (A, \sim A)$ is a possibly paraconsistent set on X.

The following observation is simple but in some sense important. It explains the idea of paraconsistency in our framework.

Remark 2.5. Note that if $\mathcal{A}$ is a possibly paraconsistent set on X, then $A \cup \sim A = X$ but $A \cap \sim A$ is not necessarily equal to $\emptyset$.

This shows that the *internal* structure of paraconsistent sets satisfies the law of the excluded middle (when $\sim A$ is treated as an *ersatz* of complement) but it does not satisfy the law of noncontradiction. Contrary to this, in the semantics of intuitionistic sets we have $A_T \cap A_F = \emptyset$ but $A_T \cup A_F$ may be different than X.

Clearly, this internal structure is a different thing than the relationship between possibly paraconsistent (or intuitonistic) sets as such. This will be discussed later.

Let us define some binary algebraic operations on possibly paraconsistent sets.

Definition 2.6. Let $X \neq \emptyset$ and assume that $\mathcal{A}$, $\mathcal{B}$ are two possibly paraconsistent sets on X. Then we define their:

(1) *Union*: $\mathcal{A} \cup \mathcal{B} = (A \cup B, \sim A \cap \sim B)$.
(2) *Strong union*: $\mathcal{A} \vee \mathcal{B} = (A \cup B, \sim A \cup \sim B)$.
(3) *Intersection*: $\mathcal{A} \cap \mathcal{B} = (A \cap B, \sim A \cup \sim B)$.

Lemma 2.7. *The operations defined in Def. 2.6 return possibly paraconsistent sets.*

Proof: Let us discuss all the cases.

(1) Check $\cup$. We need to ensure that the complement of the left component is contained in the right component. We have $(A \cup B)^c = A^c \cap B^c \subseteq \sim A \cap \sim B$.

(2) Check $\vee$. We have $(A \cup B)^c = A^c \cap B^c \subseteq \sim A \cap \sim B \subseteq \sim A \cup \sim B$.

(3) Check $\cap$. We have $(A \cap B)^c = A^c \cup B^c = \sim A \cup \sim B$.

Remark 2.8. Note that the following pair: $(A \cap B, \sim A \cap \sim B)$ is not necessarily a possibly paraconsistent set. This function can be corrected but this will be done later.

2.2. *About orderings*

Assume that our algebra is of the signature $(X, \cup, \cap)$ where $\cup$ refers to $+$ (that is, join) operation in lattices and other structures, while $\cap$ refers to $\cdot$ (that is, meet).

If so, then we can reconstruct the following two orderings:

Definition 2.9. Assume that $\mathcal{A}$, $\mathcal{B}$ are two possibly paraconsistent sets on X. We define the following relations:

(1) $\mathcal{A} \subseteq_{\cup} \mathcal{B}$ if and only if $\mathcal{A} \cup \mathcal{B} = \mathcal{B}$.

(2) $\mathcal{A} \subseteq_{\cap} \mathcal{B}$ if and only if $\mathcal{A} \cap \mathcal{B} = \mathcal{A}$.

Lemma 2.10. *Both orderings defined in Def. 2.9 are equal and they can be described as* $A \subseteq B$ *and* $\sim B \subseteq \sim A$.

Proof: Let $\mathcal{A} \subseteq_{\cup} \mathcal{B}$. Then $(A \cup B, \sim A \cap \sim B) = (B, \sim B)$. Thus, $A \cup B = B$ and $\sim A \cap \sim B = \sim B$. Hence, $A \subseteq B$ and $\sim B \subseteq \sim A$.

Now let $A \subseteq_{\cap} B$. Thus, $(A \cap B, \sim A \cup \sim B) = (A, \sim A)$. Then $A \cap B = A$ and $\sim A \cup \sim B = \sim A$. Hence, $A \subseteq B$ and $\sim B \subseteq \sim A$.

Now we can use only one symbol $\subseteq$ to denote our partial order (it is easy to check that this relation satisfies all the properties of partial order).

Remark 2.11. What about $\vee$ operator? Assume that it plays the role of join. This assumption is natural because it is based on two classical unions. Then we could write that $\mathcal{A} \subseteq_{\vee} \mathcal{B}$ if and only if $\mathcal{A} \vee \mathcal{B} = \mathcal{B}$. Thus $A \cup B = B$ and $\sim A \cup \sim B = \sim B$. Hence, $A \subseteq B$ and $\sim A \subseteq \sim B$.

2.3. *Complement*

We propose the following definition.

Definition 2.12. Let $\mathcal{A}$ be a possibly paraconsistent set on X. Then the *complement* of $\mathcal{A}$ is defined as $\mathcal{A}^c = (\sim A, A)$.

Of course this complement forms a new possibly paraconsistent set. Contrary to this, $\neg \mathcal{A} = (A^c, (\sim A)^c)$ (another hypothetical candidate for complement) is a possibly paraconsistent set if and only if $\sim A = A^c$.

2.4. *Distinguished sets and algebraic identities*

We can discuss at least three *distinguished* sets, namely: $\tilde{\emptyset} = (\emptyset, X)$, $\tilde{X} = (X, \emptyset)$ and $\overline{X} = (X, X)$.

Lemma 2.13. *Let $\mathcal{A}$ be a possibly paraconsistent set on X. Then the following properties are true:*

(1) $(\tilde{\emptyset})^c = \tilde{X}$ *and* $(\tilde{X})^c = \emptyset$.
(2) $(\overline{X})^c = (X, X)$.
(3) $\mathcal{A} \subseteq \tilde{X}$ *and* $\tilde{\emptyset} \subseteq \mathcal{A}$ *for every possibly paraconsistent set* $\mathcal{A}$.
(4) $\mathcal{A} \subseteq \overline{X}$ *if and only if* $\mathcal{A} = (A, X)$.
(5) $\mathcal{A} \subseteq_{\vee} \overline{X}$ *fo every paraconsistent set* $\mathcal{A}$.
(6) $\mathcal{A} \cap \tilde{\emptyset} = \tilde{\emptyset}$, $\mathcal{A} \cup \tilde{\emptyset} = \mathcal{A}$.
(7) $\mathcal{A} \cap \tilde{X} = \mathcal{A}$, $\mathcal{A} \cup \tilde{X} = \tilde{X}$.
(8) $\mathcal{A} \cap \overline{X} = (A, X)$ *and* $\mathcal{A} \cup \overline{X} = (X, \sim A)$.

Proof:

(1) This is obvious.
(2) This is simple too.
(3) Let $\mathcal{A} = (A, \sim A)$. Then $A \subseteq X$ and $\emptyset \subseteq \sim A$. The second fragment is analogous.
(4) Again, let $\mathcal{A} = (A, \sim A) \subseteq (X, X)$. Then $A \subseteq X$ (it is trivially true for any classical set A) and $X \subseteq \sim A$. This means that $X = \sim A$.
(5) Clearly, both A and $\sim A$ are contained in X.
(6) $\mathcal{A} \cap \tilde{\emptyset} = (A \cap \emptyset, \sim A \cup X) = (\emptyset, X) = \tilde{\emptyset}$. The second case is similar.
(7) Similar to the preceding point.
(8) $\mathcal{A} \cap \overline{X} = (A \cap X, \sim A \cup X) = (A, X)$. The second case is similar.

The next theorem deals with the essential properties of any de Morgan algebra.

Theorem 2.14. *Suppose that $X \neq \emptyset$ and $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are possibly paraconsistent sets on X. Then the following identities are true:*

(1) $\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A}$ *and* $\mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A}$ *(commutativity).*
(2) $\mathcal{A} \cap \mathcal{A} = \mathcal{A}$ *and* $\mathcal{A} \cup \mathcal{A} = \mathcal{A}$ *(idempotence).*
(3) $(\mathcal{A} \cap \mathcal{B})^c = \mathcal{A}^c \cup \mathcal{B}^c$ *and* $(\mathcal{A} \cup \mathcal{B})^c = \mathcal{A}^c \cap \mathcal{B}^c$ *(de Morgan laws).*
(4) $\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C})$ *and* $\mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C})$ *(distributivity laws).*
(5) $\mathcal{A} \cap (\mathcal{A} \cup \mathcal{B}) = \mathcal{A}$ *and* $\mathcal{A} \cup (\mathcal{A} \cap \mathcal{B}) = \mathcal{A}$ *(absorption laws).*

Proof:

(1) Obvious by the commutativity of classical union and intersection.

(2) Obvious by the idempotence of classical union and intersection.

(3) $(\mathcal{A} \cap \mathcal{B})^c = (A \cap B, \sim A \cup \sim B)^c = (\sim A \cup \sim B, A \cap B) = (\sim A \cup \sim B, A \cap B) = (\sim A, A) \cup (\sim B, B) = \mathcal{A}^c \cup \mathcal{B}^c$. The second case is similar.

(4) $\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = \mathcal{A} \cap (B \cup C, \sim B \cap \sim C) = (A \cap (B \cup C)), \sim A \cup (\sim B \cap \sim C) = ((A \cap B) \cup (A \cap C), (\sim A \cup \sim B) \cap (\sim A \cup \sim C)) = (A \cap B, \sim A \cup \sim B) \cup (A \cap C, \sim A \cup \sim C) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C})$. The second case is similar.

(5) For example, $\mathcal{A} \cap (\mathcal{A} \cup \mathcal{B}) = (A, \sim A) \cap (A \cup B, \sim A \cap \sim B) = (A \cap (A \cup B), \sim A \cup (\sim A \cap \sim B)) = (A, \sim A)$ by classical absorption laws. The second case is similar.

Remark 2.15. One can check that the law of the excluded middle is not true. For example, take $X = \{a, b, c, d, e\}$ and $\mathcal{A} = (\{a, c, d\}, \{a, b, d, e\})$. Then $\mathcal{A}^c = (\{a, b, d, e\}, \{a, c, d\})$ and their union is $(X, \{a, d\}) \neq \tilde{X}$. Analogously, take the intersection which is $(\{a, d\}, X) \neq \tilde{\emptyset}$.

3. Relationship with Çoker's intuitionistic sets

3.1. *Initial notions*

Some notions are necessary to show the correspondence between possibly paraconsistent and intuitionistic sets in a coherent way.

Definition 3.1. Let $X \neq \emptyset$ and suppose that $\mathcal{A}$ and $\mathcal{B}$ are two possibly paraconsistent sets on X. Then let us identify:

(1) $\mathcal{A}_\sim$ with $\sim A$ (that is, with the right component of $\mathcal{A}$).

(2) $(\mathcal{A} \cap \mathcal{B})_\sim$ and $(\mathcal{A} \vee \mathcal{B})_\sim$ with $\sim A \cup \sim B$.

(3) $(\mathcal{A} \cup \mathcal{B})_\sim$ with $\sim A \cap \sim B$.

(4) $(\mathcal{A}^c)_\sim$ with A.

Definition 3.2. Let $X \neq \emptyset$ and assume that $\mathcal{A}$ is a possibly paraconsistent set on X. Then let us use the following denotation: $\widehat{\mathcal{A}} = A \setminus \mathcal{A}_\sim = \{x \in X : x \notin \mathcal{A}_\sim\}$.

Obviously, $\widehat{\mathcal{A}} \subseteq A$. Moreover, it is clear that $\widehat{\mathcal{A}} = (\mathcal{A}_\sim)^c$.

Example 3.3. Let $X = \{a, b, c, d, e, f, g\}$, $\mathcal{A} = (\{a, c, d, e\}, \{b, c, d, f, g\})$ and $\mathcal{B} = (\{c, e, f\}, \{a, b, d, e, g\})$. Then (among other relationships that can be found):

(1) $\mathcal{A}_\sim = \{b, c, d, f, g\}$, $\widehat{\mathcal{B}} = B \setminus \mathcal{B}_\sim = B \setminus \sim B = \{c, f\}$.

(2) $(\mathcal{A} \cap \mathcal{B})_\sim = X$ and $(\mathcal{A} \cup \mathcal{B})_\sim = \{b, d, g\}$.

(3) $\widehat{\mathcal{A} \cup \mathcal{B}} = (A \cup B) \setminus (\mathcal{A} \cup \mathcal{B})_\sim = \{a, c, d, e, f\} \setminus \{b, d, g\} = \{a, c, e, f\}$.

Note that we write $\widehat{\mathcal{A}}$ and not $\widehat{A}$. This is because the exact form of this set depends not only on A but also on $\sim A$. And this is determined by the internal structure of $\mathcal{A}$. Due to the same reason we write $(\mathcal{A} \cap \mathcal{B})_\sim$ instead of $(A \cap B)_\sim$ or $\sim (A \cap B)$ (the same for union).

3.2. *Transforming function*

Now let us introduce the following function.

Definition 3.4. Let $X \neq \emptyset$. Assume that $\mathcal{A} = (A, \sim A)$ is a possibly paraconsistent set on X. Then let $\mathbf{f}$ be a function that assigns to $\mathcal{A}$ an intuitionistic set of the form $\mathbf{f}(\mathcal{A}) = (\widehat{\mathcal{A}}, A^c)$.

We assume that it is visible that $\mathbf{f}(\mathcal{A})$ is an intuitionistic set on X. It means that $\widehat{\mathcal{A}} \cap A^c = \emptyset$. This is true by the very definition of $\widehat{\mathcal{A}}$.

We see that in fact we divided our paraconsistent sets into three parts. The first one contains those elements that are strictly in A. The second consists of those points that are beyond A. These two parts are openly mentioned. We assume tacitly that the third part is the complement of their union, that is $(\hat{\mathcal{A}} \cup A^c)^c$. This is the area of (internal) paraconsistency.

Example 3.5. Let $X = \{a, b, c, d, e, f, g\}$, $\mathcal{A} = (\{a, b\}, \{b, c, d, e, f, g\})$ and $\mathcal{B} = (\{b, c, f, g\}, \{a, d, e, f, g\})$. Then $\mathbf{f}(\mathcal{A}) = (\{a\}, \{c, d, e, f, g\})$

Lemma 3.6. *Let $X \neq \emptyset$, $\mathcal{A}$ be a possibly paraconsistent set on X and $\mathbf{f}$ (defined as above) be our transforming function. Then $\mathbf{f}$ is one-to-one and surjective.*

Proof: Let us analyze two aspects mentioned.

(1) Suppose that $\mathcal{A} \neq \mathcal{B}$ but $\mathbf{f}(\mathcal{A}) = \mathbf{f}(\mathcal{B})$. This means that $\widehat{\mathcal{A}} = \widehat{\mathcal{B}}$ and $A^c = B^c$. The second equality implies that $A = B$. Hence, we can write that $\mathcal{A} = (A, \sim A)$ and $\mathcal{B} = (A, \sim B)$. Suppose that $\sim A \neq \sim B$. But $\sim A = A^c \cup (\widehat{\mathcal{A}})^c = B^c \cup (\widehat{\mathcal{B}})^c = \sim B$.

(2) Assume that we have an intuitionistic set on X of the form $\mathfrak{A} = (A_T, A_F)$ where $A_T \cap A_F = \emptyset$. We are looking for such possibly paraconsistent set $\mathcal{A} = (A, \sim A)$ that $\mathbf{f}(A, \sim A) = (\widehat{\mathcal{A}}, A^c) = (A_T, A_F)$. Thus $A^c = A_F$ and $A = (A_F)^c$. On the other hand $\widehat{\mathcal{A}} = A_T$. Thus $(\widehat{\mathcal{A}})^c = (A_T)^c$. This is important because now we can write that $\sim A = A^c \cup (\widehat{\mathcal{A}})^c = A_F \cup (A_T)^c$.

Now $A^c = A_F \subseteq A_F \cup (A_T)^c = \sim A$ so our set is indeed a possibly paraconsistent one.

One can check the reasoning presented above: $\mathbf{f}(\mathcal{A}) = \mathbf{f}(A_F^c, A_F \cup A_T^c) = (\widehat{\mathcal{A}}, ((A_F)^c)^c) = (\{x \in X : x \in (A_F)^c \text{ and } x \notin (A_F \cup (A_T)^c)\}, A_F) = (\{x \in X : x \notin A_F \text{ and } x \notin A_F \text{ and } x \notin (A_T)^c\}, A_F) = (\{x \in X : x \in A_T\}, A_F) = (A_T, A_F)$.

Now we should prove that our function is a homomorphism. First, there is a technical lemma.

Lemma 3.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be two possibly paraconsistent set on X. Then the following properties hold:*

(1) $\widehat{\mathcal{A} \cap \mathcal{B}} = \widehat{\mathcal{A}} \cap \widehat{\mathcal{B}}$.

(2) $\widehat{\mathcal{A} \cup \mathcal{B}} = \widehat{\mathcal{A}} \cup \widehat{\mathcal{B}}$.

(3) $\widehat{\mathcal{A}^c} = (\widehat{\mathcal{A}})^c$.

Where the symbols on the right sides are understood in a classical sense.

Proof:

(1) We have $\widehat{\mathcal{A} \cap \mathcal{B}} = \{x \in X : x \notin (\mathcal{A} \cap \mathcal{B})_\sim\} = \{x \in X : x \notin (\sim A \cup \sim B)\} = \{x \in X : x \notin \sim A \text{ and } x \notin \sim B\}$.

On the other hand: $\widehat{\mathcal{A}} \cap \widehat{\mathcal{B}} = \{x \in X : x \notin \mathcal{A}_\sim\} \cap \{x \in X : x \notin \mathcal{B}_\sim\} = \{x \in X : x \notin \sim A\} \cap \{x \in X : x \notin \sim B\} = \{x \in X : x \notin \sim A \text{ and } x \notin \sim B\} = \widehat{\mathcal{A} \cap \mathcal{B}}$.

(2) $\widehat{\mathcal{A} \cup \mathcal{B}} = \{x \in X : x \notin (\mathcal{A} \cup \mathcal{B})_\sim\} = \{x \in X : x \notin (\sim A \cap \sim B)\} = \{x \in X : x \notin \sim A \text{ or } x \notin \sim B\} = \{x \in X : x \in (\sim A)^c \text{ or } x \in (\sim B)^c\} = \{x \in X : x \in \widehat{\mathcal{A}} \text{ or } x \in \widehat{\mathcal{B}}\} = \widehat{\mathcal{A}} \cup \widehat{\mathcal{B}}$.

(3) $\widehat{\mathcal{A}^c} = \{x \in X : x \notin (\mathcal{A}^c)_\sim\} = \{x \in X : x \notin A\}$. On the other hand $(\widehat{\mathcal{A}})^c = \{x \in X : x \notin \sim A\}^c = \{x \in X : x \notin A\}$. We get the same set.

Having in mind the lemma above, we can prove the essential theorem. It says that our transforming function preserves intersection, union and complement. However, at first we should recall intuitionistic understanding of these operations.

Definition 3.8. (see [3]) Let $X \neq \emptyset$ and assume that $\mathfrak{A} = (A_T, A_F)$ and $\mathfrak{B} = (B_T, B_F)$ are two intuitionistic sets on X. Then we define their:

(1) *Union*: $\mathfrak{A} \cup \mathfrak{B} = (A_T \cup B_T, A_F \cap B_F)$.

(2) *Intersection*: $\mathfrak{A} \cap \mathfrak{B} = (A_T \cup B_T, A_F \cap B_F)$.

(3) *Complement*: $\mathfrak{A}^c = (A_F, A_T)$.

Note that we use (and we shall use) the same symbols (that is $\cup$, $\cap$, ^{c}, $\vee$ and $\wedge$) to denote operations in three frameworks: possibly paraconsistents, intuitionistic and then weak rough sets. The reader should be able to read from the context in which setting we are at a given moment.

Theorem 3.9. *Let $X \neq \emptyset$ and assume that $\mathcal{A}$, $\mathcal{B}$ are two possibly paraconsistent sets on X. Let $\mathbf{f}$ be a transforming function. Then the following properties are true:*

(1) $\mathbf{f}(\mathcal{A} \cap \mathcal{B}) = \mathbf{f}(\mathcal{A}) \cap \mathbf{f}(\mathcal{B})$.

(2) $\mathbf{f}(\mathcal{A} \cup \mathcal{B}) = \mathbf{f}(\mathcal{A}) \cup \mathbf{f}(\mathcal{B})$.

(3) $\mathbf{f}(\mathcal{A}^c) = (\mathbf{f}(\mathcal{A}))^c$.

Where the intersection, union and complement symbols on the right sides are understood in the sense of Çoker's intuitionistic sets.

Proof: Note that we shall use Lemma 3.7.

(1) We have $\mathbf{f}(\mathcal{A}\cap\mathcal{B})=\mathbf{f}(A\cap B,\sim A\cup\sim B)=(\widehat{\mathcal{A}\cap\mathcal{B}},(A\cap B)^c)=(\widehat{\mathcal{A}}\cap\widehat{\mathcal{B}},A^c\cup B^c)=(\widehat{\mathcal{A}},A^c)\cap(\widehat{\mathcal{B}},B^c)=\mathbf{f}(\mathcal{A})\cap\mathbf{f}(\mathcal{B})$.

(2) We have $\mathbf{f}(\mathcal{A}\cup\mathcal{B})=\mathbf{f}(A\cup B,\sim A\cap\sim B)=(\widehat{\mathcal{A}\cup\mathcal{B}},(A\cup B)^c)=(\widehat{\mathcal{A}}\cup\widehat{\mathcal{B}},A^c\cap B^c)=(\widehat{\mathcal{A}},A^c)\cup(\widehat{\mathcal{B}},B^c)=\mathbf{f}(\mathcal{A})\cup\mathbf{f}(\mathcal{B})$.

(3) We have $\mathbf{f}(\mathcal{A}^c)=\mathbf{f}(\sim A,A)=(\widehat{\mathcal{A}^c},(\sim A)^c)=(A^c,\widehat{\mathcal{A}})=(\widehat{\mathcal{A}},A^c)^c=(\mathbf{f}(\mathcal{A}))^c$.

Remark 3.10. One can check that:

(1) $\mathbf{f}(\emptyset,X)=(\widehat{\widetilde{\emptyset}},\emptyset^c)=(\emptyset,X)$.

(2) $\mathbf{f}(X,\emptyset)=(\widehat{\widetilde{X}},X^c)=(X,\emptyset)$.

(3) $\mathbf{f}(X,X)=(\widehat{\overline{X}},X^c)=(\emptyset,\emptyset)$.

Remark 3.11. Note that we can associate intuitionistic sets with possibly paraconsistent sets in many ways. However, not all of them are injections. For example, assume that $\mathbf{f}(\mathcal{A})=\mathbf{f}(A,\sim A)=(\widehat{\mathcal{A}},(\widehat{\mathcal{A}})^c)$. For example, if $X=\{x,y,z,p,q,r\}$, $A=\{x,y,p,q\}$ and $\sim A=\{z,p,q,r\}$, then $\mathbf{f}(\mathcal{A})=(\{x,y\},\{z,p,q,r\})$.

Now take $B=\{x,y,z,p\}$ and $\sim B=\{z,p,q,r\}$. Clearly, $\mathcal{A}\neq\mathcal{B}$ (because $A\neq B$). But $\mathbf{f}(\mathcal{B})=(\{x,y\},\{z,p,q,r\})=\mathbf{f}(\mathcal{A})$.

4. Relationship with weak rough sets

Let us recall the definition of weak rough set and essential operations on the objects of this type.

Definition 4.1. (see [18]). Let $X\neq\emptyset$. Let $\mathfrak{A}$ be an ordered pair of the form (A_1,A_2) where $A_1\subseteq A_2\subseteq X$. Then we say that $\mathfrak{A}$ is a weak rough set on X. If $\mathfrak{A}$ and $\mathfrak{B}$ are two weak rough sets on X, then we define their:

(1) *Union*: $\mathfrak{A}\cup\mathfrak{B}=(A_1\cup B_1,A_2\cup B_2)$.

(2) *Intersection*: $\mathfrak{A}\cap\mathfrak{B}=(A_1\cap B_1,A_2\cap B_2)$.

(3) *Complement*: $\mathfrak{A}^c=(A_2^c,A_1^c)$.

4.1. *Transforming function*

Definition 4.2. Let $X\neq\emptyset$. Assume that $\mathcal{A}=(A,\sim A)$ is a possibly paraconsistent set on X. Then let $\mathbf{g}$ be a function that assigns to $\mathcal{A}$ a weak rough set of the form $\mathbf{g}(\mathcal{A})=(\widehat{\mathcal{A}},A)$.

Example 4.3. (1) Let $X=\mathbb{R}$. Assume that $\mathcal{A}=([-10,100],(-\infty,-5]\cup[90,+\infty))$. Then $\mathbf{g}(\mathcal{A})=((-5,90),[-10,100])$.

(2) Let $X=\{a,b,c,d,e,f,g\}$ and $\mathcal{B}=(\{a,b,d,e\},\{a,c,f,g\})$. Then $\mathbf{g}(\mathcal{B})=(\{b,d,e\},\{a,b,d,e\})$.

Lemma 4.4. *Let $X \neq \emptyset$, $\mathcal{A}$ be a possibly paraconsistent set on X and* $\mathbf{g}$ *(defined as above) be our transforming function. Then* $\mathbf{g}$ *is one-to-one and surjective.*

Proof: We shall prove both properties.

(1) Let $\mathcal{A} \neq \mathcal{B}$ but $\mathbf{g}(\mathcal{A}) = \mathbf{g}(\mathcal{B})$. Then $\widehat{\mathcal{A}} = \widehat{\mathcal{B}}$ and $A = B$. Hence $(\widehat{\mathcal{A}})^c = (\widehat{\mathcal{B}})^c$ and $A^c = B^c$. Then $\sim A = A^c \cup (\widehat{\mathcal{A}})^c = B^c \cup (\widehat{\mathcal{B}})^c = \sim B$. So $\mathcal{A} = \mathcal{B}$ (contradiction).

(2) Assume that we have a weak rough set of the form $\mathfrak{A} = (A_1, A_2)$ where $A_1 \subseteq A_2$. We are looking for such possibly paraconsistent set $\mathcal{A} = (A, \sim A)$ that $\mathbf{g}(A, \sim A) = (\widehat{\mathcal{A}}, A) = (A_1, A_2)$. Thus $\widehat{\mathcal{A}} = A_1$ and $A = A_2$. Then $\sim A = (\widehat{\mathcal{A}})^c = A_1^c$. Finally, we obtain $\mathcal{A} = (A_1^c, A_2)$.

Note that this is indeed a possibly paraconsistent set because $(A_1^c)^c = A_1 \subseteq A_2$.

Remark 4.5. Again, note that it would not be reasonable to define $\mathbf{g}(\mathcal{A})$ as e.g. $(\widehat{\mathcal{A}}, X)$. Obviously, the resulting object is a weak rough set. But this function is not one-to-one (the reader is encouraged to find some simple counter-example).

Clearly, our function is homomorphic.

Theorem 4.6. *Let $X \neq \emptyset$ and assume that $\mathcal{A}$, $\mathcal{B}$ are two possibly paraconsistent sets. Let* $\mathbf{g}$ *be a wrs-transforming function. Then the following properties are true:*

(1) $\mathbf{g}(\mathcal{A} \cap \mathcal{B}) = \mathbf{g}(\mathcal{A}) \cap \mathbf{g}(\mathcal{B})$.
(2) $\mathbf{g}(\mathcal{A} \cup \mathcal{B}) = \mathbf{g}(\mathcal{A}) \cup \mathbf{g}(\mathcal{B})$.
(3) $\mathbf{g}(\mathcal{A}^c) = (\mathbf{g}(\mathcal{A}))^c$.

Where the intersection, union and complement symbols on the right sides are understood in the sense of weak rough sets.

Proof: For example, $\mathbf{g}(\mathcal{A} \cap \mathcal{B}) = \mathbf{g}(A \cap B, \sim A \cup \sim B) = (\widehat{\mathcal{A}} \cap \widehat{\mathcal{B}}, A \cap B) = (\widehat{\mathcal{A}}, A) \cap (\widehat{\mathcal{B}}, B) = \mathbf{g}(\mathcal{A}) \cap \mathbf{g}(\mathcal{B})$.

Other cases are similar.

5. Bisemilattice of possibly paraconsistent sets

In Remark 2.8 we pointed out that it is not reasonable to define the operation analogous to $\vee$ but with classical intersections instead of unions. However, this flaw can be easily repaired.

Definition 5.1. Let $X \neq \emptyset$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two possibly paraconsistent sets on X. Then we define *strong intersection* as $\mathcal{A} \wedge \mathcal{B} = (A \cap B, (\sim A \cap \sim B) \cup (A \cap B)^c)$.

Obviously, the operation defined above returns possibly paraconsistent set.

Let us consider the following algebra of possibly paraconsistent sets: $(X, \wedge, \vee)$. Let us analyze what is the interplay of these two operators.

Theorem 5.2. *Suppose that $X \neq \emptyset$ and $\mathcal{A}$, $\mathcal{B}$ are two possibly paraconsistent sets on X. Then the following relationships hold:*

(1) $\mathcal{A} \wedge \mathcal{B} = \mathcal{B} \wedge \mathcal{A}$ *and* $\mathcal{A} \vee \mathcal{B} = \mathcal{B} \vee \mathcal{A}$.

(2) $\mathcal{A} \wedge \mathcal{A} = \mathcal{A} \vee \mathcal{A} = \mathcal{A}$.

(3) $\mathcal{A} \wedge (\mathcal{A} \vee \mathcal{B}) = \mathcal{A}$.

(4) $\mathcal{A} \vee (\mathcal{B} \vee \mathcal{C}) = (\mathcal{A} \vee \mathcal{B}) \vee \mathcal{C}$.

(5) $\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C}) = (\mathcal{A} \wedge \mathcal{B}) \wedge \mathcal{C}$.

Proof:

(1) Commutativity of both operations is obvious.

(2) Idempotence of both operations is obvious.

(3) We have: $\mathcal{A} \wedge (\mathcal{A} \vee \mathcal{B}) = \mathcal{A} \wedge (A \cup B, \sim A \cup \sim B) = (A \cap (A \cup B), (\sim A \cap (\sim A \cup \sim B)) \cup (A \cap (A \cup B))^c) = (A, \sim A \cup (A^c \cup (A \cup B)^c)) = (A, \sim A \cup (A^c \cup (A^c \cap B^c))) = (A, \sim A \cup A^c) = (A, \sim A)$.

We used classical absorption laws and the fact that $A^c \subseteq \sim A$.

(4) We have $\mathcal{A} \vee (\mathcal{B} \vee \mathcal{C}) = \mathcal{A} \vee (B \cup C, \sim B \cup \sim C) = (A \cup (B \cup C), \sim A \cup (\sim B \cup \sim C)) = ((A \cup B) \cup C, (\sim A \cup \sim B) \cup \sim C)$.

(5) Let us calculate:

$\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C}) = \mathcal{A} \wedge (B \cap C, (\sim B \cap \sim C) \cup (B \cap C)^c) = \mathcal{A} \wedge (B \cap C, (\sim B \cap \sim C) \cup (B^c \cup C^c)) = (A \cap (B \cap C), (\sim A \cap ((\sim B \cap \sim C) \cup (B^c \cup C^c))) \cup (A \cap (B \cap C))^c)) = (A \cap B \cap C, (\sim A \cap (\sim B \cap \sim C)) \cup (\sim A \cap (B^c \cup C^c)) \cup (A \cap B \cap C)^c) = (A \cap B \cap C, (\sim A \cap \sim B \cap \sim C) \cup (\sim A \cap (B^c \cup C^c)) \cup (A \cap B \cap C)^c) = (A \cap B \cap C, X_1 \cup X_2 \cup X_3)$.

On the other hand:

$(\mathcal{A} \wedge \mathcal{B}) \wedge \mathcal{C} = (A \cap B, (\sim A \cap \sim B) \cup (A \cap B)^c) \wedge \mathcal{C} = (A \cap B, (\sim A \cap \sim B) \cup (A^c \cup B^c)) \wedge \mathcal{C} = ((A \cap B) \cap C, (((\sim A \cap \sim B) \cup (A^c \cup B^c)) \cap \sim C) \cup (A \cap B \cap C)^c) = (A \cap B \cap C, ((\sim A \cap \sim B) \cap \sim C) \cup ((A^c \cup B^c) \cap \sim C) \cup (A \cap B \cap C)^c) = (A \cap B \cap C, (\sim A \cap \sim B \cap \sim C) \cup ((A^c \cup B^c) \cap \sim C) \cup (A \cap B \cap C)^c) = (A \cap B \cap C, Y_1 \cup Y_2 \cup Y_3)$.

Both expressions are similar but not identical. Clearly, left components are the same. Now think that x belongs to the right component of $\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C})$ but it does not belong to the right component of $(\mathcal{A} \wedge \mathcal{B}) \wedge \mathcal{C}$.

The second assumption means that $x \notin Y_1$ and $x \notin Y_2$ and $x \notin Y_3$.

Let us think about the first assumption. We have a logical disjunction of statements (or set-theoretical union of sets). Suppose that $x \in X_1$. Clearly $X_1 = Y_1$. But this gives us a contradiction.

Now suppose that $x \in X_3$. Clearly, $X_3 = Y_3$. Again, contradiction.

Now let us think that $x \in X_2$. This means that $x \in \sim A \cap (B^c \cup C^c)$. In particular, it means that $x \in B^c \cup C^c$. Suppose that $x \in B^c$. But at the same time $x \notin Y_3 =$

$(A \cap B \cap C)^c = A^c \cup B^c \cup C^c$. So $x \notin A^c$ and $x \notin B^c$ and $x \notin C^c$. In particular, $x \notin B^c$. Contradiction. The same conclusion appears under the assumption that $x \in C^c$.

The reader is encouraged to use a similar reasoning to prove the second inclusion: namely, that $(\mathcal{A} \wedge \mathcal{B}) \wedge \mathcal{C} \subseteq \mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C})$.

Remark 5.3. What about the second absorption law? Take $X = \{a, b, c, d, e, f, g, h\}$ and define $\mathcal{A} = (\{a, c, d, e\}, \{b, c, d, f, g, h\})$ together with $\mathcal{B} = (\{b, d, f, h\}, \{a, b, c, f, g, h\})$.

Now $\mathcal{A} \wedge \mathcal{B} = (\{d\}, \{a, b, c, e, f, g, h\})$. Then $\mathcal{A} \vee (\mathcal{A} \wedge \mathcal{B}) = (\{a, c, d, e\}, X)$. But this set is different than $\mathcal{A}$. Thus this hypothetical absorption law (namely "$\vee - \wedge$") does not hold.

Let us recall the notion of *bisemilattice*. We say that a structure $(L, +, \cdot)$ is a bisemilattice if and only if both operations are idempotent, commutative and associative. Other requirements (like distributivity, absorption laws or existence of zero and one) are not mandatory.

Bisemilattices were analyzed by several authors (in particular, by Polish algebraists): see [7], [8], [10] or [11].

All the facts proved above allow us to say that the structure $(X, \vee, \wedge)$ (of possibly paraconsistent sets) is a bisemilattice with only one absorption law. Such specific bisemilattices were mentioned e.g. by Dudek in [6]. In this paper he proved that there exists a bisemilattice of this type being not a lattice. In fact, we may use our class as an example too (note that in the definition of lattice we require *two* absorption laws while in our case on of them is not true).

5.1. *Relationship with weak rough sets*

We want to use the function that has been already used: $\mathbf{g}(\widehat{\mathcal{A}}, A)$. It associates each possibly paraconsistent set with some weak rough set. But what about the appropriate operations?

First, let us prove the following lemma:

Lemma 5.4. *Let $X \neq \emptyset$ and $\mathcal{A}$, $\mathcal{B}$ be two possibly paraconsistent sets. Then the following properties hold:*

(1) $\widehat{\mathcal{A} \vee \mathcal{B}} = \widehat{\mathcal{A}} \cap \widehat{\mathcal{B}}$.
(2) $\widehat{\mathcal{A} \cap \mathcal{B}} = (\widehat{\mathcal{A}} \cup \widehat{\mathcal{B}}) \cap (A \cap B)$.

Proof:

(1) We have $\widehat{\mathcal{A} \vee \mathcal{B}} = \{x \in X : x \notin (\mathcal{A} \vee \mathcal{B})_{\sim}\} = \{x \in X : x \notin (\mathcal{A} \cap \mathcal{B})_{\sim}\} = \widehat{\mathcal{A}} \cap \widehat{\mathcal{B}}$.
(2) We have $\widehat{\mathcal{A} \cap \mathcal{B}} = \{x \in X : x \notin (\sim A \cap \sim B) \cup (A \cap B)^c\} = \{x \in X : x \notin (\sim A \cap \sim B)$ and $x \notin (A \cap B)^c\} = \{x \in X : x \notin (\sim A \cap \sim B)$ and $x \in A \cap B\} = (\sim A \cap \sim B)^c \cap (A \cap B) = ((\sim A)^c \cup (\sim B)^c) \cap (A \cap B) = (\widehat{\mathcal{A}} \cup \widehat{\mathcal{B}}) \cap (A \cap B)$.

Then we define appropriate operations in the framework of weak rough sets.

Definition 5.5. Let $X \neq \emptyset$ and assume that $\mathfrak{A} = (A_1, A_2)$ and $\mathfrak{B} = (B_1, B_2)$ are two weak rough sets. Then we define:

(1) $\mathfrak{A} \vee \mathfrak{B} = (A_1 \cap B_1, A_2 \cup B_2)$.
(2) $\mathfrak{A} \wedge \mathfrak{B} = ((A_1 \cup B_1) \cap (A_2 \cap B_2), A_2 \cap B_2)$.

Theorem 5.6. *Let $X \neq \emptyset$ and assume that $\mathcal{A}$, $\mathcal{B}$ are two possibly paraconsistent sets and* $\mathbf{g}$ *is a wrs-transforming function. Then the following properties are true:*

(1) $\mathbf{g}(\mathcal{A} \vee \mathcal{B}) = \mathbf{g}(\mathcal{A}) \vee \mathbf{g}(\mathcal{B})$.
(2) $\mathbf{g}(\mathcal{A} \wedge \mathcal{B}) = \mathbf{g}(\mathcal{A}) \wedge \mathbf{g}(\mathcal{B})$.

Proof:

(1) We have $\mathbf{g}(\widehat{\mathcal{A} \vee \mathcal{B}}) = (\widehat{\mathcal{A} \vee \mathcal{B}}, A \cup B) = (\widehat{\mathcal{A}} \cap \widehat{\mathcal{B}}, A \cup B) = (\widehat{\mathcal{A}}, A) \vee (\widehat{\mathcal{B}}, B) = \mathbf{g}(\mathcal{A}) \vee \mathbf{g}(\mathcal{B})$.
(2) We have: $\mathbf{g}(\widehat{\mathcal{A} \wedge \mathcal{B}}) = (\widehat{\mathcal{A} \wedge \mathcal{B}}, A \cap B) = ((\widehat{\mathcal{A}} \cup \widehat{\mathcal{B}}) \cap A \cap B, A \cap B) = (\widehat{\mathcal{A}}, A) \wedge (\widehat{\mathcal{B}}, B) = \mathbf{g}(\mathcal{A}) \wedge \mathbf{g}(\mathcal{B})$.

As for the behaviour of distinguished sets, it has been analyzed below (we omit the proof because it is simple):

Lemma 5.7. *Let $X \neq \emptyset$ and $\mathcal{A}$ be a possibly paraconsistent set on X. Then the following properties hold:*

(1) $\mathcal{A} \wedge \tilde{\emptyset} = \tilde{\emptyset}$, $\mathcal{A} \vee \tilde{\emptyset} = (A, X)$.
(2) $\mathcal{A} \wedge \tilde{X} = (A, A^c)$, $\mathcal{A} \vee \tilde{X} = (X, \sim A)$.
(3) $\mathcal{A} \wedge \overline{X} = (A, X)$, $\mathcal{A} \vee \overline{X} = \overline{X}$.

5.2. *Interpretation in terms of negotiations*

Our operations $\vee$ and $\wedge$ have somewhat interesting interpretation when they are interpreted in the framework of weak rough sets (as in Def. 5.5). In fact, we have already recognized these operations in our unpublished paper [17].

Imagine that there are two weak rough sets on some universe X. They are called $\mathfrak{A}$ and $\mathfrak{B}$. As for the operation $\vee$, it returns the intersection of necessity ranges and the union of possibility ranges. This can be interpreted as a hypothetical solution of some discussion between two decision makers (who evaluated the elements of X using weak rough sets). They compare and combine their sets to produce a new evaluation. The idea is that now *more* objects are acceptable (possible) and fewer are necessary. It is a kind of compromise. As for the necessity ranges, the approach of our decision makers is strict: they want to limit themselves only to those objects that are necessary in the eyes of both of them. Contrary to this, in case of possible objects they are ready to sum up their ranges.

The second operation, namely $\wedge$, can be described in a similar way. Clearly, the same can be said about the standard operations $\cup$ and $\cap$ but $\vee$ and $\wedge$ are less known and less typical. This makes them interesting (e.g. in the context of bisemilattices).

In [17] we proved that this bisemilattice is non-distributive. We gave appropriate counterexamples (in terms of weak rough sets). Clearly, this property is true in the setting of possibly paraconsistent sets too (by isomorphism). However, the reader is encouraged to find counterexamples that would be formulated exactly in the latter language.

6. **Conclusion and final remarks**

In this paper we introduced possibly paraconsistent sets. They can be analyzed in the context of three-valued logic. We proved that their algebra is isomorphic with the algebra of Çoker's intuitionistic sets (orthopairs) and thus with the algebra of weak rough (that is, double or. equivalently, flou) sets.

We have analyzed less typical operators that form bisemilattice with only one absorption law. We have shown that we can find corresponding operations in the setting of weak rough sets. One can think about finding appropriate operations in the setting of intuitionistic sets.

Regardless of isomorphisms, we gave exact proofs of some essential properties of possibly paraconsistent sets with all the operations introduced. We think that this is valuable from the practical point of view: even if the algebraic structure of different frameworks is the same, their meaning, interpretation and semantics can be different.

Another thing that is important, is the possible correspondence of our framework with the one that is generated by neutrosophic crisp sets of type 2. They are triples of the form (A_1, A_2, A_3) with the assumption that $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = \emptyset$ and $A_1 \cup A_2 \cup A_3 = X$. Clearly, this is just like our $A \cup (\mathcal{A} \setminus A^c) \cup A^c$ (in the possibly paraconsistent environment).

As for the neutrosophic crisp sets of type 2, they are a subclass of the wider class of neutrosophic crisp sets. The latter are defined just as triple (A_1, A_2, A_3) without any additional suppositions. Basically, their union and intersection are defined as:

(1) *Union*: $\mathfrak{A} \cup \mathfrak{B} =$
  (a) $(A_1 \cup B_1, A_2 \cap B_2, A_3 \cup B_3)$
  (b) or $(A_1 \cup B_1, A_2 \cap B_2, A_3 \cap B_3)$.
(2) *Intersection*: $\mathfrak{A} \cap \mathfrak{B} =$
  (a) $(A_1 \cap B_1, A_2 \cap B_2, A_3 \cup B_3)$
  (b) or $(A_1 \cap B_1, A_2 \cup B_2, A_3 \cap B_3)$.

Of course the class of *all* neutrosophic crisp sets is closed under all these operations. But the class of neutrosophic crisp sets of type 2 is not. This is because we do not have any guarantee that the components of the resulting set will sum up to X. For example, take $X = \{a, b, c, d, e\}$,

$\mathfrak{A} = (\{a, c\}, \{b\}, \{d, e\})$ and $\mathfrak{B} = (\{b, c\}, \{d\}, \{a, e\})$. Both these forms are neutrosophic crisp sets of type 2. Now use the first intersection: $\mathfrak{A} \cap \mathfrak{B} = (\{c\}, \emptyset, \{e\})$. Clearly, $\{c\} \cup \emptyset \cup \{e\} \neq X$.

Remark 6.1. Besides, there are some ambiguities in the theory of neutrosophic crisp sets. For example, in [14] we have Def. 3.1. where neutrosophic crisp sets *as such* are defined with the assumption that all three components are mutually exclusive. The same is repeated e.g. in [15]. But there is no such requirement in the book [13] that has been already mentioned, nor in [5] or [9] and many other papers. Thus we use the most general definition of neutrosophic crisp sets.

Hence, the first task is to introduce refined definitions of union and intersection (to make the class of NCS of type 2 closed under these operations). This will require some purely technical (but maybe a bit complicated) work. The next step would refer to the construction of an appropriate isomorphism. The whole program should be realized also with respect to the operations $\wedge$ and $\vee$.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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