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# A Study on $(\lambda - \mu)$ Zweier Sequences and Their Behaviour in Neutrosophic Normed Spaces

Vakeel A Khan and Mohd Faisal

<sup>1</sup> Department of Mathematics, Aligarh Muslim University Aligarh India; [vakhanmaths@gmail.com](mailto:vakhanmaths@gmail.com)

<sup>2</sup> Department of Mathematics, Aligarh Muslim University Aligarh India; [faisalmohd114@gmail.com](mailto:faisalmohd114@gmail.com)

**Abstract.** Ideal convergence of sequences in neutrosophic normed spaces is defined by Ömer Kışı [12]. This paper defines new sequence spaces using the Zweier matrix and neutrosophic norm. We explore  $(\lambda, \mu)$ -Zweier convergence,  $(\lambda, \mu)$  ideal convergence of double sequences in neutrosophic norm. We show that a double sequence that is  $(\lambda, \mu)$ -Zweier convergent or ideal convergent has a unique limit with respect to neutrosophic norm. Additionally, we prove that  $(\lambda, \mu)$ -Zweier ideal convergence is equivalent to  $(\lambda, \mu)$ -Zweier ideal Cauchy for a double sequence in neutrosophic normed space.

**Keywords:** Neutrosophic normed space, Statistical convergence,  $(\lambda - \mu)$  Zweier convergence,  $I$ -convergence.

## 1. Introduction and Preliminaries

The concept of fuzzy sets was introduced by Lotfi A. Zadeh [24] in 1965 as a mathematical framework to deal with uncertainty and vagueness in data. In a fuzzy set, each element is assigned a membership value ranging from 0 to 1, indicating the degree of membership of that element in the set. Kaleva, O. and Seikkala, S. (1984) [7] defined fuzzy metric space and Felbin, C. (1992) [4], Bag, T. and Samanta, S. K. (2003) [2] studied fuzzy normed linear space. Krassimir Atanassov [1] generalized the notion of fuzzy set to intuitionistic fuzzy set. Park [19] defined intuitionistic fuzzy metric space In 1995, Florentin Smarandache [23], [22] introduced the concept of neutrosophic sets as an extension of intuitionistic fuzzy sets. Neutrosophic sets aim to handle three types of indeterminacy: membership, non-membership and indeterminacy, represented by the values of truth-membership, falsehood-membership, and indeterminacy-membership respectively. Kirişci, M. and Şimşek, N. [10], [11] defined metric on neutrosophic set and also studied statistical convergence.

Concept of Statistical convergence was given H. Fast [5].  $\lambda$  statistical convergence was introduced by Mursaleen [16].  $\lambda$  statistical convergence is the extension of  $[V, \lambda]$  summability [14]. Mursaleen et al. [17] defined the concept of  $(\lambda, \mu)$  convergence for double sequence. The I-convergence [13] gives a unifying look on several types of convergence related to the statistical convergence. Mursaleen and Mohiuddine [18] defined I-convergence for double sequences in intuitionistic fuzzy normed spaces. In neutrosophic normed spaces, statistically convergent and statistically Cauchy double sequences are defined and studied by Granados C and Dhital [6]. Khan V. A. and Faisal M. [8], Khan V. A. and Ahmad M. [9] defined  $(\lambda, \mu)$  Zweier ideal convergence on neutrosophic normed space and studied Cesaro summability in neutrosophic normed spaces respectively.. Zweier-Verfahren is a German word, in which Zwei means two and Verfahren means method. Zweier matrix or operator is denoted as  $Z^\rho$ , where  $\rho \neq 1$ . Let  $x = (\xi_n) \in \omega$  where  $\omega = \{x = (\xi_n) : \xi_n \in \mathbb{R}/\mathbb{C}\}$ , then  $Z^\rho$  is defined as

$$Z^\rho = Z_{ik} = \begin{cases} \rho, & i = k \\ 1 - \rho, & i - 1 = k \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Şengönül [21], define new sequence  $y = (y_n)$  as  $Z^\rho$  transform of the sequence  $x = (\xi_n)$ ,

$$y_n = Z^\rho(\xi_n) = \rho\xi_n + (1 - \rho)\xi_{n-1} \tag{2}$$

where  $\xi_{-1} = 0$  and  $\rho \neq 1$ . Matrix

$$Z^\rho = \begin{bmatrix} \rho & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 - \rho & \rho & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 - \rho & \rho & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 - \rho & \rho & 0 & 0 & 0 & \dots \\ \cdot & \cdot & 0 & 1 - \rho & \rho & 0 & 0 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots \end{bmatrix}.$$

**Definition 1.1.** [20] A double sequence  $x = (\xi_{gh})$  is said to be convergent(Pringsheim’s Sense) to  $\mathfrak{p} > 0$ , if for every  $\kappa > 0$ , there exist  $N \in \mathbb{N}$  such that

$$|\xi_{gh} - \mathfrak{p}| < \kappa, \text{ whenever } g, h \geq N.$$

**Definition 1.2.** [20] A double sequence  $x = (\xi_{gh})$  is said to be Cauchy (Pringsheim’s Sense), if for every  $\kappa > 0$  there exist  $\mathfrak{N} \in \mathbb{N}$  such that

$$|\xi_{gh} - \xi_{pq}| < \kappa, \text{ whenever } g \geq p \geq \mathfrak{N} \text{ and } h \geq q \geq \mathfrak{N}.$$

**Definition 1.3.** Let  $\mathcal{K} \subseteq \mathbb{N} \times \mathbb{N}$ , if the sequence  $\frac{|\mathcal{K}(g, h)|}{nm}$  is convergent (Pringsheim's Sense), then we say that  $\mathcal{K}$  has double natural density and defined as

$$\delta_2(\mathcal{K}) = (P) \lim_{g, h} \frac{|\mathcal{K}(g, h)|}{gh}$$

where  $\mathcal{K}(g, h) = \{(p, q) \in \mathbb{N} \times \mathbb{N} : p \leq g \text{ and } q \leq h\}$ . If,  $g = h$  Christopher's [3] two-dimensional natural density is obtained.

**Definition 1.4.** [20] A double sequence  $x = (\xi_{gh})$  is said to be statistically convergent to  $\mathfrak{p} > 0$ , if for every  $\kappa > 0$ ,

$$\mathcal{A} = \{(g, h) : g \leq n, h \leq m \text{ such that } |\xi_{gh} - \mathfrak{p}| \geq \kappa\}$$

has  $\delta_2(\mathcal{A}) = 0$ .

**Definition 1.5.** [20] A double sequence  $x = (\xi_{gh})$  is said to be statistically Cauchy, if for every  $\kappa > 0$ , there exist  $\mathfrak{N} = \mathfrak{N}(\kappa)$  and  $\mathfrak{M} = \mathfrak{M}(\kappa)$  such that for all  $g, p \geq \mathfrak{N}$  and  $h, q \geq \mathfrak{M}$  the set

$$\mathcal{A} = \{(g, h) : g \leq \mathfrak{N}, h \leq \mathfrak{M} \text{ such that } |\xi_{gh} - \xi_{pq}| \geq \kappa\}$$

has  $\delta_2(\mathcal{A}) = 0$ .

**Definition 1.6.** [15] If continuous mapping  $\star : [0, 1] \times [0, 1] \rightarrow [0, 1]$  meets the following requirements, it is called a continuous  $t$ -norm :

(a)  $\star(\eta, \mathfrak{z}) = \star(\mathfrak{z}, \eta)$  and  $\star(\eta, \star(\mathfrak{z}, \mathfrak{c})) = \star(\star(\eta, \mathfrak{z}), \mathfrak{c})$ , for all  $\eta, \mathfrak{z}, \mathfrak{c} \in [0, 1]$ ,

(b)  $\star(\eta, 1) = \eta, \forall \eta \in [0, 1]$ ,

(c)  $\eta \leq \mathfrak{c}$  and  $\mathfrak{z} \leq \mathfrak{d} \implies \star(\eta, \mathfrak{z}) \leq \star(\mathfrak{c}, \mathfrak{d})$ , for each  $\eta, \mathfrak{z}, \mathfrak{c}, \mathfrak{d} \in [0, 1]$ .

**Definition 1.7.** [15] If continuous mapping  $\bullet : [0, 1] \times [0, 1] \rightarrow [0, 1]$  meets the following requirements, it is called continuous  $t$ -conorm if:

(a)  $\bullet(\eta, \mathfrak{z}) = \bullet(\mathfrak{z}, \eta)$  and  $\bullet(\eta, \bullet(\mathfrak{z}, \mathfrak{c})) = \bullet(\bullet(\eta, \mathfrak{z}), \mathfrak{c})$ , for all  $\eta, \mathfrak{z}, \mathfrak{c} \in [0, 1]$

(b)  $\bullet(\eta, 0) = \eta, \forall \eta \in [0, 1]$ ,

(c)  $\eta \leq \mathfrak{c}$  and  $\eta \leq \mathfrak{d} \implies \bullet(\eta, \eta) \leq \bullet(\mathfrak{c}, \mathfrak{d})$  for each  $\eta, \mathfrak{z}, \mathfrak{c}, \mathfrak{d} \in [0, 1]$ .

**Definition 1.8.** [11] Let  $V, \star$ , and  $\bullet$  be linear space, continuous  $t$ -norm and continuous  $t$ -conorm respectively. A four tuple of the form  $\{V, \phi(z, \cdot), \psi(z, \cdot), \gamma(z, \cdot) : z \in V\}$ , is called neutrosophic normed space where  $\phi, \psi$ , and  $\gamma$  are fuzzy sets on  $V \times \mathbb{R}^+$  which satisfy the following conditions:

(i)  $0 \leq \phi(\nu, p), \psi(\nu, p), \gamma(\nu, p) \leq 1$  for all  $p \in \mathbb{R}^+$ ,

(ii)  $0 \leq \phi(\nu, p) + \psi(\nu, p) + \gamma(\nu, p) \leq 3$  for all  $p \in \mathbb{R}^+$ ,

(iii)  $\phi(\nu, p) = 1$  (for  $p > 0$ ) if and only if  $\nu = 0$ ,

- (iv)  $\phi(\alpha\nu, p) = \phi(\nu, \frac{p}{|\alpha|})$  for  $\alpha \neq 0$
- (v)  $\star(\phi(\nu, p), \phi(y, q)) \leq \phi(\nu + y, p + q)$
- (vi)  $\phi(\nu, .)$  is continuous and non-decreasing
- (vii)  $\lim_{p \rightarrow \infty} \phi(\nu, p) = 1,$
- (viii)  $\psi(\nu, p) = 0$  for  $(p > 0)$  if and only if  $\nu = 0,$
- (ix)  $\psi(\alpha\nu, p) = \psi(\nu, \frac{p}{|\alpha|})$  for  $\alpha \neq 0,$
- (x)  $\bullet(\psi(\nu, p), \psi(y, q)) \geq \psi(\nu + y, p + q)$
- (xi)  $\psi(\nu, .)$  is continuous and non-increasing,
- (xii)  $\lim_{p \rightarrow \infty} \psi(\nu, p) = 0,$
- (xiii)  $\gamma(\nu, p) = 0$  (for  $p > 0$ ) if and only if  $\nu = 0,$
- (xiv)  $\gamma(\alpha\nu, p) = \gamma(\nu, \frac{p}{|\alpha|})$  if  $\alpha \neq 0,$
- (xv)  $\bullet(\gamma(\nu, p), \gamma(y, q)) \geq \gamma(\nu + y, p + q),$
- (xvi)  $\gamma(\nu, .)$  is continuous and non-increasing,
- (xvii)  $\lim_{p \rightarrow \infty} \gamma(\nu, p) = 0,$
- (xviii) If  $p \leq 0$  then  $\phi(\nu, p) = 0, \psi(\nu, p) = 1,$  and  $\gamma(\nu, p) = 1$

Then  $\mathcal{N} = (\phi, \psi, \gamma)$  is called neutrosophic norm. Throughout the paper we will use usual  $t$ -norm and usual  $t$ -conorm i.e;  $\star(\tau, \rho) = \min\{\tau, \rho\}$  and  $\bullet(\tau, \rho) = \max\{\tau, \rho\}.$

**Example 1.9.** [11] Let  $(V, \|\cdot\|)$  be a normed space. Let  $\phi, \psi, \gamma$  be Fuzzy sets on  $V \times \mathbb{R}^+$  such that, for  $t \geq \|\mathfrak{A}\|$

$$\phi(\mathfrak{A}, t) = \begin{cases} 0, & t \leq 0 \\ \frac{t}{t + \|\mathfrak{A}\|}, & t > 0, \end{cases} \quad \psi(\mathfrak{A}, t) = \begin{cases} 0, & t \leq 0 \\ \frac{\|\mathfrak{A}\|}{t + \|\mathfrak{A}\|}, & t > 0, \end{cases} \quad \text{and } \gamma(\mathfrak{A}, t) = \frac{\|\mathfrak{A}\|}{t}$$

for all  $\mathfrak{A} \in V$  and  $t \geq 0.$  If  $t \leq \|\mathfrak{A}\|$  then  $\phi(\mathfrak{A}, t) = 0, \psi(\mathfrak{A}, t) = 1$  and  $\gamma(\mathfrak{A}, t) = 1.$  Then  $(V, \mathcal{N}, \star, \bullet)$  is neutrosophic normed space.

Now we will discuss about the convergence of the sequence  $(\zeta_k)$  in neutrosophic normed space  $(V, \mathcal{N}, \star, \bullet).$

**Definition 1.10.** [11] Let  $(\xi_g)$  be sequence in NNS  $(V, \mathcal{N}, \star, \bullet)$  Then  $(\xi_g)$  is said to be convergent to  $\mathfrak{p} \in V$  if for each  $t > 0$  and  $\kappa \in (0,1)$  there exists  $g_0 \in \mathbb{N}$  such that

$$\phi(\xi_g - \mathfrak{p}, t) > 1 - \kappa, \psi(\xi_g - \mathfrak{p}, t) < \kappa \text{ and } \gamma(\xi_g - \mathfrak{p}, t) < \kappa \tag{3}$$

for all  $g \geq g_0.$

**Definition 1.11.** [11] Let  $(\xi_g)$  be sequence in NNS  $(V, \mathcal{N}, \star, \bullet)$  Then  $(\xi_g)$  is said to be Cauchy if, for each  $t > 0$  and  $\kappa \in (0,1)$  there exists  $g_0 \in \mathbb{N}$  such that

$$\phi(\xi_k - \xi_h, t) > 1 - \kappa, \psi(\xi_k - \xi_h, t) < \kappa \text{ and } \gamma(\xi_k - \xi_h, t) < \kappa \tag{4}$$

for all  $h, k \geq g_0$ .

**Definition 1.12.** [6] Let  $(V, \phi, \psi, \gamma, \star, \bullet)$  be a NNS. A double sequence  $x = (\xi_{gh})$  in  $V$  is said to be convergent statistically to  $\mathbf{p}$  if, for each  $\kappa > 0$  and  $t > 0$

$$\delta_2\{(g, h) \in \mathbb{N} \times \mathbb{N} : \phi(\xi_{gh} - \mathbf{p}, t) \leq 1 - \kappa, \psi(\xi_{gh} - \mathbf{p}, t) \geq \kappa \text{ and } \gamma(\xi_{gh} - \mathbf{p}, t) \geq \kappa\} = 0.$$

**Definition 1.13.** Let  $\Upsilon = (\Upsilon_g)$  and  $\Pi = (\Pi_h)$  be two positive non-decreasing sequences and  $\lim_{g \rightarrow \infty} \Upsilon_g = \infty, \lim_{h \rightarrow \infty} \Pi_h = \infty$ , defined as

$$\Upsilon_{g+1} \leq \Upsilon_g + 1, \quad \text{where } \Upsilon_1 = 0. \tag{5}$$

$$\Pi_{h+1} \leq \Pi_h + 1, \quad \text{where } \mu_1 = 0. \tag{6}$$

Let  $I_g = [g - \Upsilon_g + 1, g]$  and  $I_h = [h - \Pi_h + 1, h]$ . Then

$$\delta_{\Upsilon, \Pi}(\mathcal{K}) = \lim_{g, h \rightarrow \infty} \frac{1}{\Upsilon_g \mu_h} \left| \{(i, j) \in \mathbb{N} \times \mathbb{N} : (i, j) \in \mathcal{K}\} \right|$$

is the  $(\Upsilon, \Pi)$  density of the set  $\mathcal{K} \subseteq \mathbb{N} \times \mathbb{N}$ . If  $\Upsilon_g = g$  and  $\Pi_h = h$ , the  $(\Upsilon, \Pi)$  density reduces to natural double density of  $\mathcal{K}$ . The generalized double Valée Poussin mean is

$$t_{g, h} = \frac{1}{\Upsilon_g \mu_h} \sum_{i \in I_g} \sum_{j \in I_h} \xi_{ij}.$$

Throughout the paper we will denote the sequences 5,6 by  $\lambda = (\lambda_i)$  and  $\mu = (\mu_i)$  respectively.

## 2. Main Results

**Definition 2.1.** Let  $(V, \phi, \psi, \gamma, \star, \bullet)$  be a NNS and  $I^2$  be an admissible ideal in  $\mathbb{N} \times \mathbb{N}$ . A double sequence  $x = (\xi_{mn})$  in  $V$  is said to be  $I^2$ -convergent to  $\xi$  if, for each  $\epsilon > 0$  and  $t > 0$  the set

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : \phi(\xi_{mn} - \xi, t) \leq 1 - \epsilon, \psi(\xi_{mn} - \xi, t) \geq \epsilon \text{ and } \gamma(\xi_{mn} - \xi, t) \geq \epsilon\} \in I^2.$$

**Definition 2.2.** Let  $(V, \phi, \psi, \gamma, \star, \bullet)$  be a NNS. A double sequence  $x = (\xi_{mn})$  in  $V$  is said to be  $(\lambda, \mu)$ -Zweier convergent to  $\xi$  with respect to neutrosophic norm  $(\phi, \psi, \gamma)$  if, for every  $\epsilon > 0$  and  $t > 0$ , there exists a positive integer  $n_0 \in \mathbb{N}$  such that

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \phi(\mathcal{Z}^q \xi_{mn} - \xi, t) \leq 1 - \epsilon,$$

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \psi(\mathcal{Z}^q \xi_{mn} - \xi, t) \geq \epsilon,$$

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \gamma(\mathcal{Z}^q \xi_{mn} - \xi, t) \geq \epsilon$$

for all  $m, n \geq n_0$  and write it as  $(\lambda, \mu) - \lim_{m, n \rightarrow \infty} \mathcal{Z}^q \xi_{mn} = \xi$ .

**Definition 2.3.** Let  $(V, \mathcal{N}, \star, \bullet)$  be a NNS and  $I^2$  is an admissible ideal. A double sequence  $x = (\xi_{mn})$  in  $V$  is said to be  $(\lambda, \mu)$ - Zweier ideal convergent to  $\xi$  with respect to neutrosophic norm  $(\phi, \psi, \gamma)$ , if for every  $\epsilon > 0$  and for all  $t > 0$ , the set

$$\left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \phi(\mathcal{Z}^q \xi_{mn} - \xi, t) \leq 1 - \epsilon, \frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \psi(\mathcal{Z}^q \xi_{mn} - \xi, t) \geq \epsilon, \right. \\ \left. \text{and } \frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \gamma(\mathcal{Z}^q \xi_{mn} - \xi, t) \geq \epsilon \right\} \in I^2.$$

In short we write  $(\phi, \psi, \gamma) - I^2_{(\lambda, \mu)} - \lim_{m, n \rightarrow \infty} \mathcal{Z}^q \xi_{mn} = \xi$

**Definition 2.4.** Let  $(V, \mathcal{N}, \star, \bullet)$  be a NNS. A double sequence  $x = (\xi_{mn})$  in  $V$  is said to be  $(\lambda, \mu)$ - Zweier Cauchy sequence with respect to neutrosophic norm  $(\phi, \psi, \gamma)$ , if for every  $\epsilon > 0$  and for all  $t > 0$ , there exists  $N$  such that

$$\frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \phi(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t) \leq 1 - \epsilon, \\ \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \psi(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t) \geq \epsilon, \\ \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \gamma(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t) \geq \epsilon.$$

for all  $m, n, k, p \geq N$ .

**Definition 2.5.** Let  $(V, \mathcal{N}, \star, \bullet)$  be a NNS and  $I^2$  is an admissible ideal. A double sequence  $\xi = (\xi_{mn})$  in  $V$  is said to be  $(\lambda, \mu)$ -Zweier ideal Cauchy sequence if for each  $\epsilon > 0$  and  $t > 0$  there exists a positive integer  $N$  such that

$$\left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \phi(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t) \leq 1 - \epsilon, \right. \\ \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \psi(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t) \geq \epsilon, \\ \left. \text{and } \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \gamma(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t) \geq \epsilon \right\} \in I^2.$$

**Lemma 2.6.** Let  $(V, \mathcal{N}, \star, \bullet)$  be a NNS and  $I^2 \subseteq \mathbb{N} \times \mathbb{N}$  be an admissible ideal. Let  $x = (\xi_{mn})$  be a double sequence in  $V$ , then the following are equivalent

(i)  $(\phi, \psi, \gamma) - I^2_{(\lambda, \mu)} - \lim_{m, n \rightarrow \infty} \mathcal{Z}^q(\xi_{mn}) = \mathcal{L}$ .

$$(ii) \left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \phi(\mathcal{Z}^q \xi_{mn} - \mathcal{L}, t) \leq 1 - \kappa \right\} \in I^2,$$

$$\left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \psi(\mathcal{Z}^q \xi_{mn} - \mathcal{L}, t) \geq \kappa \right\} \in I^2 \text{ and}$$

$$\left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \gamma(\mathcal{Z}^q \xi_{mn} - \mathcal{L}, t) \geq \kappa \right\} \in I^2 \text{ for all } \kappa > 0 \text{ and } t > 0.$$

$$(iii) \left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \phi(\mathcal{Z}^q \xi_{mn} - \mathcal{L}, t) > 1 - \kappa \right\} \in F(I^2),$$

$$\left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \psi(\mathcal{Z}^q \xi_{mn} - \mathcal{L}, t) < \kappa \right\} \in F(I^2) \text{ and}$$

$$\left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \gamma(\mathcal{Z}^q \xi_{mn} - \mathcal{L}, t) < \kappa \right\} \in F(I^2) \text{ for all } \kappa > 0 \text{ and } t > 0.$$

$$(iv) (\phi, \psi, \gamma) - I^2_{(\lambda, \mu)} - \lim_{m, n \rightarrow \infty} \phi(\mathcal{Z}^q \xi_{mn} - \mathcal{L}, t) = 1,$$

$$(\phi, \psi, \gamma) - I^2_{(\lambda, \mu)} - \lim_{m, n \rightarrow \infty} \psi(\mathcal{Z}^q \xi_{mn} - \mathcal{L}, t) = 0 \text{ and}$$

$$(\phi, \psi, \gamma) - I^2_{(\lambda, \mu)} - \lim_{m, n \rightarrow \infty} \gamma(\mathcal{Z}^q \xi_{mn} - \mathcal{L}, t) = 0 \text{ for all } \kappa > 0 \text{ and } t > 0.$$

**Theorem 2.7.** Let  $(V, \mathcal{N}, \star, \bullet)$  be a NNS and  $I^2$  is an admissible ideal. If a double sequence  $x = (\xi_{mn})$  in  $V$  is  $(\lambda, \mu)$ - Zweier ideal convergent with respect to neutrosophic norm  $(\phi, \psi, \gamma)$ , then its limit is unique.

*Proof.* We will prove it by contradiction i.e. Let on contrary that the sequence  $x = (\xi_{mn})$  converges to two limits (say)  $\ell_1$  and  $\ell_2$ . By definition 2.3, for a given  $\epsilon > 0$ , choose  $\kappa > 0$  such that  $(1 - \kappa) \star (1 - \kappa) > 1 - \epsilon$  and  $\kappa \star \kappa < \epsilon$ . Then for  $t > 0$

$$A_{\phi,1}(\kappa, t) = \left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \phi\left(\mathcal{Z}^q \xi_{mn} - \ell_1, \frac{t}{2}\right) \leq 1 - \kappa \right\} \in I^2.$$

$$A_{\phi,2}(\kappa, t) = \left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \phi\left(\mathcal{Z}^q \xi_{mn} - \ell_2, \frac{t}{2}\right) \leq 1 - \kappa \right\} \in I^2.$$

$$A_{\psi,1}(\kappa, t) = \left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \psi\left(\mathcal{Z}^q \xi_{mn} - \ell_1, \frac{t}{2}\right) \geq \kappa \right\} \in I^2.$$

$$A_{\psi,2}(\kappa, t) = \left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \psi\left(\mathcal{Z}^q \xi_{mn} - \ell_2, \frac{t}{2}\right) \geq \kappa \right\} \in I^2.$$

$$A_{\gamma,1}(\kappa, t) = \left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \gamma\left(\mathcal{Z}^q \xi_{mn} - \ell_1, \frac{t}{2}\right) \geq \kappa \right\} \in I^2.$$



$$A_{\gamma,2}(\kappa, t) = \left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \gamma \left( \mathcal{Z}^q \xi_{mn} - \ell_2, \frac{t}{2} \right) \geq \kappa \right\} \in I^2.$$

Now the following set

$$A_{\phi,\psi,\gamma}(\kappa, t) = [A_{\phi,1}(\kappa, t) \cup A_{\phi,2}(\kappa, t)] \cap [A_{\psi,1}(\kappa, t) \cup A_{\psi,2}(\kappa, t)] \cap [A_{\gamma,1}(\kappa, t) \cup A_{\gamma,2}(\kappa, t)] \in I^2$$

$$A_{\phi,\psi,\gamma}(\kappa, t) \in I^2 \implies A_{\phi,\psi,\gamma}^{\mathbb{C}}(\kappa, t) \in F(I^2).$$

If  $(i, j) \in A_{\phi,\psi,\gamma}^{\mathbb{C}}(\kappa, t)$ , then there are three cases. Either  $(i, j) \in A_{\phi,1}^{\mathbb{C}}(\kappa, t) \cap A_{\phi,2}^{\mathbb{C}}(\kappa, t)$  or  $(i, j) \in A_{\psi,1}^{\mathbb{C}}(\kappa, t) \cap A_{\psi,2}^{\mathbb{C}}(\kappa, t)$  or  $(i, j) \in A_{\gamma,1}^{\mathbb{C}}(\kappa, t) \cap A_{\gamma,2}^{\mathbb{C}}(\kappa, t)$ . Firstly, if  $(i, j) \in A_{\phi,1}^{\mathbb{C}}(\kappa, t) \cap A_{\phi,2}^{\mathbb{C}}(\kappa, t)$ , we have

$$\frac{1}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \phi \left( \mathcal{Z}^q \xi_{mn} - \ell_1, \frac{t}{2} \right) > 1 - \kappa$$

$$\frac{1}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \phi \left( \mathcal{Z}^q \xi_{mn} - \ell_2, \frac{t}{2} \right) > 1 - \kappa$$

Now take  $(a, b) \in \mathbb{N} \times \mathbb{N}$ , such that

$$\phi \left( \mathcal{Z}^q \xi_{ab} - \ell_1, \frac{t}{2} \right) > \frac{1}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \phi \left( \mathcal{Z}^q \xi_{mn} - \ell_1, \frac{t}{2} \right) > 1 - \kappa$$

and

$$\phi \left( \mathcal{Z}^q \xi_{ab} - \ell_2, \frac{t}{2} \right) > \frac{1}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \phi \left( \mathcal{Z}^q \xi_{mn} - \ell_2, \frac{t}{2} \right) > 1 - \kappa.$$

Select  $(a, b)$  for  $(m, n)$  such that the maximum hold i.e.  $\max \left\{ \phi \left( \mathcal{Z}^q \xi_{mn} - \ell_1, \frac{t}{2} \right), \psi \left( \mathcal{Z}^q \xi_{mn} - \ell_2, \frac{t}{2} \right), \gamma \left( \mathcal{Z}^q \xi_{mn} - \ell_2, \frac{t}{2} \right) : m \in \Lambda_i, n \in \Lambda_j \right\}$ . Then, we have

$$\phi(\ell_1 - \ell_2, t) \geq \phi \left( \mathcal{Z}^q \xi_{ab} - \ell_1, \frac{t}{2} \right) \star \phi \left( \mathcal{Z}^q \xi_{ab} - \ell_2, \frac{t}{2} \right) > (1 - \kappa) \star (1 - \kappa) > 1 - \epsilon$$

Since  $\epsilon > 0$  was arbitrary, for every  $t > 0$ , we get  $\phi(\ell_1 - \ell_2, t) = 1$ , that means  $\ell_1 = \ell_2$ . For other cases if,  $(i, j) \in A_{\psi,1}^{\mathbb{C}}(\kappa, t) \cap A_{\psi,2}^{\mathbb{C}}(\kappa, t)$  or  $(i, j) \in A_{\gamma,1}^{\mathbb{C}}(\kappa, t) \cap A_{\gamma,2}^{\mathbb{C}}(\kappa, t)$ , in similar fashion we get  $\psi(\ell_1 - \ell_2, t) > \epsilon$  and  $\gamma(\ell_1 - \ell_2, t) > \epsilon$  for  $t > 0$ . Hence for each case, we get  $\ell_1 = \ell_2$ . Hence limit is unique.  $\square$

**Theorem 2.8.** *Let  $(V, \mathcal{N}, \star, \bullet)$  be a NNS. If a double sequence  $x = (\xi_{mn})$  in  $V$  is  $(\lambda, \mu)$ -Zweier convergent with respect to neutrosophic norm  $(\phi, \psi, \gamma)$ , then its limit is unique.*

*Proof.* Let  $(\lambda, \mu) - \lim_{m,n \rightarrow \infty} \mathcal{Z}^q \xi_{mn} = \xi_1$  and  $(\lambda, \mu) - \lim_{m,n \rightarrow \infty} \mathcal{Z}^q \xi_{mn} = \xi_2$ . Given  $\epsilon > 0$ , select  $\kappa > 0$  such that  $(1 - \kappa) \star (1 - \kappa) > 1 - \epsilon$  and  $\kappa \bullet \kappa < \epsilon$ . Then for any  $t > 0$ , there exists  $n_1 \in \mathbb{N}$  such that

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \phi(\mathcal{Z}^q \xi_{mn} - \xi_1, t) > 1 - \epsilon$$

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \psi(\mathcal{Z}^q \xi_{mn} - \xi_1, t) < \epsilon$$

and

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \gamma(\mathcal{Z}^q \xi_{mn} - \xi_1, t) < \epsilon$$

for all  $i, j \geq n_1$ . There also exists  $n_2$  such that

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \phi(\mathcal{Z}^q \xi_{mn} - \xi_2, t) > 1 - \epsilon$$

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \psi(\mathcal{Z}^q \xi_{mn} - \xi_2, t) < 1 - \epsilon$$

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \lambda(\mathcal{Z}^q \xi_{mn} - \xi_2, t) < \epsilon$$

for all  $i, j \geq n_2$ . Now chose  $n_0 = \max\{n_1, n_2\}$ . Then for all  $n \geq n_0$ , we have  $(a, b) \in \mathbb{N} \times \mathbb{N}$  such that

$$\phi(\mathcal{Z}^q \xi_{ab} - \xi_1, t) > \frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \phi(\mathcal{Z}^q \xi_{mn} - \xi_1, t) > 1 - \kappa$$

. and

$$\phi(\mathcal{Z}^q \xi_{ab} - \xi_2, t) > \frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \phi(\mathcal{Z}^q \xi_{mn} - \xi_2, t) > 1 - \kappa.$$

Hence, we get

$$\phi(\xi_1 - \xi_2, t) \geq \phi(\mathcal{Z}^q \xi_{mn} - \xi_1, t) \star \phi(\mathcal{Z}^q \xi_{mn} - \xi_2, t) > (1 - \kappa) \star (1 - \kappa) > 1 - \epsilon.$$

Since  $\epsilon$  was arbitrary. Hence we get  $\phi(\xi_1 - \xi_2, t) = 1$  for all  $t > 0$ . Hence  $\xi_1 = \xi_2$ . Similarly, we will prove for  $\psi$  and  $\gamma$  that  $\psi(\xi_1 - \xi_2, t) < \epsilon$  and  $\gamma(\xi_1 - \xi_2, t) < \epsilon$  for any  $\epsilon > 0$ . Hence we get  $\psi(\xi_1 - \xi_2, t) = 0$  and  $\gamma(\xi_1 - \xi_2, t) = 0$  for  $t > 0$ , this implies that  $\xi_1 = \xi_2$ .  $\square$

**Theorem 2.9.** Let  $(V, \mathcal{N}, \star, \bullet)$  be a NNS.  $(\lambda, \mu)$ -Zweier convergence of a double sequence  $x = (\xi_{mn})$  in  $V$  implies  $(\lambda, \mu)$ -Zweier ideal convergence of  $x = (\xi_{mn})$  with respect to neutrosophic norm and  $(\lambda, \mu) - \lim_{m,n \rightarrow \infty} \mathcal{Z}^q \xi_{mn} = (\phi, \psi, \gamma) - I_{(\lambda, \mu)}^2 - \lim_{m,n \rightarrow \infty} \mathcal{Z}^q \xi_{mn} = \xi$ .

*Proof.* Let sequence  $x = (\xi_{mn})$  is  $(\lambda, \mu)$ -Zweier convergent to  $\xi$ . Then for  $\epsilon > 0$  and  $t > 0$ , there exists positive integer  $N$  such that

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \phi(\mathcal{Z}^q \xi_{mn} - \xi, t) > 1 - \epsilon,$$

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \psi(\mathcal{Z}^q \xi_{mn} - \xi, t) < \epsilon,$$

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \gamma(\mathcal{Z}^q \xi_{mn} - \xi, t) < \epsilon$$

for all  $i, j \geq N$ . Then the set

$$R(\epsilon, t) = \left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \phi(\mathcal{Z}^q \xi_{mn} - \xi, t) \leq 1 - \epsilon, \right. \\ \left. \frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \psi(\mathcal{Z}^q \xi_{mn} - \xi, t) \geq \epsilon, \text{ and } \frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \gamma(\mathcal{Z}^q \xi_{mn} - \xi, t) \geq \epsilon, \right\}.$$

is contained in the set  $Q = \{(1, 1), (1, 2), (2, 1), (2, 2), \dots, (N - 1, N - 1)\}$ . Since  $I^2$  is an admissible ideal. We get,  $R(\epsilon, t) \subseteq Q \in I^2$ . Hence  $(\lambda, \mu)$ -Zweier convergence of a double sequence implies  $(\lambda, \mu)$ -Zweier ideal convergence and  $(\phi, \psi, \gamma) - I^2_{(\lambda, \mu)} - \lim_{m, n \rightarrow \infty} \mathcal{Z}^q \xi_{mn} = \xi$ .  $\square$

**Theorem 2.10.** *Let  $(V, \mathcal{N}, \star, \bullet)$  be a NNS. A double sequence  $x = (\xi_{mn})$  in  $V$  is  $(\lambda, \mu)$ -Zweier ideal convergent if and only if it is  $(\lambda, \mu)$ -Zweier ideal Cauchy.*

*Proof.* Let  $x = (\xi_{mn})$  in  $V$  is  $(\lambda, \mu)$ -Zweier ideal Cauchy sequence but not  $(\lambda, \mu)$ -Zweier ideal convergent in  $V$  with respect to neutrosophic norm  $(\phi, \psi, \gamma)$ . Now, by the definition of  $(\lambda, \mu)$ -Zweier ideal Cauchy for each  $\epsilon > 0$  and  $t > 0$ , there exists positive integer  $N$  such that

$$G(\epsilon, t) = \left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \phi(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t) \leq 1 - \epsilon, \right. \\ \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \psi(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t) \geq \epsilon, \\ \left. \text{and } \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \gamma(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t) \geq \epsilon \right\} \in I^2.$$

and

$$H(\epsilon, t) = \left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{2}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \phi\left(\mathcal{Z}^q \xi_{mn} - \xi, \frac{t}{2}\right) \leq 1 - \epsilon, \right. \\ \left. \frac{2}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \psi\left(\mathcal{Z}^q \xi_{mn} - \xi, \frac{t}{2}\right) \geq \epsilon, \text{ and} \right. \\ \left. \frac{2}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \gamma\left(\mathcal{Z}^q \xi_{mn} - \xi, \frac{t}{2}\right) \geq \epsilon \right\} \in F(I^2)$$

Since

$$\frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \phi(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t) \geq \frac{2}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \phi\left(\mathcal{Z}^q \xi_{mn} - \xi, \frac{t}{2}\right) > 1 - \epsilon, \\ \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \psi(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t) \leq \frac{2}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \psi\left(\mathcal{Z}^q \xi_{mn} - \xi, \frac{t}{2}\right) < \epsilon,$$

and

$$\frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \gamma(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t) \leq \frac{2}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \gamma\left(\mathcal{Z}^q \xi_{mn} - \xi, \frac{t}{2}\right) < \epsilon.$$

If

$$\frac{1}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \phi\left(\mathcal{Z}^q \xi_{mn} - \xi, \frac{t}{2}\right) > \frac{1 - \epsilon}{2}, \frac{1}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \psi\left(\mathcal{Z}^q \xi_{mn} - \xi, \frac{t}{2}\right) < \frac{\epsilon}{2}, \\ \frac{1}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \gamma\left(\mathcal{Z}^q \xi_{mn} - \xi, \frac{t}{2}\right) < \frac{\epsilon}{2}.$$

Then, we have

$$\delta(\lambda, \mu) \left( \left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \phi(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t) \leq 1 - \epsilon, \right. \right. \\ \left. \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \psi(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t) \geq \epsilon, \right. \\ \left. \left. \text{and } \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \gamma(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t) \geq \epsilon \right\} \right) = 0$$

That is  $G(\epsilon, t) \in F(I^2)$ , which is a contradiction. Hence, sequence  $\xi = (\xi_{mn})$  is  $(\lambda, \mu)$ -Zweier ideal convergent with respect to neutrosophic norm.

Conversely, suppose that  $(\phi, \psi, \gamma) - I^2_{(\lambda, \mu)} - \lim_{m, n \rightarrow \infty} \mathcal{Z}^q \xi_{mn} = \xi$ . Choose  $r > 0$ , such that  $(1 - r) \star (1 - r) > 1 - \epsilon$  and  $r \bullet r < \epsilon$ . For  $t > 0$  define

$$W(r, t) = \left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \phi \left( \mathcal{Z}^q \xi_{mn} - \xi, \frac{t}{2} \right) \leq 1 - r, \right. \\ \left. \frac{1}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \psi \left( \mathcal{Z}^q \xi_{mn} - \xi, \frac{t}{2} \right) \geq r, \text{ and} \right. \\ \left. \frac{1}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \gamma \left( \mathcal{Z}^q \xi_{mn} - \xi, \frac{t}{2} \right) \geq r \right\} \in I^2$$

or  $W(r, t)^c \in F(I^2)$ . Assume  $(k, p) \in W(r, t)^c$ . Then, we get

$$\frac{1}{\lambda_i \mu_j} \sum_{k \in \Lambda_i} \sum_{p \in \Lambda_j} \phi \left( \mathcal{Z}^q \xi_{mn} - \xi, \frac{t}{2} \right) > 1 - r \\ \frac{1}{\lambda_i \mu_j} \sum_{k \in \Lambda_i} \sum_{p \in \Lambda_j} \psi \left( \mathcal{Z}^q \xi_{mn} - \xi, \frac{t}{2} \right) < r \\ \frac{1}{\lambda_i \mu_j} \sum_{k \in \Lambda_i} \sum_{p \in \Lambda_j} \gamma \left( \mathcal{Z}^q \xi_{mn} - \xi, \frac{t}{2} \right) < r$$

For every  $\epsilon > 0$ , we take

$$U(\epsilon, t) = \left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \phi \left( \mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t \right) \leq 1 - \epsilon, \right. \\ \left. \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \psi \left( \mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t \right) \geq \epsilon, \right. \\ \left. \text{and } \frac{1}{\lambda_i \mu_j} \sum_{m, k \in \Lambda_i} \sum_{n, p \in \Lambda_j} \gamma \left( \mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t \right) \geq \epsilon \right\} \in I^2.$$

Now we will show that  $U(\epsilon, t) \subset W(r, t)$ . Let  $(u, v) \in U(\epsilon, t)$

$$\frac{1}{\lambda_i \mu_j} \sum_{u, k \in \Lambda_i} \sum_{v, p \in \Lambda_j} \phi \left( \mathcal{Z}^q \xi_{uv} - \mathcal{Z}^q \xi_{kp}, t \right) \leq 1 - \epsilon, \\ \frac{1}{\lambda_i \mu_j} \sum_{u, k \in \Lambda_i} \sum_{v, p \in \Lambda_j} \psi \left( \mathcal{Z}^q \xi_{uv} - \mathcal{Z}^q \xi_{kp}, t \right) \geq \epsilon, \\ \frac{1}{\lambda_i \mu_j} \sum_{u, k \in \Lambda_i} \sum_{v, p \in \Lambda_j} \gamma \left( \mathcal{Z}^q \xi_{uv} - \mathcal{Z}^q \xi_{kp}, t \right) \geq \epsilon.$$

Here we are going to divide it into three cases as follows

**Case1 :**

$$\frac{1}{\lambda_i \mu_j} \sum_{u, k \in \Lambda_i} \sum_{v, p \in \Lambda_j} \phi \left( \mathcal{Z}^q \xi_{uv} - \mathcal{Z}^q \xi_{kp}, t \right) \leq 1 - \epsilon.$$

Then

$$\frac{1}{\lambda_i \mu_j} \sum_{u \in \Lambda_i} \sum_{v \in \Lambda_j} \phi\left(\mathcal{Z}^q \xi_{uv} - \xi, \frac{t}{2}\right) \leq 1 - r \implies (u, v) \in W(r, t).$$

Otherwise, if  $\frac{1}{\lambda_i \mu_j} \sum_{u \in \Lambda_i} \sum_{v \in \Lambda_j} \phi\left(\mathcal{Z}^q \xi_{uv} - \xi, \frac{t}{2}\right) > 1 - r$ . Then, we have

$$\begin{aligned} 1 - \epsilon &\geq \frac{1}{\lambda_i \mu_j} \sum_{u, k \in \Lambda_i} \sum_{v, p \in \Lambda_j} \phi\left(\mathcal{Z}^q \xi_{uv} - \mathcal{Z}^q \xi_{kp}, t\right) \\ &\geq \frac{1}{\lambda_i \mu_j} \sum_{u \in \Lambda_i} \sum_{v \in \Lambda_j} \phi\left(\mathcal{Z}^q \xi_{uv} - \xi, \frac{t}{2}\right) \star \frac{1}{\lambda_i \mu_j} \sum_{k \in \Lambda_i} \sum_{p \in \Lambda_j} \phi\left(\mathcal{Z}^q \xi_{kp} - \xi, \frac{t}{2}\right) \\ &> (1 - r) \star (1 - r) \\ &> 1 - \epsilon. \end{aligned}$$

We reach at a contradiction. Hence  $U(\epsilon, t) \subset W(r, t)$ .

**Case2 :** Consider

$$\frac{1}{\lambda_i \mu_j} \sum_{u, k \in \Lambda_i} \sum_{v, p \in \Lambda_j} \psi\left(\mathcal{Z}^q \xi_{uv} - \mathcal{Z}^q \xi_{kp}, t\right) \geq \epsilon.$$

We get

$$\frac{1}{\lambda_i \mu_j} \sum_{u \in \Lambda_i} \sum_{v \in \Lambda_j} \psi\left(\mathcal{Z}^q \xi_{uv} - \xi, \frac{t}{2}\right) \geq r \implies (u, v) \in W(r, t).$$

Otherwise, if

$$\frac{1}{\lambda_i \mu_j} \sum_{u \in \Lambda_i} \sum_{v \in \Lambda_j} \psi\left(\mathcal{Z}^q \xi_{uv} - \xi, \frac{t}{2}\right) < r.$$

Then we obtain

$$\begin{aligned} 1 - \epsilon &\leq \frac{1}{\lambda_i \mu_j} \sum_{u, k \in \Lambda_i} \sum_{v, p \in \Lambda_j} \phi\left(\mathcal{Z}^q \xi_{uv} - \mathcal{Z}^q \xi_{kp}, t\right) \\ &\leq \frac{1}{\lambda_i \mu_j} \sum_{u \in \Lambda_i} \sum_{v \in \Lambda_j} \phi\left(\mathcal{Z}^q \xi_{uv} - \xi, \frac{t}{2}\right) \bullet \frac{1}{\lambda_i \mu_j} \sum_{k \in \Lambda_i} \sum_{p \in \Lambda_j} \phi\left(\mathcal{Z}^q \xi_{kp} - \xi, \frac{t}{2}\right) \\ &< (1 - r) \bullet (1 - r) \\ &< 1 - \epsilon. \end{aligned}$$

Which is a contradiction. Hence,  $U(\epsilon, t) \subset W(r, t)$ .

Case 3 is similar to the case 2 , in this case we will consider that

$$\frac{1}{\lambda_i \mu_j} \sum_{u, k \in \Lambda_i} \sum_{v, p \in \Lambda_j} \gamma\left(\mathcal{Z}^q \xi_{uv} - \mathcal{Z}^q \xi_{kp}, t\right) \geq \epsilon.$$

And we obtain that,  $U(\epsilon, t) \subset W(r, t)$ .

In each case we obtain that  $U(\epsilon, t) \subset W(r, t)$ . This implies that  $U(\epsilon, t) \in I^2$ . Hence double sequence is a Zweier ideal Cauchy sequence with respect to th neutrosophic norm  $(\phi, \psi, \gamma)$ . □

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