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Extension for neutrosophic vague subbisemirings of bisemirings

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Abstract. This paper introduces the idea of a neutrosophic vague subbisemiring (NSVSBS), level sets of NSVSBS, and (ρ, σ)-neutrosophic vague subbisemiring ((ρ, σ)-NSVSBS) of a bisemiring. NSVSBSs are generalizations of neutrosophic subbisemirings and SBS based on bisemirings. Let Λ be a neutrosophic vague subset in $\mathcal{B}$, we show that $\mathcal{V} = ([\mathcal{T}^-_\Lambda, \mathcal{T}^+_\Lambda], [\mathcal{I}^-_\Lambda, \mathcal{I}^+_\Lambda], [\mathcal{F}^-_\Lambda, \mathcal{F}^+_\Lambda])$ is a NSVSBS of $\mathcal{B}$ if and only if all non empty level set $\mathcal{V}^{(t_1,t_2,s)}$ is a SBS of $\mathcal{B}$ for $t_1, t_2, s \in [0, 1]$. In the case that Λ is a NSVSBS of a bisemiring $\mathcal{B}$ and V is the strongest neutrosophic vague relation of $\mathcal{B}$, we prove that Λ is a NSVSBS of $\mathcal{B} \times \mathcal{B}$. Let Λ be any NSVSBS of $\mathcal{B}$, prove that pseudo neutrosophic vague coset $(\tau\Lambda)^p$ is a NSVSBS of $\mathcal{B}$, for every $\tau \in \mathcal{B}$. Let $\Lambda_1, \Lambda_2, ..., \Lambda_n$ be the family of $NSVSBS^s$ of $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n$ respectively. We show that $\Lambda_1 \times \Lambda_2 \times ... \times \Lambda_n$ is a NSVSBS of $\mathcal{B}_1 \times \mathcal{B}_2 \times ... \times \mathcal{B}_n$. The homomorphic image of every NSVSBS is a NSVSBS. The homomorphic pre-image of every NSVSBS is a NSVSBS. Examples are provided to strengthen our results.

Keywords: subbisemiring; neutrosophic subbisemiring; neutrosophic vague bisemiring; homomorphism

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1. **Introduction**

Due to the limitations of classical mathematics, such as fuzzy set (FS) [1] and vague set (VS) [2], mathematical theories have been developed to address uncertainty and fuzziness. In the case of uncertain or vague situations, FS introduced by Zadeh [1] is the most appropriate technique. In recent years, many hybrid fuzzy models have been developed based on FS. A generalization of FS, intuitionistic fuzzy set (IFS) incorporate hesitation levels into the notion of FS, which were first proposed by Attanasov [3] in 1983. The neutrosophic set (NSS) was proposed in 1999 by Smarandache [4]. In NSS, each proposition is estimated to have a degree of truth, an indeterminacy degree, and a falsity degree. As a result of Smarandache [5], he further generalised and expanded the theory of IFSs to include the neutrosophic model as well. A study of fuzzy semirings was initiated by Ahsan et al. [6]. Palanikumar et al. [?, ?] discussed tri-quasi-ideals and bi-quasi-ideals are natural generalizations of rings such that they constitute a natural generalization of ternary semirings, semirings and ordered semirings. In 2004, Sen et al. [17] extended the study of semirings and proposed the concept of bisemiring to further develop them. The study of vague algebra was initiated by Biswas [18] through the introduction of vague groups, vague cuts and vague normal groups. In their work, Arulmozhi et al. [19] focus on the interaction between semirings, ternary semirings and other algebraic structures. A semiring $(S, +, \cdot)$ is a non-empty set in which $(S, +)$ and $(S, \cdot)$ are semigroups such that "$\cdot$" is distributive over "$+$" [20]. In 1993, Ahsan et al. [6] introduced the notion of fuzzy semirings.

An introduction to bisemirings was made in 2001 by Sen et al. [21]. A bisemiring $(\mathcal{B}, \partial, \circledcirc, \boxtimes)$ is an algebraic structure in which $(\mathcal{B}, \partial, \circledcirc)$ and $(\mathcal{B}, \circledcirc, \boxtimes)$ are semirings in which $(\mathcal{B}, \partial), (\mathcal{B}, \circledcirc)$ and $(\mathcal{B}, \boxtimes)$ are semigroups such that (a) $\zeta \circledcirc (\Im \partial \tau) = (\Re \circledcirc \Im) \partial (\Re \circledcirc \tau)$, (b) $(\Im \partial \tau) \circledcirc \Re = (\Im \circledcirc \Re) \partial (\tau \circledcirc \Re)$, (c) $\Re \boxtimes (\Im \circledcirc \tau) = (\Re \boxtimes \Im) \circledcirc (\Re \boxtimes \tau)$ and (d) $(\Im \circledcirc \tau) \boxtimes \Re = (\Im \boxtimes \Re) \circledcirc (\tau \boxtimes \Re)$ for all $\Re, \Im, \tau \in \mathcal{B}$ [17]. A non-empty subset Λ of a bisemiring $(\mathcal{B}, \partial, \circledcirc, \boxtimes)$ is a subbisemiring (SBS) if and only if $\Re \partial \Im \in \Lambda, \Re \circledcirc \Im \in \Lambda$ and $\Re \boxtimes \Im \in \Lambda$ for all $\Re, \Im \in \Lambda$ [21]. Palanikumar et al. discussed the various ideal structures of SBS theory and its applications [7]- [16]. However, numerous algebraic concepts had been generalized using FS theory. Fuzzy algebraic structures of semirings have been extensively investigated by Vandiver [22]. These are generalizations of rings requiring only a monoid, rather than a group, to achieve a particular additive structure and have been shown to be useful for a wide range of problems. Golan [20] and Glazek [23] have both extensively studied the application of semirings.

Bipolar fuzzy information has been applied to various algebraic structures over the past few years, like semigroups [?, 14, 15] and BCK/BCI algebras [24–27]. An application of bipolar fuzzy metric spaces was discussed by Zararsz et al. [28]. A vague soft hyperring and a vague soft hyper ideal were introduced by Selvachandran [29]. The bipolar fuzzy translation was introduced by Jun et al. [30] and BCK/BCI-algebra and its properties were investigated. A bipolar fuzzy regularity, bipolar fuzzy regular sub-algebra, a bipolar fuzzy filter, and a bipolar fuzzy closed quasi filter have been introduced into BCH algebras in [31]. In 2004, Sen et al. [17] contributed to the field of semirings by proposing bisemiring as a concept. Hussain et al. [32] defined the congruence relation between bisemirings and bisemiring homomorphisms. In addition to bisemiring, Hussain et al. [21, 32] described an algebraic structure called semiring and congruence relations between homomorphisms and n-semirings based on this algebraic structure.

Neutosophic vague subbisemirings (NSVSBS) are discussed here, as well as their level sets. Subbisemirings are a generalization of bisemirings, and NSVSBSs are a generalization of subbisemirings. A number of illustrative examples are provided to illustrate the theory for (ξ, τ)-NSVSBS over bisemiring theory. Following is an outline of the preliminary definitions and results presented in Section 2. The concept of a NSVSBS is introduced in Section 3. There is more information about (ξ, τ)-NSVSBS in Section 4.

2. Basic concepts

For our future studies, we will quickly review some fundamental terms in this section.

Definition 2.1. [4] A neutrosophic set (NSS) Λ in a universal set $\mathcal{U}$ is $\Lambda = \{(\Re, \mathcal{T}_\Lambda(\Re), \mathcal{I}_\Lambda(\Re), \mathcal{F}_\Lambda(\Re)) : \Re \in \mathcal{U}\}$, where $\mathcal{T}_\Lambda, \mathcal{I}_\Lambda, \mathcal{F}_\Lambda : \mathcal{U} \to [0,1]$ denotes the truth, indeterminacy and the falsity membership function, respectively. For $\langle \mathcal{T}_\Lambda, \mathcal{I}_\Lambda, \mathcal{F}_\Lambda \rangle$ is used for the NSS $\Lambda = \{(\Re, \mathcal{T}_\Lambda(\Re), \mathcal{I}_\Lambda(\Re), \mathcal{F}_\Lambda(\Re)) : \Re \in \mathcal{U}\}$.

Definition 2.2. [4] Let $\Lambda = \langle \mathcal{T}_\Lambda, \mathcal{I}_\Lambda, \mathcal{F}_\Lambda \rangle$ and $\Psi = \langle \mathcal{T}_\Psi, \mathcal{I}_\Psi, \mathcal{F}_\Psi \rangle$ be the two NSS of $\mathcal{U}$. Then

(1) $\Lambda \cap \Psi = \{(\Re, \min\{\mathcal{T}_\Lambda(\Re), \mathcal{T}_\Psi(\Re)\}, \min\{\mathcal{I}_\Lambda(\Re), \mathcal{I}_\Psi(\Re)\}, \max\{\mathcal{F}_\Lambda(\Re), \mathcal{F}_\Psi(\Re)\}) : \Re \in \mathcal{U}\}$,

(2) $\Lambda \cup \Psi = \{(\Re, \max\{\mathcal{T}_\Lambda(\Re), \mathcal{T}_\Psi(\Re)\}, \max\{\mathcal{I}_\Lambda(\Re), \mathcal{I}_\Psi(\Re)\}, \min\{\mathcal{F}_\Lambda(\Re), \mathcal{F}_\Psi(\Re)\}) : \Re \in \mathcal{U}\}$.

Definition 2.3. [4] For any NSS $\Lambda = \langle \mathcal{T}_\Lambda, \mathcal{I}_\Lambda, \mathcal{F}_\Lambda \rangle$ of $\mathcal{U}$, we defined a (ρ, σ)-cut of as the crisp subset $\{\Re \in \mathcal{U} : \mathcal{T}_\Lambda(\Re) \geq \rho, \mathcal{I}_\Lambda(\Re) \geq \rho, \mathcal{F}_\Lambda(\Re) \leq \sigma\}$ of $\mathcal{U}$.

Definition 2.4. [4] Let Λ and Ψ be two neutrosophic subsets of S. The Cartesian product of Λ and Ψ is defined as $\Lambda \times \Psi = \{((\Re, \Im), \mathcal{T}_{\Lambda\times\Psi}(\Re, \Im), \mathcal{I}_{\Lambda\times\Psi}(\Re, \Im), \mathcal{F}_{\Lambda\times\Psi}(\Re, \Im)) : \Re, \Im \in S\}$, where $\mathcal{T}_{\Lambda\times\Psi}(\Re, \Im) = \min\{\mathcal{T}_\Lambda(\Re), \mathcal{T}_\Psi(\Im)\}, \mathcal{I}_{\Lambda\times\Psi}(\Re, \Im) = \frac{\mathcal{I}_\Lambda(\Re)+\mathcal{I}_\Psi(\Im)}{2}$ and $\mathcal{F}_{\Lambda\times\Psi}(\Re, \Im) = \max\{\mathcal{F}_\Lambda(\Re), \mathcal{F}_\Psi(\Im)\}$.

Definition 2.5. [18] A vague set (VS) $\Lambda = (\mathcal{T}_\Lambda, \mathcal{F}_\Lambda)$ of $\mathcal{B}$ is said to be vague semiring if

$$\left\{\begin{array}{l} \mathcal{T}_\Lambda(\ell_1 + \ell_2) \geq \min\{\mathcal{T}_\Lambda(\ell_1), \mathcal{T}_\Lambda(\ell_2)\} \\ \mathcal{T}_\Lambda(\ell_1 \cdot \ell_2) \geq \min\{\mathcal{T}_\Lambda(\ell_1), \mathcal{T}_\Lambda(\ell_2)\} \end{array}\right\}$$

and

$$\left\{\begin{array}{l} 1 - \mathcal{F}_\Lambda(\ell_1 + \ell_2) \geq \min\{1 - \mathcal{F}_\Lambda(\ell_1), 1 - \mathcal{F}_\Lambda(\ell_2)\} \\ 1 - \mathcal{F}_\Lambda(\ell_1 \cdot \ell_2) \geq \min\{1 - \mathcal{F}_\Lambda(\ell_1), 1 - \mathcal{F}_\Lambda(\ell_2)\} \end{array}\right\}.$$

for all $\ell_1, \ell_2 \in \mathcal{B}$.

Definition 2.6. [18] A VS Λ in $\mathcal{U}$. Then

(1) A VS $\Lambda = (\mathcal{T}_\Lambda, \mathcal{F}_\Lambda)$, where $\mathcal{T}_\Lambda : \mathcal{U} \to [0,1], \mathcal{F}_\Lambda : \mathcal{U} \to [0,1]$ are mappings such that $\mathcal{T}_\Lambda(\Re)+\mathcal{F}_\Lambda(\Re) \leq 1$, for all $\Re \in \mathcal{U}$ where $\mathcal{T}_\Lambda$ and $\mathcal{F}_\Lambda$ are called true and false membership function, respectively.

(2) The interval $[\mathcal{T}_\Lambda(\Re), 1-\mathcal{F}_\Lambda(\Re)]$ is called the vague value of $\Re$ in Λ and it is denoted by $V_\Lambda(\Re)$, i.e., $V_\Lambda(\Re) = [\mathcal{T}_\Lambda(\Re), 1-\mathcal{F}_\Lambda(\Re)]$.

Definition 2.7. [18] Let Λ and Ψ be the two VSs of $\mathcal{U}$. Then

(1) Λ is contained in Ψ as $\Lambda \subseteq \Psi$ if and only if $V_\Lambda(\Re) \leq V_\Psi(\Re)$, i.e. $\mathcal{T}_\Lambda(\Re) \leq \mathcal{T}_\Psi(\Re)$ and $1-\mathcal{F}_\Lambda(\Re) \leq 1-\mathcal{F}_\Psi(\Re)$ for all $\Re \in \mathcal{U}$,

(2) the union of Λ and Ψ as $\Delta = \Lambda \cup \Psi, \mathcal{T}_\Delta = \max\{\mathcal{T}_\Lambda, \mathcal{T}_\Psi\}$ and $1-\mathcal{F}_\Delta = \max\{1-\mathcal{F}_\Lambda, 1-\mathcal{F}_\Psi\} = 1-\min\{\mathcal{F}_\Lambda, \mathcal{F}_\Psi\}$,

(3) the intersection of Λ and Ψ as $\Delta = \Lambda \cap \Psi, \mathcal{T}_\Delta = \min\{\mathcal{T}_\Lambda, \mathcal{T}_\Psi\}$ and $1-\mathcal{F}_\Delta = \min\{1-\mathcal{F}_\Lambda, 1-\mathcal{F}_\Psi\} = 1-\max\{\mathcal{F}_\Lambda, \mathcal{F}_\Psi\}$.

Definition 2.8. [18] Let Λ and Ψ be any two VSs in $\mathcal{U}$. Then

(1) $\Lambda \cap \Psi = \big\{(\Re, \min\{\mathcal{T}_\Lambda(\Re), \mathcal{T}_\Psi(\Re)\}, \min\{1-\mathcal{F}_\Lambda(\Re), 1-\mathcal{F}_\Psi(\Re)\}) : \Re \in \mathcal{U}\big\}$,

(2) $\Lambda \cup \Psi = \big\{(\Re, \max\{\mathcal{T}_\Lambda(\Re), \mathcal{T}_\Psi(\Re)\}, \max\{1-\mathcal{F}_\Lambda(\Re), 1-\mathcal{F}_\Psi(\Re)\}) : \Re \in \mathcal{U}\big\}$,

(3) $\Box\Lambda = \big\{(\Re, \mathcal{T}_\Lambda(\Re), 1-\mathcal{T}_\Lambda(\Re)) : \Re \in \mathcal{U}\big\}$,

(4) $\Diamond\Lambda = \big\{(\Re, 1-\mathcal{F}_\Lambda(\Re), \mathcal{F}_\Lambda(\Re)) : \Re \in U\big\}$.

3. Neutrosophic vague subbisemirings

In all cases, assume that $\mathcal{B}$ represents a bisemiring.

Definition 3.1. A neutrosophic VS Λ of $\mathcal{B}$ is represent a NSVSBS of $\mathcal{B}$ if

$$\begin{Bmatrix} \mathcal{V}^{\mathcal{T}}_\Lambda(\Re\Diamond_1\Im) \geq \min\{\mathcal{V}^{\mathcal{T}}_\Lambda(\Re), \mathcal{V}^{\mathcal{T}}_\Lambda(\Im)\} \\ \mathcal{V}^{\mathcal{T}}_\Lambda(\Re\Diamond_2\Im) \geq \min\{\mathcal{V}^{\mathcal{T}}_\Lambda(\Re), \mathcal{V}^{\mathcal{T}}_\Lambda(\Im)\} \\ \mathcal{V}^{\mathcal{T}}_\Lambda(\Re\Diamond_3\Im) \geq \min\{\mathcal{V}^{\mathcal{T}}_\Lambda(\Re), \mathcal{V}^{\mathcal{T}}_\Lambda(\Im)\} \end{Bmatrix} \quad \begin{Bmatrix} \mathcal{V}^{\mathcal{I}}_\Lambda(\Re\Diamond_1\Im) \geq \frac{\mathcal{V}^{\mathcal{I}}_\Lambda(\Re)+\mathcal{V}^{\mathcal{I}}_\Lambda(\Im)}{2} \\ OR \\ \mathcal{V}^{\mathcal{I}}_\Lambda(\Re\Diamond_2\Im) \geq \frac{\mathcal{V}^{\mathcal{I}}_\Lambda(\Re)+\mathcal{V}^{\mathcal{I}}_\Lambda(\Im)}{2} \\ OR \\ \mathcal{V}^{\mathcal{I}}_\Lambda(\Re\Diamond_3\Im) \geq \frac{\mathcal{V}^{\mathcal{I}}_\Lambda(\Re)+\mathcal{V}^{\mathcal{I}}_\Lambda(\Im)}{2} \end{Bmatrix}$$

$$\begin{Bmatrix} \mathcal{V}^{\mathcal{F}}_\Lambda(\Re\Diamond_1\Im) \leq \max\{\mathcal{V}^{\mathcal{F}}_\Lambda(\Re), \mathcal{V}^{\mathcal{F}}_\Lambda(\Im)\} \\ \mathcal{V}^{\mathcal{F}}_\Lambda(\Re\Diamond_2\Im) \leq \max\{\mathcal{V}^{\mathcal{F}}_\Lambda(\Re), \mathcal{V}^{\mathcal{F}}_\Lambda(\Im)\} \\ \mathcal{V}^{\mathcal{F}}_\Lambda(\Re\Diamond_3\Im) \leq \max\{\mathcal{V}^{\mathcal{F}}_\Lambda(\Re), \mathcal{V}^{\mathcal{F}}_\Lambda(\Im)\} \end{Bmatrix}.$$

That is,

$$\left\{\begin{pmatrix}\mathcal{T}^-_\Lambda(\Re\lozenge_1\Im)\geq\min\{\mathcal{T}^-_\Lambda(\Re),\mathcal{T}^-_\Lambda(\Im)\},\\1-\mathcal{F}^-_\Lambda(\Re\lozenge_1\Im)\geq\min\{1-\mathcal{F}^-_\Lambda(\Re),1-\mathcal{F}^-_\Lambda(\Im)\}\end{pmatrix}\begin{pmatrix}\mathcal{T}^-_\Lambda(\Re\lozenge_2\Im)\geq\min\{\mathcal{T}^-_\Lambda(\Re),\mathcal{T}^-_\Lambda(\Im)\},\\1-\mathcal{F}^-_\Lambda(\Re\lozenge_2\Im)\geq\min\{1-\mathcal{F}^-_\Lambda(\Re),1-\mathcal{F}^-_\Lambda(\Im)\}\end{pmatrix}\begin{pmatrix}\mathcal{T}^-_\Lambda(\Re\lozenge_3\Im)\geq\min\{\mathcal{T}^-_\Lambda(\Re),\mathcal{T}^-_\Lambda(\Im)\},\\1-\mathcal{F}^-_\Lambda(\Re\lozenge_3\Im)\geq\min\{1-\mathcal{F}^-_\Lambda(\Re),1-\mathcal{F}^-_\Lambda(\Im)\}\end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix}\mathcal{I}^+_\Lambda(\Re\lozenge_1\Im)\geq\frac{\mathcal{I}^+_\Lambda(\Re)+\mathcal{I}^+_\Lambda(\Im)}{2},\\\mathcal{I}^-_\Lambda(\Re\lozenge_1\Im)\geq\frac{\mathcal{I}^-_\Lambda(\Re)-\mathcal{I}^-_\Lambda(\Im)}{2}\end{pmatrix} OR \begin{pmatrix}\mathcal{I}^+_\Lambda(\Re\lozenge_2\Im)\geq\frac{\mathcal{I}^+_\Lambda(\Re)+\mathcal{I}^+_\Lambda(\Im)}{2},\\\mathcal{I}^-_\Lambda(\Re\lozenge_2\Im)\geq\frac{\mathcal{I}^-_\Lambda(\Re)-\mathcal{I}^-_\Lambda(\Im)}{2}\end{pmatrix} OR \begin{pmatrix}\mathcal{I}^+_\Lambda(\Re\lozenge_3\Im)\geq\frac{\mathcal{I}^+_\Lambda(\Re)+\mathcal{I}^+_\Lambda(\Im)}{2},\\\mathcal{I}^-_\Lambda(\Re\lozenge_3\Im)\geq\frac{\mathcal{I}^-_\Lambda(\Re)-\mathcal{I}^-_\Lambda(\Im)}{2}\end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix}\mathcal{F}^-_\Lambda(\Re\lozenge_1\Im)\leq\max\{\mathcal{F}^-_\Lambda(\Re),\mathcal{F}^-_\Lambda(\Im)\},\\1-\mathcal{T}^-_\Lambda(\Re\lozenge_1\Im)\leq\max\{1-\mathcal{T}^-_\Lambda(\Re),1-\mathcal{T}^-_\Lambda(\Im)\}\end{pmatrix}\begin{pmatrix}\mathcal{F}^-_\Lambda(\Re\lozenge_2\Im)\leq\max\{\mathcal{F}^-_\Lambda(\Re),\mathcal{F}^-_\Lambda(\Im)\},\\1-\mathcal{T}^-_\Lambda(\Re\lozenge_2\Im)\leq\max\{1-\mathcal{T}^-_\Lambda(\Re),1-\mathcal{T}^-_\Lambda(\Im)\}\end{pmatrix}\begin{pmatrix}\mathcal{F}^-_\Lambda(\Re\lozenge_3\Im)\leq\max\{\mathcal{F}^-_\Lambda(\Re),\mathcal{F}^-_\Lambda(\Im)\},\\1-\mathcal{T}^-_\Lambda(\Re\lozenge_3\Im)\leq\max\{1-\mathcal{T}^-_\Lambda(\Re),1-\mathcal{T}^-_\Lambda(\Im)\}\end{pmatrix}\right\}$$

for all $\Re,\Im\in\mathcal{B}$.

Example 3.2. Let $\mathcal{B}=\{\dot{a},\ddot{a},\tilde{a},\vec{a}\}$ be the bisemiring.

$\lozenge_1$	$\dot{a}$	$\ddot{a}$	$\tilde{a}$	$\vec{a}$
$\dot{a}$	$\dot{a}$	$\dot{a}$	$\dot{a}$	$\dot{a}$
$\ddot{a}$	$\dot{a}$	$\ddot{a}$	$\dot{a}$	$\ddot{a}$
$\tilde{a}$	$\dot{a}$	$\dot{a}$	$\tilde{a}$	$\tilde{a}$
$\vec{a}$	$\dot{a}$	$\ddot{a}$	$\tilde{a}$	$\vec{a}$

$\lozenge_2$	$\dot{a}$	$\ddot{a}$	$\tilde{a}$	$\vec{a}$
$\dot{a}$	$\dot{a}$	$\ddot{a}$	$\tilde{a}$	$\vec{a}$
$\ddot{a}$	$\ddot{a}$	$\ddot{a}$	$\vec{a}$	$\vec{a}$
$\tilde{a}$	$\tilde{a}$	$\vec{a}$	$\tilde{a}$	$\vec{a}$
$\vec{a}$	$\vec{a}$	$\vec{a}$	$\vec{a}$	$\vec{a}$

$\lozenge_3$	$\dot{a}$	$\ddot{a}$	$\tilde{a}$	$\vec{a}$
$\dot{a}$	$\dot{a}$	$\dot{a}$	$\dot{a}$	$\dot{a}$
$\ddot{a}$	$\dot{a}$	$\ddot{a}$	$\tilde{a}$	$\vec{a}$
$\tilde{a}$	$\vec{a}$	$\vec{a}$	$\vec{a}$	$\vec{a}$
$\vec{a}$	$\vec{a}$	$\vec{a}$	$\vec{a}$	$\vec{a}$

	$[\mathcal{T}^-_\Lambda(\varphi),\mathcal{T}^+_\Lambda(\varphi)]$	$[\mathcal{I}^-_\Lambda(\varphi),\mathcal{I}^+_\Lambda(\varphi)]$	$[\mathcal{F}^-_\Lambda(\varphi),\mathcal{F}^+_\Lambda(\varphi)]$
$\varphi=\dot{a}$	$[0.75,0.8]$	$[0.85,0.9]$	$[0.2,0.25]$
$\varphi=\ddot{a}$	$[0.65,0.75]$	$[0.8,0.85]$	$[0.25,0.35]$
$\varphi=\tilde{a}$	$[0.50,0.55]$	$[0.65,0.70]$	$[0.45,0.50]$
$\varphi=\vec{a}$	$[0.55,0.65]$	$[0.75,0.80]$	$[0.35,0.45]$

Clearly, Λ is a NSVSBS of $\mathcal{B}$.

Theorem 3.3. *The intersection of a family of every $NSVSBS^s$ of $\mathcal{B}$ is a NSVSBS of $\mathcal{B}$.*

Proof. Let $\{\mathcal{V}_i : i \in I\}$ be a collection of $NSVSBS^s$ of $\mathcal{B}$ and $\Lambda = \bigcap_{i\in I}\mathcal{V}_i$.
Let $\Re, \Im$ in $\mathcal{B}$. Then

$$\begin{aligned}\mathcal{T}^-_\Lambda(\Re\Diamond_1\Im) &= \inf_{i\in I}\ \mathcal{T}^-_{\mathcal{V}_i}(\Re\Diamond_1\Im)\\ &\geq \inf_{i\in I}\ \min\{\mathcal{T}^-_{\mathcal{V}_i}(\Re), \mathcal{T}^-_{\mathcal{V}_i}(\Im)\}\\ &= \min\left\{\inf_{i\in I}\ \mathcal{T}^-_{\mathcal{V}_i}(\Re), \inf_{i\in I}\ \mathcal{T}^-_{\mathcal{V}_i}(\Im)\right\}\\ &= \min\{\mathcal{T}^-_\Lambda(\Re), \mathcal{T}^-_\Lambda(\Im)\}.\end{aligned}$$

$$\begin{aligned}1-\mathcal{F}^-_\Lambda(\Re\Diamond_1\Im) &= \inf_{i\in I}\ 1-\mathcal{F}^-_{\mathcal{V}_i}(\Re\Diamond_1\Im)\\ &\geq \inf_{i\in I}\ \min\{1-\mathcal{F}^-_{\mathcal{V}_i}(\Re), 1-\mathcal{F}^-_{\mathcal{V}_i}(\Im)\}\\ &= \min\left\{\inf_{i\in I}\ 1-\mathcal{F}^-_{\mathcal{V}_i}(\Re), \inf_{i\in I}\ 1-\mathcal{F}^-_{\mathcal{V}_i}(\Im)\right\}\\ &= \min\{1-\mathcal{F}^-_\Lambda(\Re), 1-\mathcal{F}^-_\Lambda(\Im)\}.\end{aligned}$$

Thus $\mathcal{V}^{\mathcal{T}}_\Lambda(\Re\Diamond_1\Im) \geq \min\{\mathcal{V}_\Lambda(\Re), \mathcal{V}_\Lambda(\Im)\}$. Similarly, $\mathcal{V}^{\mathcal{T}}_\Lambda(\Re\Diamond_2\Im) \geq \min\{\mathcal{V}_\Lambda(\Re), \mathcal{V}_\Lambda(\Im)\}$ and $\mathcal{V}^{\mathcal{T}}_\Lambda(\Re\Diamond_3\Im) \geq \min\{\mathcal{V}_\Lambda(\Re), \mathcal{V}_\Lambda(\Im)\}$. Now,

$$\begin{aligned}\mathcal{I}^-_\Lambda(\Re\Diamond_1\Im) &= \inf_{i\in I-}\ \mathcal{I}^-_{\mathcal{V}_i}(\Re\Diamond_1\Im)\\ &\geq \inf_{i\in I-}\ \frac{\mathcal{I}^-_{\mathcal{V}_i}(\Re)+\mathcal{I}^-_{\mathcal{V}_i}(\Im)}{2}\\ &= \frac{\inf_{i\in I-}\ \mathcal{I}^-_{\mathcal{V}_i}(\Re)+\inf_{i\in I-}\ \mathcal{I}^-_{\mathcal{V}_i}(\Im)}{2}\\ &= \frac{\mathcal{I}^-_\Lambda(\Re)+\mathcal{I}^-_\Lambda(\Im)}{2}.\end{aligned}$$

$$\begin{aligned}\mathcal{I}^+_\Lambda(\Re\Diamond_1\Im) &= \inf_{i\in I+}\ \mathcal{I}^+_{\mathcal{V}_i}(\Re\Diamond_1\Im)\\ &\geq \inf_{i\in I+}\ \frac{\mathcal{I}^+_{\mathcal{V}_i}(\Re)+\mathcal{I}^+_{\mathcal{V}_i}(\Im)}{2}\\ &= \frac{\inf_{i\in I+}\ \mathcal{I}^+_{\mathcal{V}_i}(\Re)+\inf_{i\in I+}\ \mathcal{I}^+_{\mathcal{V}_i}(\Im)}{2}\\ &= \frac{\mathcal{I}^+_\Lambda(\Re)+\mathcal{I}^+_\Lambda(\Im)}{2}.\end{aligned}$$

Thus $\mathcal{V}^{\mathcal{I}}_\Lambda(\Re\Diamond_1\Im) \geq \min\{\mathcal{V}_\Lambda(\Re), \mathcal{V}_\Lambda(\Im)\}$. Similarly, $\mathcal{V}^{\mathcal{I}}_\Lambda(\Re\Diamond_2\Im) \geq \min\{\mathcal{V}_\Lambda(\Re), \mathcal{V}_\Lambda(\Im)\}$ and $\mathcal{V}^{\mathcal{I}}_\Lambda(\Re\Diamond_3\Im) \geq \min\{\mathcal{V}_\Lambda(\Re), \mathcal{V}_\Lambda(\Im)\}$.

Now,

$$\begin{aligned}\mathcal{F}^{-}_{\Lambda}(\Re\Diamond_1\Im) &= \sup_{i\in I} \mathcal{F}^{-}_{\mathcal{V}_i}(\Re\Diamond_1\Im)\\ &\leq \sup_{i\in I} \max\{\mathcal{F}^{-}_{\mathcal{V}_i}(\Re), \mathcal{F}^{-}_{\mathcal{V}_i}(\Im)\}\\ &= \max\left\{\sup_{i\in I} \mathcal{F}^{-}_{\mathcal{V}_i}(\Re), \sup_{i\in I} \mathcal{F}^{-}_{\mathcal{V}_i}(\Im)\right\}\\ &= \max\{\mathcal{F}^{-}_{\Lambda}(\Re), \mathcal{F}^{-}_{\Lambda}(\Im)\}.\end{aligned}$$

$$\begin{aligned}1-\mathcal{T}^{-}_{\Lambda}(\Re\Diamond_1\Im) &= \sup_{i\in I} 1-\mathcal{T}^{-}_{\mathcal{V}_i}(\Re\Diamond_1\Im)\\ &\leq \sup_{i\in I} \max\{1-\mathcal{T}^{-}_{\mathcal{V}_i}(\Re), 1-\mathcal{T}^{-}_{\mathcal{V}_i}(\Im)\}\\ &= \max\left\{\sup_{i\in I} 1-\mathcal{T}^{-}_{\mathcal{V}_i}(\Re), \sup_{i\in I} 1-\mathcal{T}^{-}_{\mathcal{V}_i}(\Im)\right\}\\ &= \max\{1-\mathcal{T}^{-}_{\Lambda}(\Re), 1-\mathcal{T}^{-}_{\Lambda}(\Im)\}.\end{aligned}$$

Thus $\mathcal{V}^{\mathcal{F}}_{\Lambda}(\Re\Diamond_1\Im) \leq \max\{\mathcal{V}_{\Lambda}(\Re), \mathcal{V}_{\Lambda}(\Im)\}$. Similarly, $\mathcal{V}^{\mathcal{F}}_{\Lambda}(\Re\Diamond_2\Im) \leq \max\{\mathcal{V}_{\Lambda}(\Re), \mathcal{V}_{\Lambda}(\Im)\}$ and $\mathcal{V}^{\mathcal{F}}_{\Lambda}(\Re\Diamond_3\Im) \leq \max\{\mathcal{V}_{\Lambda}(\Re), \mathcal{V}_{\Lambda}(\Im)\}$. Hence, Λ is a NSVSBS of $\mathcal{B}$.

Theorem 3.4. *If Λ and Ψ are the $NSVSBS^s$ of $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, then $\Lambda\times\Psi$ is a NSVSBS of $\mathcal{B}_1\times\mathcal{B}_2$.*

Proof. Let Λ and Ψ be the $NSVSBS^s$ of $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Let $\Re_1, \Re_2 \in \mathcal{B}_1$ and $\Im_1, \Im_2 \in \mathcal{B}_2$. Then $(\Re_1, \Im_1), (\Re_2, \Im_2)$ belong to $\mathcal{B}_1\times\mathcal{B}_2$. Now

$$\begin{aligned}\mathcal{T}^{-}_{\Lambda\times\Psi}[(\Re_1,\Im_1)\Diamond_1(\Re_2,\Im_2)] &= \mathcal{T}^{-}_{\Lambda\times\Psi}(\Re_1\Diamond_1\Re_2, \Im_1\Diamond_1\Im_2)\\ &= \min\{\mathcal{T}^{-}_{\Lambda}(\Re_1\Diamond_1\Re_2), \mathcal{T}^{-}_{\Psi}(\Im_1\Diamond_1\Im_2)\}\\ &\geq \min\{\min\{\mathcal{T}^{-}_{\Lambda}(\Re_1), \mathcal{T}^{-}_{\Lambda}(\Re_2)\}, \min\{\mathcal{T}^{-}_{\Psi}(\Im_1), \mathcal{T}^{-}_{\Psi}(\Im_2)\}\}\\ &= \min\{\min\{\mathcal{T}^{-}_{\Lambda}(\Re_1), \mathcal{T}^{-}_{\Psi}(\Im_1)\}, \min\{\mathcal{T}^{-}_{\Lambda}(\Re_2), \mathcal{T}^{-}_{\Psi}(\Im_2)\}\}\\ &= \min\{\mathcal{T}^{-}_{\Lambda\times\Psi}(\Re_1,\Im_1), \mathcal{T}^{-}_{\Lambda\times\Psi}(\Re_2,\Im_2)\}.\end{aligned}$$

$$\begin{aligned}1-\mathcal{F}^{-}_{\Lambda\times\Psi}[(\Re_1,\Im_1)\Diamond_1(\Re_2,\Im_2)] &= 1-\mathcal{F}^{-}_{\Lambda\times\Psi}(\Re_1\Diamond_1\Re_2, \Im_1\Diamond_1\Im_2)\\ &= \min\{1-\mathcal{F}^{-}_{\Lambda}(\Re_1\Diamond_1\Re_2), 1-\mathcal{F}^{-}_{\Psi}(\Im_1\Diamond_1\Im_2)\}\\ &\geq \min\{\min\{1-\mathcal{F}^{-}_{\Lambda}(\Re_1), 1-\mathcal{F}^{-}_{\Lambda}(\Re_2)\}, \min\{1-\mathcal{F}^{-}_{\Psi}(\Im_1), 1-\mathcal{F}^{-}_{\Psi}(\Im_2)\}\}\\ &= \min\{\min\{1-\mathcal{F}^{-}_{\Lambda}(\Re_1), 1-\mathcal{F}^{-}_{\Psi}(\Im_1)\}, \min\{1-\mathcal{F}^{-}_{\Lambda}(\Re_2), 1-\mathcal{F}^{-}_{\Psi}(\Im_2)\}\}\\ &= \min\{1-\mathcal{F}^{-}_{\Lambda\times\Psi}(\Re_1,\Im_1), 1-\mathcal{F}^{-}_{\Lambda\times\Psi}(\Re_2,\Im_2)\}.\end{aligned}$$

Thus $\mathcal{V}^{\mathcal{T}}_{\Lambda\times\Psi}(\Re\Diamond_1\Im) \geq \min\{\mathcal{V}^{\mathcal{T}}_{\Lambda\times\Psi}(\Re), \mathcal{V}^{\mathcal{T}}_{\Lambda\times\Psi}(\Im)\}$. Similarly, $\mathcal{V}^{\mathcal{T}}_{\Lambda\times\Psi}(\Re\Diamond_2\Im) \geq \min\{\mathcal{V}^{\mathcal{T}}_{\Lambda\times\Psi}(\Re), \mathcal{V}^{\mathcal{T}}_{\Lambda\times\Psi}(\Im)\}$ and $\mathcal{V}^{\mathcal{T}}_{\Lambda\times\Psi}(\Re\Diamond_3\Im) \geq \min\{\mathcal{V}^{\mathcal{T}}_{\Lambda\times\Psi}(\Re), \mathcal{V}^{\mathcal{T}}_{\Lambda\times\Psi}(\Im)\}$.

Now,

$$\begin{aligned}
\mathcal{I}^{-}_{\Lambda\times\Psi}[(\Re_1,\Im_1)\Diamond_1(\Re_2,\Im_2)] &= \mathcal{I}^{-}_{\Lambda\times\Psi}(\Re_1\Diamond_1\Re_2,\Im_1\Diamond_1\Im_2)\\
&= \frac{\mathcal{I}^{-}_{\Lambda}(\Re_1\Diamond_1\Re_2)+\mathcal{I}^{-}_{\Psi}(\Im_1\Diamond_1\Im_2)}{2}\\
&\geq \frac{1}{2}\left[\frac{\mathcal{I}^{-}_{\Lambda}(\Re_1)+\mathcal{I}^{-}_{\Lambda}(\Re_2)}{2}+\frac{\mathcal{I}^{-}_{\Psi}(\Im_1)+\mathcal{I}^{-}_{\Psi}(\Im_2)}{2}\right]\\
&= \frac{1}{2}\left[\frac{\mathcal{I}^{-}_{\Lambda}(\Re_1)+\mathcal{I}^{-}_{\Psi}(\Im_1)}{2}+\frac{\mathcal{I}^{-}_{\Lambda}(\Re_2)+\mathcal{I}^{-}_{\Psi}(\Im_2)}{2}\right]\\
&= \frac{1}{2}\Big[\mathcal{I}^{-}_{\Lambda\times\Psi}(\Re_1,\Im_1)+\mathcal{I}^{-}_{\Lambda\times\Psi}(\Re_2,\Im_2)\Big].
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}^{+}_{\Lambda\times\Psi}[(\Re_1,\Im_1)\Diamond_1(\Re_2,\Im_2)] &= \mathcal{I}^{+}_{\Lambda\times\Psi}(\Re_1\Diamond_1\Re_2,\Im_1\Diamond_1\Im_2)\\
&= \frac{\mathcal{I}^{+}_{\Lambda}(\Re_1\Diamond_1\Re_2)+\mathcal{I}^{+}_{\Psi}(\Im_1\Diamond_1\Im_2)}{2}\\
&\geq \frac{1}{2}\left[\frac{\mathcal{I}^{+}_{\Lambda}(\Re_1)+\mathcal{I}^{+}_{\Lambda}(\Re_2)}{2}+\frac{\mathcal{I}^{+}_{\Psi}(\Im_1)+\mathcal{I}^{+}_{\Psi}(\Im_2)}{2}\right]\\
&= \frac{1}{2}\left[\frac{\mathcal{I}^{+}_{\Lambda}(\Re_1)+\mathcal{I}^{+}_{\Psi}(\Im_1)}{2}+\frac{\mathcal{I}^{+}_{\Lambda}(\Re_2)+\mathcal{I}^{+}_{\Psi}(\Im_2)}{2}\right]\\
&= \frac{1}{2}\Big[\mathcal{I}^{+}_{\Lambda\times\Psi}(\Re_1,\Im_1)+\mathcal{I}^{+}_{\Lambda\times\Psi}(\Re_2,\Im_2)\Big].
\end{aligned}$$

Thus $\mathcal{V}^{\mathcal{I}}_{\Lambda\times\Psi}(\Re\Diamond_1\Im) \geq \frac{1}{2}\Big[\mathcal{V}^{\mathcal{I}}_{\Lambda\times\Psi}(\Re_1,\Im_1)+\mathcal{V}^{\mathcal{I}}_{\Lambda\times\Psi}(\Re_2,\Im_2)\Big]$. Similarly, $\mathcal{V}^{\mathcal{I}}_{\Lambda\times\Psi}(\Re\Diamond_2\Im) \geq \frac{1}{2}\Big[\mathcal{V}^{\mathcal{I}}_{\Lambda\times\Psi}(\Re_1,\Im_1)+\mathcal{V}^{\mathcal{I}}_{\Lambda\times\Psi}(\Re_2,\Im_2)\Big]$ and $\mathcal{V}^{\mathcal{I}}_{\Lambda\times\Psi}(\Re\Diamond_3\Im) \geq \frac{1}{2}\Big[\mathcal{V}^{\mathcal{I}}_{\Lambda\times\Psi}(\Re_1,\Im_1)+\mathcal{V}^{\mathcal{I}}_{\Lambda\times\Psi}(\Re_2,\Im_2)\Big]$. Now

$$\begin{aligned}
\mathcal{F}^{-}_{\Lambda\times\Psi}[(\Re_1,\Im_1)\Diamond_1(\Re_2,\Im_2)] &= \mathcal{F}^{-}_{\Lambda\times\Psi}(\Re_1\Diamond_1\Re_2,\Im_1\Diamond_1\Im_2)\\
&= \max\{\mathcal{F}^{-}_{\Lambda}(\Re_1\Diamond_1\Re_2),\mathcal{F}^{-}_{\Psi}(\Im_1\Diamond_1\Im_2)\}\\
&\leq \max\{\max\{\mathcal{F}^{-}_{\Lambda}(\Re_1),\mathcal{F}^{-}_{\Lambda}(\Re_2)\},\max\{\mathcal{F}^{-}_{\Psi}(\Im_1),\mathcal{F}^{-}_{\Psi}(\Im_2)\}\}\\
&= \max\{\max\{\mathcal{F}^{-}_{\Lambda}(\Re_1),\mathcal{F}^{-}_{\Psi}(\Im_1)\},\max\{\mathcal{F}^{-}_{\Lambda}(\Re_2),\mathcal{F}^{-}_{\Psi}(\Im_2)\}\}\\
&= \max\{\mathcal{F}^{-}_{\Lambda\times\Psi}(\Re_1,\Im_1),\mathcal{F}^{-}_{\Lambda\times\Psi}(\Re_2,\Im_2)\}.
\end{aligned}$$

$$\begin{aligned}
1-\mathcal{T}^{-}_{\Lambda\times\Psi}[(\Re_1,\Im_1)\Diamond_1(\Re_2,\Im_2)] &= 1-\mathcal{T}^{-}_{\Lambda\times\Psi}(\Re_1\Diamond_1\Re_2,\Im_1\Diamond_1\Im_2)\\
&= \max\{1-\mathcal{T}^{-}_{\Lambda}(\Re_1\Diamond_1\Re_2),1-\mathcal{T}^{-}_{\Psi}(\Im_1\Diamond_1\Im_2)\}\\
&\leq \max\{\max\{1-\mathcal{T}^{-}_{\Lambda}(\Re_1),1-\mathcal{T}^{-}_{\Lambda}(\Re_2)\},\max\{1-\mathcal{T}^{-}_{\Psi}(\Im_1),1-\mathcal{T}^{-}_{\Psi}(\Im_2)\}\}\\
&= \max\{\max\{1-\mathcal{T}^{-}_{\Lambda}(\Re_1),1-\mathcal{T}^{-}_{\Psi}(\Im_1)\},\max\{1-\mathcal{T}^{-}_{\Lambda}(\Re_2),1-\mathcal{T}^{-}_{\Psi}(\Im_2)\}\}\\
&= \max\{1-\mathcal{T}^{-}_{\Lambda\times\Psi}(\Re_1,\Im_1),1-\mathcal{T}^{-}_{\Lambda\times\Psi}(\Re_2,\Im_2)\}.
\end{aligned}$$

Thus $\mathcal{V}^{\mathcal{F}}_{\Lambda\times\Psi}(\Re\Diamond_1\Im) \leq \max\{\mathcal{V}^{\mathcal{F}}_{\Lambda\times\Psi}(\Re),\mathcal{V}^{\mathcal{F}}_{\Lambda\times\Psi}(\Im)\}$. Similarly, $\mathcal{V}^{\mathcal{F}}_{\Lambda\times\Psi}(\Re\Diamond_2\Im) \leq \max\{\mathcal{V}^{\mathcal{F}}_{\Lambda\times\Psi}(\Re),\mathcal{V}^{\mathcal{F}}_{\Lambda\times\Psi}(\Im)\}$ and $\mathcal{V}^{\mathcal{F}}_{\Lambda\times\Psi}(\Re\Diamond_3\Im) \leq \max\{\mathcal{V}^{\mathcal{F}}_{\Lambda\times\Psi}(\Re),\mathcal{V}^{\mathcal{F}}_{\Lambda\times\Psi}(\Im)\}$.

Corollary 3.5. *If $\Lambda_1, \Lambda_2, ..., \Lambda_n$ are the families of $NSVSBS^s$ of $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n$ respectively, then $\Lambda_1 \times \Lambda_2 \times ... \times \Lambda_n$ is a NSVSBS of $\mathcal{B}_1 \times \mathcal{B}_2 \times ... \times \mathcal{B}_n$.*

Definition 3.6. Let Λ be a neutrosophic VS in $\mathcal{B}$, the strongest neutrosophic vague relation (SNSVR) on $\mathcal{B}$, that is a NSVR on Λ is defined as

$$\left\{\begin{array}{l} \mathcal{V}_V^{\mathcal{T}}(\Re, \Im) = \min\{\mathcal{V}_\Lambda^{\mathcal{T}}(\Re), \mathcal{V}_\Lambda^{\mathcal{T}}(\Im)\} \\ \mathcal{V}_V^{\mathcal{I}}(\Re, \Im) = \frac{\mathcal{V}_\Lambda^{\mathcal{I}}(\Re) + \mathcal{V}_\Lambda^{\mathcal{I}}(\Im)}{2} \\ \mathcal{V}_V^{\mathcal{F}}(\Re, \Im) = \max\{\mathcal{V}_\Lambda^{\mathcal{F}}(\Re), \mathcal{V}_\Lambda^{\mathcal{F}}(\Im)\} \end{array}\right\}.$$

Theorem 3.7. *Let Λ be the NSVSBS of $\mathcal{B}$ and V be the SNSVR of $\mathcal{B}$. Then Λ is a NSVSBS of $\mathcal{B}$ if and only if V is a NSVSBS of $\mathcal{B} \times \mathcal{B}$.*

Proof. Let Λ be the NSVSBS of $\mathcal{B}$ and V be the SNSVR of $\mathcal{B}$. Then for any $\Re = (\Re_1, \Re_2)$ and $\Im = (\Im_1, \Im_2)$ are in $\mathcal{B} \times \mathcal{B}$. Now,

$$\begin{aligned} \mathcal{T}_V^-(\Re \diamondsuit_1 \Im) &= \mathcal{T}_V^-[((\Re_1, \Re_2) \diamondsuit_1 (\Im_1, \Im_2)] \\ &= \mathcal{T}_V^-(\Re_1 \diamondsuit_1 \Im_1, \Re_2 \diamondsuit_1 \Im_2) \\ &= \min\{\mathcal{T}_\Lambda^-(\Re_1 \diamondsuit_1 \Im_1), \mathcal{T}_\Lambda^-(\Re_2 \diamondsuit_1 \Im_2)\} \\ &\geq \min\{\min\{\mathcal{T}_\Lambda^-(\Re_1), \mathcal{T}_\Lambda^-(\Im_1)\}, \min\{\mathcal{T}_\Lambda^-(\Re_2), \mathcal{T}_\Lambda^-(\Im_2)\}\} \\ &= \min\{\min\{\mathcal{T}_\Lambda^-(\Re_1), \mathcal{T}_\Lambda^-(\Re_2)\}, \min\{\mathcal{T}_\Lambda^-(\Im_1), \mathcal{T}_\Lambda^-(\Im_2)\}\} \\ &= \min\{\mathcal{T}_V^-(\Re_1, \Re_2), \mathcal{T}_V^-(\Im_1, \Im_2)\} \\ &= \min\{\mathcal{T}_V^-(\Re), \mathcal{T}_V^-(\Im)\}. \end{aligned}$$

$$\begin{aligned} 1 - \mathcal{F}_V^-(\Re \diamondsuit_1 \Im) &= 1 - \mathcal{F}_V^-[((\Re_1, \Re_2) \diamondsuit_1 (\Im_1, \Im_2)] \\ &= 1 - \mathcal{F}_V^-(\Re_1 \diamondsuit_1 \Im_1, \Re_2 \diamondsuit_1 \Im_2) \\ &= \min\{1 - \mathcal{F}_\Lambda^-(\Re_1 \diamondsuit_1 \Im_1), 1 - \mathcal{F}_\Lambda^-(\Re_2 \diamondsuit_1 \Im_2)\} \\ &\geq \min\{\min\{1 - \mathcal{F}_\Lambda^-(\Re_1), 1 - \mathcal{F}_\Lambda^-(\Im_1)\}, \min\{1 - \mathcal{F}_\Lambda^-(\Re_2), 1 - \mathcal{F}_\Lambda^-(\Im_2)\}\} \\ &= \min\{\min\{1 - \mathcal{F}_\Lambda^-(\Re_1), 1 - \mathcal{F}_\Lambda^-(\Re_2)\}, \min\{1 - \mathcal{F}_\Lambda^-(\Im_1), 1 - \mathcal{F}_\Lambda^-(\Im_2)\}\} \\ &= \min\{1 - \mathcal{F}_V^-(\Re_1, \Re_2), 1 - \mathcal{F}_V^-(\Im_1, \Im_2)\} \\ &= \min\{1 - \mathcal{F}_V^-(\Re), 1 - \mathcal{F}_V^-(\Im)\}. \end{aligned}$$

Thus $\mathcal{V}_V^{\mathcal{T}}(\Re\Diamond_1\Im) \geq \min\{\mathcal{V}_V^{\mathcal{T}}(\Re), \mathcal{V}_V^{\mathcal{T}}(\Im)\}$. Similarly, $\mathcal{V}_V^{\mathcal{T}}(\Re\Diamond_2\Im) \geq \min\{\mathcal{V}_V^{\mathcal{T}}(\Re), \mathcal{V}_V^{\mathcal{T}}(\Im)\}$ and $\mathcal{V}_V^{\mathcal{T}}(\Re\Diamond_3\Im) \geq \min\{\mathcal{V}_V^{\mathcal{T}}(\Re), \mathcal{V}_V^{\mathcal{T}}(\Im)\}$. Now,

$$\begin{aligned}
\mathcal{I}_V^-(\Re\Diamond_1\Im) &= \mathcal{I}_V^-[((\Re_1,\Re_2)\Diamond_1(\Im_1,\Im_2)] \\
&= \mathcal{I}_V^-(\Re_1\Diamond_1\Im_1, \Re_2\Diamond_1\Im_2) \\
&= \frac{\mathcal{I}_\Lambda^-(\Re_1\Diamond_1\Im_1) + \mathcal{I}_\Lambda^-(\Re_2\Diamond_1\Im_2)}{2} \\
&\geq \frac{1}{2}\left[\frac{\mathcal{I}_\Lambda^-(\Re_1) + \mathcal{I}_\Lambda^-(\Im_1)}{2} + \frac{\mathcal{I}_\Lambda^-(\Re_2) + \mathcal{I}_\Lambda^-(\Im_2)}{2}\right] \\
&= \frac{1}{2}\left[\frac{\mathcal{I}_\Lambda^-(\Re_1) + \mathcal{I}_\Lambda^-(\Re_2)}{2} + \frac{\mathcal{I}_\Lambda^-(\Im_1) + \mathcal{I}_\Lambda^-(\Im_2)}{2}\right] \\
&= \frac{\mathcal{I}_V^-(\Re_1,\Re_2) + \mathcal{I}_V^-(\Im_1,\Im_2)}{2} \\
&= \frac{\mathcal{I}_V^-(\Re) + \mathcal{I}_V^-(\Im)}{2}.
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_V^+(\Re\Diamond_1\Im) &= \mathcal{I}_V^+[((\Re_1,\Re_2)\Diamond_1(\Im_1,\Im_2)] \\
&= \mathcal{I}_V^+(\Re_1\Diamond_1\Im_1, \Re_2\Diamond_1\Im_2) \\
&= \frac{\mathcal{I}_\Lambda^+(\Re_1\Diamond_1\Im_1) + \mathcal{I}_\Lambda^+(\Re_2\Diamond_1\Im_2)}{2} \\
&\geq \frac{1}{2}\left[\frac{\mathcal{I}_\Lambda^+(\Re_1) + \mathcal{I}_\Lambda^+(\Im_1)}{2} + \frac{\mathcal{I}_\Lambda^+(\Re_2) + \mathcal{I}_\Lambda^+(\Im_2)}{2}\right] \\
&= \frac{1}{2}\left[\frac{\mathcal{I}_\Lambda^+(\Re_1) + \mathcal{I}_\Lambda^+(\Re_2)}{2} + \frac{\mathcal{I}_\Lambda^+(\Im_1) + \mathcal{I}_\Lambda^+(\Im_2)}{2}\right] \\
&= \frac{\mathcal{I}_V^+(\Re_1,\Re_2) + \mathcal{I}_V^+(\Im_1,\Im_2)}{2} \\
&= \frac{\mathcal{I}_V^+(\Re) + \mathcal{I}_V^+(\Im)}{2}.
\end{aligned}$$

Thus $\mathcal{V}_V^{\mathcal{I}}(\Re\Diamond_1\Im) \geq \frac{\mathcal{V}_V(\Re)+\mathcal{V}_V(\Im)}{2}$. Similarly, $\mathcal{V}_V^{\mathcal{I}}(\Re\Diamond_2\Im) \geq \frac{\mathcal{V}_V(\Re)+\mathcal{V}_V(\Im)}{2}$ and $\mathcal{V}_V^{\mathcal{I}}(\Re\Diamond_3\Im) \geq \frac{\mathcal{V}_V(\Re)+\mathcal{V}_V(\Im)}{2}$.

Similarly, $\mathcal{V}_V^{\mathcal{F}}(\Re\Diamond_1\Im) \leq \max\{\mathcal{V}_V^{\mathcal{F}}(\Re), \mathcal{V}_V^{\mathcal{F}}(\Im)\}$, $\mathcal{V}_V^{\mathcal{F}}(\Re\Diamond_2\Im) \leq \max\{\mathcal{V}_V^{\mathcal{F}}(\Re), \mathcal{V}_V^{\mathcal{F}}(\Im)\}$ and $\mathcal{V}_V^{\mathcal{F}}(\Re\Diamond_3\Im) \leq \max\{\mathcal{V}_V^{\mathcal{F}}(\Re), \mathcal{V}_V^{\mathcal{F}}(\Im)\}$.

Conversely let us assume that V is a NSVSBS of $\mathcal{B} \times \mathcal{B}$, then for any $\Re = (\Re_1, \Re_2)$ and $\Im = (\Im_1, \Im_2)$ are in $\mathcal{B} \times \mathcal{B}$. Now,

$$\begin{aligned}
\min\{\mathcal{T}_\Lambda^-(\Re_1\Diamond_1\Im_1), \mathcal{T}_\Lambda^-(\Re_2\Diamond_1\Im_2)\} &= \mathcal{T}_V^-(\Re_1\Diamond_1\Im_1, \Re_2\Diamond_1\Im_2) \\
&= \mathcal{T}_V^-[(\Re_1, \Re_2)\Diamond_1(\Im_1, \Im_2)] \\
&= \mathcal{T}_V^-(\Re\Diamond_1\Im) \\
&\geq \min\{\mathcal{T}_V^-(\Re), \mathcal{T}_V^-(\Im)\} \\
&= \min\{\mathcal{T}_V^-(\Re_1, \Re_2)\}, \mathcal{T}_V^-(\Im_1, \Im_2)\} \\
&= \min\{\min\{\mathcal{T}_\Lambda^-(\Re_1), \mathcal{T}_\Lambda^-(\Re_2)\}, \min\{\mathcal{T}_\Lambda^-(\Im_1), \mathcal{T}_\Lambda^-(\Im_2)\}\}.
\end{aligned}$$

If $\mathcal{T}_\Lambda^-(\Re_1\Diamond_1\Im_1) \leq \mathcal{T}_\Lambda^-(\Re_2\Diamond_1\Im_2)$, then $\mathcal{T}_\Lambda^-(\Re_1) \leq \mathcal{T}_\Lambda^-(\Re_2)$ and $\mathcal{T}_\Lambda^-(\Im_1) \leq \mathcal{T}_\Lambda^-(\Im_2)$. We get $\mathcal{T}_\Lambda^-(\Re_1\Diamond_1\Im_1) \geq \min\{\mathcal{T}_\Lambda^-(\Re_1), \mathcal{T}_\Lambda^-(\Im_1)\}$ for all $\Re_1, \Im_1 \in \mathcal{B}$, and
$\min\{\mathcal{T}_\Lambda^-(\Re_1\Diamond_2\Im_1), \mathcal{T}_\Lambda^-(\Re_2\Diamond_2\Im_2)\} \geq \min\{\min\{\mathcal{T}_\Lambda^-(\Re_1), \mathcal{T}_\Lambda^-(\Re_2)\}, \min\{\mathcal{T}_\Lambda^-(\Im_1), \mathcal{T}_\Lambda^-(\Im_2)\}\}$
If $\mathcal{T}_\Lambda^-(\Re_1\Diamond_2\Im_1) \leq \mathcal{T}_\Lambda^-(\Re_2\Diamond_2\Im_2)$, then $\mathcal{T}_\Lambda^-(\Re_1\Diamond_2\Im_1) \geq \min\{\mathcal{T}_\Lambda^-(\Re_1), \mathcal{T}_\Lambda^-(\Im_1)\}$.
$\min\{\mathcal{T}_\Lambda^-(\Re_1\Diamond_3\Im_1), \mathcal{T}_\Lambda^-(\Re_2\Diamond_3\Im_2)\} \geq \min\{\min\{\mathcal{T}_\Lambda^-(\Re_1), \mathcal{T}_\Lambda^-(\Re_2)\}, \min\{\mathcal{T}_\Lambda^-(\Im_1), \mathcal{T}_\Lambda^-(\Im_2)\}\}$.
If $\mathcal{T}_\Lambda^-(\Re_1\Diamond_3\Im_1) \leq \mathcal{T}_\Lambda^-(\Re_2\Diamond_3\Im_2)$, then $\mathcal{T}_\Lambda^-(\Re_1\Diamond_3\Im_1) \geq \min\{\mathcal{T}_\Lambda^-(\Re_1), \mathcal{T}_\Lambda^-(\Im_1)\}$.

$$\begin{aligned}
&\min\{1 - \mathcal{F}_\Lambda^-(\Re_1\Diamond_1\Im_1), 1 - \mathcal{F}_\Lambda^-(\Re_2\Diamond_1\Im_2)\} \\
&= 1 - \mathcal{F}_V^-(\Re_1\Diamond_1\Im_1, \Re_2\Diamond_1\Im_2) \\
&= 1 - \mathcal{F}_V^-[(\Re_1, \Re_2)\Diamond_1(\Im_1, \Im_2)] \\
&= 1 - \mathcal{F}_V^-(\Re\Diamond_1\Im) \\
&\geq \min\{1 - \mathcal{F}_V^-(\Re), 1 - \mathcal{F}_V^-(\Im)\} \\
&= \min\{1 - \mathcal{F}_V^-(\Re_1, \Re_2)\}, 1 - \mathcal{F}_V^-(\Im_1, \Im_2)\} \\
&= \min\{\min\{1 - \mathcal{F}_\Lambda^-(\Re_1), 1 - \mathcal{F}_\Lambda^-(\Re_2)\}, \min\{1 - \mathcal{F}_\Lambda^-(\Im_1), 1 - \mathcal{F}_\Lambda^-(\Im_2)\}\}.
\end{aligned}$$

If $1 - \mathcal{F}_\Lambda^-(\Re_1\Diamond_1\Im_1) \leq 1 - \mathcal{F}_\Lambda^-(\Re_2\Diamond_1\Im_2)$, then $1 - \mathcal{F}_\Lambda^-(\Re_1) \leq 1 - \mathcal{F}_\Lambda^-(\Re_2)$ and $1 - \mathcal{F}_\Lambda^-(\Im_1) \leq 1 - \mathcal{F}_\Lambda^-(\Im_2)$. We get $1 - \mathcal{F}_\Lambda^-(\Re_1\Diamond_1\Im_1) \geq \min\{1 - \mathcal{F}_\Lambda^-(\Re_1), 1 - \mathcal{F}_\Lambda^-(\Im_1)\}$ for all $\Re_1, \Im_1 \in \mathcal{B}$, and $\min\{1 - \mathcal{F}_\Lambda^-(\Re_1\Diamond_2\Im_1), 1 - \mathcal{F}_\Lambda^-(\Re_2\Diamond_2\Im_2)\} \geq \min\{\min\{1 - \mathcal{F}_\Lambda^-(\Re_1), 1 - \mathcal{F}_\Lambda^-(\Re_2)\}, \min\{1 - \mathcal{F}_\Lambda^-(\Im_1), 1 - \mathcal{F}_\Lambda^-(\Im_2)\}\}$.
If $1 - \mathcal{F}_\Lambda^-(\Re_1\Diamond_2\Im_1) \leq 1 - \mathcal{F}_\Lambda^-(\Re_2\Diamond_2\Im_2)$, then $1 - \mathcal{F}_\Lambda^-(\Re_1\Diamond_2\Im_1) \geq \min\{1 - \mathcal{F}_\Lambda^-(\Re_1), 1 - \mathcal{F}_\Lambda^-(\Im_1)\}$.
$\min\{1 - \mathcal{F}_\Lambda^-(\Re_1\Diamond_3\Im_1), 1 - \mathcal{F}_\Lambda^-(\Re_2\Diamond_3\Im_2)\} \geq \min\{\min\{1 - \mathcal{F}_\Lambda^-(\Re_1), 1 - \mathcal{F}_\Lambda^-(\Re_2)\}, \min\{1 - \mathcal{F}_\Lambda^-(\Im_1), 1 - \mathcal{F}_\Lambda^-(\Im_2)\}\}$. If $1 - \mathcal{F}_\Lambda^-(\Re_1\Diamond_3\Im_1) \leq 1 - \mathcal{F}_\Lambda^-(\Re_2\Diamond_3\Im_2)$, then $1 - \mathcal{F}_\Lambda^-(\Re_1\Diamond_3\Im_1) \geq \min\{1 - \mathcal{F}_\Lambda^-(\Re_1), 1 - \mathcal{F}_\Lambda^-(\Im_1)\}$.
Thus $\mathcal{V}_V^{\mathcal{T}}(\Re\Diamond_1\Im) \geq \min\{\mathcal{V}_V^{\mathcal{T}}(\Re), \mathcal{V}_V^{\mathcal{T}}(\Im)\}$. Similarly, $\mathcal{V}_V^{\mathcal{T}}(\Re\Diamond_2\Im) \geq \min\{\mathcal{V}_V^{\mathcal{T}}(\Re), \mathcal{V}_V^{\mathcal{T}}(\Im)\}$ and

$\mathcal{V}_V^{\mathcal{T}}(\Re\diamondsuit_3\Im) \geq \min\{\mathcal{V}_V^{\mathcal{T}}(\Re), \mathcal{V}_V^{\mathcal{T}}(\Im)\}$. Now,

$$\begin{aligned}\frac{1}{2}\Big[\mathcal{I}_\Lambda^-(\Re_1\diamondsuit_1\Im_1) + \mathcal{I}_\Lambda^-(\Re_2\diamondsuit_1\Im_2)\Big] &= \mathcal{I}_V^-(\Re_1\diamondsuit_1\Im_1, \Re_2\diamondsuit_1\Im_2)\\ &= \mathcal{I}_V^-[(\Re_1,\Re_2)\diamondsuit_1(\Im_1,\Im_2)]\\ &= \mathcal{I}_V^-(\Re\diamondsuit_1\Im)\\ &\geq \frac{\mathcal{I}_V^-(\Re)+\mathcal{I}_V^-(\Im)}{2}\\ &= \frac{\mathcal{I}_V^-(\Re_1,\Re_2)+\mathcal{I}_V^-(\Im_1,\Im_2)}{2}\\ &= \frac{1}{2}\Bigg[\frac{\mathcal{I}_\Lambda^-(\Re_1)+\mathcal{I}_\Lambda^-(\Re_2)}{2} + \frac{\mathcal{I}_\Lambda^-(\Im_1)+\mathcal{I}_\Lambda^-(\Im_2)}{2}\Bigg].\end{aligned}$$

If $\mathcal{I}_\Lambda^-(\Re_1\diamondsuit_1\Im_1) \leq \mathcal{I}_\Lambda^-(\Re_2\diamondsuit_1\Im_2)$, then $\mathcal{I}_\Lambda^-(\Re_1) \leq \mathcal{I}_\Lambda^-(\Re_2)$ and $\mathcal{I}_\Lambda^-(\Im_1) \leq \mathcal{I}_\Lambda^-(\Im_2)$.
We get $\mathcal{I}_\Lambda^-(\Re_1\diamondsuit_1\Im_1) \geq \frac{\mathcal{I}_\Lambda^-(\Re_1)+\mathcal{I}_\Lambda^-(\Im_1)}{2}$. Similarly, $\mathcal{I}_\Lambda^-(\Re_1\diamondsuit_2\Im_1) \geq \frac{\mathcal{I}_\Lambda^-(\Re_1)+\mathcal{I}_\Lambda^-(\Im_1)}{2}$ and $\mathcal{I}_\Lambda^-(\Re_1\diamondsuit_3\Im_1) \geq \frac{\mathcal{I}_\Lambda^-(\Re_1)+\mathcal{I}_\Lambda^-(\Im_1)}{2}$.
Also, $\frac{1}{2}\Big[\mathcal{I}_\Lambda^+(\Re_1\diamondsuit_1\Im_1) + \mathcal{I}_\Lambda^+(\Re_2\diamondsuit_1\Im_2)\Big] \geq \frac{1}{2}\Bigg[\frac{\mathcal{I}_\Lambda^+(\Re_1)+\mathcal{I}_\Lambda^+(\Re_2)}{2} + \frac{\mathcal{I}_\Lambda^+(\Im_1)+\mathcal{I}_\Lambda^+(\Im_2)}{2}\Bigg]$.
If $\mathcal{I}_\Lambda^+(\Re_1\diamondsuit_1\Im_1) \leq \mathcal{I}_\Lambda^+(\Re_2\diamondsuit_1\Im_2)$, then $\mathcal{I}_\Lambda^+(\Re_1) \leq \mathcal{I}_\Lambda^+(\Re_2)$ and $\mathcal{I}_\Lambda^+(\Im_1) \leq \mathcal{I}_\Lambda^+(\Im_2)$.
We get $\mathcal{I}_\Lambda^+(\Re_1\diamondsuit_1\Im_1) \geq \frac{\mathcal{I}_\Lambda^+(\Re_1)+\mathcal{I}_\Lambda^+(\Im_1)}{2}$ and $\mathcal{I}_\Lambda^+(\Re_1\diamondsuit_2\Im_1) \geq \frac{\mathcal{I}_\Lambda^+(\Re_1)+\mathcal{I}_\Lambda^+(\Im_1)}{2}$ and $\mathcal{I}_\Lambda^+(\Re_1\diamondsuit_3\Im_1) \geq \frac{\mathcal{I}_\Lambda^+(\Re_1)+\mathcal{I}_\Lambda^+(\Im_1)}{2}$.
Thus $\mathcal{V}_V^{\mathcal{I}}(\Re\diamondsuit_1\Im) \geq \frac{\mathcal{V}_V(\Re)+\mathcal{V}_V(\Im)}{2}$. Similarly, $\mathcal{V}_V^{\mathcal{I}}(\Re\diamondsuit_2\Im) \geq \frac{\mathcal{V}_V(\Re)+\mathcal{V}_V(\Im)}{2}$ and $\mathcal{V}_V^{\mathcal{I}}(\Re\diamondsuit_3\Im) \geq \frac{\mathcal{V}_V(\Re)+\mathcal{V}_V(\Im)}{2}$. Similarly, $\max\{\mathcal{F}_\Lambda^-(\Re_1\diamondsuit_1\Im_1), \mathcal{F}_\Lambda^-(\Re_2\diamondsuit_1\Im_2)\} \leq \max\{\max\{\mathcal{F}_\Lambda^-(\Re_1), \mathcal{F}_\Lambda^-(\Re_2)\}, \max\{\mathcal{F}_\Lambda^-(\Im_1), \mathcal{F}_\Lambda^-(\Im_2)\}\}$.
If $\mathcal{F}_\Lambda^-(\Re_1\diamondsuit_1\Im_1) \geq \mathcal{F}_\Lambda^-(\Re_2\diamondsuit_1\Im_2)$, then $\mathcal{F}_\Lambda^-(\Re_1) \geq \mathcal{F}_\Lambda^-(\Re_2)$ and $\mathcal{F}_\Lambda^-(\Im_1) \geq \mathcal{F}_\Lambda^-(\Im_2)$.
We get $\mathcal{F}_\Lambda^-(\Re_1\diamondsuit_1\Im_1) \leq \max\{\mathcal{F}_\Lambda^-(\Re_1), \mathcal{F}_\Lambda^-(\Im_1)\}$.
$\max\{\mathcal{F}_\Lambda^-(\Re_1\diamondsuit_2\Im_1), \mathcal{F}_\Lambda^-(\Re_2\diamondsuit_2\Im_2)\} \leq \max\{\max\{\mathcal{F}_\Lambda^-(\Re_1), \mathcal{F}_\Lambda^-(\Re_2)\}, \max\{\mathcal{F}_\Lambda^-(\Im_1), \mathcal{F}_\Lambda^-(\Im_2)\}\}$.
If $\mathcal{F}_\Lambda^-(\Re_1\diamondsuit_2\Im_1) \geq \mathcal{F}_\Lambda^-(\Re_2\diamondsuit_2\Im_2)$, then $\mathcal{F}_\Lambda^-(\Re_1\diamondsuit_2\Im_1) \leq \max\{\mathcal{F}_\Lambda^-(\Re_1), \mathcal{F}_\Lambda^-(\Im_1)\}$.
$\max\{\mathcal{F}_\Lambda^-(\Re_1\diamondsuit_3\Im_1), \mathcal{F}_\Lambda^-(\Re_2\diamondsuit_3\Im_2)\} \leq \max\{\max\{\mathcal{F}_\Lambda^-(\Re_1), \mathcal{F}_\Lambda^-(\Re_2)\}, \max\{\mathcal{F}_\Lambda^-(\Im_1), \mathcal{F}_\Lambda^-(\Im_2)\}\}$
If $\mathcal{F}_\Lambda^-(\Re_1\diamondsuit_3\Im_1) \geq \mathcal{F}_\Lambda^-(\Re_2\diamondsuit_3\Im_2)$, then $\mathcal{F}_\Lambda^-(\Re_1\diamondsuit_3\Im_1) \leq \max\{\mathcal{F}_\Lambda^-(\Re_1), \mathcal{F}_\Lambda^-(\Im_1)\}$.
Also, Similarly to prove that $\max\{1-\mathcal{T}_\Lambda^-(\Re_1\diamondsuit_1\Im_1), 1-\mathcal{T}_\Lambda^-(\Re_2\diamondsuit_1\Im_2)\} \leq \max\{\max\{1-\mathcal{T}_\Lambda^-(\Re_1), 1-\mathcal{T}_\Lambda^-(\Re_2)\}, \max\{1-\mathcal{T}_\Lambda^-(\Im_1), 1-\mathcal{T}_\Lambda^-(\Im_2)\}\}$.
If $1-\mathcal{T}_\Lambda^-(\Re_1\diamondsuit_1\Im_1) \geq 1-\mathcal{T}_\Lambda^-(\Re_2\diamondsuit_1\Im_2)$, then $1-\mathcal{T}_\Lambda^-(\Re_1) \geq 1-\mathcal{T}_\Lambda^-(\Re_2)$ and $1-\mathcal{T}_\Lambda^-(\Im_1) \geq 1-\mathcal{T}_\Lambda^-(\Im_2)$.
We get $1-\mathcal{T}_\Lambda^-(\Re_1\diamondsuit_1\Im_1) \leq \max\{1-\mathcal{T}_\Lambda^-(\Re_1), 1-\mathcal{T}_\Lambda^-(\Im_1)\}$.
$\max\{1-\mathcal{T}_\Lambda^-(\Re_1\diamondsuit_2\Im_1), 1-\mathcal{T}_\Lambda^-(\Re_2\diamondsuit_2\Im_2)\} \leq \max\{\max\{1-\mathcal{T}_\Lambda^-(\Re_1), 1-\mathcal{T}_\Lambda^-(\Re_2)\}, \max\{1-\mathcal{T}_\Lambda^-(\Im_1), 1-\mathcal{T}_\Lambda^-(\Im_2)\}\}$.
If $1-\mathcal{T}_\Lambda^-(\Re_1\diamondsuit_2\Im_1) \geq 1-\mathcal{T}_\Lambda^-(\Re_2\diamondsuit_2\Im_2)$, then $1-\mathcal{T}_\Lambda^-(\Re_1\diamondsuit_2\Im_1) \leq \max\{1-\mathcal{T}_\Lambda^-(\Re_1), 1-\mathcal{T}_\Lambda^-(\Im_1)\}$.
$\max\{1-\mathcal{T}_\Lambda^-(\Re_1\diamondsuit_3\Im_1), 1-\mathcal{T}_\Lambda^-(\Re_2\diamondsuit_3\Im_2)\} \leq \max\{\max\{1-\mathcal{T}_\Lambda^-(\Re_1), 1-\mathcal{T}_\Lambda^-(\Re_2)\}, \max\{1-$

$\mathcal{T}_\Lambda^-(\Im_1), 1-\mathcal{T}_\Lambda^-(\Im_2)\}\}$.
If $1-\mathcal{T}_\Lambda^-(\Re_1\diamondsuit_3\Im_1) \geq 1-\mathcal{T}_\Lambda^-(\Re_2\diamondsuit_3\Im_2)$, then $1-\mathcal{T}_\Lambda^-(\Re_1\diamondsuit_3\Im_1) \leq \max\{1-\mathcal{T}_\Lambda^-(\Re_1), 1-\mathcal{T}_\Lambda^-(\Im_1)\}$. Hence, $\mathcal{V}_V^{\mathcal{F}}(\Re\diamondsuit_1\Im) \leq \max\{\mathcal{V}_V^{\mathcal{F}}(\Re), \mathcal{V}_V^{\mathcal{F}}(\Im)\}$, $\mathcal{V}_V^{\mathcal{F}}(\Re\diamondsuit_2\Im) \leq \max\{\mathcal{V}_V^{\mathcal{F}}(\Re), \mathcal{V}_V^{\mathcal{F}}(\Im)\}$ and $\mathcal{V}_V^{\mathcal{F}}(\Re\diamondsuit_3\Im) \leq \max\{\mathcal{V}_V^{\mathcal{F}}(\Re), \mathcal{V}_V^{\mathcal{F}}(\Im)\}$. Hence, Λ is a NSVSBS of $\mathcal{B}$.

Theorem 3.8. *Let Λ be a NSV subset in $\mathcal{B}$. Then $\mathcal{V} = ([\mathcal{T}_\Lambda^-, \mathcal{T}_\Lambda^+], [\mathcal{I}_\Lambda^-, \mathcal{I}_\Lambda^+], [\mathcal{F}_\Lambda^-, \mathcal{F}_\Lambda^+])$ is a NSVSBS of $\mathcal{B}$ if and only if all non empty level set $\mathcal{V}^{(t_1,t_2,s)}$ is a SBS of $\mathcal{B}$ for $t_1, t_2, s \in [0,1]$.*

Proof. Assume that $\mathcal{V}$ is a NSVSBS of $\mathcal{B}$. For $t_1, t_2, s \in [0,1]$ and $\xi_1, \xi_2 \in \mathcal{V}^{(t_1,t_2,s)}$. We have $\mathcal{T}_\Lambda^-(\xi_1) \geq t_1, \mathcal{T}_\Lambda^-(\xi_2) \geq t_1$ and $1-\mathcal{F}_\Lambda^-(\xi_1) \geq s, 1-\mathcal{F}_\Lambda^-(\xi_2) \geq s$ and $\mathcal{I}_\Lambda^-(\xi_1) \geq t_2, \mathcal{I}_\Lambda^-(\xi_2) \geq t_2$ and $\mathcal{I}_\Lambda^+(\xi_1) \geq t_2, \mathcal{I}_\Lambda^+(\xi_2) \geq t_2$, $1-\mathcal{T}_\Lambda^-(\xi_1) \leq t_1, 1-\mathcal{T}_\Lambda^-(\xi_2) \leq t_1$ and $\mathcal{F}_\Lambda^-(\xi_1) \leq s, \mathcal{F}_\Lambda^-(\xi_2) \leq s$. Now, $\mathcal{T}_\Lambda^-(\xi_1\diamondsuit_1\xi_2) \geq \min\{\mathcal{T}_\Lambda^-(\xi_1), \mathcal{T}_\Lambda^-(\xi_2)\} \geq t_1$, $1-\mathcal{F}_\Lambda^-(\xi_1\diamondsuit_1\xi_2) \geq \min\{1-\mathcal{F}_\Lambda^-(\xi_1), 1-\mathcal{F}_\Lambda^-(\xi_2)\} \geq s$ and $\mathcal{I}_\Lambda^-(\xi_1\diamondsuit_1\xi_2) \geq \frac{\mathcal{I}_\Lambda^-(\xi_1)+\mathcal{I}_\Lambda^-(\xi_2)}{2} \geq t_2$, $\mathcal{I}_\Lambda^+(\xi_1\diamondsuit_1\xi_2) \geq \frac{\mathcal{I}_\Lambda^+(\xi_1)+\mathcal{I}_\Lambda^+(\xi_2)}{2} \geq t_2$ and $\mathcal{F}_\Lambda^-(\xi_1\diamondsuit_1\xi_2) \leq \max\{\mathcal{F}_\Lambda^-(\xi_1), \mathcal{F}_\Lambda^-(\xi_2)\} \leq s$ and $1-\mathcal{T}_\Lambda^-(\xi_1\diamondsuit_1\xi_2) \leq \max\{1-\mathcal{T}_\Lambda^-(\xi_1), 1-\mathcal{T}_\Lambda^-(\xi_2)\} \leq t_1$. This implies that $\xi_1\diamondsuit_1\xi_2 \in \mathcal{V}^{(t_1,t_2,s)}$. Similarly, $\xi_1\diamondsuit_2\xi_2 \in \mathcal{V}^{(t_1,t_2,s)}$ and $\xi_1\diamondsuit_3\xi_2 \in \mathcal{V}^{(t_1,t_2,s)}$. Therefore $\mathcal{V}^{(t_1,t_2,s)}$ is a SBS of $\mathcal{B}$, where $t_1, t_2, s \in [0,1]$.

Conversely, assume that $\mathcal{V}^{(t_1,t_2,s)}$ is a SBS of $\mathcal{B}$, where $t_1, t_2, s \in [0,1]$. Suppose if there exist $\xi_1, \xi_2 \in \mathcal{B}$ such that $\mathcal{T}_\Lambda^-(\xi_1\diamondsuit_1\xi_2) < \min\{\mathcal{T}_\Lambda^-(\xi_1), \mathcal{T}_\Lambda^-(\xi_2)\}$, $1-\mathcal{F}_\Lambda^-(\xi_1\diamondsuit_1\xi_2) < \min\{1-\mathcal{F}_\Lambda^-(\xi_1), 1-\mathcal{F}_\Lambda^-(\xi_2)\}$, $\mathcal{I}_\Lambda^-(\xi_1\diamondsuit_1\xi_2) < \frac{\mathcal{I}_\Lambda^-(\xi_1)+\mathcal{I}_\Lambda^-(\xi_2)}{2}$, $\mathcal{I}_\Lambda^+(\xi_1\diamondsuit_1\xi_2) < \frac{\mathcal{I}_\Lambda^+(\xi_1)+\mathcal{I}_\Lambda^+(\xi_2)}{2}$ and $\mathcal{F}_\Lambda^-(\xi_1\diamondsuit_1\xi_2) > \max\{\mathcal{F}_\Lambda^-(\xi_1), \mathcal{F}_\Lambda^-(\xi_2)\}$. $1-\mathcal{T}_\Lambda^-(\xi_1\diamondsuit_1\xi_2) > \max\{1-\mathcal{T}_\Lambda^-(\xi_1), 1-\mathcal{T}_\Lambda^-(\xi_2)\}$. Select $t_1, t_2, s \in [0,1]$ such that $\mathcal{T}_\Lambda^-(\xi_1\diamondsuit_1\xi_2) < t_1 \leq \min\{\mathcal{T}_\Lambda^-(\xi_1), \mathcal{T}_\Lambda^-(\xi_2)\}$ and $1-\mathcal{F}_\Lambda^-(\xi_1\diamondsuit_1\xi_2) < t_1 \leq \min\{1-\mathcal{F}_\Lambda^-(\xi_1), 1-\mathcal{F}_\Lambda^-(\xi_2)\}$ and $\mathcal{I}_\Lambda^-(\xi_1\diamondsuit_1\xi_2) < t_2 \leq \frac{\mathcal{I}_\Lambda^-(\xi_1)+\mathcal{I}_\Lambda^-(\xi_2)}{2}$ and $\mathcal{I}_\Lambda^+(\xi_1\diamondsuit_1\xi_2) < t_2 \leq \frac{\mathcal{I}_\Lambda^+(\xi_1)+\mathcal{I}_\Lambda^+(\xi_2)}{2}$ and $\mathcal{F}_\Lambda^-(\xi_1\diamondsuit_1\xi_2) > s \geq \max\{\mathcal{F}_\Lambda^-(\xi_1), \mathcal{F}_\Lambda^-(\xi_2)\}$, $1-\mathcal{T}_\Lambda^-(\xi_1\diamondsuit_1\xi_2) > s \geq \max\{1-\mathcal{T}_\Lambda^-(\xi_1), 1-\mathcal{T}_\Lambda^-(\xi_2)\}$. Then $\xi_1, \xi_2 \in \mathcal{V}^{(t_1,t_2,s)}$, but $\xi_1\diamondsuit_1\xi_2 \notin \mathcal{V}^{(t_1,t_2,s)}$. This contradicts to that $\mathcal{V}^{(t_1,t_2,s)}$ is a SBS of $\mathcal{B}$. Hence, $\mathcal{T}_\Lambda^-(\xi_1\diamondsuit_1\xi_2) \geq \min\{\mathcal{T}_\Lambda^-(\xi_1), \mathcal{T}_\Lambda^-(\xi_2)\}$, $1-\mathcal{F}_\Lambda^-(\xi_1\diamondsuit_1\xi_2) \geq \min\{1-\mathcal{F}_\Lambda^-(\xi_1), 1-\mathcal{F}_\Lambda^-(\xi_2)\}$, $\mathcal{I}_\Lambda^-(\xi_1\diamondsuit_1\xi_2) \geq \frac{\mathcal{I}_\Lambda^-(\xi_1)+\mathcal{I}_\Lambda^-(\xi_2)}{2}$, $\mathcal{I}_\Lambda^+(\xi_1\diamondsuit_1\xi_2) \geq \frac{\mathcal{I}_\Lambda^+(\xi_1)+\mathcal{I}_\Lambda^+(\xi_2)}{2}$ and $\mathcal{F}_\Lambda^-(\xi_1\diamondsuit_1\xi_2) \leq \max\{\mathcal{F}_\Lambda^-(\xi_1), \mathcal{F}_\Lambda^-(\xi_2)\}$ and $1-\mathcal{T}_\Lambda^-(\xi_1\diamondsuit_1\xi_2) \leq \max\{1-\mathcal{T}_\Lambda^-(\xi_1), 1-\mathcal{T}_\Lambda^-(\xi_2)\}$. Similarly, $\diamondsuit_2$ and $\diamondsuit_3$ cases. Hence, $\mathcal{V} = ([\mathcal{T}_\Lambda^-, \mathcal{T}_\Lambda^+], [\mathcal{I}_\Lambda^-, \mathcal{I}_\Lambda^+], [\mathcal{F}_\Lambda^-, \mathcal{F}_\Lambda^+])$ is a NSVSBS of $\mathcal{B}$.

Definition 3.9. Let Λ be any NSVSBS of $\mathcal{B}$ and $\tau \in \mathcal{B}$. Then the pseudo NSV coset $(\tau\Lambda)^p$ is defined by

$$\left\{\begin{array}{l} (\tau\mathcal{V}_\Lambda^{\mathcal{T}})^p(\Re) = p(\tau)\mathcal{V}_\Lambda^{\mathcal{T}}(\Re), \\ (\tau\mathcal{V}_\Lambda^{\mathcal{I}})^p(\Re) = p(\tau)\mathcal{V}_\Lambda^{\mathcal{I}}(\Re), \\ (\tau\mathcal{V}_\Lambda^{\mathcal{F}})^p(\Re) = p(\tau)\mathcal{V}_\Lambda^{\mathcal{F}}(\Re) \end{array}\right\}.$$

That is,

$$\left\{\begin{array}{l} (\tau\mathcal{T}_\Lambda^-)^p(\Re) = p(\tau)\mathcal{T}_\Lambda^-(\Re),\ 1-(\tau\mathcal{F}_\Lambda^-)^p(\Re) = p(\tau)(1-\mathcal{F}_\Lambda^-)(\Re), \\ (\tau\mathcal{I}_\Lambda^-)^p(\Re) = p(\tau)\mathcal{I}_\Lambda^-(\Re),\ (\tau\mathcal{I}_\Lambda^+)^p(\Re) = p(\tau)\mathcal{I}_\Lambda^+(\Re), \\ (\tau\mathcal{F}_\Lambda^-)^p(\Re) = p(\tau)\mathcal{F}_\Lambda^-(\Re),\ 1-(\tau\mathcal{T}_\Lambda^-)^p(\Re) = p(\tau)(1-\mathcal{T}_\Lambda^-)(\Re) \end{array}\right\}$$

each $\Re \in \mathcal{B}$ and for any non-empty set $p \in P$.

Theorem 3.10. *Let Λ be any NSVSBS of $\mathcal{B}$, then the pseudo NSV coset $(\tau\Lambda)^p$ is a NSVSBS of $\mathcal{B}$.*

Proof. Let Λ be any NSVSBS of $\mathcal{B}$ and for each $\Re, \Im \in \mathcal{B}$. Now, $(\tau\mathcal{T}_\Lambda^-)^p(\Re\Diamond_1\Im) = p(\tau)\ \mathcal{T}_\Lambda^-(\Re\Diamond_1\Im) \geq p(\tau)\ \min\{\mathcal{T}_\Lambda^-(\Re), \mathcal{T}_\Lambda^-(\Im)\} = \min\{p(\tau)\ \mathcal{T}_\Lambda^-(\Re), p(\tau)\ \mathcal{T}_\Lambda^-(\Im)\} = \min\{(\tau\mathcal{T}_\Lambda^-)^p(\Re), (\tau\mathcal{T}_\Lambda^-)^p(\Im)\}$. Thus $(\tau\mathcal{T}_\Lambda^-)^p(\Re\Diamond_1\Im) \geq \min\{(\tau\mathcal{T}_\Lambda^-)^p(\Re), (\tau\mathcal{T}_\Lambda^-)^p(\Im)\}$ and $1-(\tau\mathcal{F}_\Lambda^-)^p(\Re\Diamond_1\Im) = p(\tau)\ (1-\mathcal{F}_\Lambda^-(\Re\Diamond_1\Im)) \geq p(\tau)\ \min\{1-\mathcal{F}_\Lambda^-(\Re), 1-\mathcal{F}_\Lambda^-(\Im)\} = \min\{p(\tau)\ (1-\mathcal{F}_\Lambda^-(\Re)), p(\tau)\ (1-\mathcal{F}_\Lambda^-(\Im))\} = \min\{1-(\tau\mathcal{F}_\Lambda^-)^p(\Re), 1-(\tau\mathcal{F}_\Lambda^-)^p(\Im)\}$. Thus $1-(\tau\mathcal{F}_\Lambda^-)^p(\Re\Diamond_1\Im) \geq \min\{1-(\tau\mathcal{F}_\Lambda^-)^p(\Re), 1-(\tau\mathcal{F}_\Lambda^-)^p(\Im)\}$. Now, $(\tau\mathcal{I}_\Lambda^-)^p(\Re\Diamond_1\Im) = p(\tau)\ \mathcal{I}_\Lambda^-(\Re\Diamond_1\Im) \geq p(\tau)\left[\frac{\mathcal{I}_\Lambda^-(\Re)+\mathcal{I}_\Lambda^-(\Im)}{2}\right] = \frac{p(\tau)\ \mathcal{I}_\Lambda^-(\Re)+p(\tau)\ \mathcal{I}_\Lambda^-(\Im)}{2} = \frac{(\tau\mathcal{I}_\Lambda^-)^p(\Re)+(\tau\mathcal{I}_\Lambda^-)^p(\Im)}{2}$. Thus $(\tau\mathcal{I}_\Lambda^-)^p(\Re\Diamond_1\Im) \geq \frac{(\tau\mathcal{I}_\Lambda^-)^p(\Re)+(\tau\mathcal{I}_\Lambda^-)^p(\Im)}{2}$ and $(\tau\mathcal{I}_\Lambda^+)^p(\Re\Diamond_1\Im) = p(\tau)\ \mathcal{I}_\Lambda^+(\Re\Diamond_1\Im) \geq p(\tau)\left[\frac{\mathcal{I}_\Lambda^+(\Re)+\mathcal{I}_\Lambda^+(\Im)}{2}\right] = \frac{p(\tau)\ \mathcal{I}_\Lambda^+(\Re)+p(\tau)\ \mathcal{I}_\Lambda^+(\Im)}{2} = \frac{(\tau\mathcal{I}_\Lambda^+)^p(\Re)+(\tau\mathcal{I}_\Lambda^+)^p(\Im)}{2}$. Thus $(\tau\mathcal{I}_\Lambda^+)^p(\Re\Diamond_1\Im) \geq \frac{(\tau\mathcal{I}_\Lambda^+)^p(\Re)+(\tau\mathcal{I}_\Lambda^+)^p(\Im)}{2}$. Now, $(\tau\mathcal{F}_\Lambda^-)^p(\Re\Diamond_1\Im) = p(\tau)\ \mathcal{F}_\Lambda^-(\Re\Diamond_1\Im) \leq p(\tau)\ \max\{\mathcal{F}_\Lambda^-(\Re), \mathcal{F}_\Lambda^-(\Im)\} = \max\{p(\tau)\ \mathcal{F}_\Lambda^-(\Re), p(\tau)\ \mathcal{F}_\Lambda^-(\Im)\} = \max\{(\tau\mathcal{F}_\Lambda^-)^p(\Re), (\tau\mathcal{F}_\Lambda^-)^p(\Im)\}$. Thus $(\tau\mathcal{F}_\Lambda^-)^p(\Re\Diamond_1\Im) \leq \max\{(\tau\mathcal{F}_\Lambda^-)^p(\Re), (\tau\mathcal{F}_\Lambda^-)^p(\Im)\}$ and $1-(\tau\mathcal{T}_\Lambda^-)^p(\Re\Diamond_1\Im) = p(\tau)\ (1-\mathcal{T}_\Lambda^-(\Re\Diamond_1\Im)) \leq p(\tau)\ \max\{1-\mathcal{T}_\Lambda^-(\Re), 1-\mathcal{T}_\Lambda^-(\Im)\} = \max\{p(\tau)\ (1-\mathcal{T}_\Lambda^-(\Re)), p(\tau)\ (1-\mathcal{T}_\Lambda^-(\Im))\} = \max\{1-(\tau\mathcal{T}_\Lambda^-)^p(\Re), 1-(\tau\mathcal{T}_\Lambda^-)^p(\Im)\}$. Thus $1-(\tau\mathcal{T}_\Lambda^-)^p(\Re\Diamond_1\Im) \leq \max\{1-(\tau\mathcal{T}_\Lambda^-)^p(\Re), 1-(\tau\mathcal{T}_\Lambda^-)^p(\Im)\}$. Similarly, $\Diamond_2$ and $\Diamond_3$ cases. Hence, $(\tau\Lambda)^p$ is a NSVSBS of $\mathcal{B}$.

Definition 3.11. Let $(\mathcal{B}_1, \varnothing_1, \varnothing_2, \varnothing_3)$ and $(\mathcal{B}_2, \eth_1, \eth_2, \eth_3)$ be the bisemirings. Let $\Upsilon : \mathcal{B}_1 \to \mathcal{B}_2$ and Λ be an NSVSBS in $\mathcal{B}_1$, V be an NSVSBS in $\Upsilon(\mathcal{B}_1) = \mathcal{B}_2$, the image of VS is defined as $\mathcal{V}_{\mho(V)}(\ell_2) = [T^-_{\mho(V)}(\ell_2), 1-F^-_{\mho(V)}(\ell_2)], [I^-_{\mho(V)}(\ell_2), I^+_{\mho(V)}(\ell_2)], [F^-_{\mho(V)}(\ell_2), 1-T^-_{\mho(V)}(\ell_2)]$ where $T^-_{\mho(V)}(\ell_2) = T^-_V\mho(\ell_2)$, $I^-_{\mho(V)}(\ell_2) = I^-_V\mho(\ell_2)$, $I^+_{\mho(V)}(\ell_2) = I^+_V\mho(\ell_2)$ and $F^-_{\mho(V)}(\ell_2) = F^-_V\mho(\ell_2)$.

Definition 3.12. Let $(\mathcal{B}_1, \varnothing_1, \varnothing_2, \varnothing_3)$ and $(\mathcal{B}_2, \eth_1, \eth_2, \eth_3)$ be the bisemirings. Let $\mho : \mathcal{B}_1 \to \mathcal{B}_2$ be any function. Let V be a VS in $\mho(\mathcal{B}_1) = \mathcal{B}_2$. Then the inverse image of V, $\mho^{-1}$ is the VS in $\mathcal{B}_1$ by $\mathcal{V}_{\mho^{-1}(V)}(\ell_1) = [T^-_{\mho^{-1}(V)}(\ell_1), 1-F^-_{\mho^{-1}(V)}(\ell_1)], [I^-_{\mho^{-1}(V)}(\ell_1), I^+_{\mho^{-1}(V)}(\ell_1)], [F^-_{\mho^{-1}(V)}(\ell_1), 1-T^-_{\mho^{-1}(V)}(\ell_1)]$, where $T^-_{\mho^{-1}(V)}(\ell_1) = T^-_V(\mho^{-1}(\ell_1))$, $I^-_{\mho^{-1}(V)}(\ell_1) = I^-_V(\mho^{-1}(\ell_1))$, $I^+_{\mho^{-1}(V)}(\ell_1) = I^+_V(\mho^{-1}(\ell_1))$, $F^-_{\mho^{-1}(V)}(\ell_1) = F^-_V(\mho^{-1}(\ell_1))$.

Theorem 3.13. *Every homomorphic image of NSVSBS of $\mathcal{B}_1$ is a NSVSBS of $\mathcal{B}_2$.*

Proof. Let $\mho : \mathcal{B}_1 \to \mathcal{B}_2$ be a homomorphism. Now, $\mho(\ell_1\varnothing_1\ell_2) = \mho(\ell_1)\eth_1\mho(\ell_2), \mho(\ell_1\varnothing_2\ell_2) = \mho(\ell_1)\eth_2\mho(\ell_2)$ and $\mho(\ell_1\varnothing_3\ell_2) = \mho(\ell_1)\eth_3\mho(\ell_2)$ for all $\ell_1, \ell_2 \in \mathcal{B}_1$. Let $V = \mho(\Lambda)$, Λ is a NSVSBS of $\mathcal{B}_1$. Let $\mho(\ell_1), \mho(\ell_2) \in \mathcal{B}_2$, $T^-_V(\mho(\ell_1)\eth_1\mho(\ell_2)) \geq T^-_\Lambda(\ell_1\varnothing_1\ell_2) \geq \min\{T^-_\Lambda(\ell_1), T^-_\Lambda(\ell_2)\} = \min\{T^-_V\mho(\ell_1), T^-_V\mho(\ell_2)\}$ and $1-F^-_V(\mho(\ell_1)\eth_1\mho(\ell_2)) \geq 1-F^-_\Lambda(\ell_1\varnothing_1\ell_2) \geq \min\{1-F^-_\Lambda(\ell_1), 1-F^-_\Lambda(\ell_2)\} =$

$\min\{1 - F_V^{-}\mho(\ell_1), 1 - F_V^{-}\mho(\ell_2)\}$. Thus $\mathcal{V}_V^{\mathcal{T}}(\mho(\ell_1)\eth_1\mho(\ell_2)) \geq \min\{\mathcal{V}_V^{\mathcal{T}}\mho(\ell_1), \mathcal{V}_V^{\mathcal{T}}\mho(\ell_2)\}$. Similarly, $\mathcal{V}_V^{\mathcal{T}}(\mho(\ell_1)\eth_2\mho(\ell_2)) \geq \min\{\mathcal{V}_V^{\mathcal{T}}\mho(\ell_1), \mathcal{V}_V^{\mathcal{T}}\mho(\ell_2)\}$ and $\mathcal{V}_V^{\mathcal{T}}(\mho(\ell_1)\eth_3\mho(\ell_2)) \geq \min\{\mathcal{V}_V^{\mathcal{T}}\mho(\ell_1), \mathcal{V}_V^{\mathcal{T}}\mho(\ell_2)\}$. Now, $I_V^{-}(\mho(\ell_1)\eth_1\mho(\ell_2)) \geq I_\Lambda^{-}(\ell_1\varnothing_1\ell_2) \geq \frac{I_\Lambda^{-}(\ell_1)+I_\Lambda^{-}(\ell_2)}{2} = \frac{I_V^{-}\mho(\ell_1)+I_V^{-}\mho(\ell_2)}{2}$ and $I_V^{+}(\mho(\ell_1)\eth_1\mho(\ell_2)) \geq I_\Lambda^{+}(\ell_1\varnothing_1\ell_2) \geq \frac{I_\Lambda^{+}(\ell_1)+I_\Lambda^{+}(\ell_2)}{2} = \frac{I_V^{+}\mho(\ell_1)+I_V^{+}\mho(\ell_2)}{2}$. Thus $\mathcal{V}_V^{\mathcal{I}}(\mho(\ell_1)\eth_1\mho(\ell_2)) \geq \frac{\mathcal{V}_V^{\mathcal{I}}\mho(\ell_1)+\mathcal{V}_V^{\mathcal{I}}\mho(\ell_2)}{2}$. Similarly, $\mathcal{V}_V^{\mathcal{I}}(\mho(\ell_1)\eth_2\mho(\ell_2)) \geq \min\{\mathcal{V}_V^{\mathcal{I}}\mho(\ell_1), \mathcal{V}_V^{\mathcal{I}}\mho(\ell_2)\}$ and $\mathcal{V}_V^{\mathcal{I}}(\mho(\ell_1)\eth_3\mho(\ell_2)) \geq \min\{\mathcal{V}_V^{\mathcal{I}}\mho(\ell_1), \mathcal{V}_V^{\mathcal{I}}\mho(\ell_2)\}$. Now, $F_V^{-}(\mho(\ell_1)\eth_1\mho(\ell_2)) \leq F_\Lambda^{-}(\ell_1\varnothing_1\ell_2) \leq \max\{F_\Lambda^{-}(\ell_1), F_\Lambda^{-}(\ell_2)\} = \max\{F_V^{-}\mho(\ell_1), F_V^{-}\mho(\ell_2)\}$ and $1 - T_V^{-}(\mho(\ell_1)\eth_1\mho(\ell_2)) \leq 1 - T_\Lambda^{-}(\ell_1\varnothing_1\ell_2) \leq \max\{1 - T_\Lambda^{-}(\ell_1), 1 - T_\Lambda^{-}(\ell_2)\} = \max\{1 - T_V^{-}\mho(\ell_1), 1 - T_V^{-}\mho(\ell_2)\}$. Thus $\mathcal{V}_V^{\mathcal{F}}(\mho(\ell_1)\eth_1\mho(\ell_2)) \leq \max\{\mathcal{V}_V^{\mathcal{F}}\mho(\ell_1), \mathcal{V}_V^{\mathcal{F}}\mho(\ell_2)\}$. Similarly, $\mathcal{V}_V^{\mathcal{F}}(\mho(\ell_1)\eth_2\mho(\ell_2)) \leq \max\{\mathcal{V}_V^{\mathcal{F}}\mho(\ell_1), \mathcal{V}_V^{\mathcal{F}}\mho(\ell_2)\}$ and $\mathcal{V}_V^{\mathcal{F}}(\mho(\ell_1)\eth_3\mho(\ell_2)) \leq \max\{\mathcal{V}_V^{\mathcal{F}}\mho(\ell_1), \mathcal{V}_V^{\mathcal{F}}\mho(\ell_2)\}$. Hence, V is a NSVSBS of $\mathcal{B}_2$.

Theorem 3.14. *Every homomorphic pre-image of NSVSBS of $\mathcal{B}_2$ is a NSVSBS of $\mathcal{B}_1$.*

Proof. Let $\mho : \mathcal{B}_1 \to \mathcal{B}_2$ and $\mho(\Re\varnothing_1\Im) = \mho(\Re)\eth_1\mho(\Im), \mho(\Re\varnothing_2\Im) = \mho(\Re)\eth_2\mho(\Im)$ and $\mho(\Re\varnothing_3\Im) = \mho(\Re)\eth_3\mho(\Im)$ for all $\Re, \Im \in \mathcal{B}_1$. Let $V = \mho(\Lambda)$, where V is any NSVSBS of $\mathcal{B}_2$. Let $\Re, \Im \in \mathcal{B}_1$. Now, $T_\Lambda^{-}(\Re\varnothing_1\Im) = T_V^{-}(\mho(\Re\varnothing_1\Im)) = T_V^{-}(\mho(\Re)\eth_1\mho(\Im)) \geq \min\{T_V^{-}\mho(\Re), T_V^{-}\mho(\Im)\} = \min\{T_\Lambda^{-}(\Re), T_\Lambda^{-}(\Im)\}$. Thus $T_\Lambda^{-}(\Re\varnothing_1\Im) \geq \min\{T_\Lambda^{-}(\Re), T_\Lambda^{-}(\Im)\}$ and $1 - F_\Lambda^{-}(\Re\varnothing_1\Im) = 1 - F_V^{-}(\mho(\Re\varnothing_1\Im)) = 1 - F_V^{-}(\mho(\Re)\eth_1\mho(\Im)) \geq \min\{1 - F_V^{-}\mho(\Re), 1 - F_V^{-}\mho(\Im)\} = \min\{1 - F_\Lambda^{-}(\Re), 1 - F_\Lambda^{-}(\Im)\}$. Thus $1 - F_\Lambda^{-}(\Re\varnothing_1\Im) \geq \min\{1 - F_\Lambda^{-}(\Re), 1 - F_\Lambda^{-}(\Im)\}$. Hence, $\mathcal{V}_V^{\mathcal{T}}(\Re\varnothing_1\Im) \geq \min\{\mathcal{V}_V^{\mathcal{T}}(\Re), \mathcal{V}_V^{\mathcal{T}}(\Im)\}$. Similarly, $\mathcal{V}_V^{\mathcal{T}}(\Re\varnothing_2\Im) \geq \min\{\mathcal{V}_V^{\mathcal{T}}(\Re), \mathcal{V}_V^{\mathcal{T}}(\Im)\}$ and $\mathcal{V}_V^{\mathcal{T}}(\Re\varnothing_3\Im) \geq \min\{\mathcal{V}_V^{\mathcal{T}}(\Re), \mathcal{V}_V^{\mathcal{T}}(\Im)\}$. Now, $I_\Lambda^{-}(\Re\varnothing_1\Im) = I_V^{-}(\mho(\Re\varnothing_1\Im)) = I_V^{-}(\mho(\Re)\eth_1\mho(\Im)) \geq \frac{I_V^{-}\mho(\Re)+I_V^{-}\mho(\Im)}{2} = \frac{I_\Lambda^{-}(\Re)+I_\Lambda^{-}(\Im)}{2}$. Thus $I_\Lambda^{-}(\Re\varnothing_1\Im) \geq \frac{I_\Lambda^{-}(\Re)+I_\Lambda^{-}(\Im)}{2}$ and $I_\Lambda^{+}(\Re\varnothing_1\Im) = I_V^{+}(\mho(\Re\varnothing_1\Im)) = I_V^{+}(\mho(\Re)\eth_1\mho(\Im)) \geq \frac{I_V^{+}\mho(\Re)+I_V^{+}\mho(\Im)}{2} = \frac{I_\Lambda^{+}(\Re)+I_\Lambda^{+}(\Im)}{2}$. Thus $I_\Lambda^{+}(\Re\varnothing_1\Im) \geq \frac{I_\Lambda^{+}(\Re)+I_\Lambda^{+}(\Im)}{2}$. Hence, $\mathcal{V}_V^{\mathcal{I}}(\Re\varnothing_1\Im) \geq \frac{\mathcal{V}_V^{\mathcal{I}}(\Re)+\mathcal{V}_V^{\mathcal{I}}(\Im)}{2}$. Similarly, $\mathcal{V}_V^{\mathcal{I}}(\Re\varnothing_2\Im) \geq \frac{\mathcal{V}_V^{\mathcal{I}}(\Re)+\mathcal{V}_V^{\mathcal{I}}(\Im)}{2}$ and $\mathcal{V}_V^{\mathcal{I}}(\Re\varnothing_3\Im) \geq \frac{\mathcal{V}_V^{\mathcal{I}}(\Re)+\mathcal{V}_V^{\mathcal{I}}(\Im)}{2}$. Now, $F_\Lambda^{-}(\Re\varnothing_1\Im) = F_V^{-}(\mho(\Re\varnothing_1\Im)) = F_V^{-}(\mho(\Re)\eth_1\mho(\Im)) \leq \max\{F_V^{-}\mho(\Re), F_V^{-}\mho(\Im)\} = \max\{F_\Lambda^{-}(\Re), F_\Lambda^{-}(\Im)\}$. Thus $F_\Lambda^{-}(\Re\varnothing_1\Im) \leq \max\{F_\Lambda^{-}(\Re), F_\Lambda^{-}(\Im)\}$ and $1 - T_\Lambda^{-}(\Re\varnothing_1\Im) = 1 - T_V^{-}(\mho(\Re\varnothing_1\Im)) = 1 - T_V^{-}(\mho(\Re)\eth_1\mho(\Im)) \leq \max\{1 - T_V^{-}\mho(\Re), 1 - T_V^{-}\mho(\Im)\} = \max\{1 - T_\Lambda^{-}(\Re), 1 - T_\Lambda^{-}(\Im)\}$. Thus $1 - T_\Lambda^{-}(\Re\varnothing_1\Im) \leq \max\{1 - T_\Lambda^{-}(\Re), 1 - T_\Lambda^{-}(\Im)\}$. Hence, $\mathcal{V}_V^{\mathcal{F}}(\Re\varnothing_1\Im) \leq \max\{\mathcal{V}_V^{\mathcal{F}}(\Re), \mathcal{V}_V^{\mathcal{F}}(\Im)\}$. Similarly, $\mathcal{V}_V^{\mathcal{F}}(\Re\varnothing_2\Im) \leq \max\{\mathcal{V}_V^{\mathcal{F}}(\Re), \mathcal{V}_V^{\mathcal{F}}(\Im)\}$ and $\mathcal{V}_V^{\mathcal{F}}(\Re\varnothing_3\Im) \leq \max\{\mathcal{V}_V^{\mathcal{F}}(\Re), \mathcal{V}_V^{\mathcal{F}}(\Im)\}$. Hence, Λ is a NSVSBS of $\mathcal{B}_1$.

Theorem 3.15. *If $\mho : \mathcal{B}_1 \to \mathcal{B}_2$ is a homomorphism, then $\mho(\Lambda_{(t_1,t_2,s)})$ is a level SBS of NSVSBS V of $\mathcal{B}_2$.*

Proof. Let $\mho : \mathcal{B}_1 \to \mathcal{B}_2$ be a homomorphism and $\mho(\Re\varnothing_1\Im) = \mho(\Re)\eth_1\mho(\Im), \mho(\Re\varnothing_2\Im) = \mho(\Re)\eth_2\mho(\Im)$ and $\mho(\Re\varnothing_3\Im) = \mho(\Re)\eth_3\mho(\Im)$ for all $\Re, \Im \in \mathcal{B}_1$. Let $V = \mho(\Lambda)$, Λ is a NSVSBS of $\mathcal{B}_1$. By Theorem 3.13, V is a NSVSBS of $\mathcal{B}_2$. Let $\Lambda_{(t_1,t_2,s)}$ be any level SBS of Λ. Suppose that $\Re, \Im \in \Lambda_{(t_1,t_2,s)}$. Then $\mho(\Re\varnothing_1\Im), \mho(\Re\varnothing_2\Im)$ and $\mho(\Re\varnothing_3\Im) \in \Lambda_{(t_1,t_2,s)}$. Now, $T_V^{-}(\mho(\Re)) =$

$T^-_\Lambda(\Re) \geq t_1, T^-_V(\mho(\Im)) = T^-_\Lambda(\Im) \geq t_1$. Thus $T^-_V(\mho(\Re)\eth_1\mho(\Im)) \geq T^-_\Lambda(\Re\varnothing_1\Im) \geq t_1$ and $1-F^-_V(\mho(\Re)) = 1-F^-_\Lambda(\Re) \geq s, 1-F^-_V(\mho(\Im)) = 1-F^-_\Lambda(\Im) \geq s$. Thus $1-F^-_V(\mho(\Re)\eth\mho(\Im)) \geq 1-F^-_\Lambda(\Re\varnothing_1\Im) \geq s$. Now, $I^-_V(\mho(\Re)) = I^-_\Lambda(\Re) \geq t_2, I^-_V(\mho(\Im)) = I^-_\Lambda(\Im) \geq t_2$. Thus $I^-_V(\mho(\Re)\eth_1\mho(\Im)) \geq I^-_\Lambda(\Re\varnothing_1\Im) \geq t_2$ and $I^+_V(\mho(\Re)) = I^+_\Lambda(\Re) \geq t_2, I^+_V(\mho(\Im)) = I^+_\Lambda(\Im) \geq t_2$. Thus $I^+_V(\mho(\Re)\eth_1\mho(\Im)) \geq I^+_\Lambda(\Re\varnothing_1\Im) \geq t_2$. Now, $F^-_V(\mho(\Re)) = F^-_\Lambda(\Re) \leq s, F^-_V(\mho(\Im)) = F^-_\Lambda(\Im) \leq s$. Thus $F^-_V(\mho(\Re)\eth_1\mho(\Im)) \leq F^-_\Lambda(\Re\varnothing_1\Im) \leq s$ and $1-T^-_V(\mho(\Re)) = 1-T^-_\Lambda(\Re) \leq t_1, 1-T^-_V(\mho(\Im)) = 1-T^-_\Lambda(\Im) \leq t_1$. Thus $1-T^-_V(\mho(\Re)\eth\mho(\Im)) \leq 1-T^-_\Lambda(\Re\varnothing_1\Im) \leq t_1$, for all $\mho(\Re), \mho(\Im) \in \mathcal{B}_2$. Similarly to prove other operations. Hence proved.

Theorem 3.16. *If $\mho : \mathcal{B}_1 \to \mathcal{B}_2$ is any homomorphism, then $\Lambda_{(t_1,t_2,s)}$ is a level SBS of NSVSBS Λ of $\mathcal{B}_1$.*

Proof. Let $\mho : \mathcal{B}_1 \to \mathcal{B}_2$ be a homomorphism and $\mho(\Re\varnothing_1\Im) = \mho(\Re)\eth_1\mho(\Im), \mho(\Re\varnothing_2\Im) = \mho(\Re)\eth_2\mho(\Im)$ and $\mho(\Re\varnothing_3\Im) = \mho(\Re)\eth_3\mho(\Im)$ for all $\Re, \Im \in \mathcal{B}_1$. Let $V = \mho(\Lambda)$, V is a NSVSBS of $\mathcal{B}_2$. By Theorem 3.14, Λ is a NSVSBS of $\mathcal{B}_1$. Let $\mho(\Lambda_{(t_1,t_2,s)})$ be a level SBS of V. Suppose that $\mho(\Re), \mho(\Im) \in \mho(\Lambda_{(t_1,t_2,s)})$. Then $\mho(\Re\varnothing_1\Im), \mho(\Re\varnothing_2\Im)$ and $\mho(\Re\varnothing_3\Im) \in \mho(\Lambda_{(t_1,t_2,s)})$. Now, $T^-_\Lambda(\Re) = T^-_V(\mho(\Re)) \geq t_1, T^-_\Lambda(\Im) = T^-_V(\mho(\Im)) \geq t_1$. Thus $T^-_\Lambda(\Re\varnothing_1\Im) \geq \min\{T^-_\Lambda(\Re), T^-_\Lambda(\Im)\} \geq t_1$ and $1-F^-_\Lambda(\Re) = 1-F^-_V(\mho(\Re)) \geq s, 1-F^-_\Lambda(\Im) = 1-F^-_V(\mho(\Im)) \geq s$. Thus $1-F^-_\Lambda(\Re\varnothing_1\Im) \geq \min\{1-F^-_\Lambda(\Re), 1-F^-_\Lambda(\Im)\} \geq s$. Now, $I^-_\Lambda(\Re) = I^-_V(\mho(\Re)) \geq t_2, I^-_\Lambda(\Im) = I^-_V(\mho(\Im)) \geq t_2$. Thus $I^-_\Lambda(\Re\varnothing_1\Im) \geq \frac{I^-_\Lambda(\Re)+I^-_\Lambda(\Im)}{2} \geq t_2$ and $I^+_\Lambda(\Re) = I^+_V(\mho(\Re)) \geq t_2, I^+_\Lambda(\Im) = I^+_V(\mho(\Im)) \geq t_2$. Thus $I^+_\Lambda(\Re\varnothing_1\Im) \geq \frac{I^+_\Lambda(\Re)+I^+_\Lambda(\Im)}{2} \geq t_2$. Now, $F^-_\Lambda(\Re) = F^-_V(\mho(\Re)) \leq s, F^-_\Lambda(\Im) = F^-_V(\mho(\Im)) \leq s$. Thus $F^-_\Lambda(\Re\varnothing_1\Im) = F^-_V(\mho(\Re)\eth_1\mho(\Im)) \leq \max\{F^-_\Lambda(\Re), F^-_\Lambda(\Im)\} \leq s$ and $1-T^-_\Lambda(\Re) = 1-T^-_V(\mho(\Re)) \leq t_1, 1-T^-_\Lambda(\Im) = 1-T^-_V(\mho(\Im)) \leq t_1$. Thus $1-T^-_\Lambda(\Re\varnothing_1\Im) = 1-T^-_V(\mho(\Re)\eth_1\mho(\Im)) \leq \max\{1-T^-_\Lambda(\Re), 1-T^-_\Lambda(\Im)\} \leq t_1$, for all $\Re, \Im \in \mathcal{B}_1$. Similarly to prove other two operations. Hence proved.

4. (ρ, σ)-Neutrosophic vague SBSs

We discuss about (ρ, σ)-NSVSBS and $(\rho, \sigma) \in [0, 1]$ be such that $0 \leq \rho < \sigma \leq 1$.

Definition 4.1. Let Λ be any NSVS of $\mathcal{B}$ is called a (ρ, σ)-NSVSBS of $\mathcal{B}$ if

$$\left\{\begin{array}{l} \max\{\mathcal{V}^{\mathcal{T}}_\Lambda(\Re\Diamond_1\Im), \rho\} \geq \min\{\mathcal{V}^{\mathcal{T}}_\Lambda(\Re), \mathcal{V}^{\mathcal{T}}_\Lambda(\Im), \sigma\} \\ \max\{\mathcal{V}^{\mathcal{T}}_\Lambda(\Re\Diamond_2\Im), \rho\} \geq \min\{\mathcal{V}^{\mathcal{T}}_\Lambda(\Re), \mathcal{V}^{\mathcal{T}}_\Lambda(\Im), \sigma\} \\ \max\{\mathcal{V}^{\mathcal{T}}_\Lambda(\Re\Diamond_3\Im), \rho\} \geq \min\{\mathcal{V}^{\mathcal{T}}_\Lambda(\Re), \mathcal{V}^{\mathcal{T}}_\Lambda(\Im), \sigma\} \end{array}\right\} \quad \left\{\begin{array}{l} \max\{\mathcal{V}^{\mathcal{I}}_\Lambda(\Re\Diamond_1\Im), \rho\} \geq \min\left\{\frac{\mathcal{V}^{\mathcal{I}}_\Lambda(\Re)+\mathcal{V}^{\mathcal{I}}_\Lambda(\Im)}{2}, \sigma\right\} \\ \qquad OR \\ \max\{\mathcal{V}^{\mathcal{I}}_\Lambda(\Re\Diamond_2\Im), \rho\} \geq \min\left\{\frac{\mathcal{V}^{\mathcal{I}}_\Lambda(\Re)+\mathcal{V}^{\mathcal{I}}_\Lambda(\Im)}{2}, \sigma\right\} \\ \qquad OR \\ \max\{\mathcal{V}^{\mathcal{I}}_\Lambda(\Re\Diamond_3\Im), \rho\} \geq \min\left\{\frac{\mathcal{V}^{\mathcal{I}}_\Lambda(\Re)+\mathcal{V}^{\mathcal{I}}_\Lambda(\Im)}{2}, \sigma\right\} \end{array}\right\}$$

$$\begin{Bmatrix} \min\{\mathcal{V}_\Lambda^{\mathcal{F}}(\Re\Diamond_1\Im),\rho\} \leq \max\{\mathcal{V}_\Lambda^{\mathcal{F}}(\Re),\mathcal{V}_\Lambda^{\mathcal{F}}(\Im),\sigma\} \\ \min\{\mathcal{V}_\Lambda^{\mathcal{F}}(\Re\Diamond_2\Im),\rho\} \leq \max\{\mathcal{V}_\Lambda^{\mathcal{F}}(\Re),\mathcal{V}_\Lambda^{\mathcal{F}}(\Im),\sigma\} \\ \min\{\mathcal{V}_\Lambda^{\mathcal{F}}(\Re\Diamond_3\Im),\rho\} \leq \max\{\mathcal{V}_\Lambda^{\mathcal{F}}(\Re),\mathcal{V}_\Lambda^{\mathcal{F}}(\Im),\sigma\} \end{Bmatrix}.$$

That is,

$$\begin{Bmatrix} \begin{pmatrix} \max\{\mathcal{T}_\Lambda^-(\Re\Diamond_1\Im),\rho\} \geq \min\{\mathcal{T}_\Lambda^-(\Re),\mathcal{T}_\Lambda^-(\Im),\sigma\}, \\ \max\{1-\mathcal{F}_\Lambda^-(\Re\Diamond_1\Im),\rho\} \geq \min\{1-\mathcal{F}_\Lambda^-(\Re),1-\mathcal{F}_\Lambda^-(\Im),\sigma\} \end{pmatrix} \\ \begin{pmatrix} \max\{\mathcal{T}_\Lambda^-(\Re\Diamond_2\Im),\rho\} \geq \min\{\mathcal{T}_\Lambda^-(\Re),\mathcal{T}_\Lambda^-(\Im),\sigma\}, \\ \max\{1-\mathcal{F}_\Lambda^-(\Re\Diamond_2\Im),\rho\} \geq \min\{1-\mathcal{F}_\Lambda^-(\Re),1-\mathcal{F}_\Lambda^-(\Im),\sigma\} \end{pmatrix} \\ \begin{pmatrix} \max\{\mathcal{T}_\Lambda^-(\Re\Diamond_3\Im),\rho\} \geq \min\{\mathcal{T}_\Lambda^-(\Re),\mathcal{T}_\Lambda^-(\Im),\sigma\}, \\ \max\{1-\mathcal{F}_\Lambda^-(\Re\Diamond_3\Im),\rho\} \geq \min\{1-\mathcal{F}_\Lambda^-(\Re),1-\mathcal{F}_\Lambda^-(\Im),\sigma\} \end{pmatrix} \end{Bmatrix}$$

$$\begin{Bmatrix} \begin{pmatrix} \max\{\mathcal{I}_\Lambda^+(\Re\Diamond_1\Im),\rho\} \geq \min\left\{\frac{\mathcal{I}_\Lambda^+(\Re)+\mathcal{I}_\Lambda^+(\Im)}{2},\sigma\right\} \\ \max\{\mathcal{I}_\Lambda^-(\Re\Diamond_1\Im),\rho\} \geq \min\left\{\frac{\mathcal{I}_\Lambda^-(\Re)-\mathcal{I}_\Lambda^-(\Im)}{2},\sigma\right\} \end{pmatrix} \\ OR \\ \begin{pmatrix} \max\{\mathcal{I}_\Lambda^+(\Re\Diamond_2\Im),\rho\} \geq \min\left\{\frac{\mathcal{I}_\Lambda^+(\Re)+\mathcal{I}_\Lambda^+(\Im)}{2},\sigma\right\} \\ \max\{\mathcal{I}_\Lambda^-(\Re\Diamond_2\Im),\rho\} \geq \min\left\{\frac{\mathcal{I}_\Lambda^-(\Re)-\mathcal{I}_\Lambda^-(\Im)}{2},\sigma\right\} \end{pmatrix} \\ OR \\ \begin{pmatrix} \max\{\mathcal{I}_\Lambda^+(\Re\Diamond_3\Im),\rho\} \geq \min\left\{\frac{\mathcal{I}_\Lambda^+(\Re)+\mathcal{I}_\Lambda^+(\Im)}{2},\sigma\right\} \\ \max\{\mathcal{I}_\Lambda^-(\Re\Diamond_3\Im),\rho\} \geq \min\left\{\frac{\mathcal{I}_\Lambda^-(\Re)-\mathcal{I}_\Lambda^-(\Im)}{2},\sigma\right\} \end{pmatrix} \end{Bmatrix}$$

$$\begin{Bmatrix} \begin{pmatrix} \min\{\mathcal{F}_\Lambda^-(\Re\Diamond_1\Im),\rho\} \leq \max\{\mathcal{F}_\Lambda^-(\Re),\mathcal{F}_\Lambda^-(\Im),\sigma\}, \\ \min\{1-\mathcal{T}_\Lambda^-(\Re\Diamond_1\Im),\rho\} \leq \max\{1-\mathcal{T}_\Lambda^-(\Re),1-\mathcal{T}_\Lambda^-(\Im),\sigma\} \end{pmatrix} \\ \begin{pmatrix} \min\{\mathcal{F}_\Lambda^-(\Re\Diamond_2\Im),\rho\} \leq \max\{\mathcal{F}_\Lambda^-(\Re),\mathcal{F}_\Lambda^-(\Im),\sigma\}, \\ \min\{1-\mathcal{T}_\Lambda^-(\Re\Diamond_2\Im),\rho\} \leq \max\{1-\mathcal{T}_\Lambda^-(\Re),1-\mathcal{T}_\Lambda^-(\Im),\sigma\} \end{pmatrix} \\ \begin{pmatrix} \min\{\mathcal{F}_\Lambda^-(\Re\Diamond_3\Im),\rho\} \leq \max\{\mathcal{F}_\Lambda^-(\Re),\mathcal{F}_\Lambda^-(\Im),\sigma\}, \\ \min\{1-\mathcal{T}_\Lambda^-(\Re\Diamond_3\Im),\rho\} \leq \max\{1-\mathcal{T}_\Lambda^-(\Re),1-\mathcal{T}_\Lambda^-(\Im),\sigma\} \end{pmatrix} \end{Bmatrix}$$

for all $\Re,\Im \in \mathcal{B}$.

Example 4.2. By Example 3.2,

	$[\mathcal{T}_\Lambda^-(\varphi),\mathcal{T}_\Lambda^+(\varphi)]$	$[\mathcal{I}_\Lambda^-(\varphi),\mathcal{I}_\Lambda^+(\varphi)]$	$[\mathcal{F}_\Lambda^-(\varphi),\mathcal{F}_\Lambda^+(\varphi)]$
$\varphi=\dot{a}$	$[0.65,0.70]$	$[0.55,0.65]$	$[0.3,0.35]$
$\varphi=\ddot{a}$	$[0.6,0.65]$	$[0.50,0.60]$	$[0.35,0.40]$
$\varphi=\tilde{a}$	$[0.35,0.40]$	$[0.25,0.30]$	$[0.60,0.65]$
$\varphi=\vec{a}$	$[0.45,0.55]$	$[0.40,0.50]$	$[0.45,0.55]$

Clearly, Λ is a $(0.25,0.85)$ NSVSBS of $\mathcal{B}$.

Theorem 4.3. *The intersection of a family of every (ρ,σ)- $NSVSBS^s$ is a (ρ,σ)-NSVSBS.*

Proof. The proof is similar to Theorem 3.3.

Theorem 4.4. *If Λ and Ψ are any two (ρ, σ)- $NSVSBS^s$ of $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, then $\Lambda \times \Psi$ is a (ρ, σ)-NSVSBS of $\mathcal{B}_1 \times \mathcal{B}_2$.*

Proof. The proof is similar to Theorem 3.4.

Corollary 4.5. *If $\Lambda_1, \Lambda_2, ..., \Lambda_n$ are the families of (ρ, σ)-$NSVSBS^s$ of $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n$ respectively, then $\Lambda_1 \times \Lambda_2 \times ... \times \Lambda_n$ is a (ρ, σ)-NSVSBS of $\mathcal{B}_1 \times \mathcal{B}_2 \times ... \times \mathcal{B}_n$.*

Definition 4.6. Let Λ be a (ρ, σ)- NSVS in $\mathcal{B}$, the (ρ, σ)-SNSVR on $\mathcal{B}$. ie) (ρ, σ)- NSVR on Λ is V given by

$$\left\{\begin{array}{l} \max\{\mathcal{V}_\Lambda^{\mathcal{T}}(\Re, \Im), \rho\} = \min\{\mathcal{V}_\Lambda^{\mathcal{T}}(\Re), \mathcal{V}_\Lambda^{\mathcal{T}}(\Im), \sigma\} \\ \max\{\mathcal{V}_\Lambda^{\mathcal{I}}(\Re, \Im), \rho\} = \min\left\{\frac{\mathcal{V}_\Lambda^{\mathcal{I}}(\Re) + \mathcal{V}_\Lambda^{\mathcal{I}}(\Im)}{2}, \sigma\right\} \\ \min\{\mathcal{V}_\Lambda^{\mathcal{F}}(\Re, \Im), \rho\} = \max\{\mathcal{V}_\Lambda^{\mathcal{F}}(\Re), \mathcal{V}_\Lambda^{\mathcal{F}}(\Im), \sigma\} \end{array}\right\}.$$

Theorem 4.7. *Let Λ be a (ρ, σ)-NSVSBS of $\mathcal{B}$ and V be the (ρ, σ)-SNSVR of $\mathcal{B}$. Then Λ is a (ρ, σ)-NSVSBS of $\mathcal{B}$ if and only if V is a (ρ, σ)-NSVSBS of $\mathcal{B} \times \mathcal{B}$.*

Proof. A similar proof is given in Theorem 3.7.

Theorem 4.8. *A homomorphic image of (ρ, σ)-NSVSBS of $\mathcal{B}_1$ is a (ρ, σ)-NSVSBS of $\mathcal{B}_2$.*

Proof. A similar proof is given in Theorem 3.13.

Theorem 4.9. *A homomorphic pre-image of (ρ, σ)-NSVSBS of $\mathcal{B}_2$ is a (ρ, σ)-NSVSBS of $\mathcal{B}_1$.*

Proof. A similar proof is given in Theorem 3.14.

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