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On λ -statistical and V_λ -statistical summability in neutrosophic-2-normed spaces

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Abstract. In the present paper, we aim to define λ -statistical summability, V_λ -statistical summability in neutrosophic-2-normed spaces (briefly called $N - 2 - NS$) and study some relationships among these notions. We give an example that shows λ -statistical summability is stronger method in neutrosophic-2-normed spaces. Finally, we define λ -statistically Cauchy sequence and λ -statistically completeness in neutrosophic-2-normed spaces and obtain the Cauchy convergence criteria in these spaces.

Keywords: λ -statistical convergence, V_λ -summable, λ -statistical Cauchy, neutrosophic-2-normed spaces.

1. Introduction

The idea of λ -statistical convergence was explored by Mursaleen [21] as a generalization of statistical convergence, initially introduced in [4] and [33] independently.

“Let $\lambda = (\lambda_n) ; \lambda_n \in \mathbb{R}^+ = (0, \infty)$ be a non decreasing sequence satisfying $\lim_{n \rightarrow \infty} \lambda_n = \infty$, $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$ and $I_n = [n - \lambda_n + 1, n]$. For $\mathfrak{A} \subseteq \mathbb{N}$, the λ -density of \mathfrak{A} is defined by $\delta_\lambda(\mathfrak{A}) = \lim_n \frac{1}{\lambda_n} |\{k \in I_n : k \in \mathfrak{A}\}|$. A sequence $\mathbf{u} = (u_k)$ of numbers is said to be λ -statistical convergent to u_0 if for each $\eta > 0$, $\lim_n \frac{1}{\lambda_n} |\{k \in I_n : |u_k - u_0| \geq \eta\}| = 0$, i.e., $\delta_\lambda(\mathfrak{A}_\eta) = 0$, where $\mathfrak{A}_\eta = \{k \in I_n : |u_k - u_0| \geq \eta\}$. We write, in this case $\mathcal{S}_\lambda - \lim_k u_k = u_0$.” Subsequently, statistical convergence and its generalizations have been developed by numerous authors including Connor [3], Fridy [6], Hazarika et al. [8, 9], Kumar et al. [15], Maddox [20], Šalát [36] and many others.

On the other hand, many problems of real life can't be modeled via crispness due to huge uncertainty in data. In view of this Zadeh [41] defined fuzzy sets as generalization of crisp sets to deal with such problems.

One of interesting generalizations of fuzzy sets is due to Atanassov [1], called intuitionistic fuzzy sets by adding the non-membership function along with the membership function to the fuzzy sets. These sets have been applied to introduce new norms (see [5], [22]), topology [31], and metric [27] and found very useful where the crisp norms are not sufficient to work due to huge uncertainty. Intuitionistic fuzzy sets are naturally used to define intuitionistic fuzzy normed spaces [34]. Recently, statistical convergence and its generalizations have been extended and developed in these spaces (see [2], [15], [25], [26] and [38]).

Another, interesting generalization of fuzzy sets is given by Smarandache [35] by introducing the indeterminacy function to the intuitionistic fuzzy sets. For ongoing development on neutrosophic set (NS) and its applications, we refer to the reader [10], [18], [23], [29-31], etc. Kirişçi and Şimşek [13] used neutrosophic sets to define neutrosophic norm and studied statistical convergence in neutrosophic normed spaces(NNS). Nowadays, the area of summability in these spaces is of much interest. For a broad view in this direction, we recommend [28], [37-39], etc. Several summability approaches have been created, including statistical convergence, lacunary statistical convergence, and ideal convergence in these spaces (see [11], [12], [14], [16], [17], and [32]). Recently, in [24] the concept of neutrosophic-2-norm is introduced where the authors studied statistical convergence in neutrosophic-2-normed spaces. In the present study, we continue to define a more general summability method, called \mathcal{S}_λ -summability in $N - 2 - NS$ and develop some of its properties.

2. Preliminaries

This section starts with a brief review on certain definitions and results needed in the sequel.

“For $\lambda = (\lambda_n)$ as defined above, the generalized de la Vallée-Poussin mean of $\mathbf{u} = (u_k)$ is defined by $t_n(\mathbf{u}) = \frac{1}{\lambda_n} \sum_{k \in I_n} u_k$. Further, $\mathbf{u} = (u_k)$ is called V_λ -summable to u_0 (see[19]) if $\lim_{n \rightarrow \infty} t_n(\mathbf{u}) = u_0$. Let

$$[V_\lambda] = \left\{ \mathbf{u} = (u_n) : \exists u_0 \in \mathbb{R}, \lim_{n \rightarrow \infty} \frac{1}{\lambda} \sum_{k \in I_n} |u_k - u_0| = 0 \right\}$$

Definition 2.1 [40] “Let $\mathfrak{I} = [0, 1]$. A binary operation $\circ : \mathfrak{I} \times \mathfrak{I} \rightarrow \mathfrak{I}$ is t -norm if $\forall \mathbf{c}, \mathbf{e}, \mathbf{g}, \mathbf{h} \in \mathfrak{I}$ we have

- 1) \circ is continuous, commutative and associative,
- 2) $\mathbf{e} = \mathbf{e} \circ 1$,
- 3) $\mathbf{c} \circ \mathbf{e} \leq \mathbf{g} \circ \mathbf{h}$ whenever $\mathbf{c} \leq \mathbf{g}$ and $\mathbf{e} \leq \mathbf{h}$.”

Definition 2.2 [40]“Let $\mathfrak{I} = [0, 1]$. A binary operation $\diamond : \mathfrak{I} \times \mathfrak{I} \rightarrow \mathfrak{I}$ is *t*-conorm if $\forall c, e, g, h \in \mathfrak{I}$ we have

- 1) \diamond is continuous, commutative and associative,
- 2) $e = e \diamond 0$,
- 3) $c \diamond e \leq g \diamond h$ whenever $c \leq g$ and $e \leq h$.”

Kirişci and Şimşek introduced the notion *NNS* in the following manner [13].

Definition 2.3 [13]“Consider \mathfrak{F} to be a vector space, $N = \{ \langle \mathfrak{Y}, G(\mathfrak{Y}), B(\mathfrak{Y}), Y(\mathfrak{Y}) \rangle : \mathfrak{Y} \in \mathfrak{F} \}$ be a normed space s.t $N : \mathfrak{F} \times \mathbb{R}^+ \rightarrow [0, 1]$. Let \circ, \diamond be *t*-norm and *t*-conorm, respectively. If the followings conditions hold, then the four tuple $\mathfrak{U} = (\mathfrak{F}, N, \circ, \diamond)$ is called *NNS*, for $\mathfrak{r}, \mathfrak{s} \in \mathfrak{F}$, $\varrho, \omega > 0$ and for each $\varsigma \neq 0$,

- (i) $0 \leq G(\mathfrak{r}, \varrho) \leq 1, 0 \leq B(\mathfrak{r}, \varrho) \leq 1, 0 \leq Y(\mathfrak{r}, \varrho) \leq 1$ for $\varrho \in \mathbb{R}^+$;
- (ii) $G(\mathfrak{r}, \varrho) + B(\mathfrak{r}, \varrho) + Y(\mathfrak{r}, \varrho) \leq 3$ for $\varrho \in \mathbb{R}^+$;
- (iii) $G(\mathfrak{r}, \varrho) = 1$ (for $\varrho > 0$) iff $\mathfrak{r} = 0$;
- (iv) $G(\varsigma\mathfrak{r}, \varrho) = G\left(\mathfrak{r}, \frac{\varrho}{|\varsigma|}\right)$;
- (v) $G(\mathfrak{r}, \omega) \circ G(\mathfrak{s}, \varrho) \leq G(\mathfrak{r} + \mathfrak{s}, \omega + \varrho)$;
- (vi) $G(\mathfrak{r}, .)$ is non-decreasing and continuous function ;
- (vii) $\lim_{\varrho \rightarrow \infty} G(\mathfrak{r}, \varrho) = 1$;
- (viii) $B(\mathfrak{r}, \varrho) = 0$ (for $\varrho > 0$) iff $\mathfrak{r} = 0$;
- (ix) $B(\varsigma\mathfrak{r}, \varrho) = B\left(\mathfrak{r}, \frac{\varrho}{|\varsigma|}\right)$;
- (x) $B(\mathfrak{r}, \omega) \diamond B(\mathfrak{s}, \varrho) \geq B(\mathfrak{r} + \mathfrak{s}, \varrho + \omega)$;
- (xi) $B(\mathfrak{r}, .)$ is non-increasing and continuous function;
- (xii) $\lim_{\lambda \rightarrow \infty} B(\mathfrak{r}, \varrho) = 0$;
- (xiii) $Y(\mathfrak{r}, \varrho) = 0$ (for $\varrho > 0$) iff $\mathfrak{r} = 0$;
- (xiv) $Y(\varsigma\mathfrak{r}, \varrho) = Y\left(\mathfrak{r}, \frac{\varrho}{|\varsigma|}\right)$;
- (xv) $Y(\mathfrak{r}, \omega) \diamond Y(\mathfrak{s}, \varrho) \geq Y(\mathfrak{r} + \mathfrak{s}, \varrho + \omega)$;
- (xvi) $Y(\mathfrak{r}, .)$ is non-increasing and continuous function;
- (xvii) $\lim_{\lambda \rightarrow \infty} Y(\mathfrak{r}, \varrho) = 0$;
- (xviii) If $\varrho \leq 0$, then $G(\mathfrak{r}, \varrho) = 0, B(\mathfrak{r}, \varrho) = 1$ and $Y(\mathfrak{r}, \varrho) = 1$.

Then $N = (G, B, Y)$ is called the neutrosophic norm.”

We now recall the idea of two norm introduced in the paper [7].

Definition 2.4 [7]“Let \mathfrak{U} be a linear space of dimension $d > 1$. A function $\|.,.\| : \mathfrak{U} \times \mathfrak{U} \rightarrow \mathbb{R}$ satisfying the prerequisites specified below: For all $\mathfrak{s}, \mathfrak{t}, \mathfrak{l} \in \mathfrak{U}$, and scalar \mathfrak{c} , we have

- (i) $\|\mathfrak{s}, \mathfrak{t}\| = 0$ iff \mathfrak{s} and \mathfrak{t} are linearly dependent;
- (ii) $\|\mathfrak{s}, \mathfrak{t}\| = \|\mathfrak{t}, \mathfrak{s}\|$;

- (iii) $\|c\mathfrak{s}, \mathfrak{t}\| = |c|\|\mathfrak{s}, \mathfrak{t}\|$ and
- (iv) $\|\mathfrak{s}, \mathfrak{t} + \mathfrak{l}\| \leq \|\mathfrak{s}, \mathfrak{t}\| + \|\mathfrak{s}, \mathfrak{l}\|$.

The pair $(\mathfrak{U}, \|\cdot, \cdot\|)$ is then called an 2-normed space.

Let $\mathfrak{U} = \mathbb{R}^2$ and for $\mathfrak{s} = (s_1, s_2)$ and $\mathfrak{t} = (t_1, t_2)$ we define $\|\mathfrak{s}, \mathfrak{t}\| = |s_1t_2 - s_2t_1|$, then $\|\mathfrak{s}, \mathfrak{t}\|$ is a 2-norm on $\mathfrak{U} = \mathbb{R}^2$.

Recently, Murtaza et al [24] defined neutrosophic-2- normed spaces as follows.

Definition 2.5 [24] “Consider \mathfrak{F} to be a vector space, $N_2 = (\{\mathfrak{r}, \mathfrak{s}\}, G(\mathfrak{r}, \mathfrak{s}), B(\mathfrak{r}, \mathfrak{s}), Y(\mathfrak{r}, \mathfrak{s})) : (\mathfrak{r}, \mathfrak{s}) \in \mathfrak{F} \times \mathfrak{F}$ be a 2–normed space s.t $N_2 : \mathfrak{F} \times \mathfrak{F} \times \mathbb{R}^+ \rightarrow [0, 1]$. Let \circ, \diamond be t -norm and t -conorm respectively. If the following conditions hold, then the four-tuple $\mathfrak{U} = (\mathfrak{F}, N_2, \circ, \diamond)$ is called a neutrosophic 2–normed spaces (briefly $N - 2 - NS$) if for each $\mathfrak{r}, \mathfrak{s}, \mathfrak{t} \in \mathfrak{U}$, $\varrho, \omega \geq 0$ and $\varsigma \neq 0$:

- (i) $0 \leq G(\mathfrak{r}, \mathfrak{s}; \varrho) \leq 1, 0 \leq B(\mathfrak{r}, \mathfrak{s}; \varrho) \leq 1$ and $0 \leq Y(\mathfrak{r}, \mathfrak{s}; \varrho) \leq 1$ for $\varrho \in \mathbb{R}^+$;
- (ii) $G(\mathfrak{r}, \mathfrak{s}; \varrho) + B(\mathfrak{r}, \mathfrak{s}; \varrho) + Y(\mathfrak{r}, \mathfrak{s}; \varrho) \leq 3$;
- (iii) $G(\mathfrak{r}, \mathfrak{s}; \varrho) = 1$ iff $\mathfrak{r}, \mathfrak{s}$ are linearly dependent;
- (iv) $G(\varsigma\mathfrak{r}, \mathfrak{s}; \varrho) = G(\mathfrak{r}, \mathfrak{s}; \frac{\varrho}{|\varsigma|})$ for each $\varsigma \neq 0$;
- (v) $G(\mathfrak{r}, \mathfrak{s}; \varrho) \circ G(\mathfrak{r}, \mathfrak{t}; \omega) \leq G(\mathfrak{r}, \mathfrak{s} + \mathfrak{t}; \varrho + \omega)$;
- (vi) $G(\mathfrak{r}, \mathfrak{s}; \cdot) : (0, \infty) \rightarrow [0, 1]$ is non-decreasing and continuous function;
- (vii) $\lim_{\varrho \rightarrow \infty} G(\mathfrak{r}, \mathfrak{s}; \varrho) = 1$;
- (viii) $G(\mathfrak{r}, \mathfrak{s}; \varrho) = G(\mathfrak{s}, \mathfrak{r}; \varrho)$
- (ix) $B(\mathfrak{r}, \mathfrak{s}; \varrho) = 0$ iff $\mathfrak{r}, \mathfrak{s}$ are linearly dependent;
- (x) $B(\varsigma\mathfrak{r}, \mathfrak{s}; \varrho) = B(\mathfrak{r}, \mathfrak{s}; \frac{\varrho}{|\varsigma|})$ for each $\varsigma \neq 0$;
- (xi) $B(\mathfrak{r}, \mathfrak{s}; \varrho) \diamond B(\mathfrak{r}, \mathfrak{t}; \omega) \geq B(\mathfrak{r}, \mathfrak{s} + \mathfrak{t}; \varrho + \omega)$;
- (xii) $B(\mathfrak{r}, \mathfrak{s}; \cdot) : (0, \infty) \rightarrow [0, 1]$ is non-increasing and continuous function;
- (xiii) $\lim_{\varrho \rightarrow \infty} B(\mathfrak{r}, \mathfrak{s}; \varrho) = 0$;
- (xiv) $B(\mathfrak{r}, \mathfrak{s}; \varrho) = B(\mathfrak{s}, \mathfrak{r}; \varrho)$;
- (xvi) $Y(\mathfrak{r}, \mathfrak{s}; \varrho) = 0$ iff $\mathfrak{r}, \mathfrak{s}$ are linearly dependent;
- (xv) $Y(\varsigma\mathfrak{r}, \mathfrak{s}; \varrho) = Y(\mathfrak{r}, \mathfrak{s}; \frac{\varrho}{|\varsigma|})$ for each $\varsigma \neq 0$;
- (xvi) $Y(\mathfrak{r}, \mathfrak{s}; \varrho) \diamond Y(\mathfrak{r}, \mathfrak{t}; \omega) \geq Y(\mathfrak{r}, \mathfrak{s} + \mathfrak{t}; \varrho + \omega)$;
- (xvii) $Y(\mathfrak{r}, \mathfrak{s}; \cdot) : (0, \infty) \rightarrow [0, 1]$ is non-increasing and continuous function;
- (xviii) $\lim_{\lambda \rightarrow \infty} Y(\mathfrak{r}, \mathfrak{s}; \varrho) = 0$;
- (xix) $Y(\mathfrak{r}, \mathfrak{s}; \varrho) = Y(\mathfrak{s}, \mathfrak{r}; \varrho)$;
- (xx) if $\varrho \leq 0$, then $G(\mathfrak{r}, \mathfrak{s}; \varrho) = 0, B(\mathfrak{r}, \mathfrak{s}; \varrho) = 1, Y(\mathfrak{r}, \mathfrak{s}; \varrho) = 1$.

In this case, $N_2 = (G, B, Y)_2$ is called a neutrosophic 2-norm. From now on wards, unless otherwise stated by \mathfrak{U} we shall denote the $N - 2 - NS (\mathfrak{F}, N_2, \circ, \diamond)$

A sequence $u = (u_k)$ in \mathfrak{U} is called convergent to u_0 if for each $\eta > 0$ and $\varrho > 0$, $\exists k_0 \in \mathbb{N}$ s.t $G(u_k - u_0, \mathfrak{w}; \varrho) > 1 - \eta$, $B(u_k - u_0, \mathfrak{w}; \varrho) < \eta$ and $Y(u_k - u_0, \mathfrak{w}; \varrho) < \eta \forall k \geq k_0$ and $\mathfrak{w} \in \mathfrak{U}$ which is equivalently to say $\lim_{k \rightarrow \infty} G(u_k - u_0, \mathfrak{w}; \varrho) = 1$, $\lim_{k \rightarrow \infty} B(u_k - u_0, \mathfrak{w}; \varrho) = 0$ and $\lim_{k \rightarrow \infty} Y(u_k - u_0, \mathfrak{w}; \varrho) = 0$. In present case, we denote $N_2 - \lim_{k \rightarrow \infty} u_k = u_0$.

A sequence $u = (u_k)$ in \mathfrak{U} is called Cauchy if for each $\eta > 0$ and $\varrho > 0$, $\exists k_0 \in \mathbb{N}$ s.t $G(u_k - u_p, \mathfrak{w}; \varrho) > 1 - \eta$, $B(u_k - u_p, \mathfrak{w}; \varrho) < \eta$ and $Y(u_k - u_p, \mathfrak{w}; \varrho) < \eta \forall k, p \geq k_0$ and $\forall \mathfrak{w} \in \mathfrak{U}$."

3. λ -Statistical Convergence in N-2-NS

In this section, we define and study λ -Statistical Convergence in $N - 2 - NS$ and develop some of its properties.

Definition 3.1 A sequence $u = (u_k)$ in $N - 2 - NS \mathfrak{U}$ is called λ -statistical convergent (or \mathcal{S}_λ -convergent) to u_0 if for each $\eta > 0$ and $\varrho > 0$

$$\lim_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : G(u_k - u_0, \mathfrak{w}; \varrho) \leq 1 - \eta \text{ or } B(u_k - u_0, \mathfrak{w}; \varrho) \geq \eta, Y(u_k - u_0, \mathfrak{w}; \varrho) \geq \eta \right\} \right| = 0 \forall \mathfrak{w} \in \mathfrak{U};$$

or equivalently, $\delta_\lambda(\mathcal{A}(\eta, \varrho)) = 0$, where

$$\mathcal{A}(\eta, \varrho) = \{k \in I_n : G(u_k - u_0, \mathfrak{w}; \varrho) \leq 1 - \eta \text{ or } B(u_k - u_0, \mathfrak{w}; \varrho) \geq \eta, Y(u_k - u_0, \mathfrak{w}; \varrho) \geq \eta\}.$$

In present case, we denote $\mathcal{S}_\lambda(N_2) - \lim_{k \rightarrow \infty} u_k = u_0$.

We now give the following Lemma:

Lemma 3.1 For any sequence $u = (u_k)$ in \mathfrak{U} , the subsequent assertions are equivalent:

- (i) $\mathcal{S}_\lambda(N_2) - \lim_{k \rightarrow \infty} u_k = u_0$;
- (ii) $\delta_\lambda\{k \in I_n : G(u_k - u_0, \mathfrak{w}; \varrho) \leq 1 - \eta\} = \delta_\lambda\{k \in I_n : B(u_k - u_0, \mathfrak{w}; \varrho) \geq \eta\} = \delta_\lambda\{k \in I_n : Y(u_k - u_0, \mathfrak{w}; \varrho) \geq \eta\} = 0$;
- (iii) $\delta_\lambda\{k \in I_n : G(u_k - u_0, \mathfrak{w}; \varrho) > 1 - \eta \text{ and } B(u_k - u_0, \mathfrak{w}; \varrho) < \eta, Y(u_k - u_0, \mathfrak{w}; \varrho) < \eta\} = 1$;
- (iv) $\delta_\lambda\{k \in I_n : G(u_k - u_0, \mathfrak{w}; \varrho) > 1 - \eta\} = \delta_\lambda\{k \in I_n : B(u_k - u_0, \mathfrak{w}; \varrho) < \eta\} = \delta_\lambda\{k \in I_n : Y(u_k - u_0, \mathfrak{w}; \varrho) < \eta\} = 1$ and
- (v) $\mathcal{S}_\lambda(N_2) - \lim_{k \rightarrow \infty} G(u_k - u_0, \mathfrak{w}; \varrho) = 1$, $\mathcal{S}_\lambda(N_2) - \lim_{k \rightarrow \infty} B(u_k - u_0, \mathfrak{w}; \varrho) = \mathcal{S}_\lambda(N_2) - \lim_{k \rightarrow \infty} Y(u_k - u_0, \mathfrak{w}; \varrho) = 0$.

We now have the following interesting implication.

Theorem 3.1 Let $u = (u_k)$ be any sequence in $N - 2 - NS \mathfrak{U}$. If $N_2 - \lim_{k \rightarrow \infty} u_k = u_0$, then $\mathcal{S}_\lambda(N_2) - \lim_{k \rightarrow \infty} u_k = u_0$.

Proof. Let $N_2 - \lim_{k \rightarrow \infty} u_k = u_0$. Then for each $\eta > 0$ and $\varrho > 0, \exists$ an integer $k_0 \in \mathbb{N}$ s.t. $G(u_k - u_0, w; \varrho) > 1 - \eta$ and $B(u_k - u_0, w; \varrho) < \eta, Y(u_k - u_0, w; \varrho) < \eta \forall k \geq k_0$ and every $w \in \mathfrak{U}$. Hence the set $\{k \in I_n : G(u_k - u_0, w; \varrho) \leq 1 - \eta \text{ or } B(u_k - u_0, w; \varrho) < \eta, Y(u_k - u_0, w; \varrho) < \eta\}$ has a finitely many terms whose λ -density is zero. Therefore, $\mathcal{S}_\lambda(N_2) - \lim_{k \rightarrow \infty} u_k = u_0$. \square

The converse of the above theorem is not true in general.

Example 3.1 Let $(\mathbb{R}^2, |\cdot|)$ be a 2-normed space. For $\epsilon, g \in [0, 1]$. Let $\epsilon \circ g = \epsilon g$ and $\epsilon \diamond g = \min\{\epsilon + g, 1\}$. Choose $s, t \in \mathfrak{F}, \varrho > 0$ and $\varrho > \|s, t\|$. Define $G(s, t; \varrho) = \frac{\varrho}{\varrho + \|s, t\|}, B(s, t; \varrho) = \frac{\|s, t\|}{\varrho + \|s, t\|}$ and $Y(s, t; \varrho) = \frac{\|s, t\|}{\varrho}$, then it is clear that $\mathfrak{U} = (\mathbb{R}^2, N_2, \circ, \diamond)$ is a $N - 2 - NS$. Define $u = (u_k)$ by

$$u_k = \begin{cases} (k, 0) & \text{if } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n, \\ (0, 0) & \text{otherwise.} \end{cases}$$

Now, for each $\eta > 0$ and $\varrho > 0$, let

$$\begin{aligned} \mathcal{A}(\eta, \varrho) &= \left\{ k \in I_n : G(u_k - u_0, w; \varrho) \leq 1 - \eta \text{ or} \right. \\ &\quad \left. B(u_k - u_0, w; \varrho) \geq \eta, Y(u_k - u_0, w; \varrho) \geq \eta \right\} \\ &= \left\{ k \in I_n : \frac{\varrho}{\varrho + \|u_k, w\|} \leq 1 - \eta \text{ or } \frac{\|u_k, w\|}{\varrho + \|u_k, w\|} \geq \eta, \frac{\|u_k, w\|}{\varrho} \geq \eta \right\} \\ &= \left\{ k \in I_n : \|u_k, w\| \geq \frac{\varrho \eta}{1 - \eta} \text{ or } \|u_k, w\| \geq \varrho \eta \right\} \\ &= \{k \in I_n : n - [\sqrt{\lambda_n}] + 1 \leq k \leq n\} \end{aligned}$$

and so we get

$$\frac{1}{\lambda_n} |\mathcal{A}(\eta, \varrho)| = \frac{1}{\lambda_n} |\{k \in I_n : n - [\sqrt{\lambda_n}] + 1 \leq k \leq n\}| \leq \frac{[\sqrt{\lambda_n}]}{\lambda_n}.$$

Taking $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\mathcal{A}(\eta, \varrho)| \leq \lim_{n \rightarrow \infty} \frac{[\sqrt{\lambda_n}]}{\lambda_n} = 0;$$

i.e., $\delta_\lambda(\mathcal{A}(\eta, \varrho)) = 0$. This shows that, $u_k \rightarrow 0(\mathcal{S}_\lambda(N_2))$ But the sequence, $u = (u_k)$ is not N_2 -convergent to 0.

Theorem 3.2 Let $u = (u_k)$ be any sequence in $N - 2 - NS \mathfrak{U}$, if $\mathcal{S}_\lambda(N_2) - \lim_{k \rightarrow \infty} u_k$ exists, then it is unique.

Proof. Let $\mathcal{S}_\lambda(N_2) - \lim_{k \rightarrow \infty} u_k = u_1$ and $\mathcal{S}_\lambda(N_2) - \lim_{k \rightarrow \infty} u_k = u_2$. Let $\eta > 0$, select $l > 0$ s.t.

$$(1 - l) \circ (1 - l) > 1 - \eta \text{ and } l \diamond l < \eta. \tag{1}$$

For $\varrho > 0$ and $\mathfrak{w} \in \mathfrak{U}$. Define the sets:

$$\begin{aligned}
 K_{G,1}(l, \varrho) &= \{k \in I_n : G(\mathbf{u}_k - \mathbf{u}_1, \mathfrak{w}; \frac{\varrho}{2}) \leq 1 - l\}, \\
 K_{G,2}(l, \varrho) &= \{k \in I_n : G(\mathbf{u}_k - \mathbf{u}_2, \mathfrak{w}; \frac{\varrho}{2}) \leq 1 - l\}; \\
 K_{B,1}(l, \varrho) &= \{k \in I_n : B(\mathbf{u}_k - \mathbf{u}_1, \mathfrak{w}; \frac{\varrho}{2}) \geq l\}, \\
 K_{B,2}(l, \varrho) &= \{k \in I_n : B(\mathbf{u}_k - \mathbf{u}_2, \mathfrak{w}; \frac{\varrho}{2}) \geq l\}; \\
 K_{Y,1}(l, \varrho) &= \{k \in I_n : Y(\mathbf{u}_k - \mathbf{u}_1, \mathfrak{w}; \frac{\varrho}{2}) \geq l\}; \\
 K_{Y,2}(l, \varrho) &= \{k \in I_n : Y(\mathbf{u}_k - \mathbf{u}_2, \mathfrak{w}; \frac{\varrho}{2}) \geq l\}.
 \end{aligned}$$

Since $\mathcal{S}_\lambda(N_2) - \lim_{k \rightarrow \infty} \mathbf{u}_k = \mathbf{u}_1$. By Lemma 3.1, we have $\delta_\lambda\{K_{G,1}(l, \varrho)\} = \delta_\lambda\{K_{B,1}(l, \varrho)\} = \delta_\lambda\{K_{Y,1}(l, \varrho)\} = 0$ and therefore $\delta_\lambda\{K_{G,1}^C(l, \varrho)\} = \delta_\lambda\{K_{B,1}^C(l, \varrho)\} = \delta_\lambda\{K_{Y,1}^C(l, \varrho)\} = 1$. Also, using $\mathcal{S}_\lambda(N_2) - \lim_{k \rightarrow \infty} \mathbf{u}_k = \mathbf{u}_2$, we get, $\delta_\lambda\{K_{G,2}(l, \varrho)\} = \delta_\lambda\{K_{B,2}(l, \varrho)\} = \delta_\lambda\{K_{Y,2}(l, \varrho)\} = 0$ and therefore $\delta_\lambda\{K_{G,2}^C(l, \varrho)\} = \delta_\lambda\{K_{B,2}^C(l, \varrho)\} = \delta_\lambda\{K_{Y,2}^C(l, \varrho)\} = 1$. Now define $K_{G,B,Y}(\mathfrak{w}, \varrho) = \{K_{G,1}(l, \varrho) \cup K_{G,2}(l, \varrho)\} \cap \{K_{B,1}(l, \varrho) \cup K_{B,2}(l, \varrho)\} \cap \{K_{Y,1}(l, \varrho) \cup K_{Y,2}(l, \varrho)\}$. Then observe that $\delta_\lambda(\{K_{G,B,Y}(\mathfrak{w}, \varrho)\}) = 0$ which implies $\delta(\{K_{G,B,Y}^C(\mathfrak{w}, \varrho)\}) = 1$. Let $m \in K_{G,B,Y}^C(\mathfrak{w}, \varrho)$, then we have

Case 1. $m \in \{K_{G,1}(l, \varrho) \cup K_{G,2}(l, \varrho)\}^C$,

Case 2. $m \in \{K_{B,1}(l, \varrho) \cup K_{B,2}(l, \varrho)\}^C$,

Case 3. $m \in \{K_{Y,1}(l, \varrho) \cup K_{Y,2}(l, \varrho)\}^C$.

Case 1: Let, $m \in \{K_{G,1}(l, \varrho) \cup K_{G,2}(l, \varrho)\}^C$, then $m \in K_{G,1}^C(l, \varrho)$ and $m \in K_{G,2}^C(l, \varrho)$.

Therefore, for any $\mathfrak{w} \in \mathfrak{U}$ we have

$$G(\mathbf{u}_m - \mathbf{u}_1, \mathfrak{w}; \frac{\varrho}{2}) > 1 - l \text{ and } G(\mathbf{u}_m - \mathbf{u}_2, \mathfrak{w}; \frac{\varrho}{2}) > 1 - l. \tag{2}$$

Now

$$\begin{aligned}
 G(\mathbf{u}_1 - \mathbf{u}_2, \mathfrak{w}; \varrho) &\geq G(\mathbf{u}_m - \mathbf{u}_1, \mathfrak{w}; \frac{\varrho}{2}) \circ G(\mathbf{u}_m - \mathbf{u}_2, \mathfrak{w}; \frac{\varrho}{2}) \\
 &> (1 - l) \circ (1 - l) \text{ by (2)} \\
 &> 1 - \mathfrak{w}. \text{ by (1)}
 \end{aligned}$$

Since $\mathfrak{w} > 0$ is arbitrary, so we have $G(\mathbf{u}_1 - \mathbf{u}_2, \mathfrak{w}; \varrho) = 1 \forall \varrho > 0$, and therefore $\mathbf{u}_1 - \mathbf{u}_2 = 0$.

This shows that $\mathbf{u}_1 = \mathbf{u}_2$.

Case 2: Let, $m \in \{K_{B,1}(l, \varrho) \cup K_{B,2}(l, \varrho)\}^C$, then $m \in K_{B,1}^C(l, \varrho)$ and $m \in K_{B,2}^C(l, \varrho)$.

Therefore, for $\mathfrak{w} \in \mathfrak{U}$ we have

$$B(\mathbf{u}_m - \mathbf{u}_1, \mathfrak{w}; \frac{\varrho}{2}) < l \text{ and } B(\mathbf{u}_m - \mathbf{u}_2, \mathfrak{w}; \frac{\varrho}{2}) < l. \tag{3}$$

Now

$$\begin{aligned} B(u_1 - u_2, \mathfrak{w}; \varrho) &\leq B(u_m - u_1, \mathfrak{w}; \frac{\varrho}{2}) \circ B(u_m - u_2, \mathfrak{w}; \frac{\varrho}{2}) \\ &< l \diamond l \text{ by (3)} \\ &< \eta. \text{ by (1)} \end{aligned}$$

Since $\eta > 0$ is arbitrary, so we have $B(u_1 - u_2, \mathfrak{w}; \varrho) = 0 \forall \varrho > 0$, and therefore $u_1 - u_2 = 0$. This shows that $u_1 = u_2$.

Case 3: Let, $m \in \{K_{Y,1}(l, \varrho) \cup K_{Y,2}(l, \varrho)\}^C$, then $m \in K_{Y,1}^C(l, \varrho)$ and $m \in K_{Y,2}^C(l, \varrho)$. Therefore, for $\mathfrak{w} \in \mathfrak{U}$ we have

$$Y(u_m - u_1, \mathfrak{w}; \frac{\varrho}{2}) < l \text{ and } Y(u_m - u_2, \mathfrak{w}; \frac{\varrho}{2}) < l. \tag{4}$$

Now

$$\begin{aligned} Y(u_1 - u_2, \mathfrak{w}; \varrho) &\leq Y(u_m - u_1, \mathfrak{w}; \frac{\varrho}{2}) \circ Y(u_m - u_2, \mathfrak{w}; \frac{\varrho}{2}) \\ &< l \diamond l \text{ by (4)} \\ &< \eta. \text{ by (1)} \end{aligned}$$

Since $\eta > 0$ is arbitrary, so we have $Y(u_1 - u_2, \mathfrak{w}; \varrho) = 0 \forall \varrho > 0$, and therefore $u_1 - u_2 = 0$. This shows that $u_1 = u_2$.

Hence, in all three cases, we have $u_1 = u_2$, i.e., λ -statistical limit of $u = (u_k)$ is unique. \square

Theorem 3.3 Let $q = (q_k)$ and $u = (u_k)$ be two sequences in $N - 2 - NS \mathfrak{U}$ s.t.

$\mathcal{S}_\lambda(N_2) - \lim_{k \rightarrow \infty} q_k = q_0$ and $\mathcal{S}_\lambda(N_2) - \lim_{k \rightarrow \infty} u_k = u_0$. Then

- (i) $\mathcal{S}_\lambda(N_2) - \lim_{k \rightarrow \infty} (q_k + u_k) = q_0 + u_0$.
- (ii) $\mathcal{S}_\lambda(N_2) - \lim_{k \rightarrow \infty} (aq_k) = aq_0$, where a is any scalar.

Proof. The proof can be obtained analogously as the proof of theorem 3.2. \square

Theorem 3.4 A sequence $u = (u_k)$ in $N - 2 - NS \mathfrak{U}$ is \mathcal{S}_λ -convergent to u_0 , iff \exists a subset $\mathfrak{R} = \{k_n : n \in \mathbb{N}\}$ of \mathbb{N} with $\delta_\lambda\{\mathfrak{R}\} = 1$ and $N_2 - \lim_{n \rightarrow \infty} u_{k_n} = u_0$.

Proof. First assume that $\mathcal{S}_\lambda(N_2) - \lim_{k \rightarrow \infty} u_k = u_0$. For $\varrho > 0, j \in \mathbb{N}$ and $\mathfrak{w} \in \mathfrak{U}$, define the set

$$\begin{aligned} \mathfrak{R}_{N_2}(j, \varrho) &= \{k \in I_n : G(u_k - u_0, \mathfrak{w}; \varrho) > 1 - \frac{1}{j} \text{ and} \\ &B(u_k - u_0, \mathfrak{w}; \varrho) < \frac{1}{j}, Y(u_k - u_0, \mathfrak{w}; \varrho) < \frac{1}{j}\}. \end{aligned}$$

Since $\mathcal{S}_\lambda(N_2) - \lim_{k \rightarrow \infty} u_k = u_0$, it is clear that for $\varrho > 0$ and $j \in \mathbb{N}$, $\mathfrak{R}_{N_2}(j+1, \varrho) \subset \mathfrak{R}_{N_2}(j, \varrho)$ and

$$\delta_\lambda(\mathfrak{R}_{N_2}(j, \varrho)) = 1. \tag{5}$$

Let $m_1 \in \mathfrak{R}_{N_2}(1, \varrho)$. Then, $\exists m_2 \in \mathfrak{R}_{N_2}(2, \varrho)$, ($m_2 > m_1$), such that for all $n \geq m_2$, $\frac{1}{\lambda_n}|\{k \in I_n : G(\mathbf{u}_k - \mathbf{u}_0, \mathbf{w}; \varrho) > 1 - \frac{1}{2} \text{ and } B(\mathbf{u}_k - \mathbf{u}_0, \mathbf{w}; \varrho) < \frac{1}{2}, Y(\mathbf{u}_k - \mathbf{u}_0, \mathbf{w}; \varrho) < \frac{1}{2}\}| > \frac{1}{2}$. Similarly, $\exists m_3 \in \mathfrak{R}_{N_2}(3, \varrho)$, ($m_3 > m_2$), such that for all $n \geq m_3$, $\frac{1}{\lambda_n}|\{k \in I_n : G(\mathbf{u}_k - \mathbf{u}_0, \mathbf{w}; \varrho) > 1 - \frac{1}{3}$ and $B(\mathbf{u}_k - \mathbf{u}_0, \mathbf{w}; \varrho) < \frac{1}{3}, Y(\mathbf{u}_k - \mathbf{u}_0, \mathbf{w}; \varrho) < \frac{1}{3}\}| > \frac{2}{3}$ and so on. Thus, we can set a sequence $(m_j)_{j \in \mathbb{N}}$ s.t $m_j \in \mathfrak{R}_{N_2}(j, \varrho)$ and $\forall n \geq m_j (j \in \mathbb{N})$, $\frac{1}{\lambda_n}|\{k \in I_n : G(\mathbf{u}_k - \mathbf{u}_0, \mathbf{w}; \varrho) > 1 - \frac{1}{j}$ and $B(\mathbf{u}_k - \mathbf{u}_0, \mathbf{w}; \varrho) < \frac{1}{j}, Y(\mathbf{u}_k - \mathbf{u}_0, \mathbf{w}; \varrho) < \frac{1}{j}\}| > \frac{j-1}{j}$. Now define $\mathfrak{R} = \{n \in \mathbb{N} : 1 < n < m_1\} \cup \{\bigcup_{j \in \mathbb{N}} \{n \in \mathfrak{R}_{N_2}(j, \varrho) : m_j \leq n < m_{j+1}\}\}$. Then for $m_j \leq n < m_{j+1}$, we have $\frac{1}{\lambda_n}|\{k \in I_n : k \in \mathfrak{R}\}| \geq \frac{1}{\lambda_n}|\{k \in I_n : G(\mathbf{u}_k - \mathbf{u}_0, \mathbf{w}; \varrho) > 1 - \frac{1}{j}$ and $B(\mathbf{u}_k - \mathbf{u}_0, \mathbf{w}; \varrho) < \frac{1}{j}, Y(\mathbf{u}_k - \mathbf{u}_0, \mathbf{w}; \varrho) < \frac{1}{j}\}| > \frac{j-1}{j}$ and hence, $\delta_\lambda(\mathfrak{R}) = 1$ as $k \rightarrow \infty$. Now we have to demonstrate that $N_2 - \lim_{n \rightarrow \infty} \mathbf{u}_{k_n} = \mathbf{u}_0$. Let $\eta > 0$ and select $j \in \mathbb{N}$ s.t $\frac{1}{j} < \eta$. Furthermore, let $n \geq m_j$ and $n \in \mathfrak{R}$. Then, by definition of \mathfrak{R} , $\exists \ell \geq j$ s.t, $m_\ell \leq n < m_{\ell+1}$ and $n \in \mathfrak{R}_{N_2}(j, \varrho)$. Thus, for each $\eta > 0$, and for $\mathbf{w} \in \mathfrak{U}$, we have $G(\mathbf{u}_n - \mathbf{u}_0, \mathbf{w}; \varrho) > 1 - \frac{1}{j} > 1 - \eta$ and $B(\mathbf{u}_n - \mathbf{u}_0, \mathbf{w}; \varrho) < \frac{1}{j} < \eta$, $Y(\mathbf{u}_n - \mathbf{u}_0, \mathbf{w}; \varrho) < \frac{1}{j} < \eta \forall n \geq m_j$ and $n \in \mathfrak{R}$. Hence $N_2 - \lim_{n \rightarrow \infty} \mathbf{u}_{k_n} = \mathbf{u}_0$.

Conversely, suppose \exists a subset $\mathfrak{R} = \{k_n\}_{n \in \mathbb{N}}$ of \mathbb{N} with $\delta_\lambda\{\mathfrak{R}\} = 1$ and $N_2 - \lim_{n \rightarrow \infty} \mathbf{u}_{k_n} = \mathbf{u}_0$. Let $\eta > 0$ or $\varrho > 0$, $\exists k_{n_0} \in \mathbb{N}$ s.t $G(\mathbf{u}_{k_n} - \mathbf{u}_0, \mathbf{w}; \varrho) > 1 - \eta$ and $B(\mathbf{u}_{k_n} - \mathbf{u}_0, \mathbf{w}; \varrho) < \eta$, $Y(\mathbf{u}_{k_n} - \mathbf{u}_0, \mathbf{w}; \varrho) < \eta \forall k_n \geq k_{n_0}$ and every $\mathbf{w} \in \mathfrak{U}$. This implies $\mathfrak{D}_{N_2}(\eta, \varrho) = \{k \in I_n : G(\mathbf{u}_{k_n} - \mathbf{u}_0, \mathbf{w}; \varrho) \leq 1 - \eta \text{ or } B(\mathbf{u}_{k_n} - \mathbf{u}_0, \mathbf{w}; \varrho) \geq \eta, Y(\mathbf{u}_{k_n} - \mathbf{u}_0, \mathbf{w}; \varrho) \geq \eta\} \subseteq \mathbb{N} - \{k_{n_0}, k_{n_0+1}, k_{n_0+2}, \dots\}$. and therefore $\delta_\lambda\{\mathfrak{D}_{N_2}(\eta, \varrho)\} \leq \delta_\lambda\{\mathbb{N}\} - \delta_\lambda\{k_{n_0}, k_{n_0+1}, k_{n_0+2}, \dots\} = 1 - 1 = 0$. This shows that $\mathcal{S}_\lambda(N_2) - \lim_{k \rightarrow \infty} \mathbf{u}_k = \mathbf{u}_0$. \square

4. V_λ -summability in N-2-NS

Definition 4.1 A sequence $\mathbf{u} = (\mathbf{u}_k)$ in $N - 2 - NS \mathfrak{U}$ is called V_λ -summable to \mathbf{u}_0 w.r.t the neutrosophic 2-norm N_2 if for each $\eta > 0, 0 < \varrho < 1$ and $\mathbf{w} \in \mathfrak{U}$

$$G\left(\left(\frac{1}{\lambda_n} \sum_{k \in I_n} \mathbf{u}_k\right) - \mathbf{u}_0, \mathbf{w}; \varrho\right) > 1 - \eta \text{ and}$$

$$B\left(\left(\frac{1}{\lambda_n} \sum_{k \in I_n} \mathbf{u}_k\right) - \mathbf{u}_0, \mathbf{w}; \varrho\right) < \eta; Y\left(\left(\frac{1}{\lambda_n} \sum_{k \in I_n} \mathbf{u}_k\right) - \mathbf{u}_0, \mathbf{w}; \varrho\right) < \eta.$$

In present case, we denote $V_\lambda(N_2) - \lim_{k \rightarrow \infty} \mathbf{u}_k = \mathbf{u}_0$ or $\mathbf{u}_k \rightarrow \mathbf{u}_0(V_\lambda(N_2))$.

Theorem 4.1 Let $\lambda = (\lambda_n)$ as defined above and $\mathbf{u} = (\mathbf{u}_k)$ be a sequence in $N - 2 - NS \mathfrak{U}$ then

- (I) $\mathbf{u}_k \rightarrow \mathbf{u}_0(V_\lambda(N_2)) \Rightarrow \mathbf{u}_k \rightarrow \mathbf{u}_0(\mathcal{S}_\lambda(N_2))$ and the inclusion $V_\lambda(N_2) \subseteq \mathcal{S}_\lambda(N_2)$ is proper.
- (II) If $\mathbf{u} \in l_\infty(\mathfrak{U})$ and $\mathbf{u}_k \rightarrow \mathbf{u}_0(\mathcal{S}_\lambda(N_2))$, then $\mathbf{u}_k \rightarrow \mathbf{u}_0(V_\lambda(N_2))$.
- (III) $\mathcal{S}_\lambda(N_2) \cap l_\infty(\mathfrak{U}) = V_\lambda(N_2) \cap l_\infty(\mathfrak{U})$, where $l_\infty(\mathfrak{U})$ is the space of all bounded sequences in \mathfrak{U} .

Proof. (I) Let $\eta > 0$ and $u_k \rightarrow u_0(V_\lambda(N_2))$. We have,

$$\begin{aligned} & \sum_{k \in I_n} (G(u_k - u_0, \mathbf{w}; \varrho) \text{ or } B(u_k - u_0, \mathbf{w}; \varrho), Y(u_k - u_0, \mathbf{w}; \varrho)) \\ = & \sum_{\substack{k \in I_n \\ G(u_k - u_0, \mathbf{w}; \varrho) \leq 1 - \eta \text{ or} \\ B(u_k - u_0, \mathbf{w}; \varrho) \geq \eta, Y(u_k - u_0, \mathbf{w}; \varrho) \geq \eta}} (G(u_k - u_0, \mathbf{w}; \varrho) \text{ or } B(u_k - u_0, \mathbf{w}; \varrho), Y(u_k - u_0, \mathbf{w}; \varrho)) \\ + & \sum_{\substack{k \in I_n \\ G(u_k - u_0, \mathbf{w}; \varrho) > 1 - \eta \text{ \&} \\ B(u_k - u_0, \mathbf{w}; \varrho) < \eta, Y(u_k - u_0, \mathbf{w}; \varrho) < \eta}} (G(u_k - u_0, \mathbf{w}; \varrho) \text{ and } B(u_k - u_0, \mathbf{w}; \varrho), Y(u_k - u_0, \mathbf{w}; \varrho)) \\ \geq & \sum_{\substack{k \in I_n \\ G(u_k - u_0, \mathbf{w}; \varrho) \leq 1 - \eta \text{ or} \\ B(u_k - u_0, \mathbf{w}; \varrho) \geq \eta, Y(u_k - u_0, \mathbf{w}; \varrho) \geq \eta}} (G(u_k - u_0, \mathbf{w}; \varrho) \text{ or } B(u_k - u_0, \mathbf{w}; \varrho), Y(u_k - u_0, \mathbf{w}; \varrho)) \\ & \geq \eta \cdot \left| \left\{ k \in I_n : G(u_k - u_0, \mathbf{w}; \varrho) \leq 1 - \eta \text{ or } B(u_k - u_0, \mathbf{w}; \varrho) \geq \eta, Y(u_k - u_0, \mathbf{w}; \varrho) \geq \eta \right\} \right|. \end{aligned}$$

Since $u_k \rightarrow u_0(V_\lambda(N_2))$. Therefore, it follows that $u_k \rightarrow u_0(\mathcal{S}_\lambda(N_2))$.

In order to show that the containment $V_\lambda(N_2) \subseteq \mathcal{S}_\lambda(N_2)$ is proper. We define a sequence $u = (u_k)$ by

$$u_k = \begin{cases} (k, 0), & \text{for } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n, \\ (0, 0), & \text{otherwise} \end{cases}$$

It is obvious that the sequence (u_k) is unbounded. Then for each $\eta \in (0, 1)$ and $\varrho > 0$ we have

$$\begin{aligned} & \frac{1}{\lambda_n} \left| \left\{ k \in I_n : G(u_k - 0, \mathbf{w}; \varrho) \leq 1 - \eta \text{ or} \right. \right. \\ & \qquad \qquad \qquad \left. \left. B(u_k - 0, \mathbf{w}; \varrho) \geq \eta, Y(u_k - 0, \mathbf{w}; \varrho) \geq \eta \right\} \right| \\ & \qquad \qquad \qquad = \frac{[\sqrt{\lambda_n}]}{\lambda_n} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

i.e., $u_k \rightarrow 0(\mathcal{S}_\lambda(N_2))$.

Further,

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \left(G(u_k - 0, \mathbf{w}; \varrho) \text{ or } B(u_k - 0, \mathbf{w}; \varrho), Y(u_k - 0, \mathbf{w}; \varrho) \right) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

This implies that $u_k \not\rightarrow 0(V_\lambda(N_2))$.

(II) Let $u = (u_k) \in l_\infty(\mathfrak{U})$ and $u_k \rightarrow u_0(\mathcal{S}_\lambda(N_2))$. Then $\exists M > 0$ s.t $G(u_k - u_0, \mathbf{w}; \varrho) \geq 1 - M$ or $B(u_k - u_0, \mathbf{w}; \varrho) \leq M, Y(u_k - u_0, \mathbf{w}; \varrho) \leq M \forall k$. Let $\eta > 0$ be arbitrary selected,

now as in case (I) we can write

$$\begin{aligned} & \frac{1}{\lambda_n} \sum_{k \in I_n} (G(u_k - u_0, \mathfrak{w}; \varrho) \text{ or } B(u_k - u_0, \mathfrak{w}; \varrho), Y(u_k - u_0, \mathfrak{w}; \varrho)) \\ &= \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ G(u_k - u_0, \mathfrak{w}; \varrho) \leq 1 - \eta \text{ or} \\ B(u_k - u_0, \mathfrak{w}; \varrho) \geq \eta, Y(u_k - u_0, \mathfrak{w}; \varrho) \geq \eta}} (G(u_k - u_0, \mathfrak{w}; \varrho) \text{ or } B(u_k - u_0, \mathfrak{w}; \varrho), Y(u_k - u_0, \mathfrak{w}; \varrho)) \\ &+ \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ G(u_k - u_0, \mathfrak{w}; \varrho) > 1 - \eta \text{ \&} \\ B(u_k - u_0, \mathfrak{w}; \varrho) < \eta, Y(u_k - u_0, \mathfrak{w}; \varrho) < \eta}} (G(u_k - u_0, \mathfrak{w}; \varrho) \text{ and } B(u_k - u_0, \mathfrak{w}; \varrho), Y(u_k - u_0, \mathfrak{w}; \varrho)) \\ &\leq \frac{M}{\lambda_n} \left| \left\{ k \in I_n : G(u_k - u_0, \mathfrak{w}; \varrho) \leq 1 - \eta \text{ or } B(u_k - u_0, \mathfrak{w}; \varrho) \geq \eta, Y(u_k - u_0, \mathfrak{w}; \varrho) \geq \eta \right\} \right| + \eta, \end{aligned}$$

which shows that $u_k \rightarrow u_0(V_\lambda(N_2))$.

(III) Follows easily from part (I) and part (II). \square

Theorem 4.2 Let $u = (u_k)$ be any sequence in $N - 2 - NS \mathfrak{U}$. Then $\mathcal{S}(N_2) \subset \mathcal{S}_\lambda(N_2)$ if

$$\liminf_{n \rightarrow \infty} \frac{\lambda_n}{n} > 0. \tag{6}$$

Proof. Given $\eta > 0$ and $\varrho > 0$, we have

$$\begin{aligned} & \{k \leq n : G(u_k - u_0, \mathfrak{w}; \varrho) \leq 1 - \eta \text{ or} \\ & \quad B(u_k - u_0, \mathfrak{w}; \varrho) \geq \eta, Y(u_k - u_0, \mathfrak{w}; \varrho) \geq \eta\} \\ & \supseteq \{k \in I_n : G(u_k - u_0, \mathfrak{w}; \varrho) \leq 1 - \eta \text{ or} \\ & \quad B(u_k - u_0, \mathfrak{w}; \varrho) \geq \eta, Y(u_k - u_0, \mathfrak{w}; \varrho) \geq \eta\} . \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{1}{n} \left| \{k \leq n : G(u_k - u_0, \mathfrak{w}; \varrho) \leq 1 - \eta \text{ or} \right. \\ & \quad \left. B(u_k - u_0, \mathfrak{w}; \varrho) \geq \eta, Y(u_k - u_0, \mathfrak{w}; \varrho) \geq \eta\} \right| \\ & \geq \frac{1}{n} \left| \{k \in I_n : G(u_k - u_0, \mathfrak{w}; \varrho) \leq 1 - \eta \text{ or} \right. \\ & \quad \left. B(u_k - u_0, \mathfrak{w}; \varrho) \geq \eta, Y(u_k - u_0, \mathfrak{w}; \varrho) \geq \eta\} \right| \\ & \geq \frac{\lambda_n}{n} \frac{1}{\lambda_n} \left| \{k \in I_n : G(u_k - u_0, \mathfrak{w}; \varrho) \leq 1 - \eta \text{ or} \right. \\ & \quad \left. B(u_k - u_0, \mathfrak{w}; \varrho) \geq \eta, Y(u_k - u_0, \mathfrak{w}; \varrho) \geq \eta\} \right|. \end{aligned}$$

Taking $n \rightarrow \infty$ and using (6), we get $u_k \rightarrow u_0(\mathcal{S}(N_2)) \Rightarrow u_k \rightarrow u_0(\mathcal{S}_\lambda(N_2))$.

5. λ -Statistical completeness in N-2-NS

Definition 5.1 A sequence $u = (u_k)$ in $N - 2 - NS \mathfrak{U}$ is called λ -statistically Cauchy (or \mathcal{S}_λ -Cauchy) if for each $\eta > 0$ and $\varrho > 0, \exists p \in \mathbb{N}$ s.t.

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : G(u_k - u_p, \mathfrak{w}; \varrho) \leq 1 - \eta \text{ or } B(u_k - u_p, \mathfrak{w}; \varrho) \geq \eta, Y(u_k - u_p, \mathfrak{w}; \varrho) \geq \eta\}| = 0 \forall \mathfrak{w} \in \mathfrak{U}$$

or equivalently, $\delta_\lambda(\mathcal{A}(\eta, \varrho)) = 0$, where

$$\mathcal{A}(\eta, \varrho) = \{k \in I_n : G(u_k - u_p, \mathfrak{w}; \varrho) \leq 1 - \eta \text{ or } B(u_k - u_p, \mathfrak{w}; \varrho) \geq \eta, Y(u_k - u_p, \mathfrak{w}; \varrho) \geq \eta\}$$

Theorem 5.1 Every \mathcal{S}_λ -convergent sequence in $N - 2 - NS \mathfrak{U}$ is \mathcal{S}_λ -Cauchy.

Proof. Let $u = (u_k)$ be a \mathcal{S}_λ -convergent sequence with $\mathcal{S}_\lambda(N_2) - \lim_{k \rightarrow \infty} u_k = u_0$. Let $\eta > 0$ and $\varrho > 0$. Choose $l > 0$ s.t. (1) is satisfied. Define a set,

$$\mathcal{A}(l, \varrho) = \{k \in I_n : G(u_k - u_0, \mathfrak{w}; \frac{\varrho}{2}) \leq 1 - l \text{ or } B(u_k - u_0, \mathfrak{w}; \frac{\varrho}{2}) \geq l, Y(u_k - u_0, \mathfrak{w}; \frac{\varrho}{2}) \geq l\},$$

then $\delta_\lambda(\mathcal{A}(l, \varrho)) = 0$ and therefore $\delta_\lambda(\mathcal{A}^C(l, \varrho)) = 1$. Let $p \in \mathcal{A}^C(l, \varrho)$ then for $\mathfrak{w} \in \mathfrak{U}$, we have $G(u_p - u_0, \mathfrak{w}; \frac{\varrho}{2}) > 1 - l$ and $B(u_p - u_0, \mathfrak{w}; \frac{\varrho}{2}) < l, Y(u_p - u_0, \mathfrak{w}; \frac{\varrho}{2}) < l$.

Now, let $\mathcal{T}(\eta, \varrho) = \{k \in I_n : G(u_k - u_p, \mathfrak{w}; \varrho) \leq 1 - \eta \text{ or } B(u_k - u_p, \mathfrak{w}; \varrho) \geq \eta, Y(u_k - u_p, \mathfrak{w}; \varrho) \geq \eta\}$. We claim that $\mathcal{T}(\eta, \varrho) \subset \mathcal{A}(l, \varrho)$. Let $m \in \mathcal{T}(\eta, \varrho)$, then we have $G(u_m - u_p, \mathfrak{w}; \varrho) \leq 1 - \eta$ or $B(u_m - u_p, \mathfrak{w}; \varrho) \geq \eta, Y(u_m - u_p, \mathfrak{w}; \varrho) \geq \eta$.

Case (i): Suppose $G(u_m - u_p, \mathfrak{w}; \varrho) \leq 1 - \eta$, then $G(u_m - u_0, \mathfrak{w}; \frac{\varrho}{2}) \leq 1 - l$ and therefore $m \in \mathcal{A}(l, \varrho)$. As otherwise, i.e, if $G(u_m - u_0, \mathfrak{w}; \frac{\varrho}{2}) > 1 - l$, then

$$\begin{aligned} 1 - \eta &\geq G(u_m - u_p, \mathfrak{w}; \varrho) \geq G(u_m - u_0, \mathfrak{w}; \frac{\varrho}{2}) \circ G(u_p - u_0, \mathfrak{w}; \frac{\varrho}{2}) \\ &> (1 - l) \circ (1 - l) \\ &> 1 - \eta. \text{(not possible)} \end{aligned}$$

Thus, $\mathcal{T}(\eta, \varrho) \subset \mathcal{A}(l, \varrho)$.

Case (ii): Suppose $B(u_m - u_p, \mathfrak{w}; \varrho) \geq \eta$, then $B(u_m - u_0, \mathfrak{w}; \frac{\varrho}{2}) \geq l$ and therefore $m \in \mathcal{A}(l, \varrho)$. As otherwise, i.e, if $B(u_m - u_0, \mathfrak{w}; \frac{\varrho}{2}) < l$, then

$$\begin{aligned} \eta &\leq B(u_m - u_p, \mathfrak{w}; \varrho) \leq B(u_m - u_0, \mathfrak{w}; \frac{\varrho}{2}) \diamond B(u_p - u_0, \mathfrak{w}; \frac{\varrho}{2}) \\ &< l \diamond l \\ &< \eta. \text{(not possible)} \end{aligned}$$

Also, suppose $Y(u_m - u_p, \mathfrak{w}; \varrho) \geq \eta$, then $Y(u_m - u_0, \mathfrak{w}; \frac{\varrho}{2}) \geq l$ and therefore $m \in \mathcal{A}(l, \varrho)$. As otherwise, i.e, if $B(u_m - u_0, \mathfrak{w}; \frac{\varrho}{2}) < l$, then

$$\begin{aligned} \eta &\leq Y(u_m - v_p, \mathfrak{w}; \varrho) \leq Y(u_m - u_0, \mathfrak{w}; \frac{\varrho}{2}) \diamond Y(u_p - u_0, \mathfrak{w}; \frac{\varrho}{2}) \\ &< l \diamond l \\ &< \eta. (\text{not possible}) \end{aligned}$$

Thus, $\mathcal{T}(\eta, \varrho) \subset \mathcal{A}(l, \varrho)$.

Hence in all cases, $\mathcal{T}(\eta, \varrho) \subset \mathcal{A}(l, \varrho)$. Since $\delta_\lambda(\mathcal{A}(l, \varrho)) = 0$, so $\delta_\lambda(\mathcal{T}(\eta, \varrho)) = 0$ and therefore $u = (u_k)$ is λ -statistically Cauchy. \square

Definition 5.2 A neutrosophic 2-normed space \mathfrak{U} is called \mathcal{S}_λ -complete if every \mathcal{S}_λ -Cauchy sequence in \mathfrak{U} is \mathcal{S}_λ -convergent in \mathfrak{U} .

Theorem 5.2 Every $N - 2 - NS$ \mathfrak{U} is \mathcal{S}_λ -complete.

Proof. Let $u = (u_k)$ be \mathcal{S}_λ -Cauchy sequence in \mathfrak{U} . Suppose on the contrary that $u = (u_k)$ is not \mathcal{S}_λ -convergent. Let $\eta > 0$ and $\varrho > 0$, then $\exists p \in \mathbb{N}$ s.t $\mathfrak{w} \in \mathfrak{U}$ if we define

$$\begin{aligned} \mathcal{A}(\eta, \varrho) &= \{k \in I_n : G(u_k - u_p, \mathfrak{w}; \varrho) \leq 1 - \eta \text{ or} \\ &B(u_k - u_p, \mathfrak{w}; \varrho) \geq \eta, Y(u_k - u_p, \mathfrak{w}; \varrho) \geq \eta\} \text{ and} \end{aligned}$$

$$\begin{aligned} \mathcal{T}(\eta, \varrho) &= \{k \in I_n : G(u_k - u_0, \mathfrak{w}; \frac{\varrho}{2}) > 1 - \eta \text{ and} \\ &B(u_k - u_0, \mathfrak{w}; \frac{\varrho}{2}) < \eta, Y(u_k - u_0, \mathfrak{w}; \frac{\varrho}{2}) < \eta\}, \end{aligned}$$

then $\delta_\lambda(\mathcal{A}(\eta, \varrho)) = \delta_\lambda(\mathcal{T}(\eta, \varrho)) = 0$ and therefore we have $\delta_\lambda(\mathcal{A}^C(\eta, \varrho)) = \delta_\lambda(\mathcal{T}^C(\eta, \varrho)) = 1$. Since $G(u_k - u_p, \mathfrak{w}; \varrho) \geq 2G(u_k - u_0, \mathfrak{w}; \frac{\varrho}{2}) > 1 - \eta$ and $B(u_k - u_p, \mathfrak{w}; \varrho) \leq 2B(u_k - u_0, \mathfrak{w}; \frac{\varrho}{2}) < \eta$, $Y(u_k - u_p, \mathfrak{w}; \varrho) \leq 2Y(u_k - u_0, \mathfrak{w}; \frac{\varrho}{2}) < \eta$, if $G(u_k - u_0, \mathfrak{w}; \frac{\varrho}{2}) > \frac{1-\eta}{2}$ and $B(u_k - u_0, \mathfrak{w}; \frac{\varrho}{2}) < \frac{\eta}{2}$, $Y(u_k - u_0, \mathfrak{w}; \frac{\varrho}{2}) < \frac{\eta}{2}$. We have $\delta_\lambda(\{k \in I_n : G(u_k - u_p, \mathfrak{w}; \varrho) > 1 - \eta \text{ and } B(u_k - u_p, \mathfrak{w}; \varrho) < \eta, Y(u_k - u_p, \mathfrak{w}; \varrho) < \eta\}) = 0$. i.e., $\delta_\lambda(\mathcal{A}^C(\eta, \varrho)) = 0$ which contradicts the fact that $\delta_\lambda(\mathcal{A}^C(\eta, \varrho)) = 1$. Therefore, $u = (u_k)$ is \mathcal{S}_λ -convergent and Hence \mathcal{S}_λ -complete. \square

Theorem 5.3 For any sequence $u = (u_k)$ in $N - 2 - NS$ \mathfrak{U} , the subsequent assertions are equivalent:

- (i) $u = (u_k)$ is \mathcal{S}_λ -Cauchy.
- (ii) \exists a subset $\mathfrak{K} = \{k_n\}$ of \mathbb{N} with $\delta_\lambda\{\mathfrak{K}\} = 1$ and $\{u_{k_n}\}_{n \in \mathbb{N}}$ is \mathcal{S}_λ -Cauchy sequence over \mathfrak{K} .

Proof. The proof of the theorem can be obtained analogously as the proof of the theorem 3.4.

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