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Properties of Co-local Function and Related Φ-operator in Ideal Neutrosophic Topological Spaces

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Abstract: In this paper, we introduce and study the notions of neutrosophic co-local function and neutrosophic complement co-local function in the context of a neutrosophic topological space equipped with a neutrosophic ideal. Moreover, we explore some new classes of neutrosophic sets defined in terms of the neutrosophic co-local function and the neutrosophic complement co-local function.

Keywords: Neutrosophic set; neutrosophic topological space; neutrosophic ideal; neutrosophic co-local function.

1. Introduction

The notion of neutrosophic set has gained much relevance in recent years due to its various applications. This notion was proposed by Smarandache [1] and has been studied by many researchers as can be seen in [2-8]. In particular, Karatas and Kuru [4] introduced new neutrosophic set operations and with them defined the concept of neutrosophic topological space. Following this line of research, Albowi and Salama [2] introduced the notion of neutrosophic ideal, which was later used by Salama and Smarandache [8] to introduce the concept of neutrosophic local function, investigate its properties and analyze the relations between different neutrosophic ideals and neutrosophic topologies. The purpose of this paper is to continue with this line of research, but this time we define the neutrosophic co-local function and the neutrosophic complement co-local function, investigate the main properties of these new neutrosophic operators with them we build new classes of neutrosophic sets in a neutrosophic topological space endowed with a neutrosophic ideal.

2. Preliminaries

Throughout this paper, let X be a nonempty set, called the universe of discourse.

Definition 2.1. [1] A neutrosophic set N on X is an object of the form

$$N = \{\langle x, \mu_N(x), \sigma_N(x), \gamma_N(x)\rangle : x \in X\},$$

where $\mu_N, \sigma_N, \gamma_N$ are functions from X to $[0,1]$ and $0 \le \mu_N(x) + \sigma_N(x) + \gamma_N(x) \le 3$.

We denote by $\mathcal{N}(X)$ the collection of all neutrosophic sets over X.

Definition 2.2. [4] For $N, M \in \mathcal{N}(X)$ we define the following:

(1) (Inclusion) N is called a neutrosophic subset of M, denoted by $N \sqsubseteq M$, if $\mu_N(x) \leq \mu_M(x)$, $\sigma_N(x) \geq \sigma_M(x)$ and $\gamma_N(x) \geq \gamma_M(x)$ for all $x \in X$. Also, we can say that M is a neutrosophic super set of N.

(2) (Equality) N is called neutrosophic equal to M, denoted by $N = M$, if $N \sqsubseteq M$ and $M \sqsubseteq N$.

(3) (Universal set) N is called the neutrosophic universal set, denoted by $\tilde{X}$, if $\mu_N(x) = 1, \sigma_N(x) = 0$ and $\gamma_N(x) = 0$ for all $x \in X$.

(4) (Empty set) N is called the neutrosophic empty set, denoted by $\tilde{\emptyset}$, if $\mu_N(x) = 0, \sigma_N(x) = 1$ and $\gamma_N(x) = 1$ for all $x \in X$.

(5) (Intersection) The neutrosophic intersection of N and M, denoted by $N \sqcap M$, is defined as

$$N \sqcap M = \{\langle x, \mu_N(x) \wedge \mu_M(x), \sigma_N(x) \vee \sigma_M(x), \gamma_N(x) \vee \gamma_M(x)\rangle : x \in X\}.$$

(6) (Union) The neutrosophic union of N and M, denoted by $N \sqcup M$, is defined as

$$N \sqcup M = \{\langle x, \mu_N(x) \vee \mu_M(x), \sigma_N(x) \wedge \sigma_M(x), \gamma_N(x) \wedge \gamma_M(x)\rangle : x \in X\}.$$

(7) (Complement) The neutrosophic complement of N, denoted by N^c, is defined as

$$N^c = \{\langle x, \gamma_N(x), 1 - \sigma_N(x), \mu_N(x)\rangle : x \in X\}.$$

Proposition 2.3. [4] If $N, M \in \mathcal{N}(X)$, then we have the following properties:

(1) $N \sqcap N = N$ and $N \sqcup N = N$.

(2) $N \sqcap M = M \sqcap N$ and $N \sqcup M = M \sqcup N$.

(3) $N \sqcap \tilde{\emptyset} = \tilde{\emptyset}$ and $N \sqcap \tilde{X} = N$.

(4) $N \sqcup \tilde{\emptyset} = N$ and $N \sqcup \tilde{X} = \tilde{X}$.

(5) $N \sqcap (M \sqcap O) = (N \sqcap M) \sqcap O$ and $N \sqcup (M \sqcup O) = (N \sqcup M) \sqcup O$.

(6) $(N^c)^c = N$.

Proposition 2.4. [6] Let $N, M \in \mathcal{N}(X)$. Then, $N \sqsubseteq M$ if and only if $M^c \sqsubseteq N^c$.

The union and intersection operations given in Definition 2.2 can be extended as follows.

Definition 2.5. [7] For $\{N_j : j \in J\} \subseteq \mathcal{N}(X)$ we define the following operations:

(1) (Arbitrary intersection) The arbitrary neutrosophic intersection of the collection $\{N_j : j \in J\}$, denoted by $\sqcap_{j \in J} N_j$, is defined as

$$\prod_{j \in J} N_j = \left\{\left\langle x, \inf_{j \in J} \mu_{N_j}(x), \sup_{j \in J} \sigma_{N_j}(x), \sup_{j \in J} \gamma_{N_j}(x)\right\rangle : x \in X\right\}.$$

(2) (Arbitrary union) The arbitrary neutrosophic union of the collecction $\{N_j : j \in J\}$, denoted by $\sqcup_{j \in J} N_j$, is defined as

$$\bigsqcup_{j \in J} N_j = \left\{\left\langle x, \sup_{j \in J} \mu_{N_j}(x), \inf_{j \in J} \sigma_{N_j}(x), \inf_{j \in J} \gamma_{N_j}(x)\right\rangle : x \in X\right\}.$$

Proposition 2.6. [4] If $\{N_j : j \in J\} \subseteq \mathcal{N}(X)$ and $M \in \mathcal{N}(X)$, then we have the following properties:

(1) $M \sqcap (\sqcup_{j \in J} N_j) = \sqcup_{j \in J} (M \sqcap N_j)$.

(2) $M \sqcup (\sqcap_{j \in J} N_j) = \sqcap_{j \in J} (M \sqcup N_j)$.

(3) $(\sqcap_{j \in J} N_j)^c = \sqcup_{j \in J} N_j^c$.

(4) $(\sqcup_{j \in J} N_j)^c = \sqcap_{j \in J} N_j^c$.

Definition 2.7. [4] A neutrosophic topology on a set X is a collection $\tau \subseteq \mathcal{NS}(X)$ which satisfies the following conditions:

(1) $\widetilde{\emptyset}$ and $\tilde{X}$ are in τ.

(2) The intersection of two neutrosophic sets belonging to τ is in τ.

(3) The union of any collection of neutrosophic sets belonging to τ is in τ.

A set X for which a neutrosophic topology τ has been defined is called a neutrosophic topological space and is denoted as a pair (X, τ). If $N \in \tau$, then N is called a neutrosophic open set and if $N^c \in \tau$, then N is called a neutrosophic closed set. We denote by τ^c the collection of all neutrosophic closed sets in the neutrosophic topological space (X, τ).

Proposition 2.8. [4] Let (X, τ) be a neutrosophic topological space. Then, the following conditions hold:

(1) $\widetilde{\emptyset}$ and $\tilde{X}$ are in τ^c.

(2) The union of two neutrosophic sets belonging to τ^c is in τ^c.

(3) The intersection of any collection of neutrosophic sets belonging to τ^c is in τ^c.

Definition 2.9. [4] Let (X, τ) be a neutrosophic topological space and $N \in \mathcal{N}(X)$. The neutrosophic closure of N, denoted by $Cl(N)$, is defined as

$$Cl(N) = \bigsqcap \{F \in \mathcal{N}(X) : N \sqsubseteq F \text{ and } F \in \tau^c\};$$

while the neutrosophic interior of N, denoted by $\text{Int}(N)$, is defined as

$$\text{Int}(N) = \bigsqcup \{U \in \mathcal{N}(X) : U \sqsubseteq N \text{ and } U \in \tau\}.$$

Proposition 2.10. [4] Let (X, τ) be a neutrosophic topological space and $N, M \in \mathcal{N}(X)$. Then, the following conditions hold:

(1) $N \sqsubseteq Cl(N)$ and $\text{Int}(N) \sqsubseteq N$.

(2) If $N \sqsubseteq M$, then $Cl(N) \sqsubseteq Cl(M)$ and $\text{Int}(N) \sqsubseteq \text{Int}(M)$.

(3) $N \in \tau^c$ if and only if $N = Cl(N)$.

(4) $N \in \tau$ if and only if $N = \text{Int}(N)$.

Definition 2.11. [5] A neutrosophic set $M = \{\langle x, \mu_M(x), \sigma_M(x), \gamma_M(x)\rangle : x \in X\}$ is called a neutrosophic point if for any element $y \in X, \mu_M(y) = a, \sigma_M(y) = b, \gamma_M(y) = c$ for $y = x$ and $\mu_M(y) = 0, \sigma_M(y) = 1, \gamma_M(y) = 1$ for $y \neq x$, where $a \in (0,1]$ and $b, c \in [0,1)$. In this case, the neutrosophic point M is denoted by $M^x_{a,b,c}$ or simply by $x_{a,b,c}$. Also, x is called the support of the neutrosophic point $x_{a,b,c}$. The neutrosophic point $x_{1,0,0}$ is called a neutrosophic crisp point.

Definition 2.12. [5] Let $N \in \mathcal{N}(X)$. A neutrosophic point $x_{a,b,c}$ is said to belong to N, denoted by $x_{a,b,c} \in N$, if $\mu_N(x) \geq a, \sigma_N(x) \leq b$ and $\gamma_N(x) \leq c$.

Lemma 2.13. [5] Let $N, M \in \mathcal{N}(X)$. Then, we have:

(1) $N = \sqcup\{x_{a,b,c} : x_{a,b,c} \in N\}$.

(2) If $x_{a,b,c} \in N$ and $N \sqsubseteq M$, then $x_{a,b,c} \in M$.

Proposition 2.14. Let $N, M \in \mathcal{N}(X)$. Then, the following properties are equivalent:

(1) $N \sqsubseteq M$.

(2) $x_{a,b,c} \in N$ implies that $x_{a,b,c} \in M$.

Proof. The proof follows directly from Lemma 2.13.

Remark 2.15. It is important to note that $\widetilde{\emptyset}$ is not the only neutrosophic set that does not have points belonging to it. For example, if $X = \{x, y\}$, then $N = \{\langle x, 0,0.5,1\rangle, \langle y, 0,0.4,1\rangle\}$ is a neutrosophic set over X for which there are no neutrosophic points belonging to it.

Let $\mathcal{N}_p(X) = \{N \in \mathcal{N}(X)$: there exists a neutrosophic point $x_{a,b,c} \in N\}$ and let $\mathcal{N}'(X) = \{\widetilde{\emptyset}\} \cup \mathcal{N}_p(X)$. In the remainder of this paper, we will use the definitions and results described previously, restricted to the collection $\mathcal{N}'(X)$.

Definition 2.16. [9] Let (X, τ) be a neutrosophic topological space and $N \in \mathcal{N}'(X)$. The neutrosophic point-kernel of N, denoted by $\mathrm{Ker}_p(N)$, is defined as

$$\mathrm{Ker}_p(N) = \bigsqcup \{x_{a,b,c} \in \mathcal{N}'(X): F \sqcap N \neq \widetilde{\emptyset} \text{ for every } F \in \tau^c(x_{a,b,c})\},$$

where $\tau^c(x_{a,b,c}) = \{F \in \tau^c : x_{a,b,c} \in F\}$.

According to [9], the collection $\tau_k = \{N \in \mathcal{N}'(X): \mathrm{Ker}_p(N^c) = N^c\}$ is a neutrosophic topology on X and Ker_p is the neutrosophic closure in the neutrosophic topological space (X, τ_k). We say that a neutrosophic set N is neutrosophic τ_k-open, if $N \in \tau_k$. The complement of a neutrosophic τ_k-open set we will call it a neutrosophic τ_k-closed set. We denote by Cok_p the neutrosophic interior in the neutrosophic topological space (X, τ_k). Let us note that M is τ_k-open neutrosophic if and only if $\mathrm{Cok}_p(M) = M$; while M is τ_k-closed neutrosophic if and only if $\mathrm{Ker}_p(M) = M$.

Definition 2.17. [2] A neutrosophic ideal on a set X is a nonempty collection $\mathcal{L} \subseteq \mathcal{N}'(X)$, which satisfies the following conditions:

(1) $N \in \mathcal{L}$ and $M \sqsubseteq N$ imply that $M \in \mathcal{L}$. (Hereditary property)

(2) $N, M \in \mathcal{L}$ imply that $N \sqcup M \in \mathcal{L}$. (Finite additivity property)

Definition 2.18. [9] An application $\Upsilon: \mathcal{N}'(X) \to \mathcal{N}'(X)$ is called a neutrosophic closure operator if it satisfies the following conditions:

(1) $N \sqsubseteq \Upsilon(N)$ (expansivity),

(2) $\Upsilon(\Upsilon(N)) = \Upsilon(N)$ (idempotency),

(3) $\Upsilon(N \sqcup M) = \Upsilon(N) \sqcup \Upsilon(M)$ (additivity),

(4) $\Upsilon(\widetilde{\emptyset}) = \widetilde{\emptyset}$ (non-spontaneous creation),

whenever $M, N \in \mathcal{N}'(X)$.

Lemma 2.19. [9] If $\Upsilon: \mathcal{N}'(X) \to \mathcal{N}'(X)$ is a neutrosophic closure operator, then the collection $\tau(\Upsilon) = \{N \in \mathcal{N}'(X): \Upsilon(N^c) = N^c\}$ is a neutrosophic topology on X and Υ is the neutrosophic closure in the neutrosophic topological space $(X, \tau(\Upsilon))$.

3. Neutrosophic co-local function and related Φ-operator

In this section, we introduce and study the concept of neutrosophic co-local function as a natural generalization of the neutrosophic point-kernel of a set in a neutrosophic topological space. Moreover, we introduce the concept of neutrosophic complement co-local function (also called neutrosophic Φ-operator) and explore some new classes of neutrosophic sets defined in terms of the neutrosophic co-local function and the neutrosophic complement co-local function.

3.1. Neutrosophic co-local function

Definition 3.1.1. Let (X,τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For each $N \in \mathcal{N}'(X)$, we define the neutrosophic co-local function of N as follows:

$$N^{\bullet}(\mathcal{L},\tau) = \bigsqcup \{x_{a,b,c} \in \mathcal{N}'(X): F \sqcap N \notin \mathcal{L} \text{ for every } F \in \tau^c(x_{a,b,c})\}.$$

We will denote $N^{\bullet}(\mathcal{L},\tau)$ by $N^{\bullet}$ or $N^{\bullet}(\mathcal{L})$. Observe that the neutrosophic co-local function can be seen as an operator from $\mathcal{N}'(X)$ to $\mathcal{N}'(X)$; that is, $(\ )^{\bullet}: \mathcal{N}'(X) \to \mathcal{N}'(X)$, defined by $N \mapsto N^{\bullet}$.

The co-local function is not a neutrosophic closure operator, since in general, it does not satisfy $N \sqsubseteq N^{\bullet}$ for each $N \in \mathcal{N}'(X)$. In the case that $N \sqsubseteq N^{\bullet}$, we say that N is a neutrosophic $\bullet$-dense in itself set. The following example shows that, in general, $\tilde{X}^{\bullet}$ is a proper neutrosophic subset of $\tilde{X}$; that is, $\tilde{X}$ is not neutrosophic $\bullet$-dense in itself.

Example 3.1.2. Let $X = \mathbb{R}$ with the neutrosophic topology $\tau = \{\widetilde{\emptyset}, \widetilde{\mathbb{R}}, A^c\}$, where $A \neq \widetilde{\emptyset}$ is any neutrosophic subset having countable support of $\mathbb{R}$ and $\mathcal{L} = \mathcal{L}_c$ the neutrosophic ideal of all neutrosophic subsets having countable support of $\mathbb{R}$. Observe that $F_1 = \widetilde{\mathbb{R}}$ and $F_2 = A$ are the only neutrosophic closed sets such that $F_1 \neq \widetilde{\emptyset}$ and $F_2 \neq \widetilde{\emptyset}$. Since $\tilde{X} \cap F_1 = F_1 \notin \mathcal{L}_c$ and $\tilde{X} \sqcap F_2 = A \in \mathcal{L}_c$, then is clear that $\tilde{X}^{\bullet} = \widetilde{\mathbb{R}}^{\bullet} = A^c \sqsubseteq \widetilde{\mathbb{R}} = \tilde{X}$, but $\tilde{X}^{\bullet} \neq \tilde{X}$.

Proposition 3.1.3. Let (X,τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For every $N \in \mathcal{N}'(X)$, the following properties hold:

(1) If $\mathcal{L} = \{\widetilde{\emptyset}\}$, then $N^{\bullet} = \mathrm{Ker}_p(N)$.

(2) If $\mathcal{L} = \mathcal{N}'(X)$, then $N^{\bullet} = \widetilde{\emptyset}$.

Lemma 3.1.4. Let (X,τ) be a neutrosophic topological space with two arbitrary neutrosophic ideals $\mathcal{L}$ and $\mathcal{L}'$ on X. If $N, M \in \mathcal{N}'(X)$, then the following properties hold:

(1) If $N \sqsubseteq M$, then $N^{\bullet} \sqsubseteq M^{\bullet}$.

(2) If $\mathcal{L} \subseteq \mathcal{L}'$, then $N^{\bullet}(\mathcal{L}') \sqsubseteq N^{\bullet}(\mathcal{L})$.

(3) $N^{\bullet} = \mathrm{Ker}_p(N^{\bullet}) \sqsubseteq \mathrm{Ker}_p(N)$ ($N^{\bullet}$ is a neutrosophic τ_k-closed set).

(4) $(N^{\bullet})^{\bullet} \sqsubseteq N^{\bullet}$.

(5) $\widetilde{\emptyset}^{\bullet} = \widetilde{\emptyset}$.

(6) $(N \sqcup M)^{\bullet} = N^{\bullet} \sqcup M^{\bullet}$.

(7) If F is a neutrosophic closed set, then $F \sqcap N^{\bullet} = F \sqcap (F \sqcap N)^{\bullet} \sqsubseteq (F \sqcap N)^{\bullet}$.

(8) If $N \in \mathcal{L}$, then $N^{\bullet} = \widetilde{\emptyset}$.

(9) If $N \sqsubseteq N^{\bullet}$, then $N^{\bullet} = \mathrm{Ker}_p(N)$.

(10) If τ_1 and τ_2 be are two neutrosophic topologies on X such that $\tau_1 \subseteq \tau_2$, then $N^{\bullet}(\mathcal{L},\tau_2) \sqsubseteq N^{\bullet}(\mathcal{L},\tau_1)$.

(11) $N^{\bullet}(\mathcal{L} \cap \mathcal{L}') = N^{\bullet}(\mathcal{L}) \sqcup N^{\bullet}(\mathcal{L}')$.

Proof. (1) Assume that $x_{a,b,c} \in N^{\bullet}$ and let $F \in \tau^c(x_{a,b,c})$. Then, $F \sqcap N \notin \mathcal{L}$ and as $N \sqsubseteq M$, we have $F \sqcap N \sqsubseteq F \sqcap M$. By the hereditary property of $\mathcal{L}$, it follows that $F \sqcap M \notin \mathcal{L}$ and hence $x_{a,b,c} \in M^{\bullet}$.

(2) Suppose that $\mathcal{L} \subseteq \mathcal{L}'$, $x_{a,b,c} \in N^{\bullet}(\mathcal{L}')$ and let $F \in \tau^c(x_{a,b,c})$ be arbitrary. Then $N \sqcap F \notin \mathcal{L}'$ and as $\mathcal{L} \subseteq \mathcal{L}'$, it follows that $N \sqcap F \notin \mathcal{L}$, which implies that $x_{a,b,c} \in N^{\bullet}(\mathcal{L})$. Thus, we conclude that $N^{\bullet}(\mathcal{L}') \sqsubseteq N^{\bullet}(\mathcal{L})$.

(3) Let $x_{a,b,c} \in \mathrm{Ker}_p(N^{\bullet})$ and $F \in \tau^c(x_{a,b,c})$ be arbitrary. Then, $F \sqcap N^{\bullet} \neq \widetilde{\emptyset}$, so there exists a neutrosophic point $y_{u,v,w} \in F \sqcap N^{\bullet}$, which implies that $y_{u,v,w} \in F$ and $y_{u,v,w} \in N^{\bullet}$. Since $F \in$

$\tau^c(y_{u,v,w})$, it follows that $F \sqcap N \notin \mathcal{L}$ and so $x_{a,b,c} \in N^{\bullet}$. On the other hand, as $N^{\cdot} \sqsubseteq \mathrm{Ker}_p(N^{\bullet})$, we conclude that $N^{\bullet} = \mathrm{Ker}_p(N^{\bullet})$. Now, since $\{\widetilde{\emptyset}\} \subseteq \mathcal{L}$, by part (1) of Proposition 3.1.3, we have $N^{\bullet} \sqsubseteq N^{\bullet}(\{\widetilde{\emptyset}\}) = \mathrm{Ker}_p(N)$.

(4) By part (3), $N^{\bullet} = \mathrm{Ker}_p(N^{\bullet}) \sqsubseteq \mathrm{Ker}_p(N)$ for every $N \in \mathcal{N}'(X)$. In particular, for $N^{\bullet}$ we have $(N^{\bullet})^{\bullet} \sqsubseteq \mathrm{Ker}_p(N^{\bullet}) = N^{\bullet}$.

(5) We have

$$\begin{aligned}\widetilde{\emptyset} &= \bigsqcup \{x_{a,b,c} \in \mathcal{N}'(X)\colon\ F \sqcap \widetilde{\emptyset} \notin \mathcal{L} \text{ for every } F \in \tau^c(x_{a,b,c})\}\\ &= \bigsqcup \{x_{a,b,c} \in \mathcal{N}'(X)\colon\ \widetilde{\emptyset} \notin \mathcal{L} \text{ for every } F \in \tau^c(x_{a,b,c})\} = \widetilde{\emptyset}.\end{aligned}$$

(6) By part (1), we have $N^{\bullet} \sqsubseteq (N \sqcup M)^{\bullet}$ and $M^{\bullet} \sqsubseteq (N \sqcup M)^{\bullet}$. Therefore, $N^{\bullet} \sqcup M^{\bullet} \sqsubseteq (N \sqcup M)^{\bullet}$. For the other inclusion, assume that $x_{a,b,c} \in (N \sqcup M)^{\bullet}$ and let $F \in \tau^c(x_{a,b,c})$ be arbitrary. Then, $(M \sqcup N) \sqcap F \notin \mathcal{L}$, i.e. $(M \sqcap F) \sqcup (N \sqcap F) \notin \mathcal{L}$. Accordingly, we have the cases $M \sqcap F \notin \mathcal{L}$ or $N \sqcap F \notin \mathcal{L}$. If $M \sqcap F \notin \mathcal{L}$, then we obtain that $x_{a,b,c} \in M^{\bullet}$, whereas if $N \sqcap F \notin \mathcal{L}$, then we have $x_{a,b,c} \in N^{\bullet}$. In both cases, it follows that $x_{a,b,c} \in M^{\bullet} \sqcup N^{\bullet}$.

(7) Let $F \in \tau^c, x_{a,b,c} \in F \sqcap N^{\bullet}$ and $G \in \tau^c(x_{a,b,c})$ be arbitrary. Then, $x_{a,b,c} \in F \sqcap G, F \sqcap G \in \tau^c$ and $x_{a,b,c} \in N^{\bullet}$, which implies that $G \sqcap (F \sqcap N) \notin \mathcal{L}$ and so $x_{a,b,c} \in (F \sqcap N)^{\bullet}$. Thus, we have $F \sqcap N^{\bullet} \sqsubseteq (F \sqcap N)^{\bullet}, F \sqcap N^{\bullet} \sqsubseteq F$ and we conclude that $F \sqcap N^{\bullet} \sqsubseteq F \sqcap (F \sqcap N)^{\bullet}$. On the other hand, the inclusion $F \sqcap N \sqsubseteq N$, means that $(F \sqcap N)^{\bullet} \sqsubseteq N^{\bullet}$ and $F \sqcap (F \sqcap N)^{\cdot} \sqsubseteq F \sqcap N^{\bullet}$. Therefore, $F \sqcap N^{\bullet} = F \sqcap (F \sqcap N)^{\bullet} \sqsubseteq (F \sqcap N)^{\bullet}$.

(8) Suppose that $N \in \mathcal{L}$ and $N^{\bullet} \neq \widetilde{\emptyset}$. Then, there exists a neutrosophic point $x_{a,b,c} \in N^{\bullet}$ and so, $N \sqcap F \notin \mathcal{L}$ for $F \in \tau^c(x_{a,b,c})$ being arbitrary. But the fact that $N \in \mathcal{L}$ implies that $N \sqcap F \in \mathcal{L}$ for each $F \in \tau^c(x_{a,b,c})$. Thus, we obtain a contradiction and hence, $N^{\bullet} = \widetilde{\emptyset}$.

(9) Assume that $N \sqsubseteq N^{\bullet}$. By part (3), $N^{\bullet} = \mathrm{Ker}_p(N^{\bullet}) \sqsubseteq \mathrm{Ker}_p(N)$ and by hypotheses, it follows that $\mathrm{Ker}_p(N) \sqsubseteq \mathrm{Ker}_p(N^{\bullet}) = N^{\bullet} \sqsubseteq \mathrm{Ker}_p(N)$ and hence, $N^{\bullet} = \mathrm{Ker}_p(N)$.

(10) Let $x_{a,b,c} \in N^{\bullet}(\mathcal{L}, \tau_2)$ and $F \in \tau_1^c(x_{a,b,c})$ be arbitrary. Since $\tau_1 \subseteq \tau_2$, we have $F \in \tau_2^c(x_{a,b,c})$ and so, $F \sqcap N \notin \mathcal{L}$. Therefore, $x_{a,b,c} \in N^{\bullet}(\mathcal{L}, \tau_1)$.

(11) Since $\mathcal{L} \cap \mathcal{L}' \subseteq \mathcal{L}$ and $\mathcal{L} \cap \mathcal{L}' \subseteq \mathcal{L}'$, by part (2), we have $N^{\bullet}(\mathcal{L}) \sqsubseteq N^{\bullet}(\mathcal{L} \cap L')$ and $N^{\bullet}(\mathcal{L}') \sqsubseteq N^{\bullet}(\mathcal{L} \cap \mathcal{L}')$. Thus, we deduce the inclusion $N^{\bullet}(\mathcal{L}) \sqcup N^{\bullet}(\mathcal{L}') \sqsubseteq N^{\bullet}(\mathcal{L} \cap \mathcal{L}')$. For the other inclusion, suppose that $x_{a,b,c} \in N^{\bullet}(\mathcal{L} \cap \mathcal{L}')$ and let $F \in \tau^c(x_{a,b,c})$ be arbitrary. Then, $N \sqcap F \notin \mathcal{L} \cap \mathcal{L}'$, which implies that $N \sqcap F \notin \mathcal{L}$ or $N \sqcap F \notin \mathcal{L}'$. If $N \sqcap F \notin \mathcal{L}$, then $x_{a,b,c} \in N^{\bullet}(\mathcal{L})$, whereas if $N \sqcap F \notin \mathcal{L}'$, then $x_{a,b,c} \in N^{\bullet}(\mathcal{L}')$. In both cases, it follows that $x_{a,b,c} \in N^{\bullet}(\mathcal{L}) \sqcup N^{\bullet}(\mathcal{L}')$. Therefore, $N^{\bullet}(\mathcal{L} \cap \mathcal{L}') \sqsubseteq N^{\bullet}(\mathcal{L}) \sqcup N^{\bullet}(\mathcal{L}')$.

Corollary 3.1.5. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. If $\{N_\alpha : \alpha \in \Delta\} \subseteq \mathcal{N}'(X)$, then the following properties hold:

(1) $(\sqcap_{\alpha \in \Delta} N_\alpha)^{\bullet} = \sqcap_{\alpha \in \Delta} {N_\alpha}^{\bullet}$.

(2) $(\sqcup_{\alpha \in \Delta} N_\alpha)^{\bullet} = \sqcup_{\alpha \in \Delta} {N_\alpha}^{\bullet}$, if Δ is finite.

Since the neutrosophic co-local function is not a neutrosophic closure operator, it is necessary to introduce a new concept that allows us to obtain a new neutrosophic topology from it.

Definition 3.1.6. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For each $N \in \mathcal{N}'(X)$, we define $Cl^{\bullet}(N) = N \sqcup N^{\bullet}$.

Remark 3.1.7. Let (X,τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For each $N \in \mathcal{N}'(X)$, the following properties hold:

(1) If $\mathcal{L} = \{\widetilde{\emptyset}\}$ then $Cl^{\bullet}(N) = N \sqcup N^{\bullet} = N \sqcup \mathrm{Ker}_p(N) = \mathrm{Ker}_p(N)$.

(2) If $\mathcal{L} = \mathcal{N}'(X)$, then $Cl^{\bullet}(N) = N \sqcup \widetilde{\emptyset} = N$.

Proposition 3.1.8. $Cl^{\bullet}$ is a neutrosophic closure operator.

Proof. The proof is an immediate consequence of Lemma 3.1.4.

According with Proposition 3.1.8 and Lemma 2.19, if (X,τ) is a neutrosophic topological space and $\mathcal{L}$ is a neutrosophic ideal on X, we denote by $\tau^{\bullet}(\mathcal{L})$ the neutrosophic topology generated by $Cl^{\bullet}$; that is $\tau^{\bullet}(\mathcal{L}) = \{N \in \mathcal{N}'(X): \quad Cl^{\bullet}(N^c) = N^c\}$. When there is no chance for confusion, we will simply write $\tau^{\bullet}$ for $\tau^{\bullet}(\mathcal{L})$. The elements of $\tau^{\bullet}$ are called neutrosophic $\tau^{\bullet}$-open sets and the complement of a neutrosophic $\tau^{\bullet}$-open set is called neutrosophic $\tau^{\bullet}$-closed set. Note that if $N \in \mathcal{N}'(X)$, then: N is neutrosophic $\tau^{\bullet}$-closed if and only if $N^c \in \tau^{\bullet}$ if and only if $Cl^{\bullet}((N^c)^c) = (N^c)^c$ if and only if $Cl^{\bullet}(N) = N$.

Remark 3.1.9. Since $N^{\bullet} = \mathrm{Ker}_p(N^{\bullet}) \sqsubseteq \mathrm{Ker}_p(N)$, then $Cl^{\bullet}(N) \sqsubseteq \mathrm{Ker}_p(N)$ for each $N \in \mathcal{N}'(X)$. Therefore, if N is a neutrosophic τ_k-closed set, then N is neutrosophic $\tau^{\bullet}$-closed. It follows that each neutrosophic τ_k-open set is neutrosophic $\tau^{\bullet}$-open; that is $\tau_k \subseteq \tau^{\bullet}$. Moreover, from Remark 3.1.7 it follows that $\tau^{\bullet}(\{\widetilde{\emptyset}\}) = \tau_k$ and $\tau^{\bullet}(\mathcal{N}'(X)) = \mathcal{N}'(X)$.

Proposition 3.1.10. Let (X,τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. If $\{N_\alpha: \alpha \in \Delta\}$ is a collection of neutrosophic $\tau^{\bullet}$-closed sets, then the following properties hold:

(1) $\sqcap \quad \{N_\alpha: \alpha \in \Delta'\}$ is a neutrosophic $\tau^{\bullet}$-closed set for any subset Δ' of Δ.

(2) $\sqcup\{N_\alpha: \alpha \in \Delta_0\}$ is a neutrosophic $\tau^{\bullet}$-closed set for any finite subset Δ_0 of Δ.

Proof. The proof is an immediate consequence of Proposition 2.6 and the duality between the notions of neutrosophic $\tau^{\bullet}$-open and neutrosophic $\tau^{\bullet}$-closed sets.

Proposition 3.1.11. Let (X,τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. Then, $N \in \mathcal{N}'(X)$ is neutrosophic $\tau^{\bullet}$-closed if and only if $N^{\bullet} \sqsubseteq N$.

Proof. Suppose that N is neutrosophic $\tau^{\bullet}$-closed. Then, $Cl^{\bullet}(N) = N$. In consequence, $N \sqcup N^{\bullet} = N$ and hence, $N^{\bullet} \sqsubseteq N$. Conversely, assume that $N^{\bullet} \sqsubseteq N$. Since $Cl^{\bullet}(N) = N \sqcup N^{\bullet}$ and $N \sqcup N^{\bullet} \sqsubseteq N$, we have $Cl^{\bullet}(N) \sqsubseteq N$. By Proposition 3.1.8, we have $N \sqsubseteq Cl^{\bullet}(N)$ and so, we conclude that $Cl^{\bullet}(N) = N$. This shows that N is neutrosophic $\tau^{\bullet}$-closed.

Proposition 3.1.12. If $\mathcal{L}$ and $\mathcal{L}'$ are neutrosophic ideals on a neutrosophic topological space (X,τ) such that $\mathcal{L} \subseteq \mathcal{L}'$, then $\tau^{\bullet}(\mathcal{L}) \subseteq \tau^{\bullet}(\mathcal{L}')$.

Proof. Consider $N \in \tau^{\bullet}(\mathcal{L})$. Then, N^c is a neutrosophic $\tau^{\bullet}(\mathcal{L})$-closed set and so, by Proposition 3.1.11, $(N^c)^{\bullet}(\mathcal{L}) \sqsubseteq N^c$. Now, by part (2) of Lemma 3.1.4, it follows that $(N^c)^{\bullet}(\mathcal{L}') \sqsubseteq (N^c)^{\bullet}(\mathcal{L}) \sqsubseteq N^c$. This shows that $(N^c)^{\bullet}(\mathcal{L}') \sqsubseteq N^c$ and N^c is a neutrosophic $\tau^{\bullet}(\mathcal{L}')$-closed set. Therefore, $N \in \tau^{\bullet}(\mathcal{L}')$.

Corollary 3.1.13. Let $\{\mathcal{J}_\alpha: \alpha \in \Delta\}$ be a collection of neutrosophic ideals on a neutrosophic topological space (X,τ). If $\mathcal{J} = \bigcap_{\alpha\in\Delta}\mathcal{J}_\alpha$ then $\tau^{\bullet}(\mathcal{J}) \subseteq \tau^{\#}$, where $\tau^{\#} = \bigcap_{\alpha\in\Delta}\tau^{\bullet}(\mathcal{J}_\alpha)$.

Proof. It is clear that $\tau^{\#}$ is a neutrosophic topology on X. Since $\mathcal{J} = \bigcap_{\alpha\in\Delta}\mathcal{J}_\alpha \subseteq \mathcal{J}_\alpha$ for every $\alpha \in \Delta$, by Proposition 3.1.12, we have $\tau^{\bullet}(\mathcal{J}) \subset \tau^{\bullet}(\mathcal{J}_\alpha)$ for every $\alpha \in \Delta$. Therefore, $\tau^{\bullet}(\mathcal{J}) \subseteq \bigcap_{\alpha\in\Delta}\tau^{\bullet}(\mathcal{J}_\alpha) = \tau^{\#}$.

Corollary 3.1.14. Suppose that (X,τ) be a neutrosophic topological space and let $\mathcal{L}$ and $\mathcal{L}'$ be two neutrosophic ideals on X. Then, $\tau^{\bullet}(\mathcal{L} \cap \mathcal{L}') = \tau^{\bullet}(\mathcal{L}) \cap \tau^{\bullet}(\mathcal{L}')$.

Proof. Let $M \in \tau^{\bullet}(\mathcal{L} \cap \mathcal{L}')$ and put $M = N^c$. Then, by part (11) of Lemma 3.4 and Proposition 3.1.11, we have:

$$M \in \tau^{\bullet}(\mathcal{L} \cap \mathcal{L}') \Leftrightarrow N \text{ is neutrosophic } \tau^{\bullet}(\mathcal{L} \cap \mathcal{L}')\text{-closed}$$
$$\Leftrightarrow N^{\bullet}(\mathcal{L}) \sqcup N^{\bullet}(\mathcal{L}') = N^{\bullet}(\mathcal{L} \cap \mathcal{L}') \sqsubseteq N$$
$$\Leftrightarrow N^{\bullet}(\mathcal{L}) \sqsubseteq N \text{ and } N^{\bullet}(\mathcal{L}') \sqsubseteq N$$
$$\Leftrightarrow M \in \tau^{\bullet}(\mathcal{L}) \text{ and } M \in \tau^{\bullet}(\mathcal{L}')$$
$$\Leftrightarrow M \in \tau^{\bullet}(\mathcal{L}) \cap \tau^{\bullet}(\mathcal{L}').$$

3.2. Neutrosophic Φ-operator and new neutrosophic sets

Definition 3.2.1. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For each $N \in \mathcal{N}'(X)$, we define the neutrosophic complement co-local function of N as $\Phi(N) = ((N^c)^{\cdot})^c$.

In Table 1 we summarize the main equalities related to the neutrosophic operator Φ, which are obtained by applying the neutrosophic complement operation or the co-local neutrosophic function from equation (1).

Table 1. Equalities related to the neutrosophic operator Φ.

(1) $\Phi(N) = ((N^c)^{\bullet})^c$	(2) $[\Phi(N)]^c = (N^c)^{\bullet}$
(3) $[\Phi(N)]^{\bullet} = (((N^c)^{\bullet})^c)^{\bullet}$	(4) $\Phi(N^c) = (N^{\bullet})^c$
(5) $[\Phi(N^c)]^c = N^{\bullet}$	(6) $[\Phi(N^c)]^{\bullet} = ((N^{\bullet})^c)^{\bullet}$
(7) $\Phi(N^{\bullet}) = (((N^{\bullet})^c)^{\bullet})^c$	(8) $[\Phi(N^{\bullet})]^c = ((N^{\bullet})^c)^{\bullet}$

Remark 3.2.2. From the equalities (6) and (8) of Table 1, we can deduce that $[\Phi(N^c)]^{\bullet} = [\Phi(N^{\bullet})]^c$.

In the following proposition, relevant properties related to the neutrosophic operator Φ (also called neutrosophic Φ-operator) are presented.

Proposition 3.2.3. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. Then, we have the following properties:

(1) If $N, M \in \mathcal{N}'(X)$ and $N \sqsubseteq M$, then $\Phi(N) \sqsubseteq \Phi(M)$. (Φ is monotone)

(2) $\Phi(N \sqcap M) = \Phi(N) \sqcap \Phi(M)$ for every $N, M \in \mathcal{N}'(X)$.

(3) $\Phi(N) \sqsubseteq \Phi(\Phi(N))$ for every $N \in \mathcal{N}'(X)$.

(4) $\Phi(\tilde{X}) = \tilde{X}$.

(5) $O \sqsubseteq \Phi(O)$ for every $O \in \tau_k$. (Φ is expansive on τ_k)

(6) $\mathrm{Cok}_p(N) \sqsubseteq \Phi(N)$ for every $N \in \mathcal{N}'(X)$.

Proof. (1) Let $N, M \in \mathcal{N}'(X)$ such that $N \sqsubseteq M$. Then, $M^c \sqsubseteq N^c$ and by part (1) of Lemma 3.1.4, $(M^c)^{\bullet} \sqsubseteq (N^c)^{\bullet}$. Therefore, $\Phi(N) = (N^c)^{\bullet})^c \sqsubseteq ((M^c)^{\bullet})^c = \Phi(M)$.

(2) If $N, M \in \mathcal{N}'(X)$, then

$$\begin{aligned}\Phi(N \sqcap M) &= (((N \sqcap M)^c)^{\bullet})^c = ((N^c \sqcup M^c)^{\bullet})^c \\ &= ((N^c)^{\bullet} \sqcup (M^c)^{\bullet})^c = ((N^c)^{\bullet})^c \sqcap ((M^c)^{\bullet})^c \\ &= \Phi(N) \sqcap \Phi(M).\end{aligned}$$

(3) Let $N \in \mathcal{N}'(X)$. By part (5) of Lemma 3.1.4, we have $((N^c)^\bullet)^\bullet \sqsubseteq (N^c)^\bullet$, which implies that $\Phi(N) = ((N^c)^\bullet)^c \sqsubseteq (((N^c)^\bullet)^\bullet)^c$. Now, by applying Definition 3.2.1 to the neutrosophic set $\Phi(N)$, we obtain that $\Phi(\Phi(N)) = (([\Phi(N)]^c)^\bullet)^c$ and by equation (2) of Table 1 , we deduce that $\Phi(\Phi(N)) = (((N^c)^\bullet)^\bullet)^c$. Hence, $\Phi(N) \sqsubseteq ((N^c)^\bullet)^\bullet)^c = \Phi(\Phi(N))$.

(4) By definition we have $\Phi(\tilde{X}) = \left((\tilde{X}^c)^\bullet\right)^c = (\tilde{\emptyset}^\bullet)^c = \tilde{\emptyset}^c = \tilde{X}$.

(5) If $O \in \tau_k$, then O^c is a neutrosophic τ_k-closed set and so $\mathrm{Ker}_p(O^c) = O^c$. By equation (2) of Table 1 and part (3) of Lemma 3.1.4, we obtain that $[\Phi(O)]^c = (O^c)^\bullet \sqsubseteq \mathrm{Ker}_p(O^c) = O^c$ and hence, $O \sqsubseteq \Phi(O)$ for every $O \in \tau_k$.

(6) Since $\mathrm{Cok}_p(N) \in \tau_k$, by part (5), we have $\mathrm{Cok}_p(N) \sqsubseteq \Phi(\mathrm{Cok}_p(N))$ and as $\mathrm{Cok}_p(N) \sqsubseteq N$, by part (1), we deuce that $\mathrm{Cok}_p(N) \sqsubseteq \Phi(\mathrm{Cok}_p(N)) \sqsubseteq \Phi(N)$.

Definition 3.2.4. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrophic ideal on X. A subset $N \in \mathcal{N}'(X)$ is said to be:

(1) neutrosophic $\bullet$-perfect, if $N = N^\bullet$

(2) neutrosophic $\bullet$-dense, if $N^\bullet = \tilde{X}$.

(3) neutrosophic $\bullet$-condensed, if $[\Phi(N)]^\bullet = N^\bullet$.

(4) neutrosophic Φ-condensed, if $\Phi(N^\bullet) = \Phi(N)$.

(5) neutrosophic $\Phi^\bullet$-condensed, if it is neutrosophic $\bullet$-condensed and neutrosophic Φ-condensed.

(6) neutrosophic non $\Phi^\bullet$-condensed, if $\Phi(N^\bullet) = \tilde{\emptyset}$.

(7) neutrosophic $\bullet$-congruent, if $[\Phi(N)]^\bullet = N$.

(8) neutrosophic Φ-congruent, if $\Phi(N^\bullet) = N$.

(9) neutrosophic $\Phi^\bullet$-congruent, if it is neutrosophic $\bullet$-congruent and neutrosophic Φ-congruent.

Proposition 3.2.5. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. If $N \in \mathcal{N}'(X)$, then we have the following properties:

(1) If N is neutrosophic $\bullet$-perfect, then it is neutrosophic Φ-condensed.

(2) N is neutrosophic Φ-condensed if and only if N^c is neutrosophic $\bullet$-condensed.

(3) N is neutrosophic $\Phi^\bullet$-condensed if and only if N^c is neutrosophic $\Phi^\bullet$-condensed.

(4) N is neutrosophic Φ-congruent if and only if N^c is neutrosophic $\bullet$-congruent.

(5) N is neutrosophic $\Phi^\bullet$-congruent if and only if N^c is neutrosophic $\Phi^\bullet$-congruent.

(6) If N neutrosophic Φ-condensed and neutrosophic non $\Phi^\bullet$-condensed, then N^c is neutrosophic $\bullet$-dense.

(7) If N neutrosophic $\bullet$-condensed and N^c is neutrosophic non $\Phi^\bullet$-condensed, then N is neutrosophic $\bullet$-dense.

(8) If N is neutrosophic non $\Phi^\bullet$-condensed and neutrosophic $\bullet$-perfect, then N^c is neutrosopic $\bullet$-dense.

Proof. (1) From Definition 3.2.4, we have:

$$\begin{aligned} N \text{ is neutrosophic } \bullet\text{-perfect} &\Leftrightarrow N = N^\bullet \\ &\Leftrightarrow \Phi(N) = \Phi(N^\bullet) \\ &\Leftrightarrow N \text{ is neutrosophic } \Phi\text{-condensed.} \end{aligned}$$

(2) By Remark 3.2.2 and equation (2) of Table 1, we get that

$$\begin{aligned} N \text{ is neutrosophic } \Phi\text{-condensed } &\Leftrightarrow \Phi(N^{\bullet}) = \Phi(N) \\ &\Leftrightarrow [\Phi(N^{\bullet})]^c = [\Phi(N)]^c \\ &\Leftrightarrow [\Phi(N^c)]^{\bullet} = (N^c)^{\bullet} \\ &\Leftrightarrow N^c \text{ is neutrosophic } \bullet\text{-condensed.} \end{aligned}$$

(3) The proof follows from (2).

(4) By Remark 3.2.2, we obtain that

$$\begin{aligned} N \text{ is neutrosophic } \Phi\text{-congruent } &\Leftrightarrow \Phi(N^{\bullet}) = N \\ &\Leftrightarrow [\Phi(N^{\bullet})]^c = N^c \\ &\Leftrightarrow [\Phi(N^c)]^{\bullet} = N^c \\ &\Leftrightarrow N^c \text{ is neutrosophic } \bullet\text{-congruent.} \end{aligned}$$

(5) The proof follows from (4).

(6) Assume that N neutrosophic Φ-condensed and neutrosophic non $\Phi^{\bullet}$-condensed. Then, $\Phi(N^{\bullet}) = \Phi(N)$ and $\Phi(N^{\bullet}) = \widetilde{\emptyset}$, which implies that $\Phi(N) = \widetilde{\emptyset}$. Thus, $[\Phi(N)]^c = \tilde{X}$ and by equation (2) of Table 1, it follows that $(N^c)^{\bullet} = \tilde{X}$. Therefore, N^c is neutrosophic $\bullet$-densed.

(7) The proof follows from (2) and (6).

(8) Suppose that N is neutrosophic non $\Phi^{\bullet}$-condensed and neutrosophic $\bullet$-perfect. Then, $\Phi(N^{\bullet}) = \widetilde{\emptyset}$ and $N^{\bullet} = N$, which implies that $\Phi(N) = \Phi(N^{\bullet}) = \widetilde{\emptyset}$. By equation (2) of Table 1 , we deduce that $(N^c)^{\bullet} = [\Phi(N)]^c = \tilde{X}$ and so, N^c is neutrosophic $\bullet$-dense.

Proposition 3.2.6. Let $N \in \mathcal{N}'(X)$ and N^c be a neutrosophic $\bullet$-perfect set. Then, the following properties are equivalent:

(1) N is neutrosophic Φ-congruent

(2) N is neutrosophic Φ-condensed.

Proof. (1) $\Longrightarrow$ (2) Suppose that N is neutrosophic Φ-congruent. Then, $\Phi(N^{\bullet}) = N$. Since N^c is neutrosophic $\bullet$-perfect, $(N^c)^{\bullet} = N^c$, which implies that $\Phi(N^{\bullet}) = N = (N^c)^c = ((N^c)^{\bullet})^c = \Phi(N)$, which shows that N is neutrosophic Φ-condensed.

(2) $\Longrightarrow$ (1) Assume that N is neutrosophic Φ-condensed. Then, $\Phi(N^{\bullet}) = \Phi(N)$. Since N^c is neutrosophic $\bullet$-perfect, $(N^c)^{\bullet} = N^c$ and by equation (2) of Table 1, it follows that $[\Phi(N)]^c = N^c$, which implies that $\Phi(N) = N$. Therefore, $\Phi(N^{\bullet}) = \Phi(N) = N$ and so, N is neutrosophic Φ congruent

Corollary 3.2.7. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. If $N \in \mathcal{N}'(X)$ is neutrosophic $\cdot$-perfect, then the following properties are equivalent:

(1) N is neutrosophic $\bullet$-congruent

(2) N is neutrosophic $\bullet$-condensed.

Proof. It is deduced from Proposition 3.2.6 by using parts (2) and (4) of Proposition 3.2.5.

Proposition 3.2.8. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For $N \in \mathcal{N}'(X)$, we have the following properties:

(1) If N is neutrosophic non $\Phi^{\bullet}$-condensed and $M \sqsubseteq N$, then M is neutrosophic non $\Phi^{\bullet}$-condensed.

(2) If N is neutrosophic non $\Phi^{\bullet}$-condensed and $M \in \mathcal{N}'(X)$, then $N \sqcap M$ is neutrosophic non $\Phi^{\bullet}$-condensed.

(3) If N is neutrosophic non $\Phi^{\bullet}$-condensed and $L \in \mathcal{L}$, then $N \sqcup L$ is neutrosophic non $\Phi^{\bullet}$-condensed.

(4) If N is neutrosophic non $\Phi^{\bullet}$-condensed, then $N^{\bullet}$ is neutrosophic non $\Phi^{\bullet}$-condensed.

(5) If N is neutrosophic non $\Phi^{\bullet}$-condensed, then for every $x_{a,b,c} \in \mathcal{N}'(X)$ and every $F \in \tau^c(x_{a,b,c})$, $\Phi(N^c) \sqcap F \neq \widetilde{\emptyset}$.

(6) If $\mathcal{J}$ is a neutrosophic ideal on X such that $\mathcal{J} \sqsubseteq \mathcal{L}$ and N is neutrosophic non $\Phi^{\bullet}$-condensed, with respect to $\mathcal{J}$, then N is neutrosophic non $\Phi^{\bullet}$-condensed with respect to $\mathcal{L}$.

Proof. (1) Suppose that N is neutrosophic non $\Phi^{\bullet}$-condensed and $M \sqsubseteq N$. Then $\Phi(N^{\bullet}) = \widetilde{\emptyset}$ and $M^{\bullet} \sqsubseteq N^{\bullet}$. Thus, $\Phi(M^{\bullet}) \sqsubseteq \Phi(N^{\bullet}) = \widetilde{\emptyset}$, which means that $\Phi(M^{\bullet}) = \widetilde{\emptyset}$ and hence, M is neutrosophic non $\Phi^{\bullet}$-condensed.

(2) Since $N \sqcap M \sqsubseteq N$ for each $M \in \mathcal{N}'(X)$, the result follows from part (1).

(3) Assume that N is neutrosophic non $\Phi^{\bullet}$-condensed and $L \in \mathcal{L}$. Then $\Phi(N^{\bullet}) = \widetilde{\emptyset}$ and $L^{\bullet} = \widetilde{\emptyset}$, which implies that $(N \sqcup L)^{\bullet} = N^{\bullet} \sqcup L^{\bullet} = N^{\bullet}$ and $\Phi((N \sqcup L)^{\bullet}) = \Phi(N^{\bullet}) = \widetilde{\emptyset}$. Therefore, $N \sqcup L$ is neutrosophic non $\Phi^{\bullet}$-condensed.

(4) Suppose that N is neutrosophic non $\Phi^{\bullet}$-condensed. Then $\Phi(N^{\bullet}) = \widetilde{\emptyset}$ and $(N^{\bullet})^{\bullet} \sqsubseteq N^{\bullet}$. Hence $\Phi((N^{\bullet})^{\bullet}) \sqsubseteq \Phi(N^{\bullet}) = \widetilde{\emptyset}$ and so $N^{\bullet}$ is neutrosophic non $\Phi^{\bullet}$-condensed.

(5) Assume that N is neutrosophic non $\Phi^{\bullet}$-condensed, i.e. $\Phi(N^{\bullet}) = \widetilde{\emptyset}$. Then $[\Phi(N^{\bullet})]^c = \tilde{X}$ and so, by Remark 3.2.2, $[\Phi(N^c)]^{\bullet} = \tilde{X}$. Therefore, for every $x_{a,b,c} \in \mathcal{N}'(X)$ and every $F \in \tau^c(x_{a,b,c})$, $F \sqcap \Phi(N^c) \notin \mathcal{L}$, which implies that $F \sqcap \Phi(N^c) \neq \emptyset$, for every $x_{a,b,c} \in \mathcal{N}'(X)$ and every $F \in \tau^c(x_{a,b,c})$.

(6) Let $\mathcal{J}$ be a neutrosophic ideal on X such that $\mathcal{J} \sqsubseteq \mathcal{L}$ and N be a neutrosophic non $\Phi^{\bullet}$-condensed set with respect to $\mathcal{J}$. Then $\Phi(N^{\bullet}(\mathcal{J})) = \widetilde{\emptyset}$ and by part (2) of Lemma 3.1.4, we have $N^{\bullet}(\mathcal{L}) \sqsubseteq N^{\bullet}(\mathcal{J})$, which implies that $\Phi(N^{\bullet}(\mathcal{L})) \sqsubseteq \Phi(N^{\bullet}(\mathcal{J})) = \widetilde{\emptyset}$. Therefore, $\Phi(N^{\bullet}(\mathcal{L})) = \widetilde{\emptyset}$ and so, N is neutrosophic non $\Phi^{\bullet}$-condensed with respect to $\mathcal{L}$.

Proposition 3.2.9. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For $N \in \mathcal{N}'(X)$, we have the following properties:

(1) N is neutrosophic non $\Phi^{\bullet}$-condensed if and only if $(N^{\bullet})^c$ is neutrosophic $\bullet$-dense.

(2) N is neutrosophic non $\Phi^{\bullet}$-condensed if and only if $\Phi(N^c)$ is neutrosophic $\bullet$-dense.

(3) N^c is neutrosophic non $\Phi^{\bullet}$-condensed if and only if $\Phi(N)$ is neutrosophic $\bullet$-dense.

Proof. The proofs of (1) and (2) are obtained from Definition 3.2.4 and equation (8) of Table 1 as follows:

$$
\begin{aligned}
N \text{ is neutrosophic non } \Phi^{\bullet}\text{-condensed} \quad &\Leftrightarrow \Phi(N^{\bullet}) = \widetilde{\emptyset} \\
&\Leftrightarrow [\Phi(N^{\bullet})]^c = \tilde{X} \\
&\Leftrightarrow ((N^{\bullet})^c)^{\bullet} = \tilde{X} \\
&\Leftrightarrow (N^{\bullet})^c \text{ is neutrosophic } \bullet\text{-dense} \\
&\Leftrightarrow \Phi(N^c) \text{ is neutrosophic } \bullet\text{-dense.}
\end{aligned}
$$

(3) The proof follows from (2) by changing N to N^c.

Corollary 3.2.10. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. If $\tilde{X}$ is a neutrosophic $\bullet$-dense in itself set, then every $L \in \mathcal{L}$ is a neutrosophic non $\Phi^{\bullet}$-condensed set.

Proof. Since $L \in \mathcal{L}$, we have $L^{\bullet} = \widetilde{\emptyset}$ and hence, $(L^{\bullet})^c = \tilde{X}$. According to equation (4) of Table 1, $\Phi(L^c) = \tilde{X}$ and as $\tilde{X}$ is neutrosophic $\bullet$-dense in itself, it follows that $[\Phi(L^c)]^{\bullet} = \tilde{X}^{\bullet} = \tilde{X}$ and so, $\Phi(L^c)$ is neutrosophic $\bullet$-dense. Now, by Theorem 3.2.9, we conclude that L is neutrosophic non $\Phi^{\bullet}$-condensed.

Definition 3.2.11. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For every $N \in \mathcal{N}'(X)$, the neutrosophic $\bullet$-frontier of N, denoted by $Fr^{\bullet}(N)$, is defined as $Fr^{\bullet}(N) = N^{\bullet} \sqcap (N^c)^{\bullet}$.

Proposition 3.2.12. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. If $N \in \mathcal{N}'(X)$ is neutrosophic $\bullet$-dense and $\Phi(Fr^{\bullet}(N)) = \widetilde{\emptyset}$, then N^c is neutrosophic non $\Phi^{\bullet}$-condensed.

Proof. Suppose that $N \in \mathcal{N}'(X)$ is neutrosophic $\bullet$-dense and $\Phi(Fr^{\bullet}(N)) = \widetilde{\emptyset}$. Then, $N^{\bullet} = \tilde{X}$ and $\Phi(N^{\bullet} \sqcap (N^c)^{\bullet}) = \widetilde{\emptyset}$. Hence, by parts (2) and (4) of Proposition 3.2.3, we have $\Phi(N^{\bullet}) \sqcap \Phi((N^c)^{\bullet}) = \widetilde{\emptyset}$ and $\Phi(N^{\bullet}) = \Phi(\tilde{X}) = \tilde{X}$, respectively. Thus, $\Phi((N^c)^{\bullet}) = \tilde{X} \sqcap \Phi((N^c)^{\bullet}) = \widetilde{\emptyset}$ and therefore, N^c is neutrosophic non $\Phi^{\bullet}$-condensed.

5. Conclusions

Neutrosophic topology is one of the most useful notions in neutrosophic set theory, because many of the topics studied in this branch of mathematics are done in the context of a neutrosophic topological space. In this work, we have used the notions of neutrosophic point and neutrosophic ideal to introduce and study the concepts of neutrosophic co-local function and neutrosophic complement co-local function of a subset of a neutrosophic topological space. We have established the most relevant properties of the concepts introduced and we have explored new classes of neutrosophic sets defined in terms of these concepts. Since various modifications of topology in neutrosophic set theory have recently been addressed, we consider that the notions and results given in this paper can be extended to the contexts of Refined Neutrosophic Topology, Refined Neutrosophic Crisp Topology, SuperHyperNeutrosophic Topology and Single-Valued Duplet Neutrosophic Topology, Single-Valued Neutrosophic Triplet Weak Topology and others highlighted in [10], which leave open a prominent field for future research.

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