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Mohamed Soueycatt

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Rashel Abu Hakmeh

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# On The Foundations of Symbolic 5-Plithogenic Number Theory

<sup>1</sup> Mohamed Soueycatt, <sup>2</sup> Barbara Charchekhandra, <sup>3</sup> Rashel Abu Hakmeh

<sup>1</sup> Department of Bioengineering, Al-Andalus Private University for Medical Sciences, Syria; [m.soueycatt55@au.edu.sy](mailto:m.soueycatt55@au.edu.sy)

<sup>2</sup> Jadavpur University, Department Of Mathematics, Kolkata, India; [Charchekhandrabar32@yahoo.com](mailto:Charchekhandrabar32@yahoo.com)

<sup>3</sup> Faculty of Science, Mutah University, Jordan

**Abstract:** This paper is dedicated to study the properties of symbolic 5-plithogenic integers and number theory, where we present many number theoretical concepts such as symbolic 5-plithogenic Diophantine equations, symbolic 5-plithogenic congruencies, and symbolic 5-plithogenic Euler's function. Also, we present many examples to explain the validity and the scientific contribution of our work.

**Keywords:** symbolic 5-plithogenic integer, symbolic 5-plithogenic Euler's function, symbolic 5-plithogenic Pythagoras triple

## Introduction

Symbolic  $n$ -plithogenic sets were defined for the first time by Smarandache in [4, 24-25], with many interesting algebraic properties.

In [1-3], the symbolic 2-plithogenic rings were defined as an extension of classical rings. Many results were obtained with respect to their ideals and homomorphisms. The symbolic 2-plithogenic rings and fields have many applications in generalizing other algebraic structures such as symbolic 2-plithogenic vector spaces, symbolic 2-plithogenic modules, and symbolic 2-plithogenic equations [5-7].

Laterally, many authors defined and studied symbolic 3-plithogenic algebraic structures, such as symbolic 3-plithogenic spaces and modules, see [8, 21-23].

In the literature, the extended integer systems were used in number theory, for example neutrosophic numbers have helped with neutrosophic number theory, refined neutrosophic numbers generated refined number theory and split-complex numbers generated split-complex number theory [9-20].

This has motivated many authors to study symbolic 2-plithogenic and symbolic 3-plithogenic number theoretical concepts such as congruencies, and Diophantine equations [26-36]. The generalized versions of number theoretical concepts are very applicable in other mathematical studies, especially in cryptography.

In this paper, we study the symbolic 5-plithogenic number theoretical concepts for the first time, and we illustrated many examples to clarify the novel approach.

**Main discussion**

**Definition:**

The ring of symbolic 5-plithogenic integers is defined as follows:

$$5 - SP_Z = \{x_0 + \sum_{i=1}^5 x_i P_i; x_i \in Z\}, \text{ where } P_i \times P_j = p_{\max(i,j)}, P_i^2 = P_i.$$

**Definition.**

Let  $X = x_0 + \sum_{i=1}^5 x_i P_i, Y = y_0 + \sum_{i=1}^5 y_i P_i, Z = z_0 + \sum_{i=1}^5 z_i P_i \in 5 - SP_Z$ , we say that:

- 1).  $X \setminus Y$  if there exists  $Z \in 5 - SP_Z$  such that  $X.Z = Y$ .
- 2).  $X \equiv Y(mod Z)$  if  $Z \setminus X - Y$ .
- 3).  $Z = gcd(X, Y)$  if  $Z \setminus X, Z \setminus Y$  and if  $T \setminus X, T \setminus Y$ , then  $T \setminus Z$ .
- 4).  $X, Y$  are relatively prime if  $gcd(X, Y) = 1$ .

**Theorem1.**

Let  $X = x_0 + \sum_{i=1}^5 x_i P_i, Y = y_0 + \sum_{i=1}^5 y_i P_i, Z = z_0 + \sum_{i=1}^5 z_i P_i \in 5 - SP_Z$ , then:

- 1).  $Z = gcd(X, Y)$  if and only if:

$$\begin{cases} z_0 = gcd(x_0, y_0) \\ \sum_{i=0}^j z_i = gcd\left(\sum_{i=0}^j x_i, \sum_{i=0}^j y_i\right); 1 \leq j \leq 5 \end{cases}$$

- 2).  $X \equiv Y(mod Z)$  if and only if  $\sum_{i=0}^j x_i \equiv \sum_{i=0}^j y_i (mod \sum_{i=0}^j z_i), 0 \leq j \leq 5$ .
- 3). If  $X \setminus Y$  then  $\sum_{i=0}^j x_i \setminus \sum_{i=0}^j y_i; 0 \leq j \leq 5$ .

**Theorem2.**

Let  $X = x_0 + \sum_{i=1}^5 x_i P_i, Y = y_0 + \sum_{i=1}^5 y_i P_i, Z = z_0 + \sum_{i=1}^5 z_i P_i, A = a_0 + \sum_{i=1}^5 a_i P_i, B = b_0 + \sum_{i=1}^5 b_i P_i, C = c_0 + \sum_{i=1}^5 c_i P_i \in 5 - SP_Z$ , then:

- 1). If  $Z \setminus X, Z \setminus Y$ , then  $Z \setminus AX + BY$ .
- 2). If  $Z = gcd(X, Y)$ , then there exists  $A, B \in 5 - SP_Z$  such that  $AX + BY = Z$ .
- 3). If  $X \equiv Y(mod Z)$ , then:

$$\begin{cases} X + C = Y + C (mod Z) & (I) \\ X - C = Y - C (mod Z) & (II) \\ X.C = Y.C (mod Z) & (III) \end{cases}$$

- 4).  $X$  is invertible modulo  $Z$  if and only if  $\sum_{i=0}^j x_i$  is invertible modulo  $\sum_{i=0}^j z_i; 0 \leq j \leq 5$ , and:

$$\begin{aligned} X^{-1}(mod Z) = & x_0^{-1}(mod z_0) + P_1[(x_0 + x_1)^{-1}(mod z_0 + z_1) - x_0^{-1}(mod z_0)] + \\ & P_2[(x_0 + x_1 + x_2)^{-1}(mod z_0 + z_1 + z_2) - (x_0 + x_1)^{-1}(mod z_0 + z_1)] + P_3[(x_0 + x_1 + \\ & x_2 + x_3)^{-1}(mod z_0 + z_1 + z_2 + z_3) - (x_0 + x_1 + x_2)^{-1}(mod z_0 + z_1 + z_2)] + \\ & P_4[(x_0 + x_1 + x_2 + x_3 + x_4)^{-1}(mod z_0 + z_1 + z_2 + z_3 + z_4) - (x_0 + x_1 + x_2 + \\ & x_3)^{-1}(mod z_0 + z_1 + z_2 + z_3)] + P_5[(x_0 + x_1 + x_2 + x_3 + x_4 + x_5)^{-1}(mod z_0 + z_1 + \\ & z_2 + z_3 + z_4 + z_5) - (x_0 + x_1 + x_2 + x_3 + x_4)^{-1}(mod z_0 + z_1 + z_2 + z_3 + z_4)]. \end{aligned}$$

**Theorem3.**

Let  $AX + BY = C$  be symbolic 5-plithogenic Diophantine equation in two variables,  $A, B, C, X, Y \in 5 - SP_Z$ , hence it is solvable if and only if:

$$\sum_{i=0}^j a_i \sum_{i=0}^j x_i + \sum_{i=0}^j b_i \sum_{i=0}^j y_i = \sum_{i=0}^j c_i; 0 \leq j \leq 5 \quad \text{are solvable, i.e.}$$

$$gcd(\sum_{i=0}^j a_i, \sum_{i=0}^j b_i) \setminus \sum_{i=0}^j c_i; 0 \leq j \leq 5.$$

**Theorem4.**

Let  $X = x_0 + \sum_{i=1}^5 x_i p_i \in 5 - SP_Z$ , then:

$$\begin{aligned}
 X^n = x_0^n + P_1 \left[ \left( \sum_{i=0}^1 x_i \right)^n - x_0^n \right] + P_2 \left[ \left( \sum_{i=0}^2 x_i \right)^n - \left( \sum_{i=0}^1 x_i \right)^n \right] \\
 + P_3 \left[ \left( \sum_{i=0}^3 x_i \right)^n - \left( \sum_{i=0}^2 x_i \right)^n \right] + P_4 \left[ \left( \sum_{i=0}^4 x_i \right)^n - \left( \sum_{i=0}^3 x_i \right)^n \right] \\
 + P_5 \left[ \left( \sum_{i=0}^5 x_i \right)^n - \left( \sum_{i=0}^4 x_i \right)^n \right]
 \end{aligned}$$

**Theorem5.**

$(X, Y, Z)$  is a symbolic 5-plithogenic Pythagoras triple i.e. it is a solution of the non linear Diophantine equation  $X^2 + Y^2 = Z^2$  , if and only if  $(\sum_{i=0}^j x_i, \sum_{i=0}^j y_i, \sum_{i=0}^j z_i); 0 \leq j \leq 5$  is a Pythagoras triple in  $Z$ .

**Theorem6.**

$(X, Y, Z, T)$  is a symbolic 5-plithogenic Pythagoras quadruple i.e. it is a solution of the non linear Diophantine equation  $X^2 + Y^2 + Z^2 = T^2$ , if and only if  $(\sum_{i=0}^j x_i, \sum_{i=0}^j y_i, \sum_{i=0}^j z_i, \sum_{i=0}^j t_i); 0 \leq j \leq 5$  is a Pythagoras quadruple in  $Z$ .

**Proof of theorem1.**

1). We put

$$\begin{aligned}
 Z = z_0 + \sum_{i=1}^5 z_i P_i, z_0 = gcd(x_0, y_0), \sum_{i=1}^1 z_i = gcd \left( \sum_{i=1}^1 x_i, \sum_{i=1}^1 y_i \right), \sum_{i=1}^2 z_i \\
 = gcd \left( \sum_{i=1}^2 x_i, \sum_{i=1}^2 y_i \right) \\
 \sum_{i=1}^3 z_i = gcd \left( \sum_{i=1}^3 x_i, \sum_{i=1}^3 y_i \right), \sum_{i=1}^4 z_i = gcd \left( \sum_{i=1}^4 x_i, \sum_{i=1}^4 y_i \right), \sum_{i=1}^5 z_i = gcd \left( \sum_{i=1}^5 x_i, \sum_{i=1}^5 y_i \right)
 \end{aligned}$$

Assume that  $T = t_0 + \sum_{i=1}^5 t_i P_i$  with  $T \setminus X, T \setminus Y$ , hence:

$$\begin{cases}
 \sum_{i=0}^j z_i \setminus \sum_{i=0}^j x_i, \sum_{i=0}^j z_i \setminus \sum_{i=0}^j y_i; 0 \leq j \leq 5 \\
 \sum_{i=0}^j t_i \setminus \sum_{i=0}^j x_i, \sum_{i=0}^j t_i \setminus \sum_{i=0}^j y_i; 0 \leq j \leq 5
 \end{cases}$$

So that  $\sum_{i=0}^j t_i \setminus \sum_{i=0}^j z_i; 0 \leq j \leq 5$ , hence  $T \setminus Z$  and  $Z = gcd(X, Y)$ .

2).  $X \equiv Y \pmod{Z}$  if and only if  $Z \setminus X - Y$ , which is equivalent to

$$\sum_{i=0}^j z_i \setminus \sum_{i=0}^j (x_i - y_i); 0 \leq j \leq 5, \text{ hence } \sum_{i=0}^j x_i \equiv \sum_{i=0}^j y_i \pmod{\sum_{i=0}^j z_i}; 0 \leq j \leq 5.$$

3). Assume that  $X \setminus Y$ , hence:

$$\left\{ \begin{array}{l} x_0 z_0 = y_0 \quad (1) \\ x_0 z_1 + x_1 z_0 + x_1 z_1 = y_1 \quad (2) \\ x_0 z_2 + x_1 z_2 + x_2 z_2 + x_2 z_0 + x_2 z_1 = y_2 \quad (3) \\ x_0 z_3 + x_1 z_3 + x_2 z_3 + x_3 z_3 + x_3 z_0 + x_3 z_1 + x_3 z_2 = y_3 \quad (4) \\ x_0 z_4 + x_1 z_4 + x_2 z_4 + x_3 z_4 + x_4 z_4 + x_4 z_0 + x_4 z_1 + x_4 z_2 + x_4 z_3 = y_4 \quad (5) \\ x_0 z_5 + x_1 z_5 + x_2 z_5 + x_3 z_5 + x_4 z_5 + x_5 z_5 + x_5 z_0 + x_5 z_1 + x_5 z_2 + x_5 z_3 + x_5 z_4 = y_5 \quad (6) \end{array} \right.$$

By adding (1) + (2), (1) + (2) + (3), (1) + (2) + (3) + (4), (1) + (2) + (3) + (4) + (5), (1) + (2) + (3) + (4) + (5) + (6), we get:

$$\left\{ \begin{array}{l} x_0 z_0 = y_0 \\ \sum_{i=1}^1 x_i \sum_{i=1}^1 z_i = \sum_{i=1}^1 y_i \\ \sum_{i=1}^2 x_i \sum_{i=1}^2 z_i = \sum_{i=1}^2 y_i \\ \sum_{i=1}^3 x_i \sum_{i=1}^3 z_i = \sum_{i=1}^3 y_i \\ \sum_{i=1}^4 x_i \sum_{i=1}^4 z_i = \sum_{i=1}^4 y_i \\ \sum_{i=1}^5 x_i \sum_{i=1}^5 z_i = \sum_{i=1}^5 y_i \end{array} \right.$$

Which means that  $\sum_{i=0}^j x_i \setminus \sum_{i=0}^j y_i; 0 \leq j \leq 5$

**Example on theorem1.**

Take  $X = 3 + 2P_1 + 2P_2 + P_3 - P_4 + 4P_5, Y = 6 + P_1 + P_2 - P_3 - P_4 + 2P_5$

$$\left\{ \begin{array}{l} gcd(x_0, y_0) = gcd(3,6) = 3 \\ gcd(x_0 + x_1, y_0 + y_1) = gcd(5,7) = 1 \\ gcd(x_0 + x_1 + x_2, y_0 + y_1 + y_2) = gcd(7,7) = 7 \\ gcd(x_0 + x_1 + x_2 + x_3, y_0 + y_1 + y_2 + y_3) = gcd(8,7) = 1 \\ gcd(x_0 + x_1 + x_2 + x_3 + x_4, y_0 + y_1 + y_2 + y_3 + y_4) = gcd(7,6) = 1 \\ gcd(x_0 + x_1 + x_2 + x_3 + x_4 + x_5, y_0 + y_1 + y_2 + y_3 + y_4 + y_5) = gcd(11,8) = 1 \end{array} \right.$$

Thus

$$z_0 = 3, z_1 = 1 - 3 = -2, z_2 = 7 - 1 = 6, z_3 = 1 - 7 = -6, z_4 = 1 - 1 = 0, z_5 = 1 - 1 = 0, \text{ hence:}$$

$$Z = \gcd(X, Y) = 3 - 2P_1 + 6P_2 - 6P_3$$

For  $L = 1 + P_1 - P_2 + 2P_5$ , we can see:

$L \setminus X - Y$ , that is because:

$$\begin{cases} 1 \setminus -3 \\ 2 \setminus -2 \\ 1 \setminus -1, \text{ thus } X \equiv Y \pmod{L}. \\ 1 \setminus 1 \\ 3 \setminus 3 \end{cases}$$

**Proof of theorem 2.**

1). Assume that  $Z \setminus X, Z \setminus Y$ , then we get:

$$\sum_{i=0}^j z_i \setminus \sum_{i=0}^j x_i, \text{ and } \sum_{i=0}^j z_i \setminus \sum_{i=0}^j y_i ; 0 \leq j \leq 5.$$

So that  $\sum_{i=0}^j z_i \setminus (\sum_{i=0}^j a_i \sum_{i=0}^j x_i + \sum_{i=0}^j b_i \sum_{i=0}^j y_i)$  for  $0 \leq j \leq 5$  and  $Z \setminus AX + BY$ .

2). Assume that  $Z = \gcd(X, Y)$ , then  $\sum_{i=0}^j z_i = \gcd(\sum_{i=0}^j x_i, \sum_{i=0}^j y_i)$  for all  $0 \leq j \leq 5$ .

According to Bezout's theorem, we can write:

$$\text{There exists } a_j, b_j \in Z \text{ such that } \sum_{i=0}^j z_i = a_j \sum_{i=0}^j x_i + b_j \sum_{i=0}^j y_i$$

by putting

$$A = a_0 + (a_1 - a_0)P_1 + (a_2 - a_1)P_2 + (a_3 - a_2)P_3 + (a_4 - a_3)P_4 + (a_5 - a_4)P_5,$$

$$B = b_0 + (b_1 - b_0)P_1 + (b_2 - b_1)P_2 + (b_3 - b_2)P_3 + (b_4 - b_3)P_4 + (b_5 - b_4)P_5, \text{ we}$$

get:

$$Z = AX + BY.$$

3). Assume that  $X \equiv Y \pmod{Z}$ , then:

$$\sum_{i=0}^j z_i \setminus \sum_{i=0}^j (x_i - y_i) \text{ for all } 0 \leq j \leq 5, \text{ hence:}$$

$$\begin{cases} \sum_{i=0}^j z_i \setminus \sum_{i=0}^j (x_i - c_i + c_i - y_i) \\ \sum_{i=0}^j z_i \setminus \sum_{i=0}^j (x_i + c_i - c_i + y_i) \end{cases}$$

Hence  $X \pm C = Y \pm C \pmod{Z}$ , also:

$$\sum_{i=0}^j z_i \setminus \sum_{i=0}^j (x_i - y_i) \sum_{i=0}^j c_i \text{ i.e. } \sum_{i=0}^j z_i \setminus \sum_{i=0}^j x_i \sum_{i=0}^j c_i - \sum_{i=0}^j y_i \sum_{i=0}^j c_i$$

Hence  $X.C \equiv Y.C \pmod{Z}$ .

4).  $X$  is invertible modulo  $Z$  If and only if there exists  $Y = y_0 + \sum_{i=1}^j y_i p_i \in 5 - SP_Z$  such that  $X.Y \equiv 1(mod Z)$ .

This equivalent to:

$\sum_{i=0}^j x_i \cdot \sum_{i=0}^j y_i \equiv 1(mod Z)$  for  $0 \leq j \leq 5$ , hence:

$\sum_{i=0}^j x_i$  is invertible modulo  $\sum_{i=0}^j z_i$  and:

$$\begin{aligned} X^{-1} = & x_0^{-1}(mod z_0) + P_1 \left[ \left( \sum_{i=0}^1 x_i \right)^{-1} \left( mod \sum_{i=0}^1 z_i \right) - x_0^{-1}(mod z_0) \right] \\ & + P_2 \left[ \left( \sum_{i=0}^2 x_i \right)^{-1} \left( mod \sum_{i=0}^2 z_i \right) - \left( \sum_{i=0}^1 x_i \right)^{-1} \left( mod \sum_{i=0}^1 z_i \right) \right] \\ & + P_3 \left[ \left( \sum_{i=0}^3 x_i \right)^{-1} \left( mod \sum_{i=0}^3 z_i \right) - \left( \sum_{i=0}^2 x_i \right)^{-1} \left( mod \sum_{i=0}^2 z_i \right) \right] \\ & + P_4 \left[ \left( \sum_{i=0}^4 x_i \right)^{-1} \left( mod \sum_{i=0}^4 z_i \right) - \left( \sum_{i=0}^3 x_i \right)^{-1} \left( mod \sum_{i=0}^3 z_i \right) \right] \\ & + P_5 \left[ \left( \sum_{i=0}^5 x_i \right)^{-1} \left( mod \sum_{i=0}^5 z_i \right) - \left( \sum_{i=0}^4 x_i \right)^{-1} \left( mod \sum_{i=0}^4 z_i \right) \right] \end{aligned}$$

**Example on theorem 2.**

Take:

$$\begin{aligned} X = 4 + 2P_1 - P_2 + 5P_3 - P_4 + P_5, Y = 2 + P_1 - P_2 + P_3 - P_4 + 4P_5, Z \\ = 2 - P_1 + P_2 - P_3 + P_4 + P_5, A = 1 + P_1, B = 2 - P_1 + 3P_2 \end{aligned}$$

we have  $Z \setminus X$ , that is because  $2 \setminus 4, 1 \setminus 6, 2 \setminus 4, 1 \setminus 9, 2 \setminus 8, 3 \setminus 9$ .

$Z \setminus Y$ , that I because  $2 \setminus 2, 1 \setminus 3, 2 \setminus 2, 1 \setminus 3, 2 \setminus 2, 3 \setminus 6$ .

On the other hand,

$$\begin{aligned} AX + BY = & (1 + P_1)(4 + 2P_1 - P_2 + 5P_3 - P_4 + P_5) \\ & + (2 - P_1 + 3P_2)(2 + P_1 - P_2 + P_3 - P_4 + 4P_5) \\ = & 4 + 4P_1 + 2P_1 + 2P_1 - 2P_2 - 2P_2 + 5P_3 + 5P_3 - P_4 - P_4 + P_5 + P_5 + 4 \\ & + 2P_1 - 2P_2 + 2P_3 - 2P_4 + 8P_5 - 2P_1 - P_1 + P_2 - P_3 + P_4 - 4P_5 + 6P_2 \\ & + 3P_2 - 3P_4 + 12P_5 = 8 + 7P_1 + P_2 + 14P_3 - 6P_4 + 18P_5 \end{aligned}$$

$Z \setminus AX + BY$ , that is because  $2 \setminus 8, 1 \setminus 15, 2 \setminus 16, 1 \setminus 30, 2 \setminus 24, 3 \setminus 42$ .



For  $T = 3 + 2P_1 - 2P_2 - P_3 - P_4$ , we can see:

$$gcd(X, T) = gcd(4, 3) + P_1[gcd(5, 6) - gcd(4, 3)] + P_2[gcd(3, 4) - gcd(5, 6)] + P_3[gcd(9, 2) - gcd(3, 4)] + P_4[gcd(8, 1) - gcd(9, 2)] + P_5[gcd(8, 1) - gcd(9, 1)]$$

hence  $X$  is invertible modulo  $T$ .

$$4^{-1}(\text{mod } 3) = 1, 6^{-1}(\text{mod } 5) = 1, 9^{-1}(\text{mod } 2) = 5, 8^{-1}(\text{mod } 1) = 1, 9^{-1}(\text{mod } 1) = 1, 4^{-1}(\text{mod } 3) = 1$$

$$X^{-1}(\text{mod } T) = 1 + P_1[1 - 1] + P_2[1 - 1] + P_3[5 - 1] + P_4[1 - 5] + P_5[1 - 1] = 1 + 4P_3 - 4P_4.$$

**Proof of theorem3.**

It is easy to check that  $AX + BY = C$  is equivalent to:

$$\sum_{i=0}^j a_i \sum_{i=0}^j x_i + \sum_{i=0}^j b_i \sum_{i=0}^j y_i = \sum_{i=0}^j c_i; 0 \leq j \leq 5$$

The previous six Diophantine equations are solvable if and only if:

$$gcd\left(\sum_{i=0}^j a_i, \sum_{i=0}^j b_i\right) \mid \sum_{i=0}^j c_i; 0 \leq j \leq 5$$

**Example on theorem3.**

Consider the following 5-plithogenic linear Diophantine equation in two variables:

$$(1 + P_2 - 3P_3 + 5P_4 + P_5)X + (1 - P_1 + P_2)Y = P_1 + P_2 - 3P_3 + 6P_4 + 2P_5$$

The equivalent system is:

$$\left\{ \begin{array}{l} x_0 + y_0 = 0 \quad (1) \\ \sum_{i=0}^1 x_i = 1 \quad (2) \\ \sum_{i=0}^2 x_i + \sum_{i=0}^2 y_i = 2 \quad (3) \\ -\sum_{i=0}^3 x_i + \sum_{i=0}^3 y_i = -1 \quad (4) \\ 4 \sum_{i=0}^4 x_i + \sum_{i=0}^4 y_i = 5 \quad (5) \\ 5 \sum_{i=0}^5 x_i + \sum_{i=0}^5 y_i = 7 \quad (6) \end{array} \right.$$

Equation (1) has a solution  $x_0 = y_0 = 0$ .

Equation (2) has a solution  $x_0 + x_1 = 1$ , hence  $x_1 = 1, y_1 = 0$ .

Equation (3) has a solution  $x_0 + x_1 + x_2 = 1, y_0 + y_1 + y_2 = 1$ , hence  $x_2 = 0, y_2 = 0$ .

Equation (4) has a solution  $x_0 + x_1 + x_2 + x_3 = 1, y_0 + y_1 + y_2 + y_3 = 0$ , hence  $x_3 = 0, y_3 = 0$ .

Equation (5) has a solution  $x_0 + x_1 + x_2 + x_3 + x_4 = 1, y_0 + y_1 + y_2 + y_3 + y_4 = 1$ , hence  $x_4 = 0, y_4 = 1$ .

Equation (6) has a solution  $x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 1, y_0 + y_1 + y_2 + y_3 + y_4 + y_5 = 1$ , hence  $x_5 = 0, y_5 = 1$ .

This means that  $X = P_1, Y = P_4 + P_5$  is a solution.

**proof on theorem4.**

For  $n = 1$ , it holds directly.

We assume that it is true for  $k$ , we prove it for  $k + 1$ .  $X^{k+1} = XX^k = (x_0 + \sum_{i=0}^5 x_i p_i) [x_0^k + P_1((\sum_{i=0}^1 x_i)^k - x_0^k) + P_2((\sum_{i=0}^2 x_i)^k - (\sum_{i=0}^1 x_i)^k) + P_3((\sum_{i=0}^3 x_i)^k - (\sum_{i=0}^2 x_i)^k) + P_4((\sum_{i=0}^4 x_i)^k - (\sum_{i=0}^3 x_i)^k) + P_5((\sum_{i=0}^5 x_i)^k - (\sum_{i=0}^4 x_i)^k)] = x_0^{k+1} + P_1[x_0^k(\sum_{i=0}^1 x_i)^k - x_0^{k+1} + x_1 x_0^k + x_1(\sum_{i=0}^1 x_i)^k - x_1 x_0^k] + P_2[x_0(\sum_{i=0}^2 x_i)^k - x_0(\sum_{i=0}^1 x_i)^k + x_1(\sum_{i=0}^2 x_i)^k - x_1(\sum_{i=0}^1 x_i)^k + x_2 x_0^k + x_1(\sum_{i=0}^1 x_i)^k - x_2 x_0^k + x_2(\sum_{i=0}^2 x_i)^k - x_2(\sum_{i=0}^1 x_i)^k] + P_3[x_0(\sum_{i=0}^3 x_i)^k - x_0(\sum_{i=0}^2 x_i)^k + x_1(\sum_{i=0}^3 x_i)^k - x_1(\sum_{i=0}^2 x_i)^k + x_2(\sum_{i=0}^3 x_i)^k - x_2(\sum_{i=0}^2 x_i)^k + x_2 x_0^k + x_3(\sum_{i=0}^1 x_i)^k - x_3 x_0^k + x_3(\sum_{i=0}^2 x_i)^k - x_2(\sum_{i=0}^1 x_i)^k + x_3(\sum_{i=0}^3 x_i)^k - x_2(\sum_{i=0}^2 x_i)^k] + \dots = x_0^{k+1} + P_1[(\sum_{i=0}^1 x_i)^{k+1} - x_0^{k+1}] + P_2[(\sum_{i=0}^2 x_i)^{k+1} - (\sum_{i=0}^1 x_i)^{k+1}] + \dots$

And the proof holds.

**Proof of theorem5.**

$X^2 + Y^2 = Z^2$  implies that:

$$\left\{ \begin{array}{l} x_0^2 + y_0^2 = z_0^2 \\ \left(\sum_{i=0}^1 x_i\right)^2 + \left(\sum_{i=0}^1 y_i\right)^2 = \left(\sum_{i=0}^1 z_i\right)^2 \\ \left(\sum_{i=0}^2 x_i\right)^2 + \left(\sum_{i=0}^2 y_i\right)^2 = \left(\sum_{i=0}^2 z_i\right)^2 \\ \left(\sum_{i=0}^3 x_i\right)^2 + \left(\sum_{i=0}^3 y_i\right)^2 = \left(\sum_{i=0}^3 z_i\right)^2 \\ \left(\sum_{i=0}^4 x_i\right)^2 + \left(\sum_{i=0}^4 y_i\right)^2 = \left(\sum_{i=0}^4 z_i\right)^2 \\ \left(\sum_{i=0}^5 x_i\right)^2 + \left(\sum_{i=0}^5 y_i\right)^2 = \left(\sum_{i=0}^5 z_i\right)^2 \end{array} \right.$$

Which implies the proof.

Theorem 6 can be proved by the same argument.

**Example on theorem5.**

Consider  $X = 3 + P_5, Y = 4 - P_5, Z = 5$ , we have:

$X^2 + Y^2 = Z^2$ , hence  $(X, Y, Z)$  is a Pythagoras triple.

We can see clearly that:

$$\left\{ \begin{array}{l} x_0 = 3, y_0 = 4, z_0 = 5 \\ \sum_{i=0}^1 x_i = 3, \sum_{i=0}^1 y_i = 4 \\ \sum_{i=0}^2 x_i = 3, \sum_{i=0}^2 y_i = 4 \\ \sum_{i=0}^3 x_i = 3, \sum_{i=0}^3 y_i = 4 \\ \sum_{i=0}^4 x_i = 3, \sum_{i=0}^4 y_i = 4 \\ \text{and } \sum_{i=0}^5 x_i = 3, \sum_{i=0}^5 y_i = 4, \sum_{i=0}^k z_i = 4 ; 1 \leq k \leq 5 \end{array} \right.$$

**Definition.**

Let  $X = x_0 + \sum_{i=0}^5 x_i P_i \in 5 - SP_Z$ , hence we say that  $X > 0$  if and only if  $x_0 > 0, \sum_{i=0}^k x_i > 0 ; 1 \leq k \leq 5$

For example:  $X = 3 + P_1 - P_2 + 2P_3 - P_4 - P_5 > 0$ , that is because:

$$3 > 0, 4 > 0, 3 > 0, 5 > 0, 4 > 0, 3 > 0.$$

If  $Y = y_0 + \sum_{i=0}^5 y_i P_i \in 5 - SP_Z$ , we say that  $X \geq Y$  if and only if  $x_0 \geq y_0, \sum_{i=0}^k x_i \geq \sum_{i=0}^k y_i; 1 \leq k \leq 5$ .

For  $X = 2 + P_1 + 2P_2 + 5P_3 + P_4 + 6P_5, Y = 1 + P_1 + P_2 + P_3 + 3P_4 + P_5, X \geq Y$ , that is because:

$$2 \geq 1, 3 \geq 2, 5 \geq 3, 10 \geq 4, 11 \geq 7, 17 \geq 8$$

**Definition.**

Let  $X = x_0 + \sum_{i=0}^5 x_i P_i, y = y_0 + \sum_{i=0}^5 y_i P_i \geq 0$ , hence:

$$\begin{aligned} X^Y = x_0^{y_0} + P_1 & \left[ \left( \sum_{i=0}^1 x_i \right)^{\sum_{i=0}^1 y_i} - x_0^{y_0} \right] + P_2 \left[ \left( \sum_{i=0}^2 x_i \right)^{\sum_{i=0}^2 y_i} - \left( \sum_{i=0}^1 x_i \right)^{\sum_{i=0}^1 y_i} \right] \\ & + P_3 \left[ \left( \sum_{i=0}^3 x_i \right)^{\sum_{i=0}^3 y_i} - \left( \sum_{i=0}^2 x_i \right)^{\sum_{i=0}^2 y_i} \right] \\ & + P_4 \left[ \left( \sum_{i=0}^4 x_i \right)^{\sum_{i=0}^4 y_i} - \left( \sum_{i=0}^3 x_i \right)^{\sum_{i=0}^3 y_i} \right] \\ & + P_5 \left[ \left( \sum_{i=0}^5 x_i \right)^{\sum_{i=0}^5 y_i} - \left( \sum_{i=0}^4 x_i \right)^{\sum_{i=0}^4 y_i} \right] \end{aligned}$$

**Example.**

Let  $X = 2 + 3P_1 - P_2 - P_3 - P_4 + P_5, Y = 1 + P_1 - P_2 + P_3 - P_4 + P_5$ , we have:

$$\left\{ \begin{array}{l} x_0 = 2, y_0 = 1, x_0^{y_0} = 2 \\ \sum_{i=0}^1 x_i = 5, \sum_{i=0}^1 y_i = 2, 5^2 = 25 \\ \sum_{i=0}^2 x_i = 4, \sum_{i=0}^2 y_i = 1, 4^1 = 4 \\ \sum_{i=0}^3 x_i = 3, \sum_{i=0}^3 y_i = 2, 3^2 = 9 \\ \sum_{i=0}^4 x_i = 2, \sum_{i=0}^4 y_i = 1, 2^1 = 2 \\ \sum_{i=0}^5 x_i = 3, \sum_{i=0}^5 y_i = 2, 3^2 = 9 \end{array} \right.$$

Hence  $X^Y = 2 + (25 - 2)P_1 + (4 - 25)P_2 + (9 - 4)P_3 + (2 - 9)P_4 + (9 - 2)P_5 = 2 + 23P_1 - 21P_2 + 5P_3 - 7P_4 + 7P_5$

**Definition.**

Let  $X = x_0 + \sum_{i=0}^5 x_i P_i > 0$ , then:

$$\begin{aligned} \varphi(X) = \varphi(x_0) + P_1 \left[ \varphi \left( \sum_{i=0}^1 x_i \right) - \varphi(x_0) \right] + P_2 \left[ \varphi \left( \sum_{i=0}^2 x_i \right) - \varphi \left( \sum_{i=0}^1 x_i \right) \right] \\ + P_3 \left[ \varphi \left( \sum_{i=0}^3 x_i \right) - \varphi \left( \sum_{i=0}^2 x_i \right) \right] + P_4 \left[ \varphi \left( \sum_{i=0}^4 x_i \right) - \varphi \left( \sum_{i=0}^3 x_i \right) \right] \\ + P_5 \left[ \varphi \left( \sum_{i=0}^5 x_i \right) - \varphi \left( \sum_{i=0}^4 x_i \right) \right] \end{aligned}$$

Where  $\varphi$  is Euler's function on  $Z$ .

**Example.**

Let  $X = 3 + 2P_1 + P_2 + P_3 - P_4 + P_5$ , then:

$$\begin{aligned} \varphi(x_0) = \varphi(3) = 2, \varphi \left( \sum_{i=0}^1 x_i \right) = \varphi(5) = 4, \varphi \left( \sum_{i=0}^2 x_i \right) = \varphi(6) = 2, \varphi \left( \sum_{i=0}^3 x_i \right) = \varphi(7) \\ = 6, \varphi \left( \sum_{i=0}^4 x_i \right) = \varphi(8) = 2, \varphi \left( \sum_{i=0}^5 x_i \right) = \varphi(7) = 6 \end{aligned}$$

$$\begin{aligned} \varphi(X) = 2 + (4 - 2)P_1 + (2 - 4)P_2 + (6 - 2)P_3 + (2 - 6)P_4 + (6 - 2)P_5 \\ = 2 + 2P_1 - 2P_2 + 4P_3 - 4P_4 + 4P_5 \end{aligned}$$

**Theorem.**

Let  $X = x_0 + \sum_{i=0}^5 x_i P_i, Y = y_0 + \sum_{i=0}^5 y_i P_i \in 5 - SP_Z, gcd(X, Y) = 1$  and  $X, Y > 0$ , hence:

$$X^{\varphi(Y)} \equiv 1 \pmod{Y}$$

**Proof.**

$gcd(x_0, y_0) = 1$ , hence  $x_0^{\varphi(y_0)} \equiv 1 \pmod{y_0}$ .

$gcd(\sum_{i=0}^1 x_i, \sum_{i=0}^1 y_i) = 1$ , hence  $(\sum_{i=0}^1 x_i)^{\varphi(\sum_{i=0}^1 y_i)} \equiv 1 \pmod{\sum_{i=0}^1 y_i}$

By a similar argument, we get:

$$\left(\sum_{i=0}^2 x_i\right)^{\varphi(\sum_{i=0}^2 y_i)} \equiv 1 \pmod{\sum_{i=0}^2 y_i}, \left(\sum_{i=0}^3 x_i\right)^{\varphi(\sum_{i=0}^3 y_i)} \equiv 1 \pmod{\sum_{i=0}^3 y_i}$$

$$\left(\sum_{i=0}^4 x_i\right)^{\varphi(\sum_{i=0}^4 y_i)} \equiv 1 \pmod{\sum_{i=0}^4 y_i}, \left(\sum_{i=0}^5 x_i\right)^{\varphi(\sum_{i=0}^5 y_i)} \equiv 1 \pmod{\sum_{i=0}^5 y_i}$$

This implies

$$X^{\varphi(Y)} \equiv 1 + (1 - 1)P_1 + (1 - 1)P_2 + (1 - 1)P_3 + (1 - 1)P_4 + (1 - 1)P_5 \equiv 1 \pmod{Y}.$$

**Example.**

Consider  $X = 5 + 2P_1 + 4P_2 + 2P_3 - 2P_4 + 2P_5, Y = 7 + 4P_1 - 4P_2 + P_3 + P_4 + P_5$ .

$$gcd(X, Y) = gcd(5, 7) + P_1[gcd(7, 11) - gcd(5, 7)] + P_2[gcd(11, 7) - gcd(7, 11)]$$

$$+ P_3[gcd(13, 9) - gcd(11, 7)] + P_4[gcd(11, 9) - gcd(13, 9)]$$

$$+ P_5[gcd(13, 10) - gcd(11, 9)] \equiv 1$$

Also, we have:

$$x_0 = 5, y_0 = 7, \varphi(y_0) = 6, x_0^{\varphi(y_0)} = 5^6 \equiv 1 \pmod{7}$$

$$\sum_{i=0}^1 x_i = 7, \sum_{i=0}^1 y_i = 11, \varphi\left(\sum_{i=0}^1 y_i\right) = 10, 7^{10} \equiv 1 \pmod{11}$$

$$\sum_{i=0}^2 x_i = 11, \sum_{i=0}^2 y_i = 7, \varphi\left(\sum_{i=0}^2 y_i\right) = 6, 11^6 \equiv 1 \pmod{7}$$

$$\sum_{i=0}^3 x_i = 13, \sum_{i=0}^3 y_i = 81, \varphi\left(\sum_{i=0}^3 y_i\right) = 4, 13^4 \equiv 1 \pmod{8}$$

$$\sum_{i=0}^4 x_i = 11, \sum_{i=0}^4 y_i = 91, \varphi\left(\sum_{i=0}^4 y_i\right) = 6, 11^6 \equiv 1 \pmod{9}$$

$$\sum_{i=0}^5 x_i = 13, \sum_{i=0}^5 y_i = 10, \varphi\left(\sum_{i=0}^5 y_i\right) = 4, 13^4 \equiv 1 \pmod{10}$$

Hence  $X^{\varphi(Y)} \equiv 1 \pmod{Y}$

### Remark.

We call the previous result by symbolic 5-plithogenic Euler's theorem.

### Conclusion

In this work, we have studied the properties of symbolic 5-plithogenic integers for the first time, where concepts such as symbolic 5-plithogenic divisors, congruencies, and linear Diophantine equations were handled by many theorems and examples.

Also, we have presented the conditions of symbolic 5-plithogenic Pythagoras triples and quadruples in the corresponding symbolic 5-plithogenic ring of integers.

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