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Nizar Altounji

Moustafa Mzher Ranna

Mohamed Bisher

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University of New Mexico

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Nizar Altounji[1], Moustafa Mazhar Ranneh[2] and Mohamed Bisher Zeina[3]

[1] Faculty of Science, Dept. of Mathematical Statistics, University of Aleppo, Aleppo, Syria; e-mail: nizar.altounji.94@hotmail.com

[2] Faculty of Science, Dept. of Mathematical Statistics, University of Aleppo, Aleppo, Syria; e-mail: mazhar.ranneh@gmail.com

[3] Faculty of Science, Dept. of Mathematical Statistics, University of Aleppo, Aleppo, Syria; e-mail: bisher.zeina@gmail.com

Abstract: In this paper, we introduce and define for the first time the plithogenic loss function, plithogenic risk function, plithogenic maximum likelihood function, and plithogenic posterior risk function, which form a base to easily define the plithogenic maximum a posteriori estimator (Plithogenic MAP) and its conditions, algebraic isomorphism was used through equations, and finally, we worked on an example of a plithogenic random variables sample exponentially distributed with a gamma prior for the parameter distribution and used the quadratic loss function, we found the posterior distribution of the parameter which is also a plithogenic gamma distribution and taken the posterior mean as an estimate of the parameter, such results are similar to the classical case of MAP taking in consideration the plithogenic parts which represent generalized indeterminacy.

Keywords: Plithogenic; Loss Function; Risk Function; Plithogenic Probability Density Function; Maximum Likelihood Function; Posterior Risk Function; Maximum a Posteriori Estimator.

1. Introduction

As Uncertainty and Ambiguity are more observed in real life applications, traditional probability theory methods of studying such applications became less

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effective with capturing all information needed, which led us to define and work with a new extension of probability theory that deals with the indeterminacy that we face in life applications to better understand its complexity, this new extension helps in many real life fields such as psychology, economics, mathematics, data analysis, artificial intelligence, etc.

Neutrosophic probability theory was first introduced in 1995, which deals with the probability as a triplet values, which represents the degree of truth, false, and indeterminacy, F. Smarandache presented neutrosophic sets and its applications in [1]–[7], M. Abobala and A. Hatip built the concept of Euclidean neutrosophic geometry which opens the world of many mathematical concepts such as real analysis and probability theory represented by using an algebraic structure depending on the indeterminacy element I that satisfies $I^2 = I$ [8]–[20].

Plithogenic probability theory is also an extension of classical probability theory that studies the indeterminacy related to the occurrence or non-occurrence of an event, it is a more generalized than neutrosophic since it deals with indeterminacy as two parts, F. Smarandache et al also presented symbolic plithogenic algebraic structures and plithogenic probability and statistics. Also, N. M. Taffach and A. Hatip gave a review on symbolic 2-plithogenic algebraic structures, as there are lots of papers related to plithogenic probability in many fields [21]–[37].

This paper deals with symbolic plithogenic numbers that take the form $a_P = a_0 + a_1P_1 + a_2P_2$; ${P_1}^2 = P_1$, ${P_2}^2 = P_2$, $P_1.P_2 = P_2.P_1 = P_2$, we will introduce important classical definitions in terms of symbolic plithogenic and focus on the definition of the plithogenic maximum a posteriori estimator (Plithogenic MAP), which can be considered as a generalization of our work in [38].

2. Preliminaries:

Definition 2.1

Let $R(P_1,P_2)=\{a+bP_1+cP_2; a,b,c\in R, {P_1}^2=P_1,\ {P_2}^2=P_2,\ P_1.P_2=P_2.P_1=P_2\}$ be the plithogenic field of reals. The one-dimensional AH-Isometry and its inverse are defined as following:

$$T: R(P_1,P_2)\longrightarrow R^3\ ; T(a+bP_1+cP_2)=\ (a,a+b,a+b+c)$$
$$T^{-1}: R^3\longrightarrow R(P_1,P_2)\ ; T^{-1}(a,b,c)=\ a+(b-a)P_1+(c-b)P_2$$

Definition 2.2

Let $f: R(P_1,P_2)\longrightarrow R(P_1,P_2); f=f(x_P), x_P\in R(P_1,P_2),\ f$ is called a plithogenic real function with one plithogenic variable.

Definition 2.3

Plithogenic random variable X_P is defined as follows:

$$X_P:\Omega_P\longrightarrow R(P_1,P_2);\ \Omega_P=\Omega_0\times\Omega_1(P_1)\times\Omega_2(P_2)$$
$$X_P=X_0+X_1P_1+X_2P_2;\ {P_1}^2=P_1, {P_2}^2=P_2, P_1.P_2=P_2.P_1=P_2$$

Where X_0,X_1,X_2 are classical random variables defined on $\Omega_0,\Omega_1,\Omega_2$ respectively.

Definition 2.4

Let $a_P=a_0+a_1P_1+a_2P_2,\ b_P=b_0+b_1P_1+b_2P_2\in R(P_1,P_2)$, we say that $a_P\geq_P b_P$ if:

$$a_0\geq b_0, a_0+a_1\geq b_0+b_1, a_0+a_1+a_2\geq b_0+b_1+b_2$$

3. Plithogenic Density, Plithogenic Conditional Density, Plithogenic Conditional Expectation, Plithogenic Loss and Risk Functions:

Theorem 3.1

Let X_P be a plithogenic random variable that has a probability density function $f(x_P;\Theta_P)$ with $\Theta_P=(\theta_{1P},\dots,\theta_{kP}); \theta_{iP}=\theta_{i0}+\theta_{i1}P_1+\theta_{i2}P_2\ ; i=1,2,\dots,k$ a vector of parameters, then $f(x_P;\Theta_P)$ is written in its formal plithogenic form as the following:

$$f(x_P;\Theta_P) = f(x_0;\Theta_0) + [f(x_0+x_1;\Theta_0+\Theta_1)\text{-}f(x_0;\Theta_0)]\ P_1 + [f(x_0+x_1+x_2;\Theta_0+\Theta_1+\Theta_2)\text{-}\ f(x_0+x_1;\Theta_0+\Theta_1)]P_2 \quad (1)$$

Where: $\Theta_0 = (\theta_{10},\dots,\theta_{k0})$, $\Theta_0+\Theta_1 = (\theta_{10}+\theta_{11},\dots,\theta_{k0}+\theta_{k1})$, $\Theta_0+\Theta_1+\Theta_2 = (\theta_{10}+\theta_{11}+\theta_{12},\dots,\theta_{k0}+\theta_{k1}+\theta_{k2})$

Proof:

See [37].

Theorem 3.2

Let X_P, Y_P be two plithogenic random variables, and let $f(x_P|y_P;\Theta_P)$ be the conditional probability density function of X_P given Y_P with $\Theta_P = (\theta_{1P},\dots,\theta_{kP})$; $\theta_{iP} = \theta_{i0}+\theta_{i1}P_1+\theta_{i2}P_2$; $i = 1,2,\dots,k$ a vector of parameters, then:

$$\begin{aligned} f(x_P|y_P;\Theta_P) = f(x_0|y_0;\Theta_0) &+ [f(x_0+x_1|y_0+y_1;\Theta_0+\Theta_1) - f(x_0|y_0;\Theta_0)]P_1 \\ &+ [f(x_0+x_1+x_2|y_0+y_1+y_2;\Theta_0+\Theta_1+\Theta_2) \\ &- f(x_0+y_1|x_0+y_1;\Theta_0+\Theta_1)\ ]P_2 \end{aligned} \quad (2)$$

Proof:

Straightforward using theorem 3.1.

Theorem 3.3

Let X_P, Y_P be two plithogenic random variables, and let $f(x_P|y_P)$ be the conditional probability density function of X_P given Y_P , then plithogenic conditional expectation is:

$$\begin{aligned} E(X_P|Y_P) = E(X_0|Y_0) &+ [E(X_0+X_1|Y_0+Y_1) - E(X_0|Y_0)]P_1 \\ &+ [E(X_0+X_1+X_2|Y_0+Y_1+Y_2) - E(X_0+X_1|Y_0+Y_1)\ ]P_2 \end{aligned} \quad (3)$$

Proof:

$$E(X_P|Y_P) = \int x_P f(x_P|y_P).dx_P$$

$$= \int (x_0 + x_1P_1 + x_2P_2) f(x_0 + x_1P_1 + x_2P_2|y_0 + y_1P_1 + y_2P_2).d(x_0 + x_1P_1 + x_2P_2)$$

Let's take the one-dimensional AH-Isometry:

$$T(E(X_P|Y_P)) = T\left(\int (x_0 + x_1P_1 + x_2P_2) f(x_0 + x_1P_1 + x_2P_2|y_0 + y_1P_1 + y_2P_2).d(x_0 + x_1P_1 + x_2P_2)\right)$$

$$= (\int x_0 f(x_0|y_0).dx_0\,, \int (x_0 + x_1) f(x_0 + x_1|y_0 + y_1).d(x_0 + x_1)\,, \int (x_0 + x_1 + x_2) f(x_0 + x_1 + x_2|y_0 + y_1 + y_2).d(x_0 + x_1 + x_2))$$

$$= (E(X_0|Y_0), E(X_0 + X_1|Y_0 + Y_1), E(X_0 + X_1 + X_2|Y_0 + Y_1 + Y_2))$$

Now we take T^{-1}:

$$\Longrightarrow E(X_P|Y_P) = E(X_0|Y_0) + [E(X_0 + X_1|Y_0 + Y_1) - E(X_0|Y_0)]P_1 + [E(X_0 + X_1 + X_2|Y_0 + Y_1 + Y_2) - E(X_0 + X_1|Y_0 + Y_1)\,]P_2$$

Theorem 3.4

Let $\theta_P = \theta_0 + \theta_1P_1 + \theta_2P_2$ be a plithogenic parameter of a probability distribution, and let $\hat{\theta}_P = \hat{\theta}_0 + \hat{\theta}_1P_1 + \hat{\theta}_2P_2$ be an estimation of θ_P, we can prove that the loss function of θ_P is:

$$Loss(\theta_P, \hat{\theta}_P) = Loss(\theta_0, \hat{\theta}_0) + \left[Loss(\theta_0 + \theta_1, \hat{\theta}_0 + \hat{\theta}_1) - Loss(\theta_0, \hat{\theta}_0)\right]P_1 + [Loss(\theta_0 + \theta_1 + \theta_2, \hat{\theta}_0 + \hat{\theta}_1 + \hat{\theta}_2) - Loss(\theta_0 + \theta_1, \hat{\theta}_0 + \hat{\theta}_1)]P_2 \quad (4)$$

Proof

Straightforward.

Remark

Classical loss could take the form: $Loss(a,\hat{a}) = |a - \hat{a}|$ or $Loss(a,\hat{a}) = (a - \hat{a})^2$, or other loss functions.

Theorem 3.5

Let $\theta_P = \theta_0 + \theta_1 P_1 + \theta_2 P_2$ be a plithogenic parameter of a probability distribution, and let $\hat{\theta}_P = \hat{\theta}_0 + \hat{\theta}_1 P_1 + \hat{\theta}_2 P_2$ be an estimation of θ_P, the risk function of θ_P is:

$$R(\theta_P,\hat{\theta}_P) = E(Loss(\theta_P,\hat{\theta}_P))$$
$$= R(\theta_0,\hat{\theta}_0) + \left[R(\theta_0+\theta_1,\hat{\theta}_0+\hat{\theta}_1) - R(\theta_0,\hat{\theta}_0)\right]P_1 + [R(\theta_0+\theta_1+\theta_2,\hat{\theta}_0+\hat{\theta}_1 + \hat{\theta}_2) - R(\theta_0+\theta_1,\hat{\theta}_0+\hat{\theta}_1)]P_2 \quad (5)$$

Proof

Straightforward.

Theorem 3.6

Let $\theta_P = \theta_0 + \theta_1 P_1 + \theta_2 P_2$ be a plithogenic parameter of a probability distribution, and let $\hat{\theta}_P = \hat{\theta}_0 + \hat{\theta}_1 P_1 + \hat{\theta}_2 P_2$ be an estimation of θ_P, the posterior risk function of θ_P given X_P is:

$$R(\theta_P,\hat{\theta}_P|X_P) = E(Loss(\theta_P,\hat{\theta}_P)|X_P)$$
$$= R(\theta_0,\hat{\theta}_0|X_0) + \left[R(\theta_0+\theta_1,\hat{\theta}_0+\hat{\theta}_1|X_0+X_1) - R(\theta_0,\hat{\theta}_0|X_0)\right]P_1 + [R(\theta_0 + \theta_1+\theta_2,\hat{\theta}_0+\hat{\theta}_1+\hat{\theta}_2|X_0+X_1+X_2) - R(\theta_0+\theta_1,\hat{\theta}_0 + \hat{\theta}_1|X_0+X_1)]P_2 \quad (6)$$

Proof

Straightforward.

Theorem 3.7

Let $\mathbb{X}_P = (X_{1P},\dots,X_{nP})$ be a sample of independent and identically distributed plithogenic random variables and $\Theta_P = (\theta_{1P},\dots,\theta_{kP}); \theta_{iP} = \theta_{i0} + \theta_{i1}P_1 + \theta_{i2}P_2\,; i = 1,2,\dots,k$ a vector of parameters. The plithogenic maximum likelihood function is given by:

$$L_P(\Theta_P) = f(\mathbb{x}_P; \Theta_P) = \prod_{i=1}^{n} f(x_{iP}; \Theta_P)$$
$$L_P(\Theta_P) = L(\Theta_0) + [L(\Theta_0 + \Theta_1) - L(\Theta_0)]P_1 + [L(\Theta_0 + \Theta_1 + \Theta_2) - L(\Theta_0 + \Theta_1)]P_2 \quad (7)$$

Where:

$$L(\Theta_0) = f(\mathbb{x}_0; \Theta_0)$$
$$L(\Theta_0 + \Theta_1) = f(\mathbb{x}_0 + \mathbb{x}_1; \Theta_0 + \Theta_1)$$
$$L(\Theta_0 + \Theta_1 + \Theta_2) = f(\mathbb{x}_0 + \mathbb{x}_1 + \mathbb{x}_2; \Theta_0 + \Theta_1 + \Theta_2)$$

And:

$$\Theta_0 = (\theta_{10}, \dots, \theta_{k0}),\ \Theta_0 + \Theta_1 = (\theta_{10} + \theta_{11}, \dots, \theta_{k0} + \theta_{k1}),\ \Theta_0 + \Theta_1 + \Theta_2 = (\theta_{10} + \theta_{11} + \theta_{12}, \dots, \theta_{k0} + \theta_{k1} + \theta_{k2})$$
$$\mathbb{X}_0 = (\mathbb{X}_{10}, \dots, \mathbb{X}_{n0}),\ \mathbb{X}_0 + \mathbb{X}_1 = (\mathbb{X}_{10} + \mathbb{X}_{11}, \dots, \mathbb{X}_{n0} + \mathbb{X}_{n1}),\ \mathbb{X}_0 + \mathbb{X}_1 + \mathbb{X}_2 = (\mathbb{X}_{10} + \mathbb{X}_{11} + \mathbb{X}_{12}, \dots, \mathbb{X}_{n0} + \mathbb{X}_{n1} + \mathbb{X}_{n2})$$

Proof

Taking one-dim AH-Isometry:

$$T(L_P(\Theta_P)) = T\left(\prod_{i=1}^{n} f(x_{i0} + x_{i1}P_1 + x_{i2}P_2; \Theta_0 + \Theta_1 P_1 + \Theta_2 P_2)\right)$$
$$= \prod_{i=1}^{n} f((x_{i0}; \Theta_0, x_{i0} + x_{i1}; \Theta_0 + \Theta_1, x_{i0} + x_{i1} + x_{i2}; \Theta_0 + \Theta_1 + \Theta_2))$$
$$= (\prod_{i=1}^{n} f(x_{i0}; \Theta_0), \prod_{i=1}^{n} f(x_{i0} + x_{i1}; \Theta_0 + \Theta_1), \prod_{i=1}^{n} f(x_{i0} + x_{i1} + x_{i2}; \Theta_0 + \Theta_1 + \Theta_2))$$
$$= (L(\Theta_0), L(\Theta_0 + \Theta_1), L(\Theta_0 + \Theta_1 + \Theta_2))$$

Again, taking T^{-1} yields:

$$L_P(\Theta_P) = L(\Theta_0) + [L(\Theta_0 + \Theta_1) - L(\Theta_0)]P_1 + [L(\Theta_0 + \Theta_1 + \Theta_2) - L(\Theta_0 + \Theta_1)]P_2$$

4. Plithogenic Maximum a Posteriori Estimation:

Theorem 4.1

Let $\mathbb{X}_P = (X_{1P}, \dots, X_{nP})$ be a sample of independent and identically distributed plithogenic random variables and $\Theta_P = (\theta_{1P}, \dots, \theta_{kP}); \theta_{iP} = \theta_{i0} + \theta_{i1}P_1 + \theta_{i2}P_2 \,; i = 1,2, \dots, k$ a vector of parameters, suppose Θ_P is a random variable follows a distribution that has a probability density function of $g(\Theta_P)$, which we call it a prior distribution, and suppose $L_P(\Theta_P)$ is the plithogenic maximum likelihood function of the sample, hence the posterior density function of Θ_P is given as follows:

$$f(\Theta_P|\mathbb{x}_P) \sim L(\Theta_0).\, g(\Theta_0) + [L(\Theta_0+\Theta_1).\, g(\Theta_0+\Theta_1) - L(\Theta_0).\, g(\Theta_0)]P_1 + [L(\Theta_0+\Theta_1+\Theta_2).\, g(\Theta_0+\Theta_1+\Theta_2) - L(\Theta_0+\Theta_1).\, g(\Theta_0+\Theta_1)]P_2 \qquad (9)$$

Proof

We write $f(\Theta_P|\mathbb{x}_P)$ by using equation (2) as the following:

$$f(\Theta_P|\mathbb{x}_P) = f(\Theta_0|\mathbb{x}_0) + [f(\Theta_0+\Theta_1|\mathbb{x}_0+\mathbb{x}_1) - f(\Theta_0|\mathbb{x}_0)]P_1 + [f(\Theta_0+\Theta_1+\Theta_2|\mathbb{x}_0+\mathbb{x}_1+\mathbb{x}_2) - f(\Theta_0+\Theta_1|\mathbb{x}_0+\mathbb{x}_1)\,]P_2$$

$$= \frac{f(\mathbb{x}_0|\Theta_0)g(\Theta_0)}{f(\mathbb{x}_0)} + \left[\frac{f(\mathbb{x}_0+\mathbb{x}_1|\Theta_0+\Theta_1)g(\Theta_0+\Theta_1)}{f(\mathbb{x}_0+\mathbb{x}_1)} - \frac{f(\mathbb{x}_0|\Theta_0)g(\Theta_0)}{f(\mathbb{x}_0)}\right]P_1 + \left[\frac{f(\mathbb{x}_0+\mathbb{x}_1+\mathbb{x}_2|\Theta_0+\Theta_1+\Theta_2)g(\Theta_0+\Theta_1+\Theta_2)}{f(\mathbb{x}_0+\mathbb{x}_1+\mathbb{x}_2)} - \frac{f(\mathbb{x}_0+\mathbb{x}_1|\Theta_0+\Theta_1)g(\Theta_0+\Theta_1)}{f(\mathbb{x}_0+\mathbb{x}_1)}\right]P_2$$

By taking one-dim AH-Isometry and excluding the denominators due to they don't affect the final shape of the distribution then retaking the inverse isometry we get:

$$f(\Theta_P|\mathbb{x}_P) \sim f(\mathbb{x}_0|\Theta_0)g(\Theta_0) + [f(\mathbb{x}_0+\mathbb{x}_1|\Theta_0+\Theta_1)g(\Theta_0+\Theta_1) - f(\mathbb{x}_0|\Theta_0)g(\Theta_0)]P_1 + [f(\mathbb{x}_0+\mathbb{x}_1+\mathbb{x}_2|\Theta_0+\Theta_1+\Theta_2)g(\Theta_0+\Theta_1+\Theta_2) - f(\mathbb{x}_0+\mathbb{x}_1|\Theta_0+\Theta_1)g(\Theta_0+\Theta_1)\,]P_2$$

$$\Rightarrow f(\Theta_P|\mathbb{x}_P) \sim L(\Theta_0).g(\Theta_0) + [L(\Theta_0+\Theta_1).g(\Theta_0+\Theta_1) - L(\Theta_0).g(\Theta_0)]P_1 + [L(\Theta_0+\Theta_1+\Theta_2).g(\Theta_0+\Theta_1+\Theta_2) - L(\Theta_0+\Theta_1).g(\Theta_0+\Theta_1)]P_2$$

Theorem 4.2

If $\widehat{\Theta}_P$ is the posterior mean of Θ_P for a plithogenic quadratic loss function, or $\widehat{\Theta}_P$ the posterior median for a plithogenic absolute loss function, then $\widehat{\Theta}_P$ is the estimator that minimizes the plithogenic posterior risk function.

Proof

The minimization of $R(\Theta_P, \widehat{\Theta}_P|\mathbb{X}_P)$ occurs when $\frac{d}{d\Theta_P} R(\Theta_P, \widehat{\Theta}_P|\mathbb{X}_P) = 0$, and by the equation (6) we see that this happens when:

$$\frac{d}{d\Theta_0} R(\Theta_0, \widehat{\Theta}_0|\mathbb{X}_0) = 0$$

$$\frac{d}{d(\Theta_0+\Theta_1)} (\Theta_0+\Theta_1, \widehat{\Theta}_0+\widehat{\Theta}_1|\mathbb{X}_0+\mathbb{X}_1) = 0$$

$$\frac{d}{d(\Theta_0+\Theta_1+\Theta_2)} (\Theta_0+\Theta_1+\Theta_2, \widehat{\Theta}_0+\widehat{\Theta}_1+\widehat{\Theta}_2|\mathbb{X}_0+\mathbb{X}_1+\mathbb{X}_2) = 0$$

We deal with these three conditions as we do in the classical case, i.e., for a quadratic loss function, $\widehat{\Theta}_P$ must equal:

$$\widehat{\Theta}_P = E(\Theta_P|\mathbb{X}_P) \Leftrightarrow \begin{cases} \widehat{\Theta}_0 = E(\Theta_0|\mathbb{X}_0) \\ \widehat{\Theta}_0+\widehat{\Theta}_1 = E(\Theta_0+\Theta_1|\mathbb{X}_0+\mathbb{X}_1) \\ \widehat{\Theta}_0+\widehat{\Theta}_1+\widehat{\Theta}_2 = E(\Theta_0+\Theta_1+\Theta_2|\mathbb{X}_0+\mathbb{X}_1+\mathbb{X}_2) \end{cases} \quad (10)$$

Which is the posterior mean.

And for the absolute loss function, we take the posterior median.

Example

Let $X_{1P}, \dots, X_{nP} \sim Exp(\theta_P)$, and let θ_P be an unknown plithogenic random variable that we want to estimate which is plithogenically gamma distributed with two known parameters r_P, λ_P ; $r_P = r_0 + r_1P_1 + r_2P_2, \lambda_P = \lambda_0 + \lambda_1P_1 + \lambda_2P_2$, then:

$$g(\theta_P) = \frac{{\lambda_P}^{r_P}}{(r_P - 1)!}{\theta_P}^{r_P-1}e^{-\lambda_P\theta_P}$$

We write $g(\theta_P)$ using equation (1) as following:

$$g(\theta_P) = \frac{{\lambda_0}^{r_0}}{(r_0 - 1)!}{\theta_0}^{r_0-1}e^{-\lambda_0\theta_0} + \left[\frac{(\lambda_0 + \lambda_1)^{r_0+r_1}}{(r_0 + r_1 - 1)!}(\theta_0 + \theta_1)^{r_0+r_1-1}e^{-(\lambda_0+\lambda_1)(\theta_0+\theta_1)} - \frac{{\lambda_0}^{r_0}}{(r_0 - 1)!}{\theta_0}^{r_0-1}e^{-\lambda_0\theta_0}\right]P_1 + [\frac{(\lambda_0 + \lambda_1 + \lambda_2)^{r_0+r_1+r_2}}{(r_0 + r_1 + r_2 - 1)!}(\theta_0 + \theta_1 + \theta_2)^{r_0+r_1+r_2-1}e^{-(\lambda_0+\lambda_1+\lambda_2)(\theta_0+\theta_1+\theta_2)} - \frac{(\lambda_0 + \lambda_1)^{r_0+r_1}}{(r_0 + r_1 - 1)!}(\theta_0 + \theta_1)^{r_0+r_1-1}e^{-(\lambda_0+\lambda_1)(\theta_0+\theta_1)}]P_2$$

Also:

$$L(\theta_P) = \prod_{i=1}^{n}\theta_P e^{-\theta_P\, x_{iP}}$$

Which can be written by (7) as:

$$L(\theta_P) = \prod_{i=1}^{n}\theta_0 e^{-\theta_0\, x_{i0}} + \left[\prod_{i=1}^{n}(\theta_0 + \theta_1)e^{-(\theta_0+\theta_1)(\,x_{i0}+x_{i1})} - \prod_{i=1}^{n}\theta_0 e^{-\theta_0\, x_{i0}}\right]P_1 + [\prod_{i=1}^{n}(\theta_0 + \theta_1 + \theta_2)e^{-(\theta_0+\theta_1+\theta_2)(\,x_{i0}+x_{i1}+x_{i2})} - \prod_{i=1}^{n}(\theta_0 + \theta_1)e^{-(\theta_0+\theta_1)(\,x_{i0}+x_{i1})}]P_2$$

Hence by (9):

$$f(\theta_P|\mathbb{x}_P) \sim {\theta_0}^{n}e^{-\theta_0\,\sum x_{i0}}\frac{{\lambda_0}^{r_0}}{(r_0 - 1)!}{\theta_0}^{r_0-1}e^{-\lambda_0\theta_0} +$$

$$[(\theta_0+\theta_1)^n e^{-(\theta_0+\theta_1)\sum(x_{i0}+x_{i1})}\frac{(\lambda_0+\lambda_1)^{r_0+r_1}}{(r_0+r_1-1)!}(\theta_0+\theta_1)^{r_0+r_1-1}e^{-(\lambda_0+\lambda_1)(\theta_0+\theta_1)}$$
$$-\theta_0{}^n e^{-\theta_0\sum x_{i0}}\frac{\lambda_0{}^{r_0}}{(r_0-1)!}\theta_0{}^{r_0-1}e^{-\lambda_0\theta_0}]P_1+$$
$$[(\theta_0+\theta_1+\theta_2)^n e^{-(\theta_0+\theta_1+\theta_2)\sum(x_{i0}+x_{i1}+x_{i2})}\frac{(\lambda_0+\lambda_1+\lambda_2)^{r_0+r_1+r_2}}{(r_0+r_1+r_2-1)!}(\theta_0+\theta_1$$
$$+\theta_2)^{r_0+r_1+r_2-1}e^{-(\lambda_0+\lambda_1+\lambda_2)(\theta_0+\theta_1+\theta_2)}$$
$$-(\theta_0+\theta_1)^n e^{-(\theta_0+\theta_1)\sum(x_{i0}+x_{i1})}\frac{(\lambda_0+\lambda_1)^{r_0+r_1}}{(r_0+r_1-1)!}(\theta_0+\theta_1)^{r_0+r_1-1}e^{-(\lambda_0+\lambda_1)(\theta_0+\theta_1)}]P_2$$

$$\Longrightarrow=\frac{\lambda_0{}^{r_0}}{(r_0-1)!}\theta_0{}^{n+r_0-1}e^{-\theta_0(\lambda_0+\sum x_{i0})}$$
$$+[\frac{(\lambda_0+\lambda_1)^{r_0+r_1}}{(r_0+r_1-1)!}(\theta_0+\theta_1)^{n+r_0+r_1-1}e^{-(\theta_0+\theta_1)(\lambda_0+\lambda_1+\sum(x_{i0}+x_{i1}))}$$
$$-\frac{\lambda_0{}^{r_0}}{(r_0-1)!}\theta_0{}^{n+r_0-1}e^{-\theta_0(\lambda_0+\sum x_{i0})}]P_1$$
$$+[\frac{(\lambda_0+\lambda_1+\lambda_2)^{r_0+r_1+r_2}}{(r_0+r_1+r_2-1)!}(\theta_0+\theta_1$$
$$+\theta_2)^{n+r_0+r_1+r_2-1}e^{-(\theta_0+\theta_1+\theta_2)(\lambda_0+\lambda_1+\lambda_2+\sum(x_{i0}+x_{i1}+x_{i2}))}$$
$$-\frac{(\lambda_0+\lambda_1)^{r_0+r_1}}{(r_0+r_1-1)!}(\theta_0+\theta_1)^{n+r_0+r_1-1}e^{-(\theta_0+\theta_1)(\lambda_0+\lambda_1+\sum(x_{i0}+x_{i1}))}]P_2$$

Excluding constants after taking suitable isomorphism and then taking its inverse yields to:

$$\Longrightarrow f(\theta_P|\mathbb{x}_P)\sim\theta_0{}^{n+r_0-1}e^{-\theta_0(\lambda_0+\sum x_{i0})}$$
$$+\left[(\theta_0+\theta_1)^{n+r_0+r_1-1}e^{-(\theta_0+\theta_1)(\lambda_0+\lambda_1+\sum(x_{i0}+x_{i1}))}\right.$$
$$\left.-\theta_0{}^{n+r_0-1}e^{-\theta_0(\lambda_0+\sum x_{i0})}\right]P_1$$
$$+[(\theta_0+\theta_1+\theta_2)^{n+r_0+r_1+r_2-1}e^{-(\theta_0+\theta_1+\theta_2)(\lambda_0+\lambda_1+\lambda_2+\sum(x_{i0}+x_{i1}+x_{i2}))}$$
$$-(\theta_0+\theta_1)^{n+r_0+r_1-1}e^{-(\theta_0+\theta_1)(\lambda_0+\lambda_1+\sum(x_{i0}+x_{i1}))}]P_2$$

Which yields that $\theta_P\sim Gamma(n+r_P,\lambda_P+\sum x_{iP})$

If we use the plithogenic quadratic loss function, then:

$$\widehat{\Theta}_P = \frac{n + r_P}{\lambda_P + \sum x_{iP}}$$

$$T(\widehat{\Theta}_P) = (\widehat{\Theta}_0, \widehat{\Theta}_0 + \widehat{\Theta}_1, \widehat{\Theta}_0 + \widehat{\Theta}_1 + \widehat{\Theta}_2) = T\left(\frac{n + r_P}{\lambda_P + \sum x_{iP}}\right) = \frac{T(n + r_P)}{T(\lambda_P + \sum x_{iP})}$$

$$= \frac{(n + r_0, n + r_0 + r_1, n + r_0 + r_1 + r_2)}{(\lambda_0 + \sum x_{i0}, \lambda_0 + \lambda_1 + \sum(x_{i0} + x_{i1}), \lambda_0 + \lambda_1 + \lambda_2 + \sum(x_{i0} + x_{i1} + x_{i2}))}$$

$$\Rightarrow \begin{cases} \widehat{\Theta}_0 = \dfrac{n + r_0}{\lambda_0 + \sum x_{i0}} \\ \widehat{\Theta}_0 + \widehat{\Theta}_1 = \dfrac{n + r_0 + r_1}{\lambda_0 + \lambda_1 + \sum(x_{i0} + x_{i1})} \\ \widehat{\Theta}_0 + \widehat{\Theta}_1 + \widehat{\Theta}_2 = \dfrac{n + r_0 + r_1 + r_2}{\lambda_0 + \lambda_1 + \lambda_2 + \sum(x_{i0} + x_{i1} + x_{i2})} \end{cases}$$

$$\Rightarrow \widehat{\Theta}_P = \frac{n + r_0}{\lambda_0 + \sum x_{i0}} + \left[\frac{n + r_0 + r_1}{\lambda_0 + \lambda_1 + \sum(x_{i0} + x_{i1})} - \frac{n + r_0}{\lambda_0 + \sum x_{i0}}\right] P_1 + \left[\frac{n + r_0 + r_1 + r_2}{\lambda_0 + \lambda_1 + \lambda_2 + \sum(x_{i0} + x_{i1} + x_{i2})} - \frac{n + r_0 + r_1}{\lambda_0 + \lambda_1 + \sum(x_{i0} + x_{i1})}\right] P_2$$

5. Conclusions and future research directions:

We found the formal definitions of the plithogenic maximum a posteriori estimator, posterior density function, and posterior risk function, which we used to find the estimation of parameter that has a gamma prior with an exponentially distributed plithogenic random variables, the results were similar to the classical case but takes into consideration the plithogeny, we also defined the plithogenic maximum likelihood function and other definitions related to our work, future researches will focus on studying more cases of conjugate priors and non-informative priors, also on finding formal definitions that deal with neutrosophic sample and plithogenic prior or vice versa.

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References

[1] F. Smarandache, "Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability," *ArXiv*, 2013.

[2] F. Smarandache, "Symbolic Neutrosophic Theory," *ArXiv*, 2015, doi: 10.5281/ZENODO.32078.

[3] F. Smarandache, "Introduction to Neutrosophic Statistics," *Branch Mathematics and Statistics Faculty and Staff Publications*, Jan. 2014, Accessed: Feb. 21, 2023. [Online]. Available: https://digitalrepository.unm.edu/math_fsp/33

[4] F. Smarandache *et al.*, "Introduction to neutrosophy and neutrosophic environment," *Neutrosophic Set in Medical Image Analysis*, pp. 3–29, 2019, doi: 10.1016/B978-0-12-818148-5.00001-1.

[5] F. Smarandache, "Neutrosophic theory and its applications," *Brussels*, vol. I, 2014.

[6] F. Smarandache, "Indeterminacy in neutrosophic theories and their applications," *International Journal of Neutrosophic Science*, vol. 15, no. 2, 2021, doi: 10.5281/zenodo.5295819.

[7] F. Smarandache, "(T, I, F)-Neutrosophic Structures," *Applied Mechanics and Materials*, vol. 811, 2015, doi: 10.4028/www.scientific.net/amm.811.104.

[8] M. B. Zeina and A. Hatip, "Neutrosophic Random Variables," *Neutrosophic Sets and Systems*, vol. 39, 2021, doi: 10.5281/zenodo.4444987.

[9] M. Abobala, M. B. Ziena, R. I. Doewes, and Z. Hussein, "The Representation of Refined Neutrosophic Matrices By Refined Neutrosophic Linear Functions," *International Journal of Neutrosophic Science*, vol. 19, no. 1, 2022, doi: 10.54216/IJNS.190131.

[10] M. Abobala, "On the Representation of Neutrosophic Matrices by Neutrosophic Linear Transformations," *Journal of Mathematics*, vol. 2021, 2021, doi: 10.1155/2021/5591576.

[11] M. B. Zeina and Y. Karmouta, "Introduction to Neutrosophic Stochastic Processes," *Neutrosophic Sets and Systems*, vol. 54, 2023.

[12] C. Granados and J. Sanabria, "On Independence Neutrosophic Random Variables," *Neutrosophic Sets and Systems*, vol. 47, 2021.

[13] M. Bisher Zeina and M. Abobala, "On The Refined Neutrosophic Real Analysis Based on Refined Neutrosophic Algebraic AH-Isometry," *Neutrosophic Sets and Systems*, vol. 54, 2023.

[14] C. Granados, "New Notions On Neutrosophic Random Variables," *Neutrosophic Sets and Systems*, vol. 47, 2021.

[15] A. Astambli, M. B. Zeina, and Y. Karmouta, "Algebraic Approach to Neutrosophic Confidence Intervals," *Journal of Neutrosophic and Fuzzy Systems*, vol. 5, no. 2, pp. 08–22, 2023, doi: 10.54216/JNFS.050201.

[16] M. Abobala and A. Hatip, "An Algebraic Approach to Neutrosophic Euclidean Geometry," *Neutrosophic Sets and Systems*, vol. 43, 2021.

[17] M. B. Zeina, M. Abobala, A. Hatip, S. Broumi, and S. Jalal Mosa, "Algebraic Approach to Literal Neutrosophic Kumaraswamy Probability Distribution," *Neutrosophic Sets and Systems*, vol. 54, pp. 124–138, 2023.

[18] A. Astambli, M. B. Zeina, and Y. Karmouta, "On Some Estimation Methods of Neutrosophic Continuous Probability Distributions Using One-Dimensional AH-Isometry," *Neutrosophic Sets and Systems*, vol. 53, 2023.

[19] M. B. Zeina and M. Abobala, "A novel approach of neutrosophic continuous probability distributions using AH-isometry with applications in medicine," *Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics*, pp. 267–286, Jan. 2023, doi: 10.1016/B978-0-323-99456-9.00014-3.

[20] M. Abobala and M. B. Zeina, “A Study of Neutrosophic Real Analysis by Using the One-Dimensional Geometric AH-Isometry,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. 3, no. 1, pp. 18–24, 2023, doi: 10.54216/GJMSA.030103).

[21] P. K. Singh, “Complex Plithogenic Set,” *International Journal of Neutrosophic Science*, vol. 18, no. 1, 2022, doi: 10.54216/IJNS.180106.

[22] S. Alkhazaleh, “Plithogenic Soft Set,” *Neutrosophic Sets and Systems*, 2020.

[23] N. Martin and F. Smarandache, “Introduction to Combined Plithogenic Hypersoft Sets,” *Neutrosophic Sets and Systems*, vol. 35, 2020, doi: 10.5281/zenodo.3951708.

[24] F. Smarandache, *Plithogeny, Plithogenic Set, Logic, Probability, and Statistics*. Belgium: Pons, 2018. Accessed: Feb. 23, 2023. [Online]. Available: http://arxiv.org/abs/1808.03948

[25] F. Smarandache, “Introduction to the Symbolic Plithogenic Algebraic Structures (revisited),” *Neutrosophic Sets and Systems* , vol. 53, Jan. 2023, doi: 10.5281/ZENODO.7536105.

[26] N. M. Taffach and A. Hatip, “A Review on Symbolic 2-Plithogenic Algebraic Structures,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. 5, no. 1, pp. 08–16, 2023, doi: 10.54216/GJMSA.050101.

[27] R. Ali and Z. Hasan, “An Introduction To The Symbolic 3-Plithogenic Modules,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. 6, no. 1, pp. 13–17, 2023, doi: 10.54216/GJMSA.060102.

[28] F. Smarandache, “Introduction to Plithogenic Logic as generalization of MultiVariate Logic,” *Neutrosophic Sets and Systems*, vol. 45, 2021.

[29] R. Ali and Z. Hasan, “An Introduction to The Symbolic 3-Plithogenic Vector Spaces,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. 6, no. 1, pp. 08–12, 2023, doi: 10.54216/GJMSA.060101.

[30] F. Smarandache, “Plithogenic Probability & Statistics are generalizations of MultiVariate Probability & Statistics,” *Neutrosophic Sets and Systems*, vol. 43, 2021.

[31] F. Smarandache, “Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets,” *Neutrosophic Sets and Systems*, vol. 21, pp. 153–166, 2018.

[32] M. Abdel-Basset and R. Mohamed, “A novel plithogenic TOPSIS- CRITIC model for sustainable supply chain risk management,” *J Clean Prod*, vol. 247, Feb. 2020, doi: 10.1016/J.JCLEPRO.2019.119586.

[33] M. Abdel-Basset, M. El-hoseny, A. Gamal, and F. Smarandache, “A novel model for evaluation Hospital medical care systems based on plithogenic sets,” *Artif Intell Med*, vol. 100, Sep. 2019, doi: 10.1016/J.ARTMED.2019.101710.

[34] N. M. Taffach and A. Hatip, “A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. 5, no. 1, pp. 36–44, 2023, doi: 10.54216/GJMSA.050103.

[35] F. Sultana *et al.*, “A study of plithogenic graphs: applications in spreading coronavirus disease (COVID-19) globally,” *J Ambient Intell Humaniz Comput*, p. 1, 2022, doi: 10.1007/S12652-022-03772-6.

[36] M. Abdel-Basset, R. Mohamed, A. E. N. H. Zaied, and F. Smarandache, “A hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics,” *Symmetry (Basel)*, vol. 11, no. 7, Jul. 2019, doi: 10.3390/SYM11070903.

[37] M. B. Zeina, N. Altounji, M. Abobala, and Y. Karmouta, “Introduction to Symbolic 2-Plithogenic Probability Theory,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. 7, no. 2, 2023.

[38] N. Altounji, M. B. Zeina, and M. M. Ranneh, “Introduction to Neutrosophic Bayes Estimation Theory,” *Galoitica: Journal of Mathematical Structures and Applications*, vol. Volume 7, no. 1, pp. 43–50, Sep. 2023, doi: 10.54216/GJMSA.070105.

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