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Enumeration of Neutrosophic Involutions over Finite Commutative Neutrosophic Rings

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Abstract:

A finite commutative ring involution is the multiplicative inverse of the element attribute R is the element itself. This classical characteristic of a finite commutative ring makes Neutrosophic involutions possible, which are counted, listed and assessed in this work. Assume that the Neutrosophic ring R(I) is the finite commutative ring with unity 1 over the ring R under the indeterminate $\boldsymbol{I}$. We first establish some useful necessary and sufficient conditions for the Neutrosophic components of the type $\boldsymbol{a+bI}$ is involutory in order to understand how to count Neutrosophic involutions of R(I). The behavior of the Neutrosophic composition table for identifying Neutrosophic involutions and counting the number of 1s that appear on the primary diagonal of the composition table of R(I) is also investigated in this work.

Keywords: Involutions, Neutrosophic Involutions, Neutrosophic Units, Pure Neutrosophic Involution, Neutrosophic Ring.

1. Introduction

An involution is a special element in any ring R with unity and it is a self-multiplicative inverse element under multiplication defined over R. For a finite ring R, let $\mathcal{J}n(R)$ denote the set of involutions of$R$, and $|\mathcal{J}n(R)|$ represents the number of involutions of R. Because R's involutions are systematically arranged mathematical objects that don't require any additional resources to implement, they have received a great deal of attention for their potential applications in security systems, coding-decoding systems, combinatorial designs, the creation of self-intelligent systems, etc. [1–11].Due to the fact that involutions have been crucial to the development, interpretation, and design of electronic devices. Every commutative ring with unity is known to contain at least one involution. However, the theory of finite commutative rings has two intriguing subcategories. One is cyclic rings and theother is non-cyclic rings. A ring R is called cyclic if the group $(R,+)$ is a cyclic group under addition defined by R. Otherwise, it is called a noncyclic ring. Every cyclic ring is commutative, and a finite cyclic ring with unity of order n is isomorphic to the ring Z_n, integersunder addition and multiplication modulo n.On the other hand, the rings $Z_m \times Z_n$ and $Z_n[i]$ are all noncyclic rings for every integer$m, n > 1$.

In 1987, Smarandache and Vasantha Kandasamy introduced a basic setup of the theory of Neutrosophic structures through indeterminacy I, because simply they had a natural and necessary role of I to play in the development of the Neutrosophic algebraic systems. Now a days, they rapidly become flourishing systems, because the structure R and indeterminate$I$ are needful in modern mathematical systems and many intelligent systems like Neutrosophic decision systems, Neutrosophic error detection systems, algorithms for digital communication systems [12-15];all these typesof systems employ the Neutrosophic structure $R(I)$.

Because they had a natural and essential role for indeterminate I to play in the growth of the Neutrosophic algebraic systems, Smarandache and Vasantha Kandasamy established the theory of Neutrosophic structures through indeterminacy I in 1987. They are now developing rapidly because many smart systems and quality systems, such as product quality systems, Neutrosophic virtual reality systems, and uncertainty systems [16–18], all use the Neutrosophic structure and logic, which is necessary in modern mathematical systems.

The classic Neutrosophic Rings, written by Florentin Smarandache and Vasantha Kandasamy, and published in 2006, sparked the growth of two contemporary mathematics fields that are closely related to one another: The mathematical concept of "Neutrosophic ring" and Neutrosophic logic. The value of the symbols T(True), F(False), and I(Indeterminate) and their corresponding laws was illustrated in Chapter 2 of Florentin Smarandache and Vasantha Kandasamy's book [19].Neutrosophic logic is interested in how we think in order to draw conclusions about mathematics. That book will help us in studying this paper. Now starts a simple introduction about the structure$R(I)$. Mathematically, a Neutrosophic ring is a system with the following components: a ring R, an indeterminate I, two Neutrosophic binary operations on R, and a set of axioms that the elements of R satisfy via the indeterminate I. For any ring R, there exists a new structure $R(I)$, called a Neutrosophic ring, and is engendered by R and I, which is represented by a Neutrosophic set "$R(I) = \langle R, I \rangle = \{a + bI : a, b \in R \ \ and \ \ I^2 = I\}$", where I is the indeterminate of the system with algebraic properties: $0I = 0$, $1I = I$, $I^2 = I$, and I^{-1} does not exist The Neutrosophic set $R(I) = \langle R, I \rangle$ of Neutrosophic elements of the form $a + bI$ forms a Neutrosophic ring under Neutrosophic addition "$(a + bI) + (c + dI) = (a + c) + (b + d)I$", and Neutrosophic multiplication "$(a + bI)(c + dI) = ac + (ad + bc + bd)I$", for every $a + bI$ and $c + dI$ in $R(I)$.`

In Neutrosophic algebra, the algebraic structures R and $R(I)$ areplaying dominant roles, and they are also specific mathematical tools for developing and studying many Neutrosophic research fields like QuadrupleNeutrosophic rings, Neutrosophic zero rings, Neutrosophic number theory, Neutrosophic Boolean rings, Neutrosophic vector spaces, Refined Neutrosophic rings, and so on. For example, see [20].

The purpose of this paper is to prepare and enumerate the Neutrosophic involutions in the Neutrosophic group of units of a finite commutative Neutrosophic ring with unity and to examine and compare the properties of the classical involutions in a group of units. For this first, we shall define involutions in various fields of mathematics and their other related algebraic concepts. Generally, in modern mathematics and other related computational systems, involution is a map f and it is equal to its inverse. This means that $f(f(x)) = x$ for all x in the domain of a function f. So, the involution is a bijection. For this reason, many fields in modern mathematics contain the term involution such as Group theory, Ring theory, and Vector spaces. Moreover, in the Euclidean and the Projective geometry, the involution is a reflection through the origin, and an involution is a projectivity of period 2, respectively. In mathematical logic, the operation of complement in Boolean algebra is called Boolean involution, and in classical logic, the negation that satisfies the law of double negation is called involution. Lastly, in Computer science, the XOR bitwise operation with a given value for one constraint is also an involution, and RC4 cryptographic cipher is involution, as encryption and decryption operations use the same map.

In [21-23],there is a classical and simple problem between the composition table and the corresponding finite ring R with unity1, that is, how many $1s$ appear in the principle diagonal of the composition table of$R$? This question produces the number of solutions of the equation$a^2 = 1$ in a finite ring R with unity1. This paper extends this procedure to Neutrosophic rings $R(I)$ for enumerating Neutrosophic involutions, and we shall show that the Neutrosophic involutions to the Neutrosophic ring $R(I)$ over the finite commutative R with unity 1come in a multiple of four through the relations $\mathcal{J}n(R(I)) = \mathcal{J}n(R) \cup (1-2I)\mathcal{J}n(R)$ and $|\mathcal{J}n(R(I))| = 2|\mathcal{J}n(R)|$.

2. Properties of Finite Neutrosophic Fields

This section introduces the concept of Neutrosophic involution and shows how to determine the number of such Neutrosophic involutions. Recall that the element a in R is involution if $a^2 = 1$, and the set of involutions of R is $\mathcal{J}n(R)$ and notated as $\mathcal{J}n(R) = \{a \in R : a = a^{-1}\}$. For conveniently, it can be defined as$\mathcal{J}n(R) = \{a \in R : a^2 = 1\}$. For example, $\mathcal{J}n(Z_8) = \{1,3,5,7\}$, $\mathcal{J}n(Z_{10}) = \{1,9\}$, and $\mathcal{J}n(Z_{12}) = \{1,5,7,11\}$. Consequently, any undefined notions and results of classical involutions are standard as in [21].

Our next definition provides a considerably more efficient variant of this classical involution of a finite commutative ring R with unity 1.

Definition 2.1. We say that a Neutrosophic element $a + bI$ of a Neutrosophic ring $R(I)$ is a Neutrosophic involution if$(a + bI)^2 = 1$, where $1 = 1 + 0I$ is the unity in$R(I)$.

The set of Neutrosophic involutions of $R(I)$ is denoted by $\mathcal{J}n(R(I))$ with the conditions

(1) $\mathcal{J}n(R) \subseteq \mathcal{J}n(R(I))$

(2) $\mathcal{J}n(R(I)) = \mathcal{J}n(U(R)) \cup \mathcal{J}n\left(U(R(I))\right)$,

where $U(R)$ and $U(R(I))$ are units and Neutrosophic units of R and $R(I)$, respectively.

Now we begin our discussion with two simple examples.

Example 2.2. Neutrosophic involution, by definition, the involutions of the Neutrosophic ring $Z_3(I) = \{a + bI: a, b \in Z_3 \;\; and \;\; I^2 = I\}$ is the set $\mathcal{I}n(Z_3(I)) = \{1,2,1-2I,2-I\}$, where $1^2 = 1$, $2^2 = 1$, $(1-2I)^2 = 1$, and $(2-I)^2 = 1$.

Example 2.3. Because $1^2 = 1$, $3^2 = 1$, $(1-2I)^2 = 1$, and $(3-2I)^2 = 1$, the involutions of the Neutrosophic ring $Z_4(I) = \{a + bI: a, b \in Z_4 \;\; and \;\; I^2 = I\}$ is the set $\mathcal{I}n(Z_3(I)) = \{1,3,1-2I,3-2I\}$.

The above examplespresent the following two confluences of involutions and Neutrosophic involutions.

(1) $(a + bI)^2 = 1$ in $R(I)$ if and only if $a^2 = 1$ in R.

(2) $a + bI$ is a Neutrosophic involution implies $b + aI$ need not be a Neutrosophic involution, and vice versa.

These two confluences proposed the following necessary and sufficient conditions on a and b for $a + bI$ is a Neutrosophic involution.

Theorem 2.4. A necessary and sufficient condition for the Neutrosophic element $a + bI$ is a Neutrosophic involution in $R(I)$ is $(a - 2aI)^2 = a^2$.

Proof. Let$a + bI$ be a nonzero element in $R(I)$. Then there exists $(a + bI)^2$ in $R(I)$ such that

$a + bI$ be a Neutrosophic involution in $R(I) \Leftrightarrow (a + bI)^2 = 1$

$\Leftrightarrow a^2 + b^2 I + 2abI = 1$

$\Leftrightarrow a^2 = 1$, and $b^2 + 2ab = 0$

$\Leftrightarrow a^2 = 1$, and $b(b + 2a) = 0$ in R.

Let us starts two cases on the element b in R.

Case 1. Suppose $b = 0$ in R. Then the Neutrosophic form reduces to classical form. This case concludes that

$a + bI$ be a Neutrosophic involution in $R(I) \Leftrightarrow a^2 = 1$, and $b = 0$ in R.

Case 1. Suppose $b \neq 0$ in R. Then

$a + bI$ be a Neutrosophic involution in $R(I) \Leftrightarrow a^2 = 1$, and $b + 2a = 0$ in R.

Therefore, $a^2 = 1$, and $b = -2a$ in R. These two conditions confirm that

$(a - 2aI)^2 = a^2,$

and clearly $a^2 = 1$ in R if and only if $(-2a)^2 \neq 1$ in R. Hence, we end up with $a - 2aI$as a Neutrosophic involution in $R(I)$ whenever a is an involution in R. This completes the proof. ■

Corollary 2.5. There is no Neutrosophic involution of the form $a + bI$, $b \neq 0$ in $R(I)$ if and only if $char(R)$ is 2. In other words, $\mathcal{I}n(R) = \mathcal{I}n(R(I))$ if and only if $char(R)$ is 2.

Proof. The widely recognized outcome makes it abundantly clear that

$char(R)$ is $2 \Leftrightarrow -2 \notin R$

$\Leftrightarrow -2aI \notin R(I)$

$\Leftrightarrow a - 2aI \notin R(I)$

$\Leftrightarrow (a - 2aI)^2 \neq a^2$ in $R(I)$.

By the Theorem [2.4],we clear that there is no Neutrosophic involution of the form $a + bI$, $b \neq 0$ in $R(I)$ if and only if $char(R)$ is 2. Hence, we conclude that

$$\mathcal{J}n(R) = \mathcal{J}n\big(R(I)\big) \text{ if and only if } char(R) \text{ is } 2,$$

because $\mathcal{J}n(R) \subseteq \mathcal{J}n\big(R(I)\big)$■

The next example establishes the correctness of the above result.

Example 2.6. Since$\mathbb{F}_4 = \{0,1,\alpha,1+\alpha: \alpha^2+\alpha+1=0\}$ is a field of characteristic 2. So, there exists a Neutrosophic field $\mathbb{F}_4(I)$same characteristic 2such that

$$\mathbb{F}_4(I) = \{a + bI: a, b \in \mathbb{F}_4 \;\; and \;\; I^2 = I\}.$$

Obviously, $\mathcal{J}n\big(\mathbb{F}_4(I)\big) = \mathcal{J}n(\mathbb{F}_4)$ because $(a+bI)^2 = 1$ in $\mathbb{F}_4(I)$ if and only if $b = 0$ in $\mathbb{F}_4$.

In the classical ring theory, it is well known that $\mathcal{J}n(R) \subseteq U(R)$ and $U(R) \nsubseteq \mathcal{J}n(R)$ for any finite commutative ring that R with unity, and similar manner, in the theory of Neutrosophic rings, these subset inclusions are both true, that is, $\mathcal{J}n\big(R(I)\big) \subseteq U\big(R(I)\big)$ and $U\big(R(I)\big) \nsubseteq \mathcal{J}n\big(R(I)\big)$, where $U\big(R(I)\big)$ is the set of Neutrosophic units of $R(I)$. However, $\mathcal{J}n\big(R(I)\big) \subseteq U\big(R(I)\big)$ and $U\big(R(I)\big) \subseteq \mathcal{J}n\big(R(I)\big)$ are both true, that is, $U\big(R(I)\big) = \mathcal{J}n\big(R(I)\big)$ if an only if $a + bI$ is in $U\big(R(I)\big)$ with $b \neq 0$. For example, if $b \neq 0$ in $a + bI$, we have $U\big(Z_8(I)\big)$ can be written as

$$U\big(Z_8(I)\big) = \mathcal{J}n\big(Z_8(I)\big) = \{1-2I, 3-6I, 5-2I, 7-6I\},$$

and which is equal to $\mathcal{J}n\big(Z_8(I)\big)_{b\neq 0}$, because $1-2I$, ,$3-6I$, $5-2I$, and $7-6I$ are all Neutrosophic involutions with $b \neq 0$, that is

$$\mathcal{J}n\big(Z_8(I)\big)_{b\neq 0} = \{1-2I, 3-6I, 5-2I, 7-6I\}.$$

This illustration supports the following definition.

Definition 2.7.A Neutrosophic involution $a + bI$ in $R(I)$ is called *pure*Neutrosophic involution if $b \neq 0$.

The set of pure Neutrosophic involutions of $R(I)$ is denoted by of

$$\mathcal{J}n\big(R(I)\big)_{b\neq 0} = \{a + bI \in R(I): \; (a+bI)^2 = a + bI \;\; and \;\; b \neq 0\}$$

The following theorem supports this observation.

Theorem 2.8. Let R be a finite commutative ring with unity 1 and let $|R| > 2$. Then $\mathcal{J}n\big(R(I)\big) = U\big(R(I)\big)$ if and only if $a + bI$ is pure in $R(I)$.

Proof. Because of $\mathcal{J}n\big(R(I)\big) \subseteq U\big(R(I)\big)$, it is enough to prove that the other subset inclusion $U\big(R(I)\big) \subseteq \mathcal{J}n\big(R(I)\big)$. For this, we shall show that every Neutrosophic unit is a Neutrosophic involution. Suppose $a + bI$is pure in $R(I)$. Then there exists $c + dI$ in $U\big(R(I)\big)$ such that $a + bI \neq c + dI$ and

$$(a+bI)(c+dI) = 1$$
$$\Leftrightarrow ac + (bc + bd + ad)I = 1$$
$$\Leftrightarrow ac = 1 \text{ and } bc + bd + ad = 0$$
$$\Leftrightarrow ac = 1 \text{ and } bc + bd + ad + ac = 1$$
$$\Leftrightarrow ac = 1 \text{ and } (a+b)(c+d) = 1$$
$$\Leftrightarrow a = 1,\; b = 0,\; c = 1 \text{ and } d = 0 \text{ in the ring } R.$$

Consequently, $R = \{0,1\}$, and $|R| = 2$, which is a contradiction to our hypothesis that$|R| > 2$. Thus $a + bI = c + dI$ is always true in $U\big(R(I)\big)$ if and only if $b \neq 0$ in R. This implies that

$$(a+bI)(a+bI) = 1 \text{ for every } a + bI \text{ in } U\big(R(I)\big).$$
$$\Leftrightarrow (a+bI)^2 = 1 \text{ for every } a + bI \text{ in } U\big(R(I)\big).$$

$\Leftrightarrow a + bI \in \mathcal{J}n(R(I))$ for every $a + bI$ in $U(R(I))$.

Therefore, $U(R(I)) \subseteq \mathcal{J}n(R(I))$ is true in $U(R(I))$, and hence $\mathcal{J}n(R(I)) = U(R(I))$. ■

The subsequent description helps us to estimate the cardinality of $\mathcal{J}n(R(I))$.

First, we construct two tableswhich are employed Neutrosophic involutions along with the earlier results.

The first table describes Neutrosophic involutions arising in the cyclic Neutrosophic ring $Z_n(I)$ from $n = 1$ to 10.

n	$\mathcal{J}n(Z_n(I))$
1	$\emptyset$
2	$\{1\}$
3	$\{1,2\} \cup \{1 - 2I, 2 - I\}$
4	$\{1,3\} \cup \{1 - 2I, 3 - 2I\}$
5	$\{1,4\} \cup \{1 - 2I, 4 - 3I\}$
6	$\{1,5\} \cup \{1 - 2I, 5 - 4I\}$
7	$\{1,6\} \cup \{1 - 2I, 6 - 5I\}$
8	$\{1,3,5,7\} \cup \{1 - 2I, 3 - 6I, 5 - 2I, 7 - 6I\}$
9	$\{1,8\} \cup \{1 - 2I, 8 - 7I\}$
10	$\{1,3,7,9\} \cup \{1 - 2I, 3 - 6I, 7 - 4I, 9 - 8I\}$

We now turn to noncyclic Neutrosophic rings over cyclic rings for determining Neutrosophic rings. For some positive integer n, there exists finite commutative ring $Z_n[i]$ such that

$$Z_n[i] = \{z = x + iy: x, y \in Z_n \quad and \quad i^2 = -1\}$$

And also for each $Z_n[i]$ there exists Neutrosophic ring $Z_n[i, I]$ such that

$$Z_n[i, I] = \{z + z'I: z, z' \in Z_n \ [i] \quad and \quad I^2 = I\}.$$

It is clear that $Z_n[i, I]$ is also non cyclic Neutrosophic ring, because

$$Z_n[i] = Z_n + iZ_n, \text{ and } Z_n[i, I] = Z_n + iZ_n + iIZ_n.$$

Here we notice that $|Z_n| = n$, $|Z_n[i]| = n^2$and $|Z_n[i, I]| = n^4$, and for more information about $Z_n[i]$ reader refer to [21].Next, the following second table illustrates the Neutrosophic Gaussian involutions from $n = 1$ to 5.

n	$\mathcal{J}n(Z_n[i, I])$
1	$\emptyset$
2	$\{1, i\}$
3	$\{1,2\} \cup \{1 - 2I, 2 - I\}$
4	$\{1,3,1 + 2i, 3 + 2i\} \cup \{1 - 2I, 3 - 2I, (1 + 2i) - 2I, (3 + 2i) - 2I\}$
5	$\{1,4,2i, 3i\} \cup \{1 - 2I, 4 - 3I, 2i - 4iI, 3i - iI\}$

By virtueofthe above tables, there are exact powers of 2 Neutrosophic involutionsthat exist in$R(I)$, these being related to the classical involutions in$R$. Also, the collection $\mathcal{J}n(R(I))$ contains a Neutrosophic element $1 - 2I$ as an element in $R(I)$ if and only if $|R| > 2$. So, one consequence of what has just been observed is that, in those finite commutative Neutrosophic ring $R(I)$ with unity cases in which a Neutrosophic involution exists, we can now state exactly how many there are.

Theorem 2.9. Let $char(R) \neq 2$. Then $\mathcal{I}n(R(I)) = (1-2I)\mathcal{I}n(R)$, where $\mathcal{I}n(R) = \{a \in R: a^2 = 1\}$.

Proof. By the theorem [2.4], it is well known that $b = 0$ in $\mathcal{I}n(R(I))$ if and only if $a^2 = 1$ in $\mathcal{I}n(R)$ if and only if $\mathcal{I}n(R) \neq \mathcal{I}n(R(I))$. Now suppose $b \neq 0$ in $\mathcal{I}n(R(I))$. Then

$$\begin{aligned}\mathcal{I}n(R(I)) &= \{a+bI \in R(I): (a+bI)^2 = 1, b \neq 0\} \\ &= \{a+bI \in R(I): a \in \mathcal{I}n(R), and \ \ b+2a = 0\} \\ &= \{a-2aI \in R(I): a \in \mathcal{I}n(R)\} \\ &= \{a(1-2I) \in R(I): a \in \mathcal{I}n(R)\} \\ &= \mathcal{I}n(R) \cup (1-2I)\mathcal{I}n(R), \text{ since } \mathcal{I}n(R) \subseteq \mathcal{I}n(R(I)). \ \blacksquare\end{aligned}$$

The next theorem illustrates an extremely useful enumerating technique for enumerating Neutrosophic involutions, often used next results. First, we notice that $\mathcal{I}n(R) = \mathcal{I}n(R(I))$ if and only if $char(R) = 2$.

Theorem 2.10. Let $char(R) \neq 2$. Then $|\mathcal{I}n(R(I))| = 2|\mathcal{I}n(R)|$.

Proof. Let $\mathcal{I}n(R) \neq \mathcal{I}n(R(I))$. Then $char(R) \neq 2$ and $charR(I) \neq 2$ but $char(R) = charR(I)$. So there exists an element $-2 \in R$ such that $-2I \in R(I)$. Therefore, $1-2I \in R(I)$, and we have

$$(1-2I)^2 = (1-2I)(1-2I) = 1-4I+4I = 1 \text{ in } R(I).$$

This yields the order $|1-2I|$ of the Neutrosophic element $1-2I$ in $R(I)$ is 2. Using $\mathcal{I}n(R(I)) = \mathcal{I}n(R) \cup (1-2I)\mathcal{I}n(R)$ and also there is a one to one correspondence $f: \mathcal{I}n(R) \longrightarrow (1-2I)\mathcal{I}n(R)$ defined by the relation

$$f(a) = (1-2I)a$$

for every element a in $\mathcal{I}n(R)$. So that $|\mathcal{I}n(R)| = |(1-2I)\mathcal{I}n(R)|$. Hence

$$\begin{aligned}|\mathcal{I}n(R(I))| &= |\mathcal{I}n(R) \cup (1-2I)\mathcal{I}n(R)| \\ &= |\mathcal{I}n(R)| + |(1-2I)\mathcal{I}n(R)|, \text{ since } \mathcal{I}n(R) \cap (1-2I)\mathcal{I}n(R) = \emptyset. \\ &= |\mathcal{I}n(R)| + |\mathcal{I}n(R)| = 2|\mathcal{I}n(R)|. \ \blacksquare\end{aligned}$$

Let's apply the aforementioned to a concrete example now.

The cardinalities of $\mathcal{I}n(R)$ and $\mathcal{I}n(R(I))$ are shown in the following brief table.

Involutions ↓	$n = 1$ →	2	3	4	5	6	7	8	9	10
$\lvert\mathcal{I}n(Z_n)\rvert$	0	1	2	2	2	2	2	4	2	4
$\lvert\mathcal{I}n(Z_n(I))\rvert$	0	1	4	4	4	4	4	8	4	8
$\lvert\mathcal{I}n(Z_n[i])\rvert$	0	2	2	4	4	4	2	8	2	8
$\lvert\mathcal{I}n(Z_n[i,I])\rvert$	0	2	4	8	8	8	4	16	4	16

Let us attention to the fact that, in the above table, it is necessary to stipulate that $|\mathcal{I}n(R(I))| \leq |\mathcal{I}n(S(I))|$ whenever R is a cyclic ring and S is a noncyclic ring. Further attention depends on finite fields.Since only finite Neutrosophic fields $\mathbb{F}_{2^n}(I)$ of characteristic2 is of even order, and in this sense the Neutrosophic equation $(a+bI)^2 = 1$ has no Neutrosophic solution in $\mathbb{F}_{2^n}(I)$, because

$-2 \notin \mathbb{F}_{2^n}(I)$. In this case $\mathcal{J}n(\mathbb{F}_{2^n}) = \mathcal{J}n(\mathbb{F}_{2^n}(I))$. Moreover, since the finite Neutrosophic field $\mathbb{F}_{p^n}(I)$ of characteristic p has order p^{2n} for some odd prime $p$and for some positive integer n. In this system $\mathbb{F}_{p^n}(I)$, the Neutrosophic equation $(a+bI)^2 = 1$ is solvable and it has Neutrosophic solutions, because $-2 \in \mathbb{F}_{p^n}(I)$. This theorem proves that there are an infinite number of solutions to the quadratic equation$(a+bI)^2 = 1$ over a finite Neutrosophic field of odd order.

Theorem 2.11. Over the finite Neutrosophic field $\mathbb{F}_{p^n}(I)$ corresponding to the odd prime p and the integer $n \geq 1$, the Neutrosophic equation

$$(a+bI)^2 = 1$$

has exactly four solutions. In particular, $\left|\mathcal{J}n\left(\mathbb{F}_{p^n}(I)\right)\right| = 4$.

Proof. For any odd prime p, there exists field $\mathbb{F}_{p^n}$ and Neutrosophic field $\mathbb{F}_{p^n}(I)$ of odd orders p^n and p^{2n}, respectively. Classically, you always the equation $a^2 = 1$ exists $\mathbb{F}_{p^n}$ and is also factorable, like

$$(a-1)(a+1) = 0$$

in $\mathbb{F}_{p^n}$. Since $\mathbb{F}_{p^n}$ is a field with no zero divisors, we must have $a = \pm 1$. Thus, $\mathcal{J}n(\mathbb{F}_{p^n}) = \{1, -1\}$. Further, since $-2 \in \mathbb{F}_{p^n}(I)$, there exists a Neutrosophic element $1-2I$ in $\mathbb{F}_{p^n}(I)$ such that

$$(1-2I)^2 = 1$$

in $\mathbb{F}_{p^n}(I)$. Using the Theorem [2.4],

$$\begin{aligned} \mathcal{J}n\left(\mathbb{F}_{p^n}(I)\right) &= \mathcal{J}n(\mathbb{F}_{p^n}) \cup (1-2I)\mathcal{J}n(\mathbb{F}_{p^n}) \\ &= \{1,-1\} \cup (1-2I)\{1,-1\} \\ &= \{1,-1,1\cdot(1-2I), -1\cdot(1-2I)\} \\ &= \{1,-1,1-2I,-1+2I\}. \end{aligned}$$

Hence, $\left|\mathcal{J}n\left(\mathbb{F}_{p^n}(I)\right)\right| = 4$. ∎

Further on the total number of Neutrosophic involutions to the $R(I)$ over R we have the subsequent result.

Theorem 2.12. Neutrosophic involutions to the ring $R(I)$ over the finite commutative R with unity 1 come in multiple of four.

Proof. Let$R$ be any finite commutative with unity 1 and let $cha(R) \neq 2$. Then there exists at least two involutions in R, namely unity u and its additive inverse $-u$ whenever $u^2 = 1$.This means that the least number of involutions in a finite commutative ring R with unity 1 is two if and only if $cha(R) \neq 2$. Consequently,

$$\mathcal{J}n(R) = \langle u, -u : u^2 = 1 \;\; in \;\; R\rangle,$$

and similarly

$$(1-2I)\mathcal{J}n(R) = \langle (1-2I)u, -(1-2I)u : u^2 = 1 \;\; in \;\; R\rangle.$$

Therefore,

$|\mathcal{I}n(R)| = |\langle u, -u: u^2 = 1\rangle| \geq 2,$ and $|(1-2I)\mathcal{I}n(R)| = |\langle (1-2I)u, -(1-2I)u: u^2 = 1\rangle| \geq 2.$

By the Theorem [2.4],the structure $\mathcal{I}n(R(I))$ can be written as

$$\mathcal{I}n(R(I)) = \mathcal{I}n(R) \cup (1-2I)\mathcal{I}n(R)$$
$$= \langle u, -u: u^2 = 1\rangle \cup \langle (1-2I)u, -(1-2I)u: u^2 = 1\rangle.$$

This shows that $|\mathcal{I}n(R(I))| \geq (2)(2) = 4,$ and also $|\mathcal{I}n(R(I))| = 2|\mathcal{I}n(R)|$. Hence, Neutrosophic involutions to the Neutrosophic ring $R(I)$ over the finite commutative R with unity 1 comes in multiple of four. ■

Corollary 2.13. The least number of Neutrosophic involutions of$R(I)$is four if and only if $cha(R) \neq 2$.

Proof. This is easily understood based on a common observation.For any $R(I)$with$char(R(I)) \neq 2$, you always have the four Neutrosophic involutions$u, -u,$ $(1-2I)u$ and $-(1-2I)u$ whenever $u^2 = 1$in R, and viceversa. ■

3. Neutrosophic Involutions of $R(I) \times S(I)$

In this section, we give some procedures of the determination of Neutrosophic involutions of $R(I) \times S(I)$ along with the involutions of$R \times S$. It is well known that if R and S are commutative rings with unity, then their Cartesian product $R \times S$ of R and S is also commutative rig with unity under the usual component-wise addition and component-wise multiplication. So for each system$R \times S$, there exists a Neutrosophic system $R(I) \times S(I)$ such that

$$R(I) \times S(I) = \{(a+bI, c+dI): a+bI \in R(I), c+dI \in S(I)\}$$

which is a commutative Neutrosophic ring with unity (1,1) under the component-wise Neutrosophic addition and Neutrosophic multiplication.

The following basic result associates the set of Neutrosophic involutions of $R(I) \times S(I)$ to Neutrosophic involutions of $R(I)$ and$S(I)$, and this association depends on component-wise Neutrosophic multiplication.

Theorem 3.1. Let R and S be commutative rigs with the same unity 1. Then

$$\mathcal{I}n(R(I) \times S(I)) = \mathcal{I}\text{n}(R(I)) \times \mathcal{I}n(S(I)).$$

Proof. It is sufficient to prove that a Neutrosophic element of $(a+bI, c+dI)$ in of $R(I) \times S(I)$ is a Neutrosophic involution if and only if of $a+bI$ is a Neutrosophic involution in of $R(I)$, and of $c+dI$ is a Neutrosophic involution in of $S(I)$. Indeed,

$$(a+bI, c+dI)^2 = (1,1) \Leftrightarrow (a+bI, c+dI)(a+bI, c+dI) = (1,1)$$
$$\Leftrightarrow ((a+bI)^2, (c+dI)^2) = (1,1)$$
$$\Leftrightarrow (a+bI)^2 = 1 \text{ and } (,(c+dI)^2) = 1$$
$$\Leftrightarrow a+bI \in \mathcal{I}\text{n}(R(I)) \text{ and } c+dI \in \mathcal{I}\text{n}(S(I)).$$ ■

The next example presents one to one corresponding involution behavior between $\mathcal{I}n(R(I) \times S(I))$ and $\mathcal{I}\text{n}(R(I)) \times \mathcal{I}n(S(I))$ for computing their corresponding cardinalities.

Example 3.2. Consider the Neutrosophic involution structures$\mathcal{I}n(Z_4(I) \times Z_8(I))and\mathcal{I}n(Z_4(I)) \times \mathcal{I}n(Z_8(I))$, where

$\mathcal{I}n(Z_4(I) \times Z_8(I)) = \{(1,1),(1,7),(1,1-2I),(1,7-6I),(3,1),(3,7),(3,1-2I),(3,7-6I),(1-2I,1),(1-2I,7),(1-2I,1-2I),(1-2I,7-6I),(3-2I,1),(3-2I,7),(3-2I,1-2I),(3-2I,7-6I)\}$,and

$$\mathcal{I}n(Z_4(I)) \times \mathcal{I}n(Z_8(I)) = \{1,\ 3,\ 1-2I, 3-2I\} \times \{1,\ 7, 1-2I, 7-6I\}$$
$$= \{(1,1),(1,7),(1,1-2I),(1,7-6I),(3,1),(3,7),(3,1-2I),(3,7-6I),$$
$$(1-2I,1),(1-2I,7),(1-2I,1-2I),(1-2I,7-6I),(3-2I,1),$$
$$(3-2I,7),(3-2I,1-2I),(3-2I,7-6I)\}.$$

Let m and n be any two positive integers greater 1. Then $Z_m \times Z_n$ is a commutative ring with unity. So the following statement associates the set of involutions of $Z_m \times Z_n$to involutions of $Z_m$andZ_n. In the light of this basic argument, the following theorem is necessary and the proof is clear.

$$(a,b) \in \mathcal{I}n(Z_m \times Z_n) \Leftrightarrow a \in \mathcal{I}n(Z_m) \text{ and } b \in \mathcal{I}n(Z_n).$$

Now the following result of the immediate consequence of the above statement.

Theorem 3.3. Let m and n be any two positive integers greater 1. Then

$$|\mathcal{I}n(Z_m(I) \times Z_n(I))| = |\mathcal{I}n(Z_m(I))||\mathcal{I}n(Z_n(I))|.$$

Proof. Define a map f: $\mathcal{I}n(Z_m(I) \times Z_n(I)) \to \mathcal{I}n(Z_m(I)) \times \mathcal{I}n(Z_n(I))$ by the relation

$$f((a+bI, c+dI)) = \begin{cases} (a,c) & if \ \ b = 0, d = 0 \\ ((1-2I)a, (1-2I)c) & if \ \ b \neq 0, d \neq 0 \end{cases}$$

Let us suppose $b = 0$ and $d = 0$. Then there is nothing to prove because the map f: $\mathcal{I}n(Z_m(I) \times Z_n(I)) \to \mathcal{I}n(Z_m(I)) \times \mathcal{I}n(Z_n(I))$ is trivially a Neutrosophic ring isomorphism. Now we can prove that this for the case $b \neq 0$ and $d \neq 0$.

***f* is one to one**. Let $(a+bI, c+dI)$, $(a'+b'I, c'+d'I) \in \mathcal{I}n(Z_m(I) \times Z_n(I))$. Then

$$f((a+bI,c+dI)) = f((a'+b'I,c'+d'I)) \Rightarrow ((1-2I)a,(1-2I)c) = ((1-2I)a',(1-2I)c')$$
$$\Rightarrow (1-2I)a = (1-2I)a',\ (1-2I)c = (1-2I)c'$$
$$\Rightarrow a - 2aI = a' - 2a'I,\ c - 2cI = c' - 2c'I$$
$$\Rightarrow a + bI = a' + b'I,\ c + dI = c' + d'I,$$

where $b = -2a,\ b' = -2a',\ d = -2c,$ and $d' = -2c'$. So, the map f is one to one.

***f* is onto.**The range of the function f: $\mathcal{I}n(Z_m(I) \times Z_n(I)) \to \mathcal{I}n(Z_m(I)) \times \mathcal{I}n(Z_n(I))$ is defined by

$$f\left(\mathcal{I}n(Z_m(I) \times Z_n(I))\right) = \{f((a+bI,c+dI)) : (a+bI,c+dI) \in \mathcal{I}n(Z_m(I) \times Z_n(I))\}$$
$$= \{f((a+bI,c+dI)) : (a+bI,c+dI) \in \mathcal{I}n(Z_m(I) \times Z_n(I))\}$$
$$= \{((1-2I)a,(1-2I)c) : a \in \mathcal{I}n(Z_m), c \in \mathcal{I}n(Z_n)\}$$
$$= \{(1-2I)a : a \in \mathcal{I}n(Z_m)\} \times \{(1-2I)c : c \in \mathcal{I}n(Z_n)\}$$
$$= \mathcal{I}n(Z_m) \cup (1-2I)\mathcal{I}n(Z_m) \times \mathcal{I}n(Z_n) \cup (1-2I)\mathcal{I}n(Z_n)$$
$$= \mathcal{I}n(Z_m(I)) \times \mathcal{I}n(Z_n(I)),$$

where $\mathcal{I}n(Z_m(I)) = \mathcal{I}n(Z_m) \cup (1-2I)\mathcal{I}n(Z_m)$ and $\mathcal{I}n(Z_n(I)) = \mathcal{I}n(Z_n) \cup (1-2I)\mathcal{I}n(Z_n)$. Consequently the map f is onto.

***f* is a Neutrosophic ring isomorphism.** For this let $\alpha = (a + bI, c + dI)$, $\beta = (a' + b'I, c' + d'I) \in \mathcal{In}\big(Z_m(I) \times Z_n(I)\big)$, then

$$f(\alpha + \beta) = f\big((a + bI, c + dI) + (a' + b'I, c' + d'I)\big)$$

$$= f\Big(\big((a + a') + (b + b')I, (c + c') + (d + d')I\big)\Big)$$

$$= \big((1 - 2I)(a + a'), (1 - 2I)(c + c')\big)$$

$$= \big((1 - 2I)a + (1 - 2I)a', (1 - 2I)c + (1 - 2I)c'\big)$$

$$= \big((1 - 2I)a, (1 - 2I)c\big) + \big((1 - 2I)a', (1 - 2I)c'\big)$$

$$= f\big((a + bI, c + dI)\big) + f(a' + b'I, c' + d'I) = f(\alpha) + f(\beta),$$

and similarly we can show that

$$f(\alpha\beta) = f(\alpha)f(\beta) \text{ for every } \alpha \text{ and } \beta \text{ in } \mathcal{In}\big(Z_m(I) \times Z_n(I)\big).$$

Thus the map f is a Neutrosophic ring isomorphism from $\mathcal{In}\big(Z_m(I) \times Z_n(I)\big)$ onto the $\mathcal{In}\big(Z_m(I)\big) \times \mathcal{In}\big(Z_n(I)\big)$ with $f(1,1) = (1 - 2I, 1 - 2I)$, and hence

$$\mathcal{In}\big(Z_m(I) \times Z_n(I)\big) \cong \mathcal{In}\big(Z_m(I)\big) \times \mathcal{In}\big(Z_n(I)\big).$$

This identity implies that

$$\big|\mathcal{In}\big(Z_m(I) \times Z_n(I)\big)\big| = \big|\mathcal{In}\big(Z_m(I)\big)\big|\big|\mathcal{In}\big(Z_n(I)\big)\big|. \ \blacksquare$$

4.Diagonal Property of Neutrosophic Elements in $R(I)$

This section introduces the diagonal property of finite commutative Neutrosophicrings.First of all we have, for any finite commutative Neutrosophic ring $R(I)$ with unity, there exists a multiplicative composition table of all elements of $R(I)$, and this table is associated to one to one correspondence of the matrix network $R(I) \times R(I)$ with the size $|R(I)| \times |R(I)|$. Classically, it is well known that there is an element 1 at the position of the entry (a, b) in the composition table of R. Then obviously $ab = 1 = ba$ in R. So automatically there a connection between $1s$ in R and entries of the composition table of R, see [22,23].Here, we can establish same theory on to Neutrosophic rings.

Definition 4.1.A Neutrosophic ring $R(I)$ with unity 1 has diagonal property if all $1s$ appeared in the main diagonal of the composition table of $R(I)$.

In [23],the author Sunil proved the following necessary and sufficient condition for $1s$ appeared in the main diagonal of the composition table of Z_n and divisors of 24.

Theorem 4.2.[23].The multiplication table for the cyclic ring Z_n contains $1s$ only on the diagonal of the multiplicative composition table of Z_n if and only if n is a divisor of 24.

Consequently, this result also obviously true in Neutrosophic rings, that is it can be stated as follows.

Theorem 4.3. The multiplication table for the Neutrosophic cyclic ring $Z_n(I)$ contains $1s$ only on the diagonal of the multiplicative composition table if and only if n is a divisor of 24.

Subsequently, if n is not a divisor of 24, then the above results are not true. For example, n is 5 which is not a divisor of 24, then there exists a Neutrosophic ring $Z_5(I) = \{a + bI: a, b \in Z_5;\ I^2 = I\}$, which does not satisfies diagonal property, because $(2 + 0I)(3 + 0I) = 6 + 0I = 1$ under modulo 5. This failure concept concludes that the condition$b = 0$exists in the form $a + bI$. However, all elements of Neutrosophic system $Z_5(I) = \{a + bI: a, b \in Z_5;\ I^2 = I\}$ is also satisfies diagonal property whenever $b \neq 0$ in the form$a + bI$, and this success illustrates the following Neutrosophic composition table for $Z_5(I)$with$b \neq 0$.

For$b \neq 0$, the pure Neutrosophic involutions of$Z_5(I)$ can be written as

$$\mathcal{I}n\big(Z_5(I)\big)_{b\neq 0} = \mathcal{I}n\Big(U\big(Z_5(I)\big)\Big) - \mathcal{I}n\big(U(Z_5)\big) = \{1-2I, 4-3I\}.$$

$\odot_5$	$1-2I$	$4-3I$
$1-2I$	**1**	4
$4-3I$	4	**1**

Consequently the following result is more eminent for satisfying diagonal property of *any*Neutrosophic ring$R(I)$.

Theorem 4.5. The multiplication table for Neutrosophic units $U\big(R(I)\big) - U(R) = \{a + bI \in R(I): a, b \in R, and \ \ b \neq 0\}$ of any finite commutative Neutrosophic ring$R(I)$ contains $1s$ only on the diagonal of the multiplicative composition table if and only if $char(R) \neq 2$.

Proof. It is clear by theTheorem [2.8],we have

$\mathcal{I}n\big(R(I)\big) = U\big(R(I)\big)$ if and only if $a + bI$ is in $U\big(R(I)\big)$ with $b \neq 0$.

$\mathcal{I}n\big(R(I)\big) = U\big(R(I)\big)$ if and only if $a + bI$ is a pure Neutrosophic involutions of $R(I)$.

Hence, the number of 1s appear on the principal diagonal of the table of

$$U\big(R(I)\big) = \{a + bI \in R(I): a, b \in R, and \ \ b \neq 0\}$$

is equal to 2^k for some integer $k \geq 1$, because the cardinality of $U\big(R(I)\big) = \{a + bI \in R(I): a, b \in R, and \ \ b \neq 0\}$ is even and it is greater than or equal to the power of2. ■

5. Conclusions

In this paper, we have analytically studied Neutrosophic involutory behavior of the Neutrosophic elements of the finite commutative Neutrosophic ring $R(I)$. A necessary and sufficient for the Neutrosophic element$a + bI$ being a Neutrosophic involution has been obtained. From this criterion we have developed a general procedure to enumerate Neutrosophic involutions of the form $a + bI$ over $R(I)$ from given classical involutions over the corresponding finite commutative ring R. The proposed technique can be used to determine more desired Neutrosophic involutions of $R(I)$.

6. Future Work

A Neutrosophic involution over a finite commutative Neutrosophic ring $R(I)$ is an element property whose multiplicative inverse is itself. Owing to this property, we will prepare and produce techniques for enumerating Neutrosophic involutions which are applied in Computational systems like the XOR bitwise operations with a given value for one parameter with indeterminate, and develop RC4 cryptographic ciphers, further we will use these Neutrosophic involutions for studying liminalities and minimalities of Reversible Rings.

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References

[1]. Charpin, P.; Mesnager,S. .; Sarkar, S. On involutions of finite fields. *2015 IEEE International Symposium on Information Theory (ISIT)*, Hong Kong, China, 2015, pp. 186-190

[2]. Charpin, P.; Mesnager, S. .; Sarkar, S. Dickson polynomials that are involutions. Finite Fields and Applications - Fq12 - Saratoga, USA, 2015, pp.13-17.

[3]. R. S. Coulter and S. Mesnager, "Bent Functions From Involutions Over F2n," in *IEEE Transactions on Information Theory*,018, 64(4), pp. 2979-2986.

[4]. Biryukov, A. Analysis of Involutional ciphers: Khazad and Anubis in Fast Software Encryption (Lecture Notes in Computer Science. T. Johansson, Eds. Berlin, Germany: Springer.2003, 2887, pp.,45–53.

[5]. Charpin, P.; Mesnager, S. .; Sarkar, S. Involutions over the Galois field Fn 2. IEEE Trans. Inf. Theor.2016, 62, pp.2266–2276.

[6]. Zheng, D.; Yuan, M.; li, N.; Lei h,; and Xiangyong Z.Constructions of Involutions over Finite Fields. IEEE, Transactions on Information Theory, 2019, 65,pp.1-8.

[7]. Chacron,M .; Herstein, I. N.; Montgomery, S. Structure of a Certain Class of Rings with Involution. Can. J. math., 1975, 5,pp.1114-1126.

[8]. Amitsur, S. A. Rings with involution. Israel J. Math.1968, 6, pp. 99-106.

[9]. Seiler, W.M. Involution: The Formal Theory of Differential Equations and Its Applications in Computer Algebra. Springer,2009, 15, pp.1- 650.

[10]. Dugger,D. Involutions on surfaces. *J. Homotopy Relat. Struct.* **14**, 919–992 (2019).

[11]. Mahalingam, K. Involution Codes with Application to DNA Strand Design. University of South Florida 2004,pp.1-76.

[12]. Majumdar, P.Neutrosophic Sets and Its Applications to Decision Making. Computational Intelligence for Big Data Analysis, Springer International Publishing Switzerland 2015, 19, 97-115.

[13]. Amin,B.; Salama, A. A.; El-Henawy, I.M.; Mahfouz,K.;Gafar,M.G. Intelligent Neutrosophic Diagnostic System for Cardiotocography Data.Comput Intell Neurosci. 2021 doi: 10.1155/2021/6656770.

[14]. Radwan, N.M.; Hanna,W.K.ANeutrosophic Intelligent System for Heart Disease Diagnosis: Case Study in Developing Country. International Journal of Fuzzy System Applications (IJFSA), 2022, 11(1) pp.1-13.

[15]. Şahin,R. Neutrosophic Hierarchical Clustering Algorithms. Neutrosophic Sets and Systems, 2014, 2, pp.1-7.

[16]. Maissam, J.; Smarandache, F.; Broumi, S. Inspection Assignment Form for Product Quality Control Using Neutrosophic Logic. Neutrosophic Systems with Applications, 2023, 1, pp. 4–13.

[17]. Ahmed, S.; Nehal, M.; Ibrahim, E. Neutrosophic CRITIC MCDM Methodology for Ranking Factors and Needs of Customers in Product's Target Demographic in Virtual Reality Metaverse. Neutrosophic Systems with Applications, 2023, 2, pp. 55–65.

[18]. Ibrahim, M. H. An Intelligent Decision Support Model for Optimal Selection of Machine Tool under Uncertainty: Recent Trends. Neutrosophic Systems with Applications,2023, 3, pp. 35–44.

[19]. Vasantha Kandasamy W.B.; Smarandache,F. Neutrosophic rings, Hexis, Phoenix, Arizona, 2006.

[20]. Chalapathi, T.; Madhavi, L. Neutrosophic Boolean Rings. Neutrosophic Sets and Systems, 2020,30,pp. 57-66.

[21]. Chalapathi,T.; Sajana,S. Enumeration of Involutions of Finite Rings. Journal of New Theory, 2021,36, pp. 64-74.

[22]. Sunil, K. C. what is special about the divisors of 24? .Math. Mag. 2012, 85(5), pp.366–372

[23]. Sunil K. C.;Mayers,M. What Is Special about the Divisors of 12? .Mathematics Magazine, 2013,86, pp.143-146.

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