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# Research on a Class of Special Quasi TA-Neutrosophic Extended Triplet: TA-Groups

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**Abstract**. Tarski associative groupoid (TA-groupoid) and Tarski associative neutrosophic extended triplet groupoid (TA-NET-groupoid) are two interesting structures in non-associative algebra. In this paper, a new concept of TA-group is proposed based on TA-groupoid, as a special quasi TA-Neutrosophic extended triplet, its related properties are investigated and the relationship between TA-group and regular TA-groupoid is described in more detail. Moreover, the decomposition theorem of inverse TA-groupoid is proved. Finally, some concrete examples are provided to reveal that the relations among all kinds of TA-groupoids.

**Keywords:**Tarski associative groupoid (TA-groupoid); Tarski associative neutrosophic extended triplet groupoid (TA-NET-groupoid); Tarski associative group (TA-group)

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## 1. **Introduction**

Associative law is a kind of operation law describing symmetry in algebraic systems. Groups and semigroups are two typical algebraic systems which satisfy associative law ([1–3]). In recent years, with the wide application of algebraic systems in various fields ([4–6]), many kinds of non-associative algebraic structures have been studied in order to explore more generalized symmetries and operation laws in algebras. Among them, Abel-Grassmann's groupoid (AG-groupoid), Cyclic associative groupoid (CA-groupoid) and TA-groupoid have been widely discussed.

AG-groupoid([7]) is a non-associative groupoid satisfying the condition $(xy)z = (zy)x$. Based on this research, a series of AG-groupoid satisfying different conditions have been proposed([8–10]). In 1954, the term "cyclic associative law" was used in Sholander's article([11]) to represent the operating law: $(xy)z = (yz)x$. Subsequently, many scholars systematically studied the relevant algebraic structures satisfying cyclic associative law.

If a semigroup satisfies $x(yz) = x(zy)$, it can be called right commutative. On this basis, the association law is added, then the following equation holds: $(xy)z = x(zy)$ (Tarski associative law). Tarski associative law is actually a special case of generalized associative law proposed by Suschkewitsch ([12]) as early as 1929. Accordingly, Xiaohong Zhang proposed the concept of TA-groupoid in 2020 and studied its related properties([13]).

In addition, based on the relevant theory of Neutrosophic set([14]) proposed in 1995, Smarandache put forward a new algebraic structure of the neutrosophic extended triplet group (NETG) ([15]). Subsequently, domestic and foreign scholars carried out a lot of research on this basis. Among them, Xiaohong Zhang clarified some theoretical knowledge of TA-NET-groupoid in 2020 by adding local identity elements and local inverse elements to the NETGs and combining with TA-groupoid, laying a theoretical foundation for the research of related algebraic structures. Xiaogang An et al. concluded that TA-NET-groupoid is a semigroup, and explained the relationship between regular TA-groupoid and Tarski associative neutrosophic extended triplet groupoid (TA-NET-groupoid)([16]). The research of these scholars greatly promoted the further development of algebra. In order to further study the structure of regular TA-groupoid, the TA-group and inverse TA-groupoid are proposed, properties of TA-group and the relations between different TA-groupoids are studied in detail in this paper.

This paper is organized as follows. In Section 2, some basic definitions and properties of TA-groupoid and TA-NET-groupoid are recalled. In TA-groupoid, there is a special class of groupoid with several right identity elements, which we call TA-group. Thus, the concept of TA-group is put forward first in Section 3, which is followed with the discussion of some structural properties of TA-group and the relationship between TA-group and other algebraic structures. We then summarize our paper and indicate the next research direction at last.

## 2. Preliminaries

**Definition 2.1** ([13]). If a groupoid $(S, *)$ satisfies Tarski associative law: $\forall x, y, z \in S$, $(x * y) * z = x * (z * y)$, then $S$ is said as a Tarski associative groupoid (shortly TA-groupoid).

A TA-groupoid $(S, *)$ is called locally associative ([13]) if $\forall m \in S$, $(m*m)*m = m*(m*m)$. Then TA-groupoid is locally associative.

**Proposition 2.1** ([13]). Let $(S, *)$ be a TA-groupoid. Then $\forall a, b, c, d, e, f \in S$,
(1) $(a * b) * (c * d) = (a * d) * (c * b)$;
(2) $((a * b) * (c * d)) * (e * f) = (a * d) * ((e * f) * (c * b))$.

**Definition 2.2** ([15, 17]). If $S$ is a non-empty set under the binary operation $*$, for any $m \in S$, there are neut($m$) and anti($m$), s.t. $neut(m) \in S$, $anti(m) \in S$, and $m * neut(m) =$

$neut(m) * m = m$; $m * anti(m) = anti(m) * m = neut(m)$. Then $S$ is said as a neutrosophic extended triplet set.

Annotation: For any $m \in S$, neither $neut(m)$ nor $anti(m)$ is unique. Thus $\{neut(m)\}$ and $\{anti(m)\}$ are used to denote the sets of $neut(m)$ and $anti(m)$, respectively.

**Definition 2.3** ([13])**.** Let $(G, *)$ be a neutrosophic extended triplet set. If
(1) $\forall x, y \in G$, $x * y \in G$;
(2) $\forall x, y, z \in G$, $(x * y) * z = x * (z * y)$.

Then, we say that $(G, *)$ is a Tarski associative neutrosophic extended triplet groupoid (or TA-NET-groupoid). A TA-NET-groupoid satisfying the commutative law is a commutative TA-NET-groupoid.

**Theorem 2.1** ([13])**.** Let $(G, *)$ be a TA-NET-groupoid. Then $\forall w \in G$,
(1) $neut(w) * neut(w) = neut(w)$;
(2) $neut(neut(w)) = neut(w)$;
(3) $anti(neut(w)) \in \{anti(neut(w))\}, w = anti(neut(w)) * w$.

**Theorem 2.2** ([13])**.** Let $(G, *)$ be a TA-NET-groupoid. Then $\forall w \in G$, $\forall p, q \in \{anti(w)\}, \forall anti(w) \in \{anti(w)\}$,
(1) $p * (neut(w)) = neut(w) * q$;
(2) $anti(neut(w)) * anti(w) \in \{anti(w)\}$;
(3) $neut(w) * anti(q) = w * neut(q)$;
(4) $neut(p) * neut(w) = neut(w) * neut(p) = neut(w)$;
(5) $(q * neut(w)) * w = w * (neut(w) * q) = neut(w)$;
(6) $neut(q) * w = w$.

**Theorem 2.3** ([13])**.** If $(G, *)$ is a TA-NET-groupoid. $E(G)$ represents the set composed of all different neutral elements in $G$, for all $e \in E(G)$, $G(e) = \{a \in G | neut(a) = e\}$. Then,
(1) $G(e)$ is a subgroup of $G$.
(2) for $\forall e_1, e_2 \in E(G), e_1 \neq e_2 \Rightarrow G(e_1) \cap G(e_2) = \varnothing$.
(3) $G = \cup_{e \in E(G)} G(e)$.

**Theorem 2.4** ([16])**.** A TA-NET-groupoid is a semigroup.

**Definition 2.4** ([13])**.** A TA-groupoid $(G, *)$ is said to be left cancellative, if $x \in G$, $a, b \in G$, $x * a = x * b$ implies $a = b$.

**Definition 2.5** ([13])**.** A TA-groupoid $(G, *)$ is said to be a right cancellative TA-groupoid, if $x \in G$, $a, b \in G$, $a * x = b * x$ implies $a = b$. A groupoid is a cancellative TA-groupoid which is both a left and right cancellative.

**Theorem 2.5** ([13])**.** Let $(G, *)$ be a TA-groupoid. Then
(1) A left cancellative element is a right cancellative element;
(2) Two left cancellative elements are still left cancellative after * operation;
(3) A left cancellative element and a right cancellative element are left cancellative after * operation;
(4) For any $x, y \in G$, if $x * y$ is right cancellative, then $y$ is right cancellative.

**Definition 2.6** ([16])**.** Assume that $(G, *)$ is a TA-groupoid, $a \in G$. Then $a$ is a regular element of $G$ if there exists $x \in G$ such that $a * (x * a) = a$. The TA-groupoid $G$ is said to be regular if all its elements are regular.

**Definition 2.7** ([2])**.** A semigroup $S$ is said to be an inverse semigroup, if there is a unary operation $a \mapsto a^{-1}$ satisfying

$$(a^{-1})^{-1} = a, aa^{-1}a = a,$$

and for all $x, y \in S$,

$$(xx^{-1})(yy^{-1}) = (yy^{-1})(xx^{-1}).$$

**Theorem 2.6** ([2])**.** Let $S$ be a semigroup. It is an inverse semigroup iff all its elements have a unique inverse.

## 3. **TA-Group and Inverse TA-Groupoid**

In the following, we propose two new concepts of TA-group and inverse TA-groupoid, and investigate their properties and structures.

**Definition 3.1.** Let $(S, *)$ be a TA-groupoid. Then, $S$ is called a TA-(r,l)-loop, if for any $a \in S$, exist two elements $neut_{(r,l)}(a)$ and $anti_{(r,l)}(a)$ in $S$ satisfying the condition: $a * neut_{(r,l)}(a) = a$, $anti_{(r,l)}(a) * a = neut_{(r,l)}(a)$. That is, $a * (anti_{(r,l)}(a) * a) = a$.

**Definition 3.2.** Assume that $(G, *)$ is a TA-groupoid. $G$ is said to be a Tarski associative group (or simply TA-group), if
(1) there is a right identity element in $G$, that is to say, $\exists e \in G$, for all element $a \in G$, $a * e = a$;
(2) there is a certain right identity element $e \in G$, for any $a \in G$, there exists an element $a' \in G$ such that $a' * a = e$.

Obviously, by definition 3.1 and 3.2 we know that TA-group is a special TA-(r,l)-loop.

**Exmaple 3.1.** Let $G = \{1, 2, 3, 4\}$. In Table 1, the TA-group $(G, *)$ is given. And

$$1 * 1 = 1, 2 * 1 = 2, 3 * 1 = 3, 4 * 1 = 4;$$

$$1 * 1 = 1, 1 * 2 = 1, 3 * 3 = 1, 3 * 4 = 1.$$

At this time, right identity element is 1.

$$1 * 2 = 1, 2 * 2 = 2, 3 * 2 = 3, 4 * 2 = 4;$$

$$2 * 1 = 2, 2 * 2 = 2, 4 * 3 = 2, 4 * 4 = 2.$$

At this time, right identity element is 2.

TABLE 1. This is a TA-group.

| * | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| 1 | 1 | 1 | 3 | 3 |
| 2 | 2 | 2 | 4 | 4 |
| 3 | 3 | 3 | 1 | 1 |
| 4 | 4 | 4 | 2 | 2 |

**Theorem 3.1.** Let $(G, *)$ be a TA-group, $e$ is a right identity element in $G$. Then
(1) $(a, a' \in G,\ a' * a = e) \Rightarrow e * a' = a'$;
(2) $(a, a' \in G,\ a' * a = e) \Rightarrow (a * a') * (a * a') = a * a'$;
(3) $(a, a' \in G,\ a' * a = e) \Rightarrow x * (a * a' = x)$ for all $x \in G$.

*Proof.* (1) In order to obtain the conclusion, suppose that $a, a' \in G$, $a' * a = e$. We have

$$e * a' = (a' * a) * a' = a' * (a' * a) = a' * e = a'.$$

(2) If $a, a' \in G$, $a' * a = e$. Then by (1),

$$(a * a) * a' = (a * a) * (e * a') = (a * a') * (e * a) = ((a * a') * a) * e = (a * a') * a = a * (a * a');$$

$$a = a * e = a * (a' * a) = (a * a) * a'.$$

It follows that $a = (a * a) * a' = a * (a * a')$. On the other hand,

$$a' * a = a' * (a * (a * a')) = (a' * (a * a')) * a;$$

$$\begin{aligned} a' &= e * a' = (a' * a) * a' = ((a' * (a * a')) * a) * a' \\ &= (a' * (a * a')) * (a' * a) = (a' * (a * a')) * e \\ &= a' * (a * a'). \end{aligned}$$

Therefore,

$$\begin{aligned} a * a' &= ((a * a) * a') * (a' * (a * a')) = ((a * a) * (a * a')) * (a' * a') \\ &= ((a * a') * (a * a)) * (a' * a') = (a * a') * ((a' * a') * (a * a)) \\ &= (a * a') * ((a' * a) * (a * a')) = (a * a') * (e * (a * a')) \\ &= ((a * a') * (a * a')) * e = (a * a') * (a * a'). \end{aligned}$$

(3) Assume that $a, a' \in G$, $a' * a = e$. For any $x \in G$, applying (1) we get that $x * (a * a') = (x * (a * a')) * e = x * (e * (a * a')) = x * ((e * a') * a) = x * (a' * a) = x * e = x$. □

**Theorem 3.2.** Let $(G, *)$ be a TA-groupoid with right identity element. Then it is a semigroup.

*Proof.* Let $(G, *)$ be a TA-groupoid with right identity element. $e$ is right identity element in $G$, for any $a, b, c \in G$, there is,

$$\begin{aligned} a * (b * c) &= [a * (b * c)] * e \\ &= a * [e * (b * c)] \\ &= (a * b) * (e * c) \\ &= (a * c) * (e * b)(By Proposition 2.1) \\ &= a * [(e * b) * c] \\ &= a * [e * (c * b)] \\ &= [a * (c * b)] * e \\ &= a * (c * b). \end{aligned}$$

Then according to Tarski associative law, $a * (b * c) = a * (c * b) = (a * b) * c$. That is to say, $G$ satisfies associative law. So $G$ is a semigroup. □

**Theorem 3.3.** Let $(G, *)$ be a TA-group. Then it is a regular semigroup.

*Proof.* Because TA-group is a TA-groupoid with right identity element, according to Theorem 3.2, $G$ is a semigroup. Then according to definition of TA-group, there exists $x \in G$ such that $a * (x * a) = a$. So $G$ is regular semigroup. □

But not every regular semigroup is TA-group, see Example 3.2.

**Exmaple 3.2.** Let $G = \{1, 2, 3, 4\}$. In Table 2, a regular semigroup $(G, *)$ is given.

TABLE 2. This is a regular semigroup.

| $*$ | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 3 | 4 |
| 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 |

But it isn't TA-group since $(2 * 1) * 3 = 1 \neq 3 = 2 * (3 * 1)$.

**Theorem 3.4.** TA-group is TA-NET-groupoid.

*Proof.* Let $G$ be a TA-group and $e$ is right identity element of $G$. Then for all $a \in G$, there exists $x \in G$ such that $x * a = e$. That is, $a * (x * a) = a$. On the basis of Theorem 3.2, $a * (a * x) = a$. Assume that $a * x = neut(a)$ and $x = anti(a)$, there is, $a * neut(a) = a$ and $a * anti(a) = neut(a) = anti(a) * a$. Then $neut(a) * a = (a * x) * a = a * (a * x) = a$. So $neut(a) * a = a = a * neut(a)$ and $anti(a) * a = neut(a) = a * anti(a)$. So $G$ is a TA-NET-groupoid. □

But not every TA-NET-groupoid is TA-group, see Example 3.3.

**Exmaple 3.3.** Let $G = \{1, 2, 3, 4\}$. Consider a TA-NET-groupoid in Table 3.

$$neut(1) = 1, anti(1) = 1; neut(2) = 2, anti(2) = 2;$$

$$neut(3) = 3, anti(3) = 3; neut(4) = 4, anti(4) = 4.$$

Table 3. This is a TA-NET-groupoid.

| $*$ | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 3 | 2 |
| 3 | 4 | 4 | 3 | 4 |
| 4 | 4 | 4 | 4 | 4 |

But it isn't TA-group since there isn't right identity element.

**Theorem 3.5.** Let $(S, *)$ be a TA-group, $a, b, c, d, f \in S$, $e$ is the right identity element in $S$. There is,

(1) if $a * b = e$, $e$ is identity element of $a$;

(2) $((a * b) * c) * d = a * (d * (c * b))$;

(3) if $a * b = c * d$, then $a * (d^{-1} * b) = c$.

*Proof.* (1) If $a * b = e$, i.e. $a$ is left inverse element of $b$. Then

$$a = a * e = a * (a * b) = (a * b) * a = e * a.$$

That means that $e$ is an identity element of $a$.

(2) According to Proposition 2.1, $((a * b) * c) * d = (a * b) * (d * c) = (a * c) * (d * b) = a * ((d * b) * c) = a * (d * (b * c))$.

(3) If $a * b = c * d$, there exists $d^{-1} \in S$ s.t. $d^{-1} * d = e$. Then according to Tarski associative law,

$$(a * b) * d^{-1} = (c * d) * d^{-1} = c * (d^{-1} * d) = c * e = c.$$

That is to say, $a * (d^{-1} * b) = c$. □

But right identity element of TA-group isn't unique, see Example 3.4.

**Exmaple 3.4.** Let $G = \{1, 2, 3, 4\}$. Consider a TA-group in Table 1.

Right identity elements are 1 and 2.

**Theorem 3.6.** Let $(G, *)$ be a TA-group, $e$ be right identity element of $G$. Then for any $a \in G$, left inverse element of $a$ relative to $e$ is unique.

*Proof.* Let $a \in G$ and $e$ is right identity element in $G$. Assume that left inverse element of $a$ relative to $e$ isn't unique, that is, there exist $b, c \in G$ s.t. $b * a = e$ and $c * a = e$. Then

$$b = b * e = b * (c * a) = (b * a) * c = (c * a) * c = c * (c * a) = c * e = c.$$

So $b = c$ and left inverse element is unique. □

**Proposition 3.1.** Assume that $(G, *)$ is a TA-group. There is,

(1) $G$ is right cancellative;

(2) if $a * b = e$ is a right identity element, then $b * a = e_1$ also is a right identity element.

*Proof.* (1) Assume that $(G, *)$ is a TA-group and $e$ is a right identity element in $G$. For any $a, b \in G$ and there exists $y \in G$ s.t. $a * y = b * y$. And there exists $y' \in G$ s.t. $a * e = a, b * e = b, y * e = y, y' * y = e$. Then

$$a = a * e = a * (y' * y) = (a * y) * y' = (b * y) * y' = b * (y' * y) = b * e = b.$$

So $G$ is right cancellative.

(2) According to Theorem 3.2, TA-group satisfies right commutative law, then for any $c \in G$, there is,

$$c * (b * a) = c * (a * b) = c * e = c.$$

Then $b * a$ is a right identity element in $G$, that is, $b * a = e_1$ is a right identity element in $G$. □

**Theorem 3.7.** Let $G$ be semigroup. Then it is a TA-group if and only if it satisfies:

(1) $\forall a, b \in G$, there is unique solution to equation $x * a = b$;

(2) $\forall a, b, c \in G$, $(a * b) * c = a * (c * b)$.

*Proof.* $(\Rightarrow)$ Assume that $c, d$ are solutions to equation $x * a = b$, then $c * a = d * a$. Because $G$ is TA-group, according to Proposition 3.1(1), $c = d$. So there is unique a solution to equation $x * a = b$.

($\Leftarrow$) For given $a \in G$, there exists $e \in G$ s.t. $e * a = a$. Then $e^2 * a = e * (e * a) = e * a = a$, because $x * a = b$ has a unique solution, and $e^2 = e$. If there exists $e'$ s.t. $e' * a = a * e$, then $(e' * a) * e = (a * e) * e = a * e^2 = a * e$. So $e' * a = a * e = (e' * a) * e$, then $a = e' * a = a * e$. For all $b \in G$, because $x * a = e$ has unique solution, there exists $c \in G$ s.t. $c * a = b$. And $b * e = (c * a) * e = c * (a * e) = c * a = b$, so $e$ is right identity element of $G$.

Let $b = e$, there exists $c \in G$ s.t. $c * a = e$, that is, $c$ is left inverse element of $a$ relative to $a$. So $G$ is TA-group. □

**Theorem 3.8.** Assume that $G$ is TA-groupoid. Then it is TA-group if and only if it satisfies:
(1) $e$ is right identity element in $G$, that is, $\forall a \in G$, there is, $a * e = a$;
(2) $e$ is right identity element in $G$, and there exists right inverse element $b \in G$ such that $a * b = e$.

*Proof.* ($\Rightarrow$) Let $G$ be TA-groupoid. According to the definition of TA-group and $e$ is right identity element in $G$, $\forall a \in G$, there exist $a' \in G$, s.t. $a * e = a, a' * a = e$. According to Proposition 3.1, $a * a' = e_1$ is also a right identity element. So for any $a \in G$, there exist $a', e_1$ s.t. $a * a' = e_1$. According to Theorem 3.3, for all $a, b, c \in G$, there is, $(a * b) * c = a * (b * c)$.

($\Leftarrow$) Let $G$ be a TA-groupoid. Then for any $a \in G$, there exist $e, c \in G$ s.t. $a*e = a, a*c = e$. Accoring to Theorem 3.2, it satisfies right commutative law.

$$a * (c * a) = a * (a * c) = a * e = a.$$

That is to say, $c * a$ is local right identity element of $a$. And $\forall b \in G$,

$$b * (c * a) = b * (a * c) = b * e = a.$$

So $c * a$ is right identity element in $G$, and $c$ is left inverse element of $a$ relative to $c * a$. Thus $G$ is TA-group. □

In the following, we proposed the notion of TA-subgroup and gave the equivalent characterization of TA-subgroup.

**Definition 3.3.** Let $(G, *)$ be TA-group and $S$ be non-empty subset of $G$. If $S$ is a TA-group under operation $*$ on $G$, then $S$ is called TA-subgroup of $G$.

**Theorem 3.9.** The non-empty subset $S$ of $G$ is TA-subgroup if and only if
(1) $\forall a, b \in S$, there is, $a * b \in S$;
(2) $e$ is a right identity element of $S$, and for all $a \in S$, there is $a' \in S$ s.t. $a' * a = e$.

*Proof.* (⇒) According to Definition 3.3, (1) and (2) hold.

(⇐) $\forall a, b \in S$, there is, $a * b \in S$, then $S$ is a TA-groupoid. Because $e$ is right identity element of $S$, then for any $a \in S$, there is, $a * e = a$. And there exists $a' \in S$ such that $a' * a = e$. So $S$ is TA-group. Thus, $S$ is TA-subgroup of $G$. □

**Theorem 3.10.** Commutative TA-group is Abelian group.

*Proof.* Let $(G, *)$ be a TA-group, $e$ is a right identity element in $G$. According to Theorem 3.3, $G$ is a commutative semigroup. Then for any $a \in G$, there is $x \in G$ s.t. $a * e = a, x * a = e$. Then $e * a = a * e = a$ and $a * x = x * a = e$. So $e$ is identity element of $G$ and $x$ is inverse element of $a$. Assume that $e'$ also is identity element in $G$, there is, $e = e * e' = e'$. That is to say, identity element is unique. Assume that there exist $x, y \in G$ such that $x * a = e = y * a$, according to Proposition 3.1, $G$ is right cancellative, and $x = y$, thus the inverse element is unique. So $G$ is Abelian group. □

**Theorem 3.11.** If right identity element of TA-group is unique, then
(1) left inverse element is right inverse element;
(2) right identity element is left identity element;
(3) identity element is unique;
(4) inverse element is unique;
(5) it is a group.

*Proof.* (1) Suppose that $(G, *)$ is a TA-group. $\forall a \in G$, $\exists a', a'' \in G$, s.t. $a * e = a$, $a' * a = e$. $a' * e = a'$, $a'' * a' = e$. $a'' * e = a''$. Then

$$e * a = a'' * a' * a = a'' * (a' * a) = a'' * e = a''.$$

Then

$$e * a' = a'' * a' * a' = e * a * a' * a',$$

And

$$e * a' = e * a * a' * a'.$$

$$e * a' * a = e * (a' * a) = e * e = e.$$

$$e * a * a' * a' * a = e * a * a' * e = e * a * (a' * e) = e * a * a'.$$

So $e = e * a * a'$. Because $e$ is unique, then $a * a' = e$.

(2) By(1), $e * a = (a * a') * a = a * (a * a') = a * e = a$.

(3) According to (2), right identity element is left identity element and right identity element is unique, then there exists identity element and it is unique.

(4) By (1), left inverse element and right inverse element are unique and they are equivalent to each other. Assume that inverse element of $a$ isn't unique and there exists $y \in G$ s.t. $a * y = e$. So $a' = a' * e = a' * (a * y) = (a' * a) * y = e * y = y$. That is, inverse element of $a$ is unique.

(5) By (3) and (4), there are identity element and inverse element in $G$, and they are unique. And TA-group satiefies associative law, then $G$ is a group. □

Example 3.5 shows that TA-group whose right identity element is unique, and it is a group.

**Exmaple 3.5.** Let $G = \{a, b, c, d\}$, in Table 4, the operation $*$ on $G$ is given. It is both a TA-group and a group. And

$$a * a = a, b * a = b = a * b, c * a = c = a * c, d * a = d = a * d.$$

$$a * a = a, b * b = a, c * d = a = d * c.$$

Table 4. This is a group.

| * | **a** | **b** | **c** | **d** |
|---|---|---|---|---|
| a | a | b | c | d |
| b | b | a | d | c |
| c | c | d | b | a |
| d | d | c | a | b |

**Remark 3.1.** TA-group is regular TA-groupoid.

But not every regular TA-groupoid is TA-group, see Example 3.6.

**Exmaple 3.6.** Let $G = \{a, b, c, d\}$, consider the regular TA-groupoid in Table 5. Since there is no right identity element, so it isn't TA-group.

Table 5. This is a regular TA-groupoid.

| * | **a** | **b** | **c** | **d** |
|---|---|---|---|---|
| $a$ | $a$ | $a$ | $b$ | $b$ |
| $b$ | $a$ | $a$ | $b$ | $b$ |
| $c$ | $c$ | $c$ | $c$ | $c$ |
| $d$ | $d$ | $d$ | $d$ | $d$ |

In the next part, the concept and property of inverse TA-groupoid are given.

**Definition 3.4.** Let $G$ be a regular TA-groupoid. Then for any $a \in G$, there exists $x \in G$ s.t. $a * (x * a) = a$. $G$ is called an inverse TA-groupoid if $x$ is unique.

The following Theorem shows that the decomposition theorem of inverse TA-groupoid.

**Theorem 3.12.** Let $G$ be inverse TA-groupoid. Then it is disjoint union of groups.

*Proof.* Let $x * a = e$, then $a * e = a$. Assume that $G_e$ is composed of element whose local right identity element is $e$.

For any $a, b, c \in G_e$, there is,

$$\begin{aligned} a*(b*c) &= [a*(b*c)]*e = a*[e*(b*c)] \\ &= (a*b)*(e*c) = (a*c)*(e*b)(By Proposition 2.1) \\ &= a*[(e*b)*c] = a*[e*(c*b)] \\ &= [a*(c*b)]*e = a*(c*b) \end{aligned}$$

According to Tarski associative law, $(a * b) * c = a * (c * b) = a * (b * c)$. Then it satisfies associative law.

For any $a \in G_e$, there exists $x \in G$ such that $a * (x * a) = a$. Assume that $x * a = e$, then $a * e = a$, $a * ((x * e) * a) = a * (x * (a * e)) = a * (x * a) = a$. Because $x$ is unique, then $x * e = x$. That is to say, $x \in G_e$.

For any $a \in G_e$, $a * e = a$. Then $a * (e * e) = (a * e) * e = a * e = a$. And there exists $x \in G_e$ such that $x * a = e$. $(e * e) * e = (e * e) * (x * a) = (e * a) * (x * e) = (e * a) * x = e * (x * a) = e * e$. That is to say, $e * e \in G_e$ and $e * e$ is right identity element of $G_e$. According to Theorem 3.2, $e * e = (x * a) * (x * a) = (x * a) * (a * x) = x * ((a * x) * a) = x * (a * (x * a)) = x * a = e$. So $e \in G_e$.

For any $a, b \in G_e$, according to associative law, there is, $(a * b) * e = a * (b * e) = a * b$. So $a * b \in G_e$. That is to say, $G_e$ is TA-groupoid.

Above all, $G_e$ satisfies associative law, $e \in G_e$ and for any $a \in G_e$, there exists $x \in G_e$ such that $x * a = e$.

Because $x$ is unique and $e$ is unique, we know $G_e$ is a TA-group with unique right identity element, then it is a group.

Then $G$ is union of group, and $x$ is unique and $x * a = e$ is unique. That is to say, for any local right identity element $e \in G$, every subgroup $G_e$ of $G$ is disjoint, $G$ is disjoint union of groups. □

According to Definition 2.3 and Theorem 3.12, we know inverse TA-groupoid is TA-NET-groupoid, but whether a TA-NET-groupoid is a inverse TA-groupoid? see Example 3.7. The example shows that not every TA-NET-groupoid is inverse TA-groupoid.

**Exmaple 3.7.** Let $G = \{1, 2, 3, 4\}$. In Table 6, the TA-NET-groupoid $(G, *)$ is shown. And

$$1 * 1 = 1; 2 * 2 = 2;$$

$$1 * 3 = 3 = 3 * 1, 3 * 3 = 1;$$

$$4 * 4 = 4; 5 * 5 = 5.$$

Table 6. This is a TA-NET-groupoid.

| $*$ | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 3 | 4 | 4 |
| 2 | 1 | 2 | 3 | 4 | 5 |
| 3 | 3 | 3 | 1 | 4 | 4 |
| 4 | 4 | 4 | 4 | 4 | 4 |
| 5 | 4 | 5 | 4 | 4 | 5 |

It isn't inverse TA-groupoid since $1 * (1 * 1) = 1$ , $1 * (2 * 1) = 1$ and $1 \neq 2$.

We know both completely regular semigroup and inverse TA-groupoid are disjoint union of groups, and completely regular semigroup satisfies associative law, whether a completely regular semigroup is a TA-groupoid? See Example 3.8. The example shows that not every completely regular semigroup is TA-groupoid.

**Exmaple 3.8.** Let $G = \{1, 2, 3, 4\}$. In Table 7, a completely regular semigroup $(G, *)$ is given.

Table 7. This is a completely regular semigroup.

| $*$ | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| 1 | 1 | 4 | 4 | 4 |
| 2 | 1 | 2 | 3 | 4 |
| 3 | 1 | 3 | 3 | 4 |
| 4 | 1 | 4 | 4 | 4 |

It isn't TA-groupoid since $(1 * 1) * 2 = 4 \neq 1 = 1 * (2 * 1)$.

Because TA-NET-groupoid is semigroup and inverse TA-groupoid is TA-NET-groupoid, inverse TA-groupoid is semigroup. According to Definition 2.7, Theorem 2.6 and Definition 3.4, inverse TA-groupoid is inverse semigroup.

The following figure shows relationships among various TA-groupoid.

## 4. Conclusions

In this paper, we proposed the concept of TA-group and inverse TA-groupoid. Some results are obtained as follows: (1) if right identity element of TA-group is unique, then it is a group; (2) the equivalent characterization of TA-group is given; (3) inverse TA-groupoid is disjoint union of group; (4) commutative TA-groupoid is group. Figure 1 shows their relations.

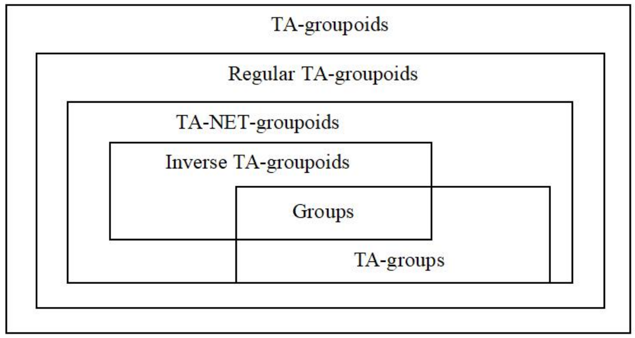


Figure 1. The relationships among various TA-groupoid.

As a future direction for further research, we can discuss the relationships among TA-groupoid, AG-groupoid and hyper logical algebras(see [18–20]).

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