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Inayat Rasool Ganaie

Archana Sharma

Vijay Kumar

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On \mathcal{S}_θ -summability in neutrosophic soft normed linear spaces

Inayat Rasool Ganaie^{1,*}, Archana Sharma¹ and Vijay Kumar¹

¹Department Of Mathematics, Chandigarh University, Mohali-140413, Punjab, India;

inayatrasool.maths@gmail.com, dr.archanasharma1022@gmail.com and kaushikvjy@gmail.com

*Correspondence: inayatrasool.maths@gmail.com; Tel.: (+919622643253)

Abstract. For any lacunary sequence $\theta = (k_s)$, the aim of the present paper is to define \mathcal{S}_θ -convergence, \mathcal{S}_θ -Cauchy and \mathcal{S}_θ -completeness via neutrosophic soft norm. We study certain properties of these notions and give an important characterization of \mathcal{S}_θ -convergence in neutrosophic soft normed linear spaces (briefly *NSNLS*). We provide examples that shows \mathcal{S}_θ -convergence is a more general method of summability in these spaces.

Keywords: \mathcal{S}_θ -convergence, \mathcal{S}_θ -Cauchy, soft sets, soft normed linear spaces.

1. Introduction

Statistical convergence was first introduced by Fast [8] and linked with the summability theory by Schoenberg [10]. Later, The idea is developed by Maddox [9], Fridy [12], Connor[13], Mursaleen and Edely [18], Šalát [32], Kumar and mursaleen [35] and many others.

Friday and Orhan [11] used lacunary sequences to define a new kind of statistical convergence as follows. “By a lacunary sequence we mean an increasing integer sequence $\theta = (k_s)$ with $k_0 = 0$ and $h_s = k_s - k_{s-1} \rightarrow \infty$ as $s \rightarrow \infty$. The intervals determined by θ will be denoted by $I_s = (k_{s-1}, k_s]$ and the ratio $\frac{k_s}{k_{s-1}}$ will be abbreviated as q_s . For $K \subseteq \mathbb{N}$, the number $\delta_\theta(K) = \lim_{s \rightarrow \infty} \frac{1}{h_s} |\{k \in I_s : k \in K\}|$ is called θ -density of K , provided the limit exists. A sequence $x = (x_k)$ of numbers is said to be lacunary statistically convergent (briefly \mathcal{S}_θ -convergent) to x_0 if for every $\epsilon > 0$, $\lim_{s \rightarrow \infty} \frac{1}{h_s} |\{k \in I_s : |x_k - x_0| \geq \epsilon\}| = 0$ or equivalently, the set $K(\epsilon)$ has θ -density zero, where $K(\epsilon) = \{k \in \mathbb{N} : |x_k - x_0| \geq \epsilon\}$. In this case, we write $\mathcal{S}_\theta - \lim_{k \rightarrow \infty} x_k = x_0$.” Some further interesting works on lacunary statistical convergence can be found in [4], [19], [25], [34], [36], etc.

Zadeh [16] proposed the theory of fuzzy sets in 1965 as a more convenient tool for handling issues that cannot be modelled via crisp set theory. Atanassov [15] observed that fuzzy sets need more modification to handle problems in a time domain and therefore he introduced the intuitionistic fuzzy sets. After the introduction of intuitionistic fuzzy sets, a progressive development is made in this field. For instance, intuitionistic fuzzy metric spaces were introduced by Park [14], intuitionistic fuzzy topological spaces by Saadati and Park [26], etc.

The neutrosophic sets were initially introduced by Smarandache[7] as a generalization of fuzzy sets and intuitionistic fuzzy sets to avoid the complexity arising from uncertainty in settling many practical challenges in real-world activities. Kirişçi and Şimşek[17] defined neutrosophic norm and studied statistical convergence in neutrosophic normed spaces(*NNS*). For a broad view in this direction, we recommend to the reader [1], [2], [3], [20], [21], [22], [33].

Many approaches discussed above to minimize the uncertainty have their own drawbacks due to the inadequacy of the parametrization. In view of this, Molodtsov[6] proposed a new theory, called soft set theory to reduce the uncertainty during mathematical modelling. These sets turn out very useful tools in many areas of engineering and medical sciences. For instance: Maji et al [23] applied the theory of soft sets to decision-making problems. Kong et al.[39] presented a heuristic algorithm of normal parameter reduction of soft sets. Zou and Xiao[38] presented a data analysis approach of soft sets under incomplete information. Yuksel et al.[30] applied soft set theory to diagnose the prostate cancer risk in human beings whereas Çelik and Yamak[37] applied fuzzy soft set theory for medical diagnosis using fuzzy arithmetic operations.

Maji [24] presented a combined concept of Neutrosophic soft sets in 2013. Recently, Bera and Mahapatra [31] defined a generalized norm and called it a neutrosophic soft norm. They also studied some properties of *NSNLS* and developed fundamental concepts of sequences in these spaces. In this article, we develop and study the concept of \mathcal{S}_θ -convergence in *NSNLS*. We also introduce the concepts of \mathcal{S}_θ -Cauchy sequence, \mathcal{S}_θ -completeness and develop some of their properties.

2. Preliminaries

This section starts with a brief information on soft sets, soft vector spaces and neutrosophic soft normed spaces. We begin with the following notations and definitions.

Throughout this work, \mathbb{N} will denote the set of positive integers, \mathbb{R} the set of reals and \mathbb{R}^+ the set of positive real numbers.

Definition 2.1 [5] A binary operation $\circ : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is continuous t -norm if \circ satisfies the following conditions:

- (i) $d \circ e = e \circ d$ and $d \circ (e \circ f) = (d \circ e) \circ f$.
- (ii) \circ is continuous.

- (iii) $d \circ 1 = 1 \circ d = d$ for all $d \in [0, 1]$.
- (iv) $d \circ e \leq f \circ g$ if $d \leq f, e \leq g$ with $d, e, f, g \in [0, 1]$.

Definition 2.2 [5] A binary operation $\diamond : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is continuous t -conorm(s -norm) if \diamond satisfies the following conditions:

- (i) $d \diamond e = e \diamond d$ and $d \diamond (e \diamond f) = (d \diamond e) \diamond f$.
- (ii) \diamond is continuous.
- (iii) $d \diamond 0 = 0 \diamond d =$ for all $d \in [0, 1]$.
- (iv) $d \diamond e \leq f \circ g$ if $d \leq f, e \leq g$ with $d, e, f, g \in [0, 1]$.

For any universe set U and the set E of the parameters, the soft set is defined as follows:

Definition 2.3 [6] A pair (H, E) is called a soft set over U if and only if H is a mapping of E into the set of all subsets of the set U . i.e., the soft set is a parametrized family of subsets of the set U .

Moreover, every set $H(\epsilon), \epsilon \in E$, from this family may be considered as the set of ϵ -elements of the soft set (H, E) , or as the set of ϵ -approximate elements of the set.

Definition 2.4 [6] A soft set (H, E) over U is said to be absolute soft set if for all $\epsilon \in E, H(\epsilon) = U$. We will denote it by \tilde{U} .

Definition 2.5 [27] Let \mathbb{R} be the set of real numbers, $B(\mathbb{R})$ be the collection of all non-empty bounded subsets of \mathbb{R} and E taken as a set of parameters. Then a mapping $F : E \rightarrow B(\mathbb{R})$ is called a soft real set. If a soft real set is a singleton soft set, then it is called a soft real number and denoted by $\tilde{r}, \tilde{s}, \tilde{t}$, etc. $\tilde{0}, \tilde{1}$ are the soft real numbers where $\tilde{0}(e) = 0, \tilde{1}(e) = 1$ for all $e \in E$ respectively.

Let $\mathbb{R}(E)$ and $\mathbb{R}^+(E)$ respectively denote the sets of all soft real numbers and all positive soft real numbers.

Definition 2.6 [28] Let (H, E) be a soft set over U . The set (H, E) is said to be a soft point, denoted by H_e^u if there is exactly one $e \in E$ s.t $H(e) = \{u\}$ for some $u \in U$ and $H(e') = \phi$ for all $e' \in E - \{e\}$.

Two soft points $H_e^u, H_{e'}^w$ are said to be equal if $e = e'$ and $u = w$. Let $\Delta_{\tilde{U}}$ denotes the set of all soft points on \tilde{U} .

In case U is a vector space over \mathbb{R} and the parameter set $E = \mathbb{R}$, the soft point is called a soft vector.

Soft vector spaces are used to define soft norm as follows:

Definition 2.7 [29] Let \tilde{U} be a absolute soft vector space. Then a mapping $\|\cdot\| : \tilde{U} \rightarrow \mathbb{R}^+(E)$ is said to be a soft norm on \tilde{U} , if $\|\cdot\|$ satisfies the following conditions:

- (i) $\|u_e\| \geq \tilde{0}$ for all $u_e \in \tilde{U}$ and $\|u_e\| = \tilde{0} \Leftrightarrow u_e = \tilde{\theta}_0$ where $\tilde{\theta}_0$ denotes the zero element of \tilde{U} .
- (ii) $\|\tilde{\alpha} u_e\| = |\tilde{\alpha}| \|u_e\|$ for all $u_e \in \tilde{U}$ and for every soft scalar $\tilde{\alpha}$.
- (iii) $\|u_e + v_{e'}\| \leq \|u_e\| + \|v_{e'}\|$ for all $u_e, v_{e'} \in \tilde{U}$.

(iv) $\|u_e \cdot v_{e'}\| = \|u_e\| \|v_{e'}\|, \forall u_e, v_{e'} \in \tilde{U}$.

The soft vector space \tilde{U} with a soft norm $\|\cdot\|$ on \tilde{U} is said to be a soft normed linear space and is denoted by $(\tilde{U}, \|\cdot\|)$.

We now recall the definition of neutrosophic soft normed linear spaces and the convergence structure in these spaces.

Definition 2.8 [31] Let \tilde{U} be a soft linear space over the field F and $\mathbb{R}(E), \Delta_{\tilde{U}}$ denote respectively, the set of all soft real numbers and the set of all soft points on \tilde{U} . Then a neutrosophic subset N over $\Delta_{\tilde{U}} \times \mathbb{R}(E)$ is called a neutrosophic soft norm on \tilde{U} if for $u_e, v_{e'} \in \tilde{U}$ and $\tilde{\alpha} \in F$ ($\tilde{\alpha}$ being soft scalar), the following conditions hold.

- (i) $0 \leq G_N(u_e, \tilde{\eta}_1), B_N(u_e, \tilde{\eta}_1), Y_N(u_e, \tilde{\eta}_1) \leq 1, \forall \tilde{\eta}_1 \in \mathbb{R}(E)$.
- (ii) $0 \leq G_N(u_e, \tilde{\eta}_1) + B_N(u_e, \tilde{\eta}_1) + Y_N(u_e, \tilde{\eta}_1) \leq 3, \forall \tilde{\eta}_1 \in \mathbb{R}(E)$.
- (iii) $G_N(u_e, \tilde{\eta}_1) = 0$ with $\tilde{\eta}_1 \leq \tilde{0}$.
- (iv) $G_N(u_e, \tilde{\eta}_1) = 1$, with $\tilde{\eta}_1 > \tilde{0}$ if and only if $u_e = \tilde{\theta}$, the null soft vector.
- (v) $G_N(\tilde{\alpha} u_e, \tilde{\eta}_1) = G_N\left(u_e, \frac{\tilde{\eta}_1}{|\tilde{\alpha}|}\right), \forall \tilde{\alpha} (\neq \tilde{0}), \tilde{\eta}_1 > \tilde{0}$.
- (vi) $G_N(u_e, \tilde{\eta}_1) \circ G_N(v_{e'}, \tilde{\eta}_2) \leq G_N(u_e \oplus v_{e'}, \tilde{\eta}_1 \oplus \tilde{\eta}_2), \forall \tilde{\eta}_1, \tilde{\eta}_2 \in \mathbb{R}(E)$
- (vii) $G_N(u_e, \cdot)$ is continuous non-decreasing function for $\tilde{\eta}_1 > \tilde{0}$ and $\lim_{\tilde{\eta}_1 \rightarrow \infty} G_N(u_e, \tilde{\eta}_1) = 1$.
- (viii) $B_N(u_e, \tilde{\eta}_1) = 1$ with $\tilde{\eta}_1 \leq \tilde{0}$.
- (ix) $B_N(u_e, \tilde{\eta}_1) = 0$, with $\tilde{\eta}_1 > \tilde{0}$ if and only if $u_e = \tilde{\theta}$, the null soft vector.
- (x) $B_N(\tilde{\alpha} u_e, \tilde{\eta}_1) = B_N\left(u_e, \frac{\tilde{\eta}_1}{|\tilde{\alpha}|}\right), \forall \tilde{\alpha} (\neq \tilde{0}), \tilde{\eta}_1 > \tilde{0}$.
- (xi) $B_N(u_e, \tilde{\eta}_1) \diamond B_N(v_{e'}, \tilde{\eta}_2) \geq B_N(u_e \oplus v_{e'}, \tilde{\eta}_1 \oplus \tilde{\eta}_2) \forall \tilde{\eta}_1, \tilde{\eta}_2 \in \mathbb{R}(E)$.
- (xii) $B_N(u_e, \cdot)$ is continuous non-increasing function for $\tilde{\eta}_1 > \tilde{0}$ and $\lim_{\tilde{\eta}_1 \rightarrow \infty} B_N(u_e, \tilde{\eta}_1) = 0$.
- (xiii) $Y_N(u_e, \tilde{\eta}_1) = 0$ with $\tilde{\eta}_1 \leq \tilde{0}$.
- (xiv) $Y_N(u_e, \tilde{\eta}_1) = 0$, with $\tilde{\eta}_1 > \tilde{0}$ if and only if $u_e = \tilde{\theta}$, the null soft vector.
- (xv) $Y_N(\tilde{\alpha} u_e, \tilde{\eta}_1) = Y_N\left(u_e, \frac{\tilde{\eta}_1}{|\tilde{\alpha}|}\right), \forall \tilde{\alpha} (\neq \tilde{0}), \tilde{\eta}_1 > \tilde{0}$.
- (xvi) $Y_N(u_e, \tilde{\eta}_1) \diamond Y_N(v_{e'}, \tilde{\eta}_2) \geq Y_N(u_e \oplus v_{e'}, \tilde{\eta}_1 \oplus \tilde{\eta}_2) \forall \tilde{\eta}_1, \tilde{\eta}_2 \in \mathbb{R}(E)$.
- (xvii) $Y_N(u_e, \cdot)$ is continuous non-increasing function for $\tilde{\eta}_1 > \tilde{0}$ and $\lim_{\tilde{\eta}_1 \rightarrow \infty} Y_N(u_e, \tilde{\eta}_1) = 0$.

In this case, $\mathcal{N} = (G_N, B_N, Y_N)$ is called the neutrosophic soft norm and $(\tilde{U}(F), G_N, B_N, Y_N, \circ, \diamond)$ is the neutrosophic soft normed linear space (NSNLS briefly).

Let $(\tilde{U}, \|\cdot\|)$ be a soft normed space. Take the operations \circ and \diamond as $x \circ y = xy; x \diamond y = x + y - xy$. For $\tilde{\eta} > \tilde{0}$, define

$$G_N(u_e, \tilde{\eta}) = \begin{cases} \frac{\tilde{\eta}}{\tilde{\eta} + \|u_e\|} & \text{if } \tilde{\eta} > \|u_e\| \\ 0 & \text{otherwise} \end{cases}$$

$$B_N(u_e, \tilde{\eta}) = \begin{cases} \frac{\|u_e\|}{\tilde{\eta} + \|u_e\|} & \text{if } \tilde{\eta} > \|u_e\| \\ 0 & \text{otherwise} \end{cases}$$

$$Y_N(u_e, \tilde{\eta}) = \begin{cases} \frac{\|u_e\|}{\tilde{\eta}} & \text{if } \tilde{\eta} > \|u_e\| \\ 0 & \text{otherwise,} \end{cases}$$

then $(\tilde{U}(F), G_N, B_N, Y_N, \circ, \diamond)$ is an *NSNLS*. From now onwards, unless otherwise stated by \tilde{V} we shall denote the *NSNLS* $(\tilde{U}(F), G_N, B_N, Y_N, \circ, \diamond)$.

Definition 2.9 [31] A sequence $v = (v_{e_k}^k)$ of soft points in \tilde{V} is said to be convergent to a soft point $v_e \in \tilde{V}$ if for $0 < \epsilon < 1$ and $\tilde{\eta} > \tilde{0} \exists n_0 \in \mathbb{N}$ s.t $G_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) > 1 - \epsilon$, $B_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) < \epsilon$, $Y_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) < \epsilon$. In this case, we write $\lim_{k \rightarrow \infty} v_{e_k}^k = v_e$.

Definition 2.10 [31] A sequence $v = (v_{e_k}^k)$ of soft points in \tilde{V} is said to be cauchy sequence if for $0 < \epsilon < 1$ and $\tilde{\eta} > \tilde{0} \exists n_0 \in \mathbb{N}$ s.t for all $k, p \geq n_0$ $G_N(v_{e_k}^k \ominus v_{e_p}^p, \tilde{\eta}) > 1 - \epsilon$, $B_N(v_{e_k}^k \ominus v_{e_p}^p, \tilde{\eta}) < \epsilon$, $Y_N(v_{e_k}^k \ominus v_{e_p}^p, \tilde{\eta}) < \epsilon$.

3. Lacunary statistical convergence in NSNLS

In this section, we define \mathcal{S}_θ -convergence in neutrosophic soft normed linear spaces and develop some of its properties.

Definition 3.1 A sequence $v = (v_{e_k}^k)$ of soft points in \tilde{V} is said to be lacunary statistical convergent or \mathcal{S}_θ -convergent to a soft point v_e in \tilde{V} w.r.t neutrosophic soft norm- (G_N, B_N, Y_N) if for each $\epsilon > 0$ and $\tilde{\eta} > \tilde{0}$,

$$\lim_{s \rightarrow \infty} \frac{1}{h_s} \left| \left\{ k \in I_s : G_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \geq \epsilon, Y_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \geq \epsilon \right\} \right| = 0,$$

i.e., $\delta_\theta(A) = 0$ where

$$A = \{k \in \mathbb{N} : G_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \geq \epsilon, Y_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \geq \epsilon\}.$$

In this case, we write $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k \rightarrow \infty} v_{e_k}^k = v_e$.

Let, $S_\theta(G_N, B_N, Y_N)$ denotes the set of all sequences of soft points in \tilde{V} which are \mathcal{S}_θ -convergent with respect to the neutrosophic soft norm (G_N, B_N, Y_N) .

Definition 3.1 together with the property of θ -density, we have the following lemma.

Lemma 3.1 For any sequence $v = (v_{e_k}^k)$ of soft points in \tilde{V} , the following statements are equivalent:

- (i) $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k \rightarrow \infty} v_{e_k}^k = v_e$;
- (ii) $\delta_\theta\{k \in \mathbb{N} : G_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \leq 1 - \epsilon\} = \delta_\theta\{k \in \mathbb{N} : B_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \geq \epsilon\} = \delta_\theta\{k \in \mathbb{N} : Y_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \geq \epsilon\} = 0$;
- (iii) $\delta_\theta\{k \in \mathbb{N} : G_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) > 1 - \epsilon \text{ and } B_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) < \epsilon, Y_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) < \epsilon\} = 1$;
- (iv) $\delta_\theta\{k \in \mathbb{N} : G_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) > 1 - \epsilon\} = \delta_\theta\{k \in \mathbb{N} : B_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) < \epsilon\} = \delta_\theta\{k \in \mathbb{N} : Y_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) < \epsilon\} = 1$;
- (v) $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k \rightarrow \infty} G_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) = 1$ and $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k \rightarrow \infty} B_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) = 0$, $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k \rightarrow \infty} Y_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) = 0$.

Theorem 3.1 Let $\theta = (k_s)$ be a lacunary sequence and $v = (v_{e_k}^k)$ be any sequence in \tilde{V} . If $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k \rightarrow \infty} v_{e_k}^k$ exists, then it is unique.

Proof. Suppose that $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{n \rightarrow \infty} v_{e_k}^k = v_{e_1}$ and $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{n \rightarrow \infty} v_{e_k}^k = v'_{e_2}$, where $v_{e_1} \neq v'_{e_2}$. Let $\epsilon > 0$ and $\tilde{\eta} > \tilde{0}$. Choose $\varrho > 0$ s.t.

$$(1 - \varrho) \circ (1 - \varrho) > 1 - \epsilon \text{ and } \varrho \diamond \varrho < \epsilon \tag{1}$$

Define the following sets:

$$H_{G_N,1}(\varrho, \tilde{\eta}) = \left\{ k \in \mathbb{N} : G_N\left(v_{e_k}^k \ominus v_{e_1}, \frac{\tilde{\eta}}{2}\right) \leq 1 - \varrho \right\}.$$

$$H_{G_N,2}(\varrho, \tilde{\eta}) = \left\{ k \in \mathbb{N} : G_N\left(v_{e_k}^k \ominus v'_{e_2}, \frac{\tilde{\eta}}{2}\right) \leq 1 - \varrho \right\}.$$

$$H_{B_N,1}(\varrho, \tilde{\eta}) = \left\{ k \in \mathbb{N} : B_N\left(v_{e_k}^k \ominus v_{e_1}, \frac{\tilde{\eta}}{2}\right) \geq \varrho \right\}.$$

$$H_{B_N,2}(\varrho, \tilde{\eta}) = \left\{ k \in \mathbb{N} : B_N\left(v_{e_k}^k \ominus v'_{e_2}, \frac{\tilde{\eta}}{2}\right) \geq \varrho \right\}.$$

$$H_{Y_N,1}(\varrho, \tilde{\eta}) = \left\{ k \in \mathbb{N} : Y_N\left(v_{e_k}^k \ominus v_{e_1}, \frac{\tilde{\eta}}{2}\right) \geq \varrho \right\}.$$

$$H_{Y_N,2}(\varrho, \tilde{\eta}) = \left\{ k \in \mathbb{N} : Y_N\left(v_{e_k}^k \ominus v'_{e_2}, \frac{\tilde{\eta}}{2}\right) \geq \varrho \right\}.$$

Since $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k \rightarrow \infty} v_{e_k}^k = v_{e_1}$, then using lemma 3.1, we have

$$\delta_\theta\{H_{G_N,1}(\varrho, \tilde{\eta})\} = \delta_\theta\{H_{B_N,1}(\varrho, \tilde{\eta})\} = \delta_\theta\{H_{Y_N,1}(\varrho, \tilde{\eta})\} = 0 \text{ and therefore } \delta_\theta\{H_{G_N,1}^C(\varrho, \tilde{\eta})\} = \delta_\theta\{H_{B_N,1}^C(\varrho, \tilde{\eta})\} = \delta_\theta\{H_{Y_N,1}^C(\varrho, \tilde{\eta})\} = 1.$$

Further, $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k \rightarrow \infty} v_{e_k}^k = v'_{e_2}$, so

$$\delta_\theta\{H_{G_N,2}(\varrho, \tilde{\eta})\} = \delta_\theta\{H_{B_N,2}(\varrho, \tilde{\eta})\} = \delta_\theta\{H_{Y_N,2}(\varrho, \tilde{\eta})\} = 0 \text{ and therefore } \delta_\theta\{H_{G_N,2}^C(\varrho, \tilde{\eta})\} = \delta_\theta\{H_{B_N,2}^C(\varrho, \tilde{\eta})\} = \delta_\theta\{H_{Y_N,2}^C(\varrho, \tilde{\eta})\} = 1 \text{ for all } \tilde{\eta} > \tilde{0}. \text{ Now define}$$

$$K_{G_N, B_N, Y_N}(\epsilon, \tilde{\eta}) = \{H_{G_N,1}(\varrho, \tilde{\eta}) \cup H_{G_N,2}(\varrho, \tilde{\eta})\} \\ \cap \{H_{B_N,1}(\varrho, \tilde{\eta}) \cup H_{B_N,2}(\varrho, \tilde{\eta})\} \cap \{H_{Y_N,1}(\varrho, \tilde{\eta}) \cup H_{Y_N,2}(\varrho, \tilde{\eta})\},$$

then $\delta_\theta\{K_{G_N, B_N, Y_N}(\epsilon, \tilde{\eta})\} = 0$ and therefore, $\delta_\theta\{K_{G_N, B_N, Y_N}^C(\epsilon, \tilde{\eta})\} = 1$. Let $m \in K_{G_N, B_N, Y_N}^C(\epsilon, \tilde{\eta})$, then we have following possibilities.

1. $m \in \left\{H_{G_N,1}(\varrho, \tilde{\eta}) \cup H_{G_N,2}(\varrho, \tilde{\eta})\right\}^C$; or
2. $m \in \left\{H_{B_N,1}(\varrho, \tilde{\eta}) \cup H_{B_N,2}(\varrho, \tilde{\eta})\right\}^C$; or
3. $m \in \left\{H_{Y_N,1}(\varrho, \tilde{\eta}) \cup H_{Y_N,2}(\varrho, \tilde{\eta})\right\}^C$.

Case 1: Let $m \in \left\{H_{G_N,1}(\varrho, \tilde{\eta}) \cup H_{G_N,2}(\varrho, \tilde{\eta})\right\}^C$, then $m \in H_{G_N,1}^C(\varrho, \tilde{\eta})$ and $m \in H_{G_N,2}^C(\varrho, \tilde{\eta})$ and therefore,

$$G_N\left(v_{e_m}^m \ominus v_{e_1}, \frac{\tilde{\eta}}{2}\right) > 1 - \varrho \text{ and } G_N\left(v_{e_m}^m \ominus v'_{e_2}, \frac{\tilde{\eta}}{2}\right) > 1 - \varrho. \tag{2}$$

Now

$$\begin{aligned} G_N(v_{e_1} \ominus v'_{e_2}, \tilde{\eta}) &= G_N\left(v_{e_m}^m \ominus v_{e_m}^m \oplus v_{e_1} \ominus v'_{e_2}, \frac{\tilde{\eta}}{2} \oplus \frac{\tilde{\eta}}{2}\right) \\ &\geq G_N\left(v_{e_m}^m \ominus v_{e_1}, \frac{\tilde{\eta}}{2}\right) \circ G_N\left(v_{e_m}^m \ominus v'_{e_2}, \frac{\tilde{\eta}}{2}\right) \\ &> (1 - \varrho) \circ (1 - \varrho) \quad \text{by (2)} \\ &> 1 - \epsilon. \quad \text{by (1)} \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, so we have $G_N(v_{e_1} \ominus v'_{e_2}, \tilde{\eta}) = 1$ for all $\tilde{\eta} > \tilde{0}$, which gives $v_{e_1} = v'_{e_2}$.

Case 2: Let $m \in \left\{H_{B_N,1}(\varrho, \tilde{\eta}) \cup H_{B_N,2}(\varrho, \tilde{\eta})\right\}^C$, then $m \in H_{B_N,1}^C(\varrho, \tilde{\eta})$ and $m \in H_{B_N,2}^C(\varrho, \tilde{\eta})$ and therefore,

$$B_N\left(v_{e_m}^m \ominus v_{e_1}, \frac{\tilde{\eta}}{2}\right) < \varrho \text{ and } B_N\left(v_{e_m}^m \ominus v'_{e_2}, \frac{\tilde{\eta}}{2}\right) < \varrho. \tag{3}$$

Now

$$\begin{aligned} B_N(v_{e_1} \ominus v'_{e_2}, \tilde{\eta}) &= B_N\left(v_{e_m}^m \ominus v_{e_m}^m \oplus v_{e_1} \ominus v'_{e_2}, \frac{\tilde{\eta}}{2} \oplus \frac{\tilde{\eta}}{2}\right) \\ &\leq B_N\left(v_{e_m}^m \ominus v_{e_1}, \frac{\tilde{\eta}}{2}\right) \diamond B_N\left(v_{e_m}^m \ominus v'_{e_2}, \frac{\tilde{\eta}}{2}\right) \\ &< \varrho \diamond \varrho \quad \text{by (3)} \\ &< \epsilon. \quad \text{by (1)} \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, so we have $B_N(v_{e_1} \ominus v'_{e_2}, \tilde{\eta}) = 0$ for all $\tilde{\eta} > \tilde{0}$, which gives $v_{e_1} = v'_{e_2}$.

Case 3: Let $m \in \left\{H_{Y_N,1}(\varrho, \tilde{\eta}) \cup H_{Y_N,2}(\varrho, \tilde{\eta})\right\}^C$, then $m \in H_{Y_N,1}^C(\varrho, \tilde{\eta})$ and $m \in H_{Y_N,2}^C(\varrho, \tilde{\eta})$ and therefore,

$$Y_N\left(v_{e_m}^m \ominus v_{e_1}, \frac{\tilde{\eta}}{2}\right) < \varrho \text{ and } Y_N\left(v_{e_m}^m \ominus v'_{e_2}, \frac{\tilde{\eta}}{2}\right) < \varrho. \tag{4}$$

Now

$$\begin{aligned} Y_N(v_{e_1} \ominus v'_{e_2}, \tilde{\eta}) &= Y_N\left(v_{e_m}^m \ominus v_{e_m}^m \oplus v_{e_1} \ominus v'_{e_2}, \frac{\tilde{\eta}}{2} \oplus \frac{\tilde{\eta}}{2}\right) \\ &\leq Y_N\left(v_{e_m}^m \ominus v_{e_1}, \frac{\tilde{\eta}}{2}\right) \diamond Y_N\left(v_{e_m}^m \ominus v'_{e_2}, \frac{\tilde{\eta}}{2}\right) \\ &< \varrho \diamond \varrho \quad \text{by (4)} \\ &< \epsilon. \quad \text{by (1)} \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, so we have $Y_N(v_{e_1} \ominus v'_{e_2}, \tilde{\eta}) = 0$ for all $\tilde{\eta} > \tilde{0}$, which gives $v_{e_1} = v'_{e_2}$. Hence, in all cases we have $v_{e_1} = v'_{e_2}$, i.e., $\mathcal{S}_\theta(G_N, B_N, Y_N)$ -limit of $(v_{e_k}^k)$ is unique. \square

Theorem 3.2 Let $\theta = (k_s)$ be a lacunary sequence and $v = (v_{e_k}^k)$ be any sequence in \tilde{V} . If $(G_N, B_N, Y_N) - \lim_{k \rightarrow \infty} v_{e_k}^k = v_e$, then $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k \rightarrow \infty} v_{e_k}^k = v_e$.

Proof. Let $(G_N, B_N, Y_N) - \lim_{k \rightarrow \infty} v_{e_k}^k = v_e$. Then for each $\epsilon > 0$ and $\eta > 0$, \exists positive integers $k_0 \in \mathbb{N}$ s.t $G_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) > 1 - \epsilon$ and $B_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) < \epsilon, Y_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) < \epsilon \forall k > k_0$. Hence, the set

$$\begin{aligned} A = \{k \in \mathbb{N} : G_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \leq 1 - \epsilon \text{ or} \\ B_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \geq \epsilon, Y_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \geq \epsilon\} \end{aligned}$$

has finite number of terms. Since every finite subset of \mathbb{N} has θ -density zero and hence

$$\begin{aligned} \delta_\theta(\{k \in \mathbb{N} : G_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \leq 1 - \epsilon \text{ or} \\ B_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \geq \epsilon, Y_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \geq \epsilon\}) = 0. \end{aligned}$$

Therefore, $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k \rightarrow \infty} v_{e_k}^k = v_e$. \square

The following example shows that the converse of the above theorem need not be true.

Example 3.1 Let $(\tilde{\mathbb{R}}, \|\cdot\|)$ be a soft normed linear space. For v_e in $\tilde{\mathbb{R}}$ and $\tilde{\eta} > \tilde{0}$, if we define

$$G_N(v_e, \tilde{\eta}) = \frac{\tilde{\eta}}{\tilde{\eta} \oplus \|v_e\|}, \quad B_N(v_e, \tilde{\eta}) = \frac{\|v_e\|}{\tilde{\eta} \oplus \|v_e\|}, \quad Y_N(v_e, \tilde{\eta}) = \frac{\|v_e\|}{\tilde{\eta}}$$

$x \circ y = xy$ and $x \diamond y = \min\{x+y, 1\}$, then it is easy to see that $\tilde{V} = (\tilde{\mathbb{R}}, G_N, B_N, Y_N, \circ, \diamond) \forall x, y \in [0, 1]$ is a neutrosophic soft normed linear space.

Now define a sequence $v = (v_{e_k}^k)$ in \tilde{V} by

$$v_{e_k}^k = \begin{cases} \tilde{k} & \text{if } k_s - [\sqrt{h_s}] + 1 \leq k \leq k_s, s \in \mathbb{N} \\ \tilde{0} & \text{otherwise.} \end{cases}$$

Now, for each $\epsilon > 0$ and $\tilde{\eta} > \tilde{0}$, let

$$\begin{aligned} A(\epsilon, \tilde{\eta}) &= \left\{ k \in I_s : G_N(v_{e_k}^k, \tilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v_{e_k}^k, \tilde{\eta}) \geq \epsilon, Y_N(v_{e_k}^k, \tilde{\eta}) \geq \epsilon \right\} \\ &= \left\{ k \in I_s : \frac{\tilde{\eta}}{\tilde{\eta} \oplus \|v_{e_k}^k\|} \leq 1 - \epsilon \text{ or } \frac{\|v_{e_k}^k\|}{\tilde{\eta} \oplus \|v_{e_k}^k\|} \geq \epsilon, \frac{\|v_{e_k}^k\|}{\tilde{\eta}} \geq \epsilon \right\} \\ &= \left\{ k \in I_s : \|v_{e_k}^k\| \geq \frac{\tilde{\eta} \epsilon}{1 - \epsilon} \text{ or } \|v_{e_k}^k\| \geq \tilde{\eta} \epsilon \right\} \\ &\subseteq \left\{ k \in I_s : v_{e_k}^k = \tilde{k} \right\} \\ &= \left\{ k \in I_s : k_s - [\sqrt{h_s}] + 1 \leq k \leq k_s, s \in \mathbb{N} \right\} \end{aligned}$$

and so we get

$$\frac{1}{h_s} |A(\epsilon, \tilde{\eta})| \leq \frac{1}{h_s} |\{k \in I_s : k_s - [\sqrt{h_s}] + 1 \leq k \leq k_s\}| \leq \frac{\sqrt{h_s}}{h_s}.$$

Taking $s \rightarrow \infty$,

$$\lim_{s \rightarrow \infty} \frac{1}{h_s} |A(\epsilon, \tilde{\eta})| \leq \lim_{s \rightarrow \infty} \frac{\sqrt{h_s}}{h_s} = 0, \text{ i.e., } \delta_\theta(A(\epsilon, \tilde{\eta})) = 0.$$

This shows that, $v = (v_{e_k}^k)$ is $\mathcal{S}_\theta(G_N, B_N, Y_N)$ -convergent to $\tilde{0}$. But by the structure of the sequence, $v = (v_{e_k}^k)$ is not convergent to $\tilde{0}$ w.r.t (G_N, B_N, Y_N) .

Theorem 3.3 Let $\theta = (k_s)$ be a lacunary sequence and let $u = (u_{e_k}^k)$ and $v = (v_{e_k}^k)$ be any two sequences in \tilde{V} s.t $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k \rightarrow \infty} (u_{e_k}^k) = u_{e_1}$ and $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k \rightarrow \infty} (v_{e_k}^k) = v_{e_2}$. Then

- (i) $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k \rightarrow \infty} (u_{e_k}^k \oplus v_{e_k}^k) = u_{e_1} \oplus v_{e_2}$
- (ii) $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k \rightarrow \infty} (\tilde{\alpha} u_{e_k}^k) = \tilde{\alpha} u_{e_1}$, where $\tilde{0} \neq \tilde{\alpha} \in F$.

Proof. The proof of the theorem can be obtained as the proof of theorem 3.1, so omitted. \square

Theorem 3.4 Let $\theta = (k_s)$ be a lacunary sequence. A sequence $v = (v_{e_k}^k)$ in \tilde{V} is $\mathcal{S}_\theta(G_N, B_N, Y_N)$ -convergent to v_e , if and only if \exists a subset $K = \{k_1, k_2, \dots\}$ of \mathbb{N} s.t $\delta_\theta(K) = 1$ and $(G_N, B_N, Y_N) - \lim_{\substack{k \in K \\ k \rightarrow \infty}} v_{e_k}^k = v_e$.

Proof. First suppose that $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k \rightarrow \infty} v_{e_k}^k = v_e$. For $\tilde{\eta} > \tilde{0}$ and $\beta \in \mathbb{N}$, define the set

$$\begin{aligned} K_{G_N, B_N, Y_N}(\beta, \tilde{\eta}) &= \left\{ k \in \mathbb{N} : G_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) > 1 - \frac{1}{\beta} \text{ and} \right. \\ &\quad \left. B_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) < \frac{1}{\beta}, Y_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) < \frac{1}{\beta} \right\} \text{ and} \\ K_{G_N, B_N, Y_N}^C(\beta, \tilde{\eta}) &= \left\{ k \in \mathbb{N} : G_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \leq 1 - \frac{1}{\beta} \text{ or} \right. \\ &\quad \left. B_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \geq \frac{1}{\beta}, Y_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \geq \frac{1}{\beta} \right\}. \end{aligned}$$

Since $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k \rightarrow \infty} v_{e_k}^k = v_e$, it follows that $\delta_\theta(K_{G_N, B_N, Y_N}^C(\beta, \tilde{\eta})) = 0$. Furthermore, for $\tilde{\eta} > \tilde{0}$ and $\beta \in \mathbb{N}$, we observe $K_{G_N, B_N, Y_N}(\beta, \tilde{\eta}) \supset K_{G_N, B_N, Y_N}(\beta + 1, \tilde{\eta})$ and

$$\delta_\theta(K_{G_N, B_N, Y_N}(\beta, \tilde{\eta})) = 1. \tag{5}$$

Now, we have to show that, for $k \in K_{G_N, B_N, Y_N}(\beta, \tilde{\eta})$, $(G_N, B_N, Y_N) - \lim_{\substack{k \in K \\ k \rightarrow \infty}} v_{e_k}^k = v_e$. Suppose for $k \in K_{G_N, B_N, Y_N}(\beta, \tilde{\eta})$, $(v_{e_k}^k)$ is not convergent to v_e w.r.t (G_N, B_N, Y_N) . Then \exists some $\xi > 0$ and a +ve integer k_0 s.t $G_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \leq 1 - \xi$ or $B_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \geq \xi$, $Y_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \geq \xi \forall k > k_0$. Let $G_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) > 1 - \xi$ and $B_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) < \xi$, $Y_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) < \xi \forall k < k_0$. Then

$$\delta_\theta(\{k \in \mathbb{N} : G_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) > 1 - \xi \text{ and } B_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) < \xi, Y_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) < \xi\}) = 0.$$

Since $\xi > \frac{1}{\beta}$ where $\beta \in \mathbb{N}$, we have $\delta_\theta(K_{G_N, B_N, Y_N}(\beta, \tilde{\eta})) = 0$. In this way we obtained a contradiction to (5) as $\delta_\theta(K_{G_N, B_N, Y_N}(\beta, \tilde{\eta})) = 1$. Hence, $(G_N, B_N, Y_N) - \lim_{\substack{k \in K \\ k \rightarrow \infty}} v_{e_k}^k = v_e$.

Conversely, Suppose that \exists a subset $K = \{k_1, k_2, \dots, k_j, \dots\}$ of \mathbb{N} with $\delta_\theta(K) = 1$ and $(G_N, B_N, Y_N) - \lim_{k \rightarrow \infty} v_{e_k}^k = v_e$ over K i.e., $(G_N, B_N, Y_N) - \lim_{\substack{k \in K \\ k \rightarrow \infty}} v_{e_k}^k = v_e$. Let $\epsilon > 0$ and $\tilde{\eta} > \tilde{0}$, $\exists k_{j_0} \in \mathbb{N}$ s.t for all $k_j \geq k_{j_0}$, $G_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) > 1 - \epsilon$ and $B_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) < \epsilon$, $Y_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) < \epsilon$. So if we consider the set

$$T_{G_N, B_N, Y_N}(\epsilon, \tilde{\eta}) = \left\{ k \in \mathbb{N} : G_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \geq \epsilon, Y_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \geq \epsilon \right\},$$

then it is easy to see that $T_{G_N, B_N, Y_N}(\epsilon, \tilde{\eta}) \subset \mathbb{N} - \{k_{j_0+1}, k_{j_0+2}, \dots\}$. This immediately implies that $\delta_\theta(T_{G_N, B_N, Y_N}(\epsilon, \tilde{\eta})) \leq \delta_\theta(\mathbb{N}) - \delta_\theta(\{k_{j_0+1}, k_{j_0+2}, \dots\}) = 1 - 1 = 0$ and therefore $\delta_\theta(T_{G_N, B_N, Y_N}(\epsilon, \tilde{\eta})) = 0$ as $\delta_\theta(T_{G_N, B_N, Y_N}(\epsilon, \tilde{\eta}))$ can not be negative. This shows that $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{n \rightarrow \infty} v_{e_k}^k = v_e$. \square

4. Lacunary statistical completeness in NSNLS

Definition 4.1 A sequence $v = (v_{e_k}^k)$ of soft points in \tilde{V} is said to be lacunary statistically Cauchy (or \mathcal{S}_θ -Cauchy) w.r.t neutrosophic soft norm (G_N, B_N, Y_N) if for each $\epsilon > 0$ and $\tilde{\eta} > \tilde{0}$, $\exists p \in \mathbb{N}$ s.t

$$\lim_{s \rightarrow \infty} \frac{1}{h_s} \left| \left\{ k \in I_s : G_N(v_{e_k}^k \ominus v_{e_p}^p, \tilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v_{e_k}^k \ominus v_{e_p}^p, \tilde{\eta}) \geq \epsilon, Y_N(v_{e_k}^k \ominus v_{e_p}^p, \tilde{\eta}) \geq \epsilon \right\} \right| = 0,$$

or equivalently, the θ -density of the set K is zero, i.e., $\delta_\theta(K) = 0$ where

$$K = \{k \in \mathbb{N} : G_N(v_{e_k}^k \ominus v_{e_p}^p, \tilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v_{e_k}^k \ominus v_{e_p}^p, \tilde{\eta}) \geq \epsilon, Y_N(v_{e_k}^k \ominus v_{e_p}^p, \tilde{\eta}) \geq \epsilon\}.$$

Theorem 4.1 Let $\theta = (k_s)$ be any lacunary sequence. If a sequence $v = (v_{e_k}^k)$ of soft points in \tilde{V} is $\mathcal{S}_\theta(G_N, B_N, Y_N)$ -convergent, then it is $\mathcal{S}_\theta(G_N, B_N, Y_N)$ Cauchy.

Proof. Let $v = (v_{e_k}^k)$ be any lacunary statistically convergent sequence with $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k \rightarrow \infty} v_{e_k}^k = v_e$. Let $\epsilon > 0$ and $\tilde{\eta} > \tilde{0}$. Choose $\varrho > 0$ s.t (1) is satisfied. Define a set,

$$M(\varrho, \tilde{\eta}) = \left\{ k \in \mathbb{N} : G_N(v_{e_k}^k \ominus v_e, \frac{\tilde{\eta}}{2}) \leq 1 - \varrho \text{ or } B_N(v_{e_k}^k \ominus v_e, \frac{\tilde{\eta}}{2}) \geq \varrho, Y_N(v_{e_k}^k \ominus v_e, \frac{\tilde{\eta}}{2}) \geq \varrho \right\},$$

then

$$M^C(\varrho, \tilde{\eta}) = \left\{ k \in \mathbb{N} : G_N(v_{e_k}^k \ominus v_e, \frac{\tilde{\eta}}{2}) > 1 - \varrho \text{ and } B_N(v_{e_k}^k \ominus v_e, \frac{\tilde{\eta}}{2}) < \varrho, Y_N(v_{e_k}^k \ominus v_e, \frac{\tilde{\eta}}{2}) < \varrho \right\}.$$

Since $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{n \rightarrow \infty} v_{e_k}^k = v_e$, so $\delta_\theta(M(\varrho, \tilde{\eta})) = 0$ and $\delta_\theta(M^C(\varrho, \tilde{\eta})) = 1$. Let $p \in M^C(\varrho, \tilde{\eta})$, then

$$G_N\left(v_{e_p}^p \ominus v_e, \frac{\tilde{\eta}}{2}\right) > 1 - \varrho \text{ and } B_N\left(v_{e_p}^p \ominus v_e, \frac{\tilde{\eta}}{2}\right) < \varrho, Y_N\left(v_{e_p}^p \ominus v_e, \frac{\tilde{\eta}}{2}\right) < \varrho. \tag{6}$$

Now, let $T(\epsilon, \tilde{\eta}) = \{k \in \mathbb{N} : G_N(v_{e_k}^k \ominus v_{e_p}^p, \tilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v_{e_k}^k \ominus v_{e_p}^p, \tilde{\eta}) \geq \epsilon, Y_N(v_{e_k}^k \ominus v_{e_p}^p, \tilde{\eta}) \geq \epsilon\}$, then we have to show that $T(\epsilon, \tilde{\eta}) \subseteq M(\varrho, \tilde{\eta})$. Let $m \in T(\epsilon, \tilde{\eta})$, then

$$G_N(v_{e_m}^m \ominus v_{e_p}^p, \tilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v_{e_m}^m \ominus v_{e_p}^p, \tilde{\eta}) \geq \epsilon, Y_N(v_{e_m}^m \ominus v_{e_p}^p, \tilde{\eta}) \geq \epsilon. \tag{7}$$

Case 1: If $G_N(v_{e_m}^m \ominus v_{e_p}^p, \tilde{\eta}) \leq 1 - \epsilon$, then $G_N\left(v_{e_m}^m \ominus v_e, \frac{\tilde{\eta}}{2}\right) \leq 1 - \varrho$ and therefore $m \in M(\varrho, \tilde{\eta})$.

As otherwise i.e., if $G_N\left(v_{e_m}^m \ominus v_e, \frac{\tilde{\eta}}{2}\right) > 1 - \varrho$, then by (1), (6) and (7) we get

$$\begin{aligned} 1 - \epsilon &\geq G_N(v_{e_m}^m \ominus v_{e_p}^p, \tilde{\eta}) = G_N\left(v_{e_m}^m \ominus v_e \oplus v_e \ominus v_{e_p}^p, \frac{\tilde{\eta}}{2} \oplus \frac{\tilde{\eta}}{2}\right) \\ &\geq G_N\left(v_{e_m}^m \ominus v_e, \frac{\tilde{\eta}}{2}\right) \circ G_N\left(v_{e_p}^p \ominus v_e, \frac{\tilde{\eta}}{2}\right) \\ &> (1 - \varrho) \circ (1 - \varrho) \\ &> 1 - \epsilon, \end{aligned}$$

which is impossible. Thus, $T(\epsilon, \tilde{\eta}) \subseteq M(\varrho, \tilde{\eta})$.

Case 2: If $B_N(v_{e_m}^m \ominus v_{e_p}^p, \tilde{\eta}) \geq \epsilon$, then $B_N\left(v_{e_m}^m \ominus v_e, \frac{\tilde{\eta}}{2}\right) \geq \varrho$ and therefore $m \in M(\varrho, \tilde{\eta})$. As otherwise i.e., if $B_N\left(v_{e_m}^m \ominus v_e, \frac{\tilde{\eta}}{2}\right) < \varrho$, then by (1), (6) and (7) we get

$$\begin{aligned} \epsilon &\leq B_N(v_{e_m}^m \ominus v_{e_p}^p, \tilde{\eta}) = B_N\left(v_{e_m}^m \ominus v_e \oplus v_e \ominus v_{e_p}^p, \frac{\tilde{\eta}}{2} \oplus \frac{\tilde{\eta}}{2}\right) \\ &\leq B_N\left(v_{e_m}^m \ominus v_e, \frac{\tilde{\eta}}{2}\right) \diamond B_N\left(v_{e_p}^p \ominus v_e, \frac{\tilde{\eta}}{2}\right) \\ &< \varrho \diamond \varrho \\ &< \epsilon, \end{aligned}$$

which is impossible.

Also, If $Y_N(v_{e_m}^m \ominus v_{e_p}^p, \tilde{\eta}) \geq \epsilon$, then $Y_N\left(v_{e_m}^m \ominus v_e, \frac{\tilde{\eta}}{2}\right) \geq \varrho$ and therefore $m \in M(\varrho, \tilde{\eta})$. As otherwise i.e., if $Y_N\left(v_{e_m}^m \ominus v_e, \frac{\tilde{\eta}}{2}\right) < \varrho$, then by (1), (6) and (7) we get

$$\begin{aligned} \epsilon &\leq Y_N(v_{e_m}^m \ominus v_{e_p}^p, \tilde{\eta}) = Y_N\left(v_{e_m}^m \ominus v_e \oplus v_e \ominus v_{e_p}^p, \frac{\tilde{\eta}}{2} \oplus \frac{\tilde{\eta}}{2}\right) \\ &\leq Y_N\left(v_{e_m}^m \ominus v_e, \frac{\tilde{\eta}}{2}\right) \diamond Y_N\left(v_{e_p}^p \ominus v_e, \frac{\tilde{\eta}}{2}\right) \\ &< \varrho \diamond \varrho \\ &< \epsilon, \end{aligned}$$

which is impossible. Thus, $T(\epsilon, \tilde{\eta}) \subseteq M(\varrho, \tilde{\eta})$.

Hence in all cases, $T(\epsilon, \tilde{\eta}) \subseteq M(\varrho, \tilde{\eta})$. Since $\delta_\theta(M(\varrho, \tilde{\eta})) = 0$, so $\delta_\theta(T(\epsilon, \tilde{\eta})) = 0$, and therefore $v = (v_{e_k}^k)$ is $\mathcal{S}_\theta(G_N, B_N, Y_N)$ Cauchy. \square

Definition 4.2 A NSNLS \tilde{V} is said to be \mathcal{S}_θ -complete if every \mathcal{S}_θ -Cauchy sequence in \tilde{V} w.r.t neutrosophic soft norm- (G_N, B_N, Y_N) is \mathcal{S}_θ -convergent w.r.t neutrosophic soft norm- (G_N, B_N, Y_N) .

Theorem 4.2 Let $\theta = (k_s)$ be any lacunary sequence. Then every NSNLS \tilde{V} is \mathcal{S}_θ -complete but not complete in general.

Proof. Let $v = (v_{e_k}^k)$ be \mathcal{S}_θ -Cauchy but not \mathcal{S}_θ -convergent w.r.t neutrosophic soft norm- (G_N, B_N, Y_N) . For a given $\epsilon > 0$ and $\tilde{\eta} > \tilde{0}$. Choose $\varrho > 0$ s.t (1) is satisfied. Now

$$\begin{aligned} G_N(v_{e_k}^k \ominus v_{e_p}^p, \tilde{\eta}) &\geq G_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \circ G_N(v_{e_p}^p \ominus v_e, \tilde{\eta}) > (1 - \varrho) \circ (1 - \varrho) > 1 - \epsilon \\ B_N(v_{e_k}^k \ominus v_{e_p}^p, \tilde{\eta}) &\leq B_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \diamond B_N(v_{e_p}^p \ominus v_e, \tilde{\eta}) < \varrho \diamond \varrho < \epsilon \\ Y_N(v_{e_k}^k \ominus v_{e_p}^p, \tilde{\eta}) &\leq Y_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \diamond Y_N(v_{e_p}^p \ominus v_e, \tilde{\eta}) < \varrho \diamond \varrho < \epsilon. \end{aligned}$$

Since $v = (v_{e_k}^k)$ is not \mathcal{S}_θ -convergent w.r.t neutrosophic soft norm- (G_N, B_N, Y_N) . Therefore $\delta_\theta(H^C(\varrho, \tilde{\eta})) = 0$, where

$$H(\varrho, \tilde{\eta}) = \{k \in \mathbb{N} : \mathcal{B}_{v_{e_k}^k \ominus v_{e_p}^p}(\varrho) \leq 1 - \epsilon\}$$

and so $\delta_\theta(H(\varrho, \tilde{\eta})) = 1$ which is a contradiction, since $v = (v_{e_k}^k)$ was \mathcal{S}_θ -cauchy w.r.t neutrosophic soft norm- (G_N, B_N, Y_N) . So $v = (v_{e_k}^k)$ must be \mathcal{S}_θ -convergent w.r.t neutrosophic soft norm- (G_N, B_N, Y_N) . Hence every $NSNLS \tilde{V}$ is \mathcal{S}_θ -complete.

The following example demonstrates that $NSNLS$ is not complete in general:

Example 4.1[26] Let $\tilde{U} = (0, 1]$ and $G_N(v, \tilde{\eta}) = \frac{\tilde{\eta}}{\tilde{\eta} \oplus |v|}$, $B_N(v, \tilde{\eta}) = \frac{|v|}{\tilde{\eta} \oplus |v|}$, $Y_N(v, \tilde{\eta}) = \frac{|v|}{\tilde{\eta}}$ for all $v \in \tilde{U}$. Then $\tilde{V} = (\tilde{U}, G_N, B_N, Y_N, \min, \max)$ is $NSNLS$ but not complete, since the sequence of soft points $(\frac{1}{k})$ is cauchy w.r.t (G_N, B_N, Y_N) but not convergent w.r.t (G_N, B_N, Y_N) .

Theorem 4.3 If every \mathcal{S}_θ -cauchy sequence of soft points in \tilde{V} has a \mathcal{S}_θ -convergent subsequence then \tilde{V} is \mathcal{S}_θ -complete.

Proof. Let $v = (v_{e_k}^k)$ be any \mathcal{S}_θ -cauchy sequence of soft points in \tilde{V} which has a \mathcal{S}_θ -convergent subsequence $(v_{e_{k(j)}}^{k(j)})$ i.e., $\mathcal{S}_\theta - \lim_{j \rightarrow \infty} v_{e_{k(j)}}^{k(j)} = v_e$ for some v_e in \tilde{V} . Let $\epsilon > 0$ and $\tilde{\eta} > \tilde{0}$. Choose $\varrho > 0$ s.t (1) is satisfied. Since $v = (v_{e_k}^k)$ is \mathcal{S}_θ -cauchy, so $\exists n_0 \in \mathbb{N}$ s.t $\forall k, p \geq n_0$ $\delta_\theta(A) = 0$ where

$$A = \left\{ k \in \mathbb{N} : G_N \left(v_{e_k}^k \ominus v_{e_p}^p, \frac{\tilde{\eta}}{2} \right) \leq 1 - \varrho \text{ or } \right. \\ \left. B_N \left(v_{e_k}^k \ominus v_{e_p}^p, \frac{\tilde{\eta}}{2} \right) \geq \varrho, Y_N \left(v_{e_k}^k \ominus v_{e_p}^p, \frac{\tilde{\eta}}{2} \right) \geq \varrho \right\}.$$

Again since $\mathcal{S}_\theta - \lim_{j \rightarrow \infty} v_{e_{k(j)}}^{k(j)} = v_e$. So we have $\delta_\theta(B) = 0$, where

$$B = \left\{ k(j) \in \mathbb{N} : G_N \left(v_{e_{k(j)}}^{k(j)} \ominus v_e, \frac{\tilde{\eta}}{2} \right) \leq 1 - \varrho \text{ or } \right. \\ \left. B_N \left(v_{e_{k(j)}}^{k(j)} \ominus v_e, \frac{\tilde{\eta}}{2} \right) \geq \varrho, Y_N \left(v_{e_{k(j)}}^{k(j)} \ominus v_e, \frac{\tilde{\eta}}{2} \right) \geq \varrho \right\}.$$

Now define

$$D = \{k \in \mathbb{N} : G_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \leq 1 - \epsilon \text{ or } \\ B_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \geq \epsilon, Y_N(v_{e_k}^k \ominus v_e, \tilde{\eta}) \geq \epsilon\}.$$

We now show that $A^C \cap B^C \subseteq D^C$. Let $m \in A^C \cap B^C$. As $m \in A^C$, so

$$G_N \left(v_{e_m}^m \ominus v_{e_p}^p, \frac{\tilde{\eta}}{2} \right) > 1 - \varrho \text{ and} \\ B_N \left(v_{e_m}^m \ominus v_{e_p}^p, \frac{\tilde{\eta}}{2} \right) < \varrho, Y_N \left(v_{e_m}^m \ominus v_{e_p}^p, \frac{\tilde{\eta}}{2} \right) < \varrho, \tag{8}$$

and since $m \in B^C$, so $m = k(j_0)$ for $j_0 \in \mathbb{N}$ and

$$\begin{aligned}
 G_N\left(v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\tilde{\eta}}{2}\right) &> 1 - \varrho \text{ and} \\
 B_N\left(v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\tilde{\eta}}{2}\right) &< \varrho, Y_N\left(v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\tilde{\eta}}{2}\right) < \varrho.
 \end{aligned}
 \tag{9}$$

Now

$$\begin{aligned}
 G_N(v_{e_m}^m \ominus v_e, \tilde{\eta}) &= G_N\left(v_{e_m}^m \ominus v_{e_{k(j_0)}}^{k(j_0)} \oplus v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\tilde{\eta}}{2} \oplus \frac{\tilde{\eta}}{2}\right) \\
 &\geq G_N\left(v_{e_m}^m \ominus v_{e_{k(j_0)}}^{k(j_0)}, \frac{\tilde{\eta}}{2}\right) \circ G_N\left(v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\tilde{\eta}}{2}\right) \\
 &> (1 - \varrho) \circ (1 - \varrho) \text{ for } p = k(j_0) \\
 &> 1 - \epsilon
 \end{aligned}$$

and

$$\begin{aligned}
 B_N(v_{e_m}^m \ominus v_e, \tilde{\eta}) &= B_N\left(v_{e_m}^m \ominus v_{e_{k(j_0)}}^{k(j_0)} \oplus v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\tilde{\eta}}{2} \oplus \frac{\tilde{\eta}}{2}\right) \\
 &\leq B_N\left(v_{e_m}^m \ominus v_{e_{k(j_0)}}^{k(j_0)}, \frac{\tilde{\eta}}{2}\right) \diamond B_N\left(v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\tilde{\eta}}{2}\right) \\
 &< \varrho \diamond \varrho \text{ for } p = k(j_0) \\
 &< \epsilon,
 \end{aligned}$$

$$\begin{aligned}
 Y_N(v_{e_m}^m \ominus v_e, \tilde{\eta}) &= Y_N\left(v_{e_m}^m \ominus v_{e_{k(j_0)}}^{k(j_0)} \oplus v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\tilde{\eta}}{2} \oplus \frac{\tilde{\eta}}{2}\right) \\
 &\leq Y_N\left(v_{e_m}^m \ominus v_{e_{k(j_0)}}^{k(j_0)}, \frac{\tilde{\eta}}{2}\right) \diamond Y_N\left(v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\tilde{\eta}}{2}\right) \\
 &< \varrho \diamond \varrho \text{ for } p = k(j_0) \\
 &< \epsilon,
 \end{aligned}$$

by (1), (8) and (9)

which implies that $m \in D^C$, so $A^C \cap B^C \subseteq D^C$ or $D \subseteq A \cup B$. Therefore, $\delta_\theta(D) \leq \delta_\theta(A \cup B) = 0$. This shows that $v = (v_{e_k}^k)$ is \mathcal{S}_θ -convergent and therefore, \tilde{V} is \mathcal{S}_θ -complete. \square

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