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Softmax function based neutrosophic aggregation operators and application in multi-attribute decision making problem

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Abstract. The softmax function is a well-known generalization of the logistic function. It has been extensively applied in various probabilistic classification methods such as softmax regression, linear discriminant analysis, naive Bayes classifiers, and artificial neural networks. Inspired by the advantages of the softmax function, we have developed the softmax function-based single-valued neutrosophic aggregation operators. Then we have established some essential properties of aggregation operators based on the softmax function with the neutrosophic set. Additionally, we have defined a multi-attribute decision-making method based on the proposed aggregation operators. Using the proposed MCDM method, we have developed a novel algorithm. This algorithm helps to examine FD-risk assessment problems. Also, the proposed algorithm process is a reasonable strategy for the decision-making problem. It is easy to recognize when choosing a neutrosophic set of information for a practical decision problem. We used this proposed MADM method to exercise a realistic MADM problem with neutrosophic information. Finally, we have considered one numerical illustration to show the validity and reliability of the proposed methods.

Keywords: Softmax function; Single valued neutrosophic set, Aggregation operator, Multi attribute decision making strategy

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1. **Introduction**

An intuitionistic fuzzy set (Atanassov 1986) is an effectual generalization of the fuzzy set (Zadeh 1965). But single-valued neutrosophic (SVN) set (Wang 2010) is a successful generalization of the fuzzy set (Zadeh 1965). SVN set each element is expressed by a triplet of membership degrees which are membership, indeterminacy, and falsity degrees. The Sum of the membership degrees value lies between 0 and 3. Day by day, SVNS has received more intent from the researchers due to its structure formation. In the decision of some modern science, real problems depend on multi-attribute decision making (MADM). Because MADM provide the best choice option to select the alternatives with respect to the attributes. Expressing the attribute's value in the decision-making problem is a significantly important factor. Sometimes in the decision-making problem, so many uncertainties and complexity occur. In this case, the SVN set has a significant role in expressing this information. The possiblistic mean-variance of the SVN set was developed by Garai et al. (2020a). Recently many researchers proposed various strategies for their work considered by the SVN set, such as Jun (2013), Garai & Garg (2022a), Sod (2018), Wei (2018), Ren (2017), Biswas (2016), Pramanik (2017), Garai & Garg (2022b) and so on.

In an uncertain environment, some decision-making (DM) problems handle by the aggregation operator. Using the different aggregation operators, many researchers recently have on the DM problem under the SVN environment. For instance, Garg and Nancy (2018) developed some new hybrid aggregation operators using arithmetic and geometric aggregation operators. They also solved one MADM problem in the SVN environment. Ji et al. (2016) proposed the SVNS-Frank normalized Bonferroni mean (SVNFNPBM) operator to aggregate all values. This SVNFNPBM operator applied to choose the third-party logistics example. Some arithmetic operations of SVN numbers use frank norm operators as defined by Nancy and Garg (2016). Also, it applied to MADM problems. Sodenkamp et al. (2018) present an aggregation strategy for multi-attribute group decision making (MAGDM) problems under an SVN environment. Liu et al. (2014) defined some aggregation operators by combining Hamacher operations and generalized aggregation operators in the SVN environment. Recently, Chen and Ye (2017) considered two operators: Dombi weighted geometric average and Dombi weighted arithmetic average operators under an SVN environment.

Liu et al. (2019) developed a new single-valued neutrosophic Schweizer-sklar prioritized weighted averaging (SVNSSPRWA) operator. After that, he studied some basic properties of the proposed aggregation operators. It also gave the two decision-making models for showing the effectiveness of these novel operators. Further, Lui et al. (2020) developed the novel weighted single-valued neutrosophic power dual muirhead mean (WSVNPDMM) operator and single-valued neutrosophic power dual muirhead mean (SVNPDMM) operator. Further, they proposed a new technique for the MAGDM problem based on these aggregation operators. Tan and Zang (2020) defined a new distance measure, similarity measure, and neutrosophic entropy for straight SVN sets. Rong et al. (2020) defined several new operational laws of SVN number depending on Archimedean copula and co-copula (ACC) and discussed their related properties. They proposed some novel power aggregation operators (AOs) to merge SVN information, i.e., SVN copula power geometric (SVNCPG), weighted SVNCPA (WSVNCPA) operator, etc. Also, he has proposed MADM problems with SVN information using these operators.

Nowadays, many researchers are developing some operators in the SVN environment. Based on the dombi t-norm and t-conorm, Chen and Ye (2017) developed the SVNDWGAA operator to deal with the aggregation of SVN numbers in the MADM process. Li et al. (2016) improved a generalized weighted geometric heronian mean (IGWGHM) operator. Also, Li et al. (2016) proposed the improved weighted heronian mean (NNIGWHM) operator and improved generalized weighted geometric heronian mean (NNIGWGHM) operator for neutrosophic numbers. And these operators applied to MADM problems. Garai et al. (2020b) proposed the new ranking of SVN-number and used it for the MADM problem. Recently, Wei and Wei (2018) presented some SVN-dombi prioritized average (SVNDPA) operators and SVN-Dombi prioritized geometric (SVNDPG) operators. They utilized these operators to solve MADM problems in SVNS environment.

The paper is structured as follows: In Section 2, some basic concepts and definitions related to NS, and SVNS are discussed, and also presented score function and accuracy function of SVNNs. In Section 3, SVNWA, SVNWG, GSSVNWA, and GSSVNWG are defined and introduced as some basic properties and examples. Section 4 presented a MADM strategy based on the proposed aggregation operators. In section 5, we solved a numerical model to check the validity and applicability of the proposed method. Finally, this study's conclusions and future research direction are presented in Section 6.

1.1. *Motivation*

So, the above discussion says that many aggregation operators are extended with the different single-valued neutrosophic information. Then some researchers are successfully applied to many MADM problems and multi-attribute group decision making (MAGDM) problems under SVNS environments. But in this weighted aggregation, operators have certain restrictions because most of the aggregations are not applicable without SVNNs. Hence some operators cannot be relevant in some real-life problems. The softmax function handles these types of restrictions.

Previously, many researchers worked on MADM under a fuzzy environment using different usual aggregation operators. Torres et al. (2014) applied the softmax function in decision-making problems under an uncertain fuzzy set environment. He proposes a series of aggregation operators based on the softmax function. Later, Yu (2016) extended the softmax function-based aggregation operator in an intuitionistic fuzzy set environment. He developed the series of aggregation operators and applied these to a MADM problem. When we ranked the different alternatives to real MADM problems under the SVN environment, some had difficulties raised that time. We cannot organize the alternatives to the MADM problem using standard ranking methods like a fuzzy number. Now, how can we rank the alternatives with single-valued neutrosophic information? Also, how can we apply the softmax function-based aggregation operator in MADM? What is the usefulness of the softmax function-based aggregation operator in the MADM problem? When we studied some articles related to this research, a few questions arose in our minds. Therefore from that place, we try to establish a best ranking method with the help of a softmax function-based aggregation operator.

This paper has developed the softmax function-based single-valued neutrosophic aggregation operators. This aggregation operator is an extension of IF aggregation operators. We have proposed a softmax SVN weighted average (SVNWA) operator; Softmax SVN weighted geometric (SVNWG). In addition, we also developed some aggregation operators: generalized softmax single-valued neutrosophic weighted average (GSSVNWA) operator and generalized softmax SVN weighted geometric (GSSVNIFWG) operator. Some fundamental properties of softmax-based aggregation operators are developed here. We have introduced a novel MADM method using the proposed softmax-based aggregation operators. Finally, To check the importance of the proposed MADM method numerically.

1.2. *Novelty*

This paper extends the softmax function-based intuitionistic fuzzy (IF) aggregation operators to softmax function-based SVN aggregation operators. Additionally, some softmax function-based aggregation operators are developed here, which are the softmax SVN weighted average (SVNWA) operator, Softmax SVN weighted geometric (SVNWG) operator, and generalized softmax single-valued neutrosophic weighted average (GSSVNWA) operator and generalized softmax SVN weighted geometric (GSSVNIFWG) operator. Then, we proposed the essential properties of the proposed softmax-based aggregation operators. Further, this decision-making technique is applied to real MADM problems.

The main contributions of the paper is that:

- We extend the SIFWA operator to SSVNWA operator.
- We extend the SIFWG operator to SSVNWG operator.
- We extend the GSIFWA operator to GSSVNWA operator.
- We extend the GSIFWG operator to GSSVNWG operator.
- We develop a MADM strategy based on the proposed operators.
- To check the validity of MADM strategy we solved one real MADM problem.

2. Basic Preliminaries

Let X be a universe set. A nutrosophic (Smarandache 1998) sates $\tilde{E}$ over X is defined by $\tilde{E} = \{\langle x, (T_{\tilde{E}}(x), I_{\tilde{E}}(x),\ F_{\tilde{E}}(x))\rangle : x \in X\}$, where $T_{\tilde{E}}(x), I_{\tilde{E}}(x)$ and $F_{\tilde{E}}(x)$ are called truth membership function, indeterminacy-membership function and falsity membership functions respectively. They are defined as

$$T_{\tilde{E}} : X \to]^{-}0, 1^{+}[,\ I_{\tilde{E}} : X \to]^{-}0, 1^{+}[,\ F_{\tilde{E}} : X \to]^{-}0, 1^{+}[$$

such that $0^{-} \leq T_{\tilde{E}}(x) + I_{\tilde{E}}(x) + F_{\tilde{E}}(x) \leq 3^{+}$ Let X be a universe set (Wang 2010). A SVN-set $\tilde{E}$ over X is a neutroophic set, but the truth-membership function, indeterminacy-membership function and falsity-membership function are respectively defined by

$$T_{\tilde{E}} : X \to [0, 1],\ I_{\tilde{E}} : X \to [0, 1],\ F_{\tilde{E}} : X \to [0, 1]$$

such that $0 \leq T_{\tilde{E}}(x) + I_{\tilde{E}}(x) + F_{\tilde{E}}(x) \leq 3$. For convenience, a SVNN can be expressed to be $\tilde{E} = (T_{\tilde{E}}, I_{\tilde{E}}, F_{\tilde{E}}), T_{\tilde{E}} \in [0, 1], I_{\tilde{E}} \in [0, 1], F_{\tilde{E}} \in [0, 1]$ and $0 \leq T_{\tilde{E}} + I_{\tilde{E}} + F_{\tilde{E}} \leq 3$. Let $\tilde{C} = \{\langle x, (T_{\tilde{C}}(x), I_{\tilde{C}}(x),\ F_{\tilde{C}}(x))\rangle : x \in X\}$ and $\tilde{E} = \{\langle x, (T_{\tilde{E}}(x), I_{\tilde{E}}(x),\ F_{\tilde{E}}(x))\rangle : x \in X\}$ be two SVN-sets in X, then operations between them defined (Wang 2010) as follows:

(i) $\tilde{C} \subseteq \tilde{E}$ iff $T_{\tilde{C}}(x) \leq T_{\tilde{E}}(x), I_{\tilde{C}}(x) \geq I_{\tilde{E}}(x), T_{\tilde{C}}(x) \geq T_{\tilde{E}}(x)$ for all $x \in X$.
(ii) $\tilde{C} = \tilde{E}$ iff $\tilde{C} \subseteq \tilde{E}$ and $\tilde{E} \subseteq \tilde{C}$ for all $x \in X$.
(iii) $\tilde{E}^{c} = \{\langle x, (F_{\tilde{E}}(x), 1 - I_{\tilde{E}}(x),\ T_{\tilde{E}}(x))\rangle : x \in X\}$ for all $x \in X$.
(iv) $\tilde{C} \cup \tilde{E} = \{\langle x, max(T_{\tilde{C}}(x), T_{\tilde{E}}(x)), min(I_{\tilde{C}}(x), I_{\tilde{E}}(x)), min(F_{\tilde{C}}(x), F_{\tilde{E}}(x))\rangle : x \in X\}$ for all $x \in X$.
(v) $\tilde{C} \cap \tilde{E} = \{\langle x, min(T_{\tilde{C}}(x), T_{\tilde{E}}(x)), max(I_{\tilde{C}}(x), I_{\tilde{E}}(x)), max(F_{\tilde{C}}(x), F_{\tilde{E}}(x))\rangle : x \in X\}$ for all $x \in X$.

Let $\tilde{E}, \tilde{E}_1, \tilde{E}_2$ be three SVN-sets in X. Then, the arithmetic (Wang 10) operations are defined as follows:

(i) $\tilde{E}_1 + \tilde{E}_2 = \{\langle x, T_{\tilde{E}_1}(x) + T_{\tilde{E}_2}(x) - T_{\tilde{E}_1}(x).T_{\tilde{E}_2}(x), I_{\tilde{E}_1}(x).I_{\tilde{E}_2}(x), F_{\tilde{E}_1}(x).F_{\tilde{E}_2}(x)\rangle : x \in X\}$ for all $x \in X$.
(ii) $\tilde{E}_1.\tilde{E}_2 = \{\langle x, T_{\tilde{E}_1}(x).T_{\tilde{E}_2}(x), I_{\tilde{E}_1}(x) + I_{\tilde{E}_2}(x) - I_{\tilde{E}_1}(x).I_{\tilde{E}_2}(x), F_{\tilde{E}_1}(x) + F_{\tilde{E}_2}(x) - F_{\tilde{E}_1}(x).F_{\tilde{E}_2}(x)\rangle : x \in X\}$ for all $x \in X$.
(iii) $\lambda.\tilde{E} = \{\langle x, (1 - (1 - T_{\tilde{E}}(x))^{\lambda}, (I_{\tilde{E}}(x))^{\lambda}, (F_{\tilde{E}}(x))^{\lambda}\rangle : x \in X\}$ for all $x \in X$.
(iv) $\tilde{E}^{\lambda} = \{\langle x, (T_{\tilde{E}}(x))^{\lambda}, 1 - (1 - I_{\tilde{E}}(x))^{\lambda}, 1 - (1 - F_{\tilde{E}}(x))^{\lambda}\rangle : x \in X\}$ for all $x \in X$, Where $\lambda > 0$ is a parameter.

For any SVN set, the ranking method is very significant and many research results have been received (Zhang et al. 2014, Wang et al. 2010). Zhang et al. 2014 given a method based on score function and accuracy function. For any SVN-set $A = (T_A, I_A, F_A)$, the accuracy and

score function defined as:
The score function of $\tilde{E}$ is

$$S(\tilde{E}) = \frac{2 + T_{\tilde{E}} - I_{\tilde{E}} - F_{\tilde{E}}}{3}, S(\tilde{E}) \in [0, 1] \tag{1}$$

and the accuracy function of $\tilde{E}$ is

$$H(\tilde{E}) = T_{\tilde{E}} - F_{\tilde{E}}, H(\tilde{E}) \in [-1, 1] \tag{2}$$

Zhang et al. 2014 gave an order relation between two SVN numbers, which is defined as follows:
Let $\tilde{C} = (T_{\tilde{C}}, I_{\tilde{C}}, F_{\tilde{C}})$ and $\tilde{E} = (T_{\tilde{E}}, I_{\tilde{E}}, F_{\tilde{E}})$ be two SVNNs.
Now, if $S(A) > S(B)$, then $\tilde{C} \succ \tilde{E}$. Again if $S(\tilde{C}) = S(\tilde{E})$, then

(i) *If $H(\tilde{C}) = H(\tilde{C})$, then $\tilde{C} \approx \tilde{E}$.*
(ii) *If $H(\tilde{C}) > H(\tilde{E})$, then $\tilde{C} \succ \tilde{E}$.*

3. Softmax function based aggregation operators

This section has discussed the softmax function and its essential properties. Here we established some rigorous methods related to the softmax function-based aggregation operator.

3.1. *Softmax function*

A softmax function is a generalization form of the logistic process in the area of mathematics. It has been progressively applied to many research fields, for instance, machine learning (Jacobs 1991, Torres 2003) and decision making (Torres 14, Yu 16). The mathematical form of the softmax function is represented as follows:

$$\phi_k(j, \vartheta_1, \vartheta_2, ..., \vartheta_n) = \phi_k^j = \frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{n} \exp(\vartheta_j/k)}, k > 0. \tag{3}$$

For the SVN-sets $\alpha_j (j = 1, 2, 3, ..., n)$, S_j is the score value of SVN-number α_j. Every ϑ_j is formulated by given the equation

$$\vartheta_j = \begin{cases} \prod_{i=1}^{j-1} S_i, & \text{j} = 2, 3, ..., n \\ 1 & \text{j} = 1 \end{cases} \tag{4}$$

where k is the modulation parameter. *Some properties of softmax function (Yu 2016) are defined as follows:*

(i) $0 \leq \phi_k^j \leq 1$.
(ii) $\sum_{j=1}^{n} \phi_k^j = 1$.

Softmax function Torres 2014 has the non linear characteristic, monotonous and boundedness properties.

3.2. *SVN-sets aggregation operators based on softmax function*

In this section, we have extended the softmax function based on IF aggregation operators, such as softmax IF weighted average operator (SIFWA), softmax IF weighted geometric (SIFWG) operator, generalized softmax IF weighted average (GSIFWA) operator, and generalized softmax IF weighted geometric (GSIFWG) operator to softmax SVN weighted average (SVNWA) operator; softmax SVN weighted geometric (SVNWG) operator, generalized softmax SVN weighted average (GSSVNWA) operator, and generalized softmax SVN weighted geometric (GSSVNIFWG) operator, respectively. *Let $\alpha_j (j = 1, 2, ..., n)$ be a collection of SVNNs. Then softmax single valued neutrosophic weighted average (SSVNWA) operator is a function from $\alpha^n \to \alpha$ such that*

$$SSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) = \oplus_{j=1}^{n}(\phi_k^j \alpha_j) = (\phi_k^1 \alpha_1) \oplus (\phi_k^2 \alpha_2) \oplus ... \oplus (\phi_k^n \alpha_n)$$

where, $\phi_k^j = \frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{n}\exp(\vartheta_j/k)}, \vartheta_j = \begin{cases} \prod_{i=1}^{j-1} S_i, & \text{j} = 2, 3, ..., n \\ 1 & \text{j} = 1 \end{cases}$, S_i is the score function of the SVN-number α_i.

Theorem 3.1. Let $\alpha_j (j = 1, 2, ..., n)$ be a collection of SVN-numbers, then aggregated value of SVN-numbers using the SSVNWA operation is also a SVN-number. The SSVNWA operator can be generated as:

$$SSVNWA\left(\alpha_1, \alpha_2, ..., \alpha_n\right) = \left(1 - \prod_{j=1}^{n}\left(1 - T_j\right)^{\frac{\exp\left(\vartheta_j/k\right)}{\sum_{j=1}^{n}\exp\left(\vartheta_j/k\right)}}, \prod_{j=1}^{n}\left(I_j\right)^{\frac{\exp\left(\vartheta_j/k\right)}{\sum_{j=1}^{n} exp\left(\vartheta_j/k\right)}}, \prod_{j=1}^{n}\left(F_j\right)^{\frac{\exp\left(\vartheta_j/k\right)}{\sum_{j=1}^{n}\exp\left(\vartheta_j/k\right)}}\right) \tag{5}$$

Proof. : *We proof the above theorem 1 by using mathematical induction. For $n = 1$, we have:*

$$SSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) = \left(1 - \prod_{j=1}^{1}(1 - T_j), \prod_{j=1}^{1}(I_j), \prod_{j=1}^{1}(F_j)\right) = \left(T_1, I_1, F_1\right).$$

Since for $n = 1$, $\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{n}\exp(\vartheta_j/k)} = 1$. Thus Eq.(5) holds for n = 1. Assume that the Eq. (5) holds for n = m,

$$SSVNWA(\alpha_1, \alpha_2, ..., \alpha_m) = \left(1 - \prod_{j=1}^{m}(1 - T_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}}, \prod_{j=1}^{m}(I_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}}, \prod_{j=1}^{m}(F_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}}\right)$$

Now we prove that the Eq. (5) holds for $n = m + 1$.

$$\begin{aligned} SSVNWA(\alpha_1, \alpha_2, ..., \alpha_m, \alpha_{m+1}) &= (\phi_k^1 \alpha_1) \oplus (\phi_k^2 \alpha_2) ... \oplus (\phi_k^m \alpha_m) \oplus (\phi_k^{m+1} \alpha_{m+1}) \\ &= \Bigg(\Bigg(1 - \prod_{j=1}^{m} (1 - T_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m} \exp(\vartheta_j/k)}} \Bigg) + \Bigg(1 - (1 - T_{m+1})^{\frac{\exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1} \exp(\vartheta_j/k)}} \Bigg) \\ &- \Bigg(\Bigg(1 - \prod_{j=1}^{m} (1 - T_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m} \exp(\vartheta_j/k)}} \Bigg) \times \Bigg(1 - (1 - T_{m+1})^{\frac{\exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1} \exp(\vartheta_j/k)}} \Bigg) \Bigg), \\ &\prod_{j=1}^{m+1} (I_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m+1} \exp(\vartheta_j/k)}}, \prod_{j=1}^{m+1} (F_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m+1} \exp(\vartheta_j/k)}} \Bigg) \\ &= \Bigg(1 - \prod_{j=1}^{m+1} (1 - T_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m+1} \exp(\vartheta_j/k)}}, \prod_{j=1}^{m+1} (I_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m+1} \exp(\vartheta_j/k)}}, \prod_{j=1}^{m+1} (F_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m+1} \exp(\vartheta_j/k)}} \Bigg) \end{aligned}$$

Therefore, Eq. (5) holds for n = m+1, hence the Eq. (5) holds for all positive integer by principle of mathematical induction. Hence, the proof of the theorem is completed. □

Example 3.2. Let$\alpha_1 = (0.6, 0.4, 0.5), \alpha_2 = (0.7, 0.3, 0.5), \alpha_3 = (0.8, 0.3, 0.4)$ and $\alpha_4 = (0.7, 0.4, 0.5)$ be the four SVN-numbers. Rank the four SVN-numbers using SSVNWA operator.

Solution: Here we have used the SSVNWA operator to aggregate the four SVN-numbers.
At first, we calculated the score values of four SVN-numbers using Eq. (1).
$S(\alpha_1) = 0.567, S(\alpha_2) = 0.633, S(\alpha_3) = 0.700, S(\alpha_4) = 0.600$
$\vartheta_1 = 1, \vartheta_2 = 0.567, \vartheta_3 = 0.359, \vartheta_4 = 0.251.$
To calculate $\exp(\vartheta_j/k)$ we take k =1, then $\exp(\vartheta_1/k) = 2.718, \exp(\vartheta_2/k) = 1.763, \exp(\vartheta_3/k) = 1.432, \exp(\vartheta_4/k) = 1.285$
and $\frac{exp(\vartheta_1/k)}{\sum_{j=1}^{4} \exp(\vartheta_j/k)} = 0.378, \frac{exp(\vartheta_2/k)}{\sum_{j=1}^{4} \exp(\vartheta_j/k)} = 0.245, \frac{exp(\vartheta_3/k)}{\sum_{j=1}^{4} \exp(\vartheta_j/k)} = 0.199, \frac{exp(\vartheta_4/k)}{\sum_{j=1}^{4} \exp(\vartheta_j/k)} = 0.178$

$$\begin{aligned} SSVNWA(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= \Big(\Big(1 - (1 - 0.6)^{0.378} \times (1 - 0.7)^{0.245} \times (1 - 0.8)^{0.199} \times (1 - 0.7)^{0.178} \Big), \\ &\Big((0.4)^{0.378} \times (0.3)^{0.245} \times (0.3)^{0.199} \times (0.4)^{0.178}\Big), \\ &\Big((0.5)^{0.378} \times (0.5)^{0.245} \times (0.4)^{0.199} \times (0.5)^{0.178}\Big) \Big) \\ &= (0.691, 0.352, 0.478) \end{aligned}$$

3.2.1. *Properties of $SSVNWA$ operator*

Property 1: Idem-potency

If $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha(say)$, then $SSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) = \alpha$

Proof. : Let $\alpha_j = \langle T_j, I_j, F_j \rangle, (j = 1, 2, 3..., n)$ and $\alpha = \langle T, I, F \rangle$.
Since all α_j are equal based on Theorem (1), we get

$SSVNWA(\alpha, \alpha, ..., \alpha) = \left(1 - (1-T)^{\sum_{j=1}^{n} \phi_k^j}, (I)^{\sum_{j=1}^{n} \phi_k^j}, (F)^{\sum_{j=1}^{n} \phi_k^j}\right) = \langle T, I, F \rangle = \alpha.$

Since, $\phi_k^j = \frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{n} \exp(\vartheta_j/k)}, k > 0$ *and* $\sum_{j=1}^{n} \phi_k^j = 1.$ □

Property 2: Monotonicity

Let $\alpha_j (j = 1, 2, ..., n)$ and $\beta_j (j = 1, 2, ..., n)$ be any two sets of SVN-numbers. If $\alpha_j \leq \beta_j$ for any j,

then $SSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \leq SSVNWA(\beta_1, \beta_2, ..., \beta_n)$.

Proof. Based on the Theorem (1), we get

$$SSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) = \left(1 - \prod_{j=1}^{n} (1 - T_{\alpha_j})^{\frac{\exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} \exp(\vartheta_{\alpha_j}/k)}}, \prod_{j=1}^{n} (I_{\alpha_j})^{\frac{\exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} \exp(\vartheta_{\alpha_j}/k)}}, \prod_{j=1}^{n} (F_{\alpha_j})^{\frac{\exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} \exp(\vartheta_{\alpha_j}/k)}}\right)$$

and

$$SSVNWA(\beta_1, \beta_2, ..., \beta_n) = \left(1 - \prod_{j=1}^{n} (1 - T_{\beta_j})^{\frac{\exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} \exp(\vartheta_{\beta_j}/k)}}, \prod_{j=1}^{n} (I_{\beta_j})^{\frac{\exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} \exp(\vartheta_{\beta_j}/k)}}, \prod_{j=1}^{n} (F_{\beta_j})^{\frac{\exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} \exp(\vartheta_{\beta_j}/k)}}\right)$$

Since all $\alpha_j \leq \beta_j (j = 1, 2, ..., n)$. *Therefore,*

$$\begin{aligned}
T_{\alpha_j} \leq T_{\beta_j} \quad &\Rightarrow \quad (1 - T_{\alpha_j}) \geq (1 - T_{\beta_j}) \\
&\Rightarrow \quad (1 - T_{\alpha_j})^{\frac{\exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} \exp(\vartheta_{\alpha_j}/k)}} \geq (1 - T_{\beta_j})^{\frac{\exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} \exp(\vartheta_{\beta_j}/k)}} \\
&\Rightarrow \quad \prod_{j=1}^{n} (1 - T_{\alpha_j})^{\frac{\exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} \exp(\vartheta_{\alpha_j}/k)}} \geq \prod_{j=1}^{n} (1 - T_{\beta_j})^{\frac{\exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} \exp(\vartheta_{\beta_j}/k)}} \\
&\Rightarrow \quad \left(1 - \prod_{j=1}^{n} (1 - T_{\alpha_j})^{\frac{\exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} \exp(\vartheta_{\alpha_j}/k)}}\right) \leq \left(1 - \prod_{j=1}^{n} (1 - T_{\beta_j})^{\frac{\exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} \exp(\vartheta_{\beta_j}/k)}}\right)
\end{aligned}$$

Further,

$$\begin{aligned}
I_{\alpha_j} \geq I_{\beta_j} \quad &\Rightarrow \quad (I_{\alpha_j})^{\frac{\exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} \exp(\vartheta_{\alpha_j}/k)}} \geq (I_{\beta_j})^{\frac{\exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} \exp(\vartheta_{\beta_j}/k)}} \\
&\Rightarrow \quad \prod_{j=1}^{n} (I_{\alpha_j})^{\frac{\exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} \exp(\vartheta_{\alpha_j}/k)}} \geq \prod_{j=1}^{n} (I_{\beta_j})^{\frac{\exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} \exp(\vartheta_{\beta_j}/k)}}
\end{aligned}$$

Similarly, we have also

$$\prod_{j=1}^{n} (F_{\alpha_j})^{\frac{\exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} \exp(\vartheta_{\alpha_j}/k)}} \geq (F_{\beta_j})^{\frac{\exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} \exp(\vartheta_{\beta_j}/k)}}$$

Hence the proof is complete. □

Property 3: Boundedness

Let $\alpha_j (j = 1, 2, ..., n)$ be any set of SVN-number. If $\alpha^- = \min \{\alpha_j\}$ and $\alpha^+ = \max \{\alpha_j\}$, then $\alpha^- \leq SSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \leq \alpha^+$.

Proof. : *Let* $\alpha^+ = \max\{\alpha_1, \alpha_2, ..., \alpha_n\}$ *and* $\alpha^- = min\{\alpha_1, \alpha_2, ..., \alpha_n\}$. *According to properties 1 and 2, we have*
$SSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \geq SSVNWA(\alpha^-, \alpha^-, ..., \alpha^-) = \alpha^-$ *and*
$SSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \leq SSVNWA(\alpha^+, \alpha^+, ..., \alpha^+) = \alpha^+$
So, we have $\alpha^- \leq SSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \leq \alpha^+$.
Hence the proof is complete. □

Let $\alpha_j (j = 1, 2, ..., n)$ *be a collection of SVN-numbers. Then softmax single valued neutrosophic weighted geometric (SSVNWG) operator is a function from* $\alpha^n \rightarrow \alpha$ *such that*

$$SSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) = \oplus_{j=1}^{n}(\phi_k^j \alpha_j) = \otimes_{j=1}^{n} (\alpha_j)^{\phi_k^j} = (\alpha_1)^{\phi_k^1} \otimes (\alpha_2)^{\phi_k^2} \otimes ... \otimes (\alpha_n)^{\phi_k^n} \quad (6)$$

where $\phi_k^j = \frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{n}\exp(\vartheta_j/k)}, \vartheta_j = \begin{cases} \prod_{i=1}^{j-1} S_i, & \text{j} = 2, 3, ..., n \\ 1 & \text{j} = 1 \end{cases}$ and S_i is the score function of the SVN-number α_i.

Theorem 3.3. Let $\alpha_j (j = 1, 2, ..., n)$, be a collection of SVNNs, then aggregated value of SVN-numbers using the SSVNWG operation is also a SVN-number and

$$SSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) = \left(\prod_{j=1}^{n} (T_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{n}\exp(\vartheta_j/k)}}, 1 - \prod_{j=1}^{n} (1 - I_j)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n}\exp(\vartheta_j/k)}}, 1 - \prod_{j=1}^{n} (1 - F_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{n}\exp(\vartheta_j/k)}} \right) \quad (7)$$

Proof. : *We proof the above Theorem 2 by using mathematical induction.*
For n = 1, from the Eq. (7) we have

$$SSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) = \left(\prod_{j=1}^{1} (T_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{n}\exp(\vartheta_j/k)}}, 1 - \prod_{j=1}^{1} (1 - I_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{n}\exp(\vartheta_j/k)}}, 1 - \prod_{j=1}^{1} (1 - F_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{n}\exp(\vartheta_j/k)}} \right) = \langle T_1, I_1, F_1 \rangle$$

Since, for $n = 1$, $\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)} = 1$. *Thus Eq.(7) holds for n = 1. Assume that the Eq. (7) holds for* $n = m$,

$$SSVNWG(\alpha_1, \alpha_2, ..., \alpha_m) = \left(\prod_{j=1}^{m} (T_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}}, 1 - \prod_{j=1}^{m} (1 - I_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}}, 1 - \prod_{j=1}^{m} (1 - F_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}} \right)$$

Now we have prove that the Eq. (7) hold for $n = m + 1$. Then

$$
\begin{aligned}
SSVNWG(\alpha_1, \alpha_2, ..., \alpha_m, \alpha_{m+1}) &= (\alpha_1)^{\phi_k^1} \otimes (\alpha_2)^{\phi_k^2} \otimes ... \otimes (\alpha_m)^{\phi_k^m} \otimes (\alpha_{m+1})^{\phi_k^{m+1}} \\
&= \Bigg(\prod_{j=1}^{m} (T_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}},\ 1 - \prod_{j=1}^{m} (1 - I_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}}, \\
&\quad 1 - \prod_{j=1}^{m} (1 - F_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}} \Bigg) \\
&= \Bigg(\prod_{j=1}^{m} (T_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}} \times (T_{m+1})^{\frac{\exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}}, \\
&\quad \prod_{j=1}^{m} (I_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}} + (I_{m+1})^{\frac{\exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}} - \prod_{j=1}^{m} (I_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}} \\
&\quad \times (I_{m+1})^{\frac{\exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}},\ \prod_{j=1}^{m} (F_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}} + (F_{m+1})^{\frac{\exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}} \\
&\quad - \prod_{j=1}^{m} (F_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m}\exp(\vartheta_j/k)}} \times (F_{m+1})^{\frac{\exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}} \Bigg) \\
&= \Bigg(\prod_{j=1}^{m+1} (T_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}},\ 1 - \prod_{j=1}^{m+1} (1 - I_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}}, \\
&\quad 1 - \prod_{j=1}^{m+1} (1 - F_j)^{\frac{\exp(\vartheta_j/k)}{\sum_{j=1}^{m+1}\exp(\vartheta_j/k)}} \Bigg)
\end{aligned}
$$

Therefore, Eq. (7) holds for $n = m+1$, hence the Eq. (7) holds for all positive integer by principle of mathematical induction. Hence, the proof of the theorem is completed. □

Example 3.4. Let $\alpha_1 = (0.6, 0.4, 0.5), \alpha_2 = (0.7, 0.3, 0.5), \alpha_3 = (0.8, 0.3, 0.4)$ and $\alpha_4 = (0.7, 0.4, 0.5)$ be the four SVN-numbers. Rank the four SVN-numbers using $SSVNWG$ operator.

Solution: In the following, we use the $SSVNWG$ operator to aggregate these SVN-numbers. At first we have calculated the score values of four SVNNs using Eq. (1)

$S(\alpha_1) = 0.567, S(\alpha_2) = 0.633, S(\alpha_3) = 0.700, S(\alpha_4) = 0.600$

then $\vartheta_1 = 1, \vartheta_2 = 0.567, \vartheta_3 = 0.359, \vartheta_4 = 0.251$.

To calculate the $\exp(\vartheta_j/k)$ we take $k = 1$, then

$exp(\vartheta_1/k) = 2.718, exp(\vartheta_2/k) = 1.763, exp(\vartheta_3/k) = 1.432,$

$exp(\vartheta_4/k) = 1.285$ and

$$\frac{exp(\vartheta_1/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.378, \frac{exp(\vartheta_2/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.245,$$
$$\frac{exp(\vartheta_3/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.199, \frac{exp(\vartheta_4/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.178,$$

$$\begin{aligned} SSVNWG(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= \Big((0.6)^{0.378} \times (0.7)^{0.245} \times (0.8)^{0.199} \times (0.7)^{0.178}, \\ &1 - (1-0.4)^{0.378} \times (1-0.3)^{0.245} \times (1-0.3)^{0.199} \times (1-0.4)^{0.178}, \\ &1 - (1-0.5)^{0.378} \times (1-0.5)^{0.245} \times (1-0.4)^{0.199} \times (1-0.5)^{0.178} \Big) \\ &= (0.678, 0.357, 0.482). \end{aligned}$$

3.2.2. *Properties of SSVNWG operator*

Property 1: Idem-potency

If $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha(say)$, then
the$SSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) = \alpha$

Proof. : Let $\alpha_j = \langle T_j, I_j, F_j \rangle, (j = 1, 2, 3..., n)$ and $\alpha = \langle T, I, F \rangle$.
Since all α_j are equal, based on Theorem (2), we get
$SSVNWG(\alpha, \alpha, ..., \alpha)$
$= \left((T)^{\sum_{j=1}^{n} \phi_k^j}, 1 - (1-I)^{\sum_{j=1}^{n} \phi_k^j}, 1 - (1-F)^{\sum_{j=1}^{n} \phi_k^j} \right)$
$= \langle T, I, F \rangle = \alpha.$
Since, $\phi_k^j = \frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)}, k > 0$ and $\sum_{j=1}^{n} \phi_k^j = 1$. Hence the proof is completed. □

Property 2: Monotonicity

Let $\alpha_j (j = 1, 2, ..., n)$ and $\beta_j (j = 1, 2, ..., n)$ be any two sets of SVN-numbers. If $\alpha_j \leq \beta_j$ for any j, then $SSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \leq SSVNWG(\beta_1, \beta_2, ..., \beta_n)$.

Proof. : *Based on the Theorem (2), we get*

$$\begin{aligned} SSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) &= \Bigg(\prod_{j=1}^{n} (T_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}, \\ &(1 - \prod_{j=1}^{n} (1 - I_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}), (1 - \prod_{j=1}^{n} (1 - T_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}) \Bigg) \end{aligned}$$

and

$$SSVNWA(\beta_1, \beta_2, ..., \beta_n) = \Big(\prod_{j=1}^{n} (T_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}), (1 - \prod_{j=1}^{n} (1 - I_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}, (1 - \prod_{j=1}^{n} (1 - F_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}) \Big)$$

Since all $\alpha_j \leq \beta_j (j = 1, 2, ..., n)$. *Therefore,*

$$T_{\alpha_j} \leq T_{\beta_j} \Rightarrow (T_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \leq (T_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}$$

$$\Rightarrow \prod_{j=1}^{n} (T_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \leq \prod_{j=1}^{n} (T_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}$$

Further,

$$I_{\alpha_j} \geq I_{\beta_j} \Rightarrow (1 - I_{\alpha_j}) \leq (1 - I_{\beta_j})$$

$$\Rightarrow (1 - I_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \leq (1 - I_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}$$

$$\Rightarrow \prod_{j=1}^{n} (1 - I_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \leq \prod_{j=1}^{n} (1 - I_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}$$

$$\Rightarrow \left(1 - \prod_{j=1}^{n} (1 - I_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \right) \geq \left(1 - \prod_{j=1}^{n} (1 - I_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}} \right)$$

Similarly, we have also

$$\Rightarrow \left(1 - \prod_{j=1}^{n} (1 - F_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \right) \geq \left(1 - \prod_{j=1}^{n} (1 - F_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}} \right)$$

.

Hence the proof is completed. □

Property 3: Boundedness

Let $\alpha_j (j = 1, 2, ..., n)$ be any set of SVN-number. If $\alpha^- = \min\{\alpha_j\}$ and $\alpha^+ = \max\{\alpha_j\}$, then $\alpha^- \leq SSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \leq \alpha^+$.

Proof. : *Let* $\alpha^+ = \max\{\alpha_1, \alpha_2, ..., \alpha_n\}$ *and* $\alpha^- = \min\{\alpha_1, \alpha_2, ..., \alpha_n\}$.
According to properties 1 and 2, we have
$SSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \geq SSVNWG(\alpha^-, \alpha^-, ..., \alpha^-) = \alpha^-$ *and*
$SSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \leq SSVNWG(\alpha^+, \alpha^+, ..., \alpha^+) = \alpha^+$
So we have $\alpha^- \leq SSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \leq \alpha^+$.
Hence the proof is completed. □

Let $\alpha_j (j = 1, 2, ..., n)$ be a collection of SVN-numbers. Then generalized softmax single valued neutrosophic weighted average (GSSVNWA) operator is a function $\alpha^n \to \alpha$ such that

$$GSSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) = \left(\oplus_{j=1}^{n} \phi_k^j \alpha_j^\lambda\right)^{\frac{1}{\lambda}} = \left(\phi_k^1 \alpha_1^\lambda \oplus \phi_k^2 \alpha_2^\lambda \oplus ... \oplus \phi_k^n \alpha_n^\lambda\right)^{\frac{1}{\lambda}} \quad (8)$$

Let $\alpha_j (j = 1, 2, ..., n)$ be a collection of SVN-numbers. Then generalized softmax single valued neutrosophic weighted geometric (GSSVNWG) operator is a function $\alpha^n \to \alpha$ such that

$$GSSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) = \frac{1}{\lambda}\left(\otimes_{j=1}^{n} (\lambda\phi_j)^{\phi_k^j}\right) = \frac{1}{\lambda}\left((\lambda\alpha_1)^{\phi_k^1} \otimes (\lambda\alpha_2)^{\phi_k^2} \otimes ... \otimes (\lambda\alpha_n)^{\phi_k^n}\right) \quad (9)$$

Theorem 3.5. Let $\alpha_j (j = 1, 2, ..., n)$, be a collection of SVN-numbers, then aggregated value of SVN-numbers using the GSSVNWA operation is also a SVNN, and

$$\begin{aligned} GSSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \;=\; & \left\langle \left(1 - \prod_{j=1}^{n}(1 - T_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} \exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, 1 - \left(1 - \prod_{j=1}^{n}\left(1 - (1 - I_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, \right. \\ & \left. 1 - \left(1 - \prod_{j=1}^{n}\left(1 - (1 - F_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}} \right\rangle \end{aligned} \quad (10)$$

Proof. : *We proof the above theorem 3 by using mathematical induction.*
For n = 1, we have:

$$\begin{aligned} GSSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \;=\; & \left\langle \left(1 - \prod_{j=1}^{1}(1 - T_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{1} \exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, 1 - \left(1 - \prod_{j=1}^{1}\left(1 - (1 - I_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, \right. \\ & \left. 1 - \left(1 - \prod_{j=1}^{1}\left(1 - (1 - F_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}} \right\rangle \\ =\; & \langle T_1, I_1, F_1 \rangle \end{aligned}$$

Since, for $n = 1$, $\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)} = 1$.
Thus Eq.(10) holds for n = 1, we assume that the Eq. (10) holds for n = m,

$$\begin{aligned} GSSVNWA(\alpha_1, \alpha_2, ..., \alpha_m) \;=\; & \left\langle \left(1 - \prod_{j=1}^{m}(1 - T_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, 1 - \left(1 - \prod_{j=1}^{m}\left(1 - (1 - I_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, \right. \\ & \left. 1 - \left(1 - \prod_{j=1}^{m}\left(1 - (1 - F_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}} \right\rangle \end{aligned}$$

Now, we prove that the Eq. (10) holds for $n = m + 1$.

$$GSSVNWA(\alpha_1, \alpha_2, ..., \alpha_m, \alpha_{m+1}) = ((\phi_k^1 \alpha_1^\lambda \oplus \phi_k^2 \alpha_2^\lambda \oplus ... \oplus \phi_k^m \alpha_m^\lambda) \oplus \phi_k^{m+1} \alpha_{m+1}^\lambda)^{\frac{1}{\lambda}}$$

$$= \Bigg\langle \left(1 - \prod_{j=1}^{m} (1 - T_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}} + \left(1 - (1 - T_{m+1}^\lambda)^{\frac{exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m=+1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}$$

$$- \left(1 - \prod_{j=1}^{m} (1 - T_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}} \times \left(1 - (1 - T_{m+1}^\lambda)^{\frac{exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m=+1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}},$$

$$\left(1 - \left(1 - \prod_{j=1}^{m} \left(1 - (1 - I_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\right) \times$$

$$\left(1 - \left(1 - \left(1 - (1 - I_{m+1})^\lambda\right)^{\frac{exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\right),$$

$$\left(1 - \left(1 - \prod_{j=1}^{m} \left(1 - (1 - F_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\right) \times$$

$$\left(1 - \left(1 - \left(1 - (1 - F_{m+1})^\lambda\right)^{\frac{exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\right) \Bigg\rangle$$

$$= \Bigg\langle \left(1 - \prod_{j=1}^{m+1} (1 - T_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m+1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}},$$

$$1 - \left(1 - \prod_{j=1}^{m+1} \left(1 - (1 - I_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m+1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}},$$

$$1 - \left(1 - \prod_{j=1}^{m+1} \left(1 - (1 - F_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m+1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}} \Bigg\rangle$$

Therefore, Eq. (10) holds for $n = m+1$, *hence the Eq. (10) holds for all positive integer by principle of mathematical induction. Hence the proof of the theorem is completed.* □

Example 3.6. Let $\alpha_1 = (0.6, 0.4, 0.5), \alpha_2 = (0.7, 0.3, 0.5), \alpha_3 = (0.8, 0.3, 0.4)$ and $\alpha_4 = (0.7, 0.4, 0.5)$ be the four SVN-numbers. Rank the four SVN-numbers using the GSSVNWA operator.

Solution: In the following, we use the GSSVNWA operator to aggregate these SVN-numbers. At first we calculate the score values of four SVN-numbers using Eq. (1). $S(\alpha_1) = 0.567, S(\alpha_2) = 0.633, S(\alpha_3) = 0.700, S(\alpha_4) = 0.600$ then $\vartheta_1 = 1, \vartheta_2 = 0.567, \vartheta_3 = 0.359, \vartheta_4 = 0.251$. To calculate $\exp(\vartheta_j/k)$ we take $k = 1$, then

$$exp(\vartheta_1/k) = 2.718, exp(\vartheta_2/k) = 1.763, exp(\vartheta_3/k) = 1.432, exp(\vartheta_4/k) = 1.285$$

and

$$\frac{exp(\vartheta_1/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.378, \frac{exp(\vartheta_2/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.245,$$

$$\frac{exp(\vartheta_3/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.199, \frac{exp(\vartheta_4/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.178$$

Taking$\lambda = 1$ GSSVNWA reduces to SSVNWA and we get

$$GSSVNWA(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.691, 0.352, 0.478).$$

Taking $\lambda = 2$ we obtain

$$\begin{aligned} GSSVNWA(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= \Bigg(\Big(1-\big(1-(0.6)^2\big)^{0.378} \times \big(1-(0.7)^2\big)^{0.245} \times \big(1-(0.8)^2\big)^{0.199} \times \big(1-(0.7)^2\big)^{0.178}\Big)^{\frac{1}{2}}, \\ & 1-\Big(1-\big(1-(1-0.6)^2\big)^{0.378} \times \big(1-(1-0.3)^2\big)^{0.245} \times \\ & \big(1-(1-0.3)^2\big)^{0.199} \times \big(1-(1-0.4)^2\big)^{0.178}\Big)^{\frac{1}{2}}, \\ & 1-\Big(1-\big(1-(1-0.5)^2\big)^{0.378} \times \big(1-(1-0.5)^2\big)^{0.245} \times \\ & \big(1-(1-0.4)^2\big)^{0.199} \times \big(1-(1-0.5)^2\big)^{0.178}\Big)^{\frac{1}{2}} \Bigg) \\ &= (0.694, 0.401, 0.477) \end{aligned}$$

Taking $\lambda = 3$, we obtain

$$\begin{aligned} GSSVNWA(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= \Bigg(\Big(1-\big(1-(0.6)^3\big)^{0.378} \times \big(1-(0.7)^3\big)^{0.245} \times \big(1-(0.8)^3\big)^{0.199} \times \big(1-(0.7)^3\big)^{0.178}\Big)^{\frac{1}{3}}, \\ & 1-\Big(1-\big(1-(1-0.6)^3\big)^{0.378} \times \big(1-(1-0.3)^3\big)^{0.245} \times \\ & \big(1-(1-0.3)^3\big)^{0.199} \times \big(1-(1-0.4)^3\big)^{0.178}\Big)^{\frac{1}{3}}, \\ & 1-\Big(1-\big(1-(1-0.5)^3\big)^{0.378} \times \big(1-(1-0.5)^3\big)^{0.245} \times \\ & \big(1-(1-0.4)^3\big)^{0.199} \times \big(1-(1-0.5)^3\big)^{0.178}\Big)^{\frac{1}{3}} \Bigg) \\ &= (0.697, 0.392, 0.476) \end{aligned}$$

Similarly we can also checked for $\lambda = 4, 5, ...$ and so on

3.2.3. *Properties of $GSSVNWA$ operator*

Property 1: Idem-potency If $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$ (say), then the $GSSVNWAO(\alpha_1, \alpha_2, ..., \alpha_n) = \alpha$

Proof. *Let $\alpha_j = \langle T_j, I_j, F_j \rangle, (j = 1, 2, 3..., n)$ and $\alpha = \langle T, I, F \rangle$.*
Since all α_j are equal, based on Theorem (3), we get

$$SSVNWG(\alpha,\alpha,...,\alpha) = \left\langle \left(1-(1-T^{\lambda})^{\sum_{j=1}^{n}\phi_k^j}\right)^{\frac{1}{\lambda}}, 1-\left(1-\left(1-(1-I)^{\lambda}\right)^{\sum_{j=1}^{n}\phi_k^j}\right)^{\frac{1}{\lambda}}, 1-\left(1-\left(1-(1-F)^{\lambda}\right)^{\sum_{j=1}^{n}\phi_k^j}\right)^{\frac{1}{\lambda}} \right\rangle = \langle T, I, F\rangle = \alpha$$

Since, $\phi_k^j = \frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)}, k > 0$ *and* $\sum_{j=1}^{n}\phi_k^j = 1$.
Hence the proof is completed. □

Property 2: Monotonicity

Let $\alpha_j(j = 1, 2, ..., n)$ and $\beta_j(j = 1, 2, ..., n)$ be any two SVN-numbers. If $\alpha_j \leq \beta_j$ for any j, then
$GSSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \leq GSSVNWA(\beta_1, \beta_2, ..., \beta_n)$.

Proof. : Based on the Theorem (3), we get

$$GSSVNWA(\alpha_1,\alpha_2,...,\alpha_n) = \left\langle \left(1-\prod_{j=1}^{n}(1-T_{\alpha_j}^{\lambda})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}}, 1-\left(1-\prod_{j=1}^{n}\left(1-(1-I_{\alpha_j})^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}}, 1-\left(1-\prod_{j=1}^{n}\left(1-(1-F_{\alpha_j})^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}}\right.$$

and

$$GSSVNWA(\beta_1,\beta_2,...,\beta_n) = \left\langle \left(1-\prod_{j=1}^{n}(1-T_{\beta_j}^{\lambda})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}}, 1-\left(1-\prod_{j=1}^{n}\left(1-(1-I_{\beta_j})^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}}, 1-\left(1-\prod_{j=1}^{n}\left(1-(1-F_{\beta_j})^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}} \right\rangle$$

Since all $\alpha_j \leq \beta_j (j = 1, 2, ..., n)$.
Therefore for $T_{\alpha_j} \leq T_{\beta_j}$ we have

$$\begin{aligned} T^{\lambda}_{\alpha_j} &\leq T^{\lambda}_{\beta_j} \Rightarrow (1 - T^{\lambda}_{\alpha_j}) \geq (1 - T^{\lambda}_{\beta_j}) \\ &\Rightarrow (1 - T^{\lambda}_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \geq (1 - T^{\lambda}_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}} \\ &\Rightarrow \prod_{j=1}^{n} (1 - T^{\lambda}_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \geq \prod_{j=1}^{n} (1 - T^{\lambda}_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}} \\ &\Rightarrow \left(1 - \prod_{j=1}^{n} (1 - T^{\lambda}_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right) \leq \left(1 - \prod_{j=1}^{n} (1 - T^{\lambda}_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right) \\ &\Rightarrow \left(1 - \prod_{j=1}^{n} (1 - T^{\lambda}_{\alpha_j})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}} \leq \left(1 - \prod_{j=1}^{n} (1 - T^{\lambda}_{\beta_j})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}} \end{aligned}$$

Further for

$$\begin{aligned} I_{\alpha_j} \geq I_{\beta_j} &\Rightarrow (1 - I_{\alpha_j}) \leq (1 - I_{\beta_j}) \\ &\Rightarrow (1 - I_{\alpha_j})^{\lambda} \leq (1 - I_{\beta_j})^{\lambda} \\ &\Rightarrow \left(1 - (1 - I_{\alpha_j})^{\lambda}\right) \geq \left(1 - (1 - I_{\beta_j})^{\lambda}\right) \\ &\Rightarrow \left(1 - (1 - I_{\alpha_j})^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \geq \left(1 - (1 - I_{\beta_j})^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}} \\ &\Rightarrow \prod_{j=1}^{n} \left(1 - (1 - I_{\alpha_j})^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \geq \prod_{j=1}^{n} \left(1 - (1 - I_{\beta_j})^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}} \\ &\Rightarrow \left(1 - \prod_{j=1}^{n} \left(1 - (1 - I_{\alpha_j})^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}} \leq \left(1 - \prod_{j=1}^{n} \left(1 - (1 - I_{\beta_j})^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}} \\ &\Rightarrow 1 - \left(1 - \prod_{j=1}^{n} \left(1 - (1 - I_{\alpha_j})^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}} \geq 1 - \left(1 - \prod_{j=1}^{n} \left(1 - (1 - I_{\beta_j})^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}} \end{aligned}$$

Similarly, we can also show that

$$\Rightarrow 1 - \left(1 - \prod_{j=1}^{n} \left(1 - (1 - F_{\alpha_j})^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}} \geq 1 - \left(1 - \prod_{j=1}^{n} \left(1 - (1 - F_{\beta_j})^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}}.$$

Hence the proof is completed. □

Property 3: Boundedness

Let $\alpha_j (j = 1, 2, ..., n)$ be any set of SVNN. If $\alpha^- = \min\{\alpha_j\}$ and $\alpha^+ = \max\{\alpha_j\}$, then $\alpha^- \leq GSSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \leq \alpha^+$.

Proof. : Let $\alpha^+ = \max\{\alpha_1, \alpha_2, ..., \alpha_n\}$ and $\alpha^- = \min\{\alpha_1, \alpha_2, ..., \alpha_n\}$. According to properties 1 and 2, we have

$GSSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \geq GSSVNWA(\alpha^-, \alpha^-, ..., \alpha^-) = \alpha^-$ and

$GSSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \leq GSSVNWA(\alpha^+, \alpha^+, ..., \alpha^+) = \alpha^+$

So, we have $\alpha^- \leq GSSVNWA(\alpha_1, \alpha_2, ..., \alpha_n) \leq \alpha^+$.

Hence the proof is completed. □

Theorem 3.7. Let $\alpha_j (j = 1, 2, ..., n)$, be a collection of SVN-numbers. The aggregated value of SVN-numbers using the GSSVNWG operator is also a SVN-number and

$$GSSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) = \left\langle 1 - \left(1 - \prod_{j=1}^{n} \left(1 - (1 - T_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{j=1}^{n} (1 - I_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{j=1}^{n} (1 - F_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}} \right\rangle \quad (11)$$

Proof. : *We proof the above Theorem 4 by using mathematical induction.*
For n = 1, we have:

$$GSSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) = \left\langle 1 - \left(1 - \prod_{j=1}^{1} \left(1 - (1 - T_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{j=1}^{1} (1 - I_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{j=1}^{1} (1 - F_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{1} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}} \right\rangle = \langle T_1, I_1, F_1 \rangle$$

Since, for $n = 1$, $\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)} = 1$.
Thus Eq.(11) holds for n = 1. Assume that the Eq. (11) holds for n = m,

$$GSSVNWG(\alpha_1, \alpha_2, ..., \alpha_m) = \left\langle 1 - \left(1 - \prod_{j=1}^{m} \left(1 - (1 - T_j)^\lambda\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{j=1}^{m} (1 - I_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}} \left(1 - \prod_{j=1}^{m} (1 - F_j^\lambda)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m} exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}} \right\rangle.$$

Now, we prove that the Eq. (12) holds for n = m + 1.

$$GSSVNWG(\alpha_1,\alpha_2,...,\alpha_m,\alpha_{m+1}) = \frac{1}{\lambda}\left((\lambda\alpha_1)^{\phi_k^1}\otimes(\lambda\alpha_2)^{\phi_k^2}\otimes...\otimes(\lambda\alpha_m)^{\phi_k^m}\otimes(\lambda\alpha_{m+1})^{\phi_k^{m+1}}\right)$$

$$= \Bigg\langle \left(1-\left(1-\prod_{j=1}^{m}\left(1-(1-T_j)^{\lambda}\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\right)$$

$$\times\left(1-\left(1-\left(1-(1-T_{m+1})^{\lambda}\right)^{\frac{exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\right),$$

$$\left(1-\prod_{j=1}^{m}(1-I_j^{\lambda})^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}+\left(1-(1-I_{m+1}^{\lambda})^{\frac{exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}$$

$$-\left(1-\prod_{j=1}^{m}(1-I_j^{\lambda})^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\times\left(1-(1-I_{m+1}^{\lambda})^{\frac{exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}},$$

$$\left(1-\prod_{j=1}^{m}(1-F_j^{\lambda})^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}+\left(1-(1-F_{m+1}^{\lambda})^{\frac{exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}$$

$$-\left(1-\prod_{j=1}^{m}(1-F_j^{\lambda})^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\times\left(1-(1-F_{m+1}^{\lambda})^{\frac{exp(\vartheta_{m+1}/k)}{\sum_{j=1}^{m+1}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\Bigg\rangle$$

$$= \Bigg\langle 1-\left(1-\prod_{j=1}^{m+1}\left(1-(1-T_j)^{\lambda}\right)^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m+1}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}},$$

$$\left(1-\prod_{j=1}^{m+1}(1-I_j^{\lambda})^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m+1}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}},$$

$$\left(1-\prod_{j=1}^{m+1}(1-F_j^{\lambda})^{\frac{exp(\vartheta_j/k)}{\sum_{j=1}^{m+1}exp(\vartheta_j/k)}}\right)^{\frac{1}{\lambda}}\Bigg\rangle$$

Therefore, Eq. (11) holds for n = m+1, hence the Eq. (11) holds for all positive integer by principle of mathematical induction. Hence the proof of the theorem is completed. □

Example 3.8. Let $\alpha_1 = (0.6, 0.4, 0.5), \alpha_2 = (0.7, 0.3, 0.5), \alpha_3 = (0.8, 0.3, 0.4)$ and $\alpha_4 = (0.7, 0.4, 0.5)$ be the four SVN-numbers. Rank the four SVN-numbers using GSSVNWG operator.

Solution : In the following, we use the GSSVNWG operator to aggregate these SVN-numbers. At first, we calculate the score values of four SVN-numbers using Eq. (1).

$S(\alpha_1) = 0.567, S(\alpha_2) = 0.633, S(\alpha_3) = 0.700, S(\alpha_4) = 0.600$, then

$\vartheta_1 = 1, \vartheta_2 = 0.567, \vartheta_3 = 0.359, \vartheta_4 = 0.251$.

To calculate $exp(\vartheta_j/k)$ we take k =1, then

$exp(\vartheta_1/k) = 2.718, exp(\vartheta_2/k) = 1.763, exp(\vartheta_3/k) = 1.432, exp(\vartheta_4/k) = 1.285$
and

$$\frac{exp(\vartheta_1/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.378, \frac{exp(\vartheta_2/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.245$$

,

$$\frac{exp(\vartheta_3/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.199, \frac{exp(\vartheta_4/k)}{\sum_{j=1}^{4} exp(\vartheta_j/k)} = 0.178,$$

Taking $\lambda = 1$, GSSVNWG reduces to SSVNWG and we get $GSSVNWG(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.678, 0.357, 0.482)$.
Again for $\lambda = 2$, we obtain

$$\begin{aligned} GSSVNWG(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= \Bigg(1 - \Big(1 - \left(1 - (1 - 0.6)^2\right)^{0.378} \times \left(1 - (1 - 0.7)^2\right)^{0.245} \\ &\quad \times \left(1 - (1 - 0.8)^2\right)^{0.199} \times \left(1 - (1 - 0.7)^2\right)^{0.178}\Big)^{\frac{1}{2}}, \\ &\quad \left(1 - \left(1 - (0.6)^2\right)^{0.378} \times \left(1 - (0.3)^2\right)^{0.245} \times \left(1 - (0.3)^2\right)^{0.199} \times \left(1 - (0.4)^2\right)^{0.178}\right)^{\frac{1}{2}}, \\ &\quad \left(1 - \left(1 - (0.5)^2\right)^{0.378} \times \left(1 - (0.5)^2\right)^{0.245} \times \left(1 - (0.4)^2\right)^{0.199} \times \left(1 - (0.5)^2\right)^{0.178}\right)^{\frac{1}{2}}\Bigg) \\ &= (0.671, 0.464, 0.482). \end{aligned}$$

Taking $\lambda = 3$, we obtain

$$\begin{aligned} GSSVNWG(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= \Bigg(1 - \Big(1 - \left(1 - (1 - 0.6)^3\right)^{0.378} \times \left(1 - (1 - 0.7)^3\right)^{0.245} \\ &\quad \times \left(1 - (1 - 0.8)^3\right)^{0.199} \times \left(1 - (1 - 0.7)^3\right)^{0.178}\Big)^{\frac{1}{3}}, \\ &\quad \left(1 - \left(1 - (0.6)^3\right)^{0.378} \times \left(1 - (0.3)^3\right)^{0.245} \times \left(1 - (0.3)^3\right)^{0.199} \times \left(1 - (0.4)^3\right)^{0.178}\right)^{\frac{1}{3}}, \\ &\quad \left(1 - \left(1 - (0.5)^3\right)^{0.378} \times \left(1 - (0.5)^3\right)^{0.245} \times \left(1 - (0.4)^3\right)^{0.199} \times \left(1 - (0.5)^3\right)^{0.178}\right)^{\frac{1}{3}}\Bigg) \\ &= (0.662, 0.480, 0.484). \end{aligned}$$

Similarly, we can also show that for $\lambda = 4, 5, ...$ and so on.

3.2.4. *Properties of $GSSVNWG$ operator*

Property 1: Idem-potency

If $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha(say)$, then $GSSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) = \alpha$.

Proof. Let $\alpha_j = \langle T_j, I_j, F_j\rangle, (j = 1, 2, 3..., n)$ and $\alpha = \langle T, I, F\rangle$.
Since all α_j are equal, based on Theorem (4), we get

$$\begin{aligned} GSSVNWG(\alpha,\alpha,...,\alpha) &= \Big\langle 1-\Big(1-\big(1-(1-T)^{\lambda}\big)^{\sum_{j=1}^{n}\phi_k^j}\Big)^{\frac{1}{\lambda}}, \Big(1-(1-I^{\lambda})^{\sum_{j=1}^{n}\phi_k^j}\Big)^{\frac{1}{\lambda}}, \\ &\quad \Big(1-(1-F^{\lambda})^{\sum_{j=1}^{n}\phi_k^j}\Big)^{\frac{1}{\lambda}} \Big\rangle \\ &= \langle T, I, F\rangle = \alpha \end{aligned}$$

Since, $\phi_k^j = \frac{exp(\vartheta_j/k)}{\sum_{j=1}^{n} exp(\vartheta_j/k)}, k > 0$ *and* $\sum_{j=1}^{n}\phi_k^j = 1$. *Hence the proof is completed.* □

2. Monotonicity:

Let $\alpha_j(j = 1, 2, ..., n)$ and $\beta_j(j = 1, 2, ..., n)$ be any two sets of SVN-numbers. If $\alpha_j \leq \beta_j$ for any j, then $GSSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \leq GSSVNWG(\beta_1, \beta_2, ..., \beta_n)$.

Proof. : Based on the Theorem (4), we get

$$\begin{aligned} GSSVNWG(\alpha_1,\alpha_2,...,\alpha_n) &= \Bigg(1-\Bigg(1-\prod_{j=1}^{n}\Big(1-\big(1-T_{\alpha_j}\big)^{\lambda}\Big)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n}exp(\vartheta_{\alpha_j}/k)}}\Bigg)^{\frac{1}{\lambda}}, \\ &\quad \Bigg(1-\prod_{j=1}^{n}(1-I_{\alpha_j}^{\lambda})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n}exp(\vartheta_{\alpha_j}/k)}}\Bigg)^{\frac{1}{\lambda}}, \\ &\quad \Bigg(1-\prod_{j=1}^{n}(1-F_{\alpha_j}^{\lambda})^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n}exp(\vartheta_{\alpha_j}/k)}}\Bigg)^{\frac{1}{\lambda}} \Bigg) \end{aligned}$$

and

$$\begin{aligned} GSSVNWG(\beta_1,\beta_2,...,\beta_n) &= \Bigg(1-\Bigg(1-\prod_{j=1}^{n}\Big(1-\big(1-T_{\beta_j}\big)^{\lambda}\Big)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n}exp(\vartheta_{\beta_j}/k)}}\Bigg)^{\frac{1}{\lambda}}, \\ &\quad \Bigg(1-\prod_{j=1}^{n}(1-I_{\beta_j}^{\lambda})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n}exp(\vartheta_{\beta_j}/k)}}\Bigg)^{\frac{1}{\lambda}}, \\ &\quad \Bigg(1-\prod_{j=1}^{n}(1-F_{\beta_j}^{\lambda})^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n}exp(\vartheta_{\beta_j}/k)}}\Bigg)^{\frac{1}{\lambda}} \Bigg) \end{aligned}$$

Since all $\alpha_j \leq \beta_j (j = 1, 2, ..., n)$. Therefore,

$$\begin{aligned}
T_{\alpha_j} \leq T_{\beta_j} \quad & \Rightarrow \quad \left(1 - T_{\alpha_j}\right) \geq \left(1 - T_{\beta_j}\right), \\
& \Rightarrow \quad \left(1 - T_{\alpha_j}\right)^{\lambda} \geq \left(1 - T_{\beta_j}\right)^{\lambda}, \\
& \Rightarrow \quad \left(1 - \left(1 - T_{\alpha_j}\right)^{\lambda}\right) \leq \left(1 - \left(1 - T_{\beta_j}\right)^{\lambda}\right) \\
& \Rightarrow \quad \left(1 - \left(1 - T_{\alpha_j}\right)^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \leq \left(1 - \left(1 - T_{\beta_j}\right)^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}} \\
& \Rightarrow \quad \prod_{j=1}^{n} \left(1 - \left(1 - T_{\alpha_j}\right)^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \leq \prod_{j=1}^{n} \left(1 - \left(1 - T_{\beta_j}\right)^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}} \\
& \Rightarrow \quad 1 - \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - T_{\alpha_j}\right)^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}} \\
& \leq \quad 1 - \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - T_{\beta_j}\right)^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}}
\end{aligned}$$

Further,

$$\begin{aligned}
I_{\alpha_j} \geq I_{\beta_j} \quad & \Rightarrow \quad I_{\alpha_j}^{\lambda} \geq I_{\beta_j}^{\lambda} \Rightarrow \left(1 - I_{\alpha_j}^{\lambda}\right) \leq \left(1 - I_{\beta_j}^{\lambda}\right) \\
& \Rightarrow \quad \left(1 - I_{\alpha_j}^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \leq \left(1 - I_{\beta_j}^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}} \\
& \Rightarrow \quad \prod_{j=1}^{n} \left(1 - I_{\alpha_j}^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}} \leq \prod_{j=1}^{n} \left(1 - I_{\beta_j}^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}} \\
& \Rightarrow \quad \left(1 - \prod_{j=1}^{n} \left(1 - I_{\alpha_j}^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right) \geq \left(1 - \prod_{j=1}^{n} \left(1 - I_{\beta_j}^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right) \\
& \Rightarrow \quad \left(1 - \prod_{j=1}^{n} \left(1 - I_{\alpha_j}^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}} \geq \left(1 - \prod_{j=1}^{n} \left(1 - I_{\beta_j}^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}}
\end{aligned}$$

Similarly, we can also show that

$$\Rightarrow \left(1 - \prod_{j=1}^{n} \left(1 - F_{\alpha_j}^{\lambda}\right)^{\frac{exp(\vartheta_{\alpha_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\alpha_j}/k)}}\right)^{\frac{1}{\lambda}} \geq \left(1 - \prod_{j=1}^{n} \left(1 - F_{\beta_j}^{\lambda}\right)^{\frac{exp(\vartheta_{\beta_j}/k)}{\sum_{j=1}^{n} exp(\vartheta_{\beta_j}/k)}}\right)^{\frac{1}{\lambda}}$$

Hence the proof is completed. □

Property 3: Boundedness

Let $\alpha_j (j = 1, 2, ..., n)$ be any set of SVN-numbers. If $\alpha^- = \min\{\alpha_j\}$ and $\alpha^+ = \max\{\alpha_j\}$, then $\alpha^- \leq GSSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \leq \alpha^+$.

Proof. : *Let* $\alpha^+ = max\{\alpha_1, \alpha_2, ..., \alpha_n\}$ *and* $\alpha^- = min\{\alpha_1, \alpha_2, ..., \alpha_n\}$.
According to the Property 1 and 2, we have
$GSSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \geq GSSVNWG(\alpha^-, \alpha^-, ..., \alpha^-) = \alpha^-$
and $GSSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \leq GSSVNWG(\alpha^+, \alpha^+, ..., \alpha^+) = \alpha^+$
So, we have $\alpha^- \leq GSSVNWG(\alpha_1, \alpha_2, ..., \alpha_n) \leq \alpha^+$.
Hence the proof is completed. □

4. MADM under SVN environment based on proposed operators

The main goal of a MADM strategy is to find the one or more alternative which satisfies the objective of decision maker from a set of possible alternatives w.r.t significant attributes. Using the proposed aggregation operators a MADM strategy under SVN environment is considered. A MADM strategy is presented here ton show the application of proposed approach.

4.1. *Decision making approach based on proposed operators*

Let $\Psi = \{\psi_1, \psi_2, ..., \psi_m\}$and $C = \{C_1, C_2, ..., C_n\}$ be the possible set of alternatives and attributes respectively. Let $W = \{w_1, w_2, ..., w_n\}$ be the weight vector of attributes C_j (j = 1, 2, 3, . . . , n), where $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$. Now, we have described the steps of proposed MADM strategy by following algorithm.

Algorithm:

Step 1: Formulate the decision matrix

For MADM with SVN–number information, the rating values of the alternative $\psi_i(i = 1, 2, ..., m)$ on the basis of attribute $C_j(j = 1, 2, ..., n)$can be expressed in SVN–number as a_{ij} where (i = 1, 2, 3, . . . , m; j = 1, 2, 3, . . . , n).

The decision matrix is represented as follows:

$$[\ A_{ij}]_{m\times n} = \begin{array}{c} \\ \psi_1 \\ \psi_2 \\ \vdots \\ \psi_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \tag{12}$$

is called an decision making matrix.

Step 2: Compute the score matrix and ϑ_{ij} value matrix

Using Eq. (1) and Eq. (2), We calculate the score value of each alternative for different attribute and represent as matrix form as:

$$[\ S_{ij}]_{m\times n} = \begin{array}{c} \\ \psi_1 \\ \psi_2 \\ \vdots \\ \psi_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & \cdots & s_{mn} \end{pmatrix} \end{array} \tag{13}$$

and calculate ϑ_{ij} value using Eq. (3), represent as follows:

$$[\vartheta_{ij}]_{m\times n} = \begin{array}{c} \\ \psi_1 \\ \psi_2 \\ \vdots \\ \psi_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} \vartheta_{11} & \vartheta_{12} & \cdots & \vartheta_{1n} \\ \vartheta_{21} & \vartheta_{22} & \cdots & \vartheta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \vartheta_{m1} & \vartheta_{m2} & \cdots & \vartheta_{mn} \end{pmatrix} \end{array} \tag{14}$$

Step 3: Compute weighted matrix

We calculate weight of each alterntive for each attribute by the Eq. (3) and represent in matrix form as:

$$[\phi_k^{ij}]_{m\times n} = \begin{array}{c} \\ \psi_1 \\ \psi_2 \\ \vdots \\ \psi_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} \phi_k^{11} & \phi_k^{12} & \cdots & \phi_k^{1n} \\ \phi_k^{21} & \phi_k^{22} & \cdots & \phi_k^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_k^{m1} & \phi_k^{m2} & \cdots & \phi_k^{mn} \end{pmatrix} \end{array} \tag{15}$$

Where, $\phi_k^{ij} = \frac{exp(\vartheta_{ij}/k)}{\sum_{j=1}^{n} exp(\vartheta_{ij}/k)}, \vartheta_{ij} = \begin{cases} \prod_{i=1}^{j-1} S_i, & \text{j} = 2, 3, ..., n \\ 1 & \text{j} = 1 \end{cases}$, S_i is the score function of the SVNN α_i.

Step 4: Aggregate the all attributes

Using aggregation operators we aggragate the all attribute values for respective alternative and results are shown in table form as:

TABLE 1. The aggregated SVNNs and score values of aggregated SVNNs

Alternatives	Aggregated SVNNs	Score values
ψ_1	$\tilde{a_1}$	$\tilde{S_1}$
ψ_2	$\tilde{a_2}$	$\tilde{S_2}$
.	.	.
.	.	.
ψ_m	$\tilde{a_m}$	$\tilde{S_m}$

Step 5: Ranking the alternatives

Based on the score value (From Table 1) of alternative, we arranged the ranking odrer of alternatives using Eq. (1) and Eq. (2).

Step 6: End the algorithm.

5. Numerical illustration

Every year worldwide, many peoples are affected by various natural disasters. These disasters are Hurricanes and Tropical Storms, Drought, Wildfires, Floods, Earthquakes, Tornadoes, severe storms, etc. The most common thoughtful nature disaster is the Flood disaster. Flood disaster problems can handle by MADM strategy according to the given information Yu (2016). In Flood disaster control and mitigation, risk decision and evaluation are significant steps. According to our knowledge (Ya 2012), we have composed four essential attributes to evaluate the risk of Flood disaster, which are:

i) Disaster-inducing factors (C_1),

ii) Hazard-formative environment (C_2),

iii) Characters of hazard affected body (C_3), and

iv) Social disaster bearing capacity (C_4).

Apparently, these evaluation attribute are complicated and difficult to characterize quantitatively. We can handle this type of difficulties considering the attributes information by SVN set.

Let us assume that ψ_1, ψ_2, ψ_3, and ψ_4 are the four maritime cities in India. Our aim is to find the best city according to the four attributes. We expressed the appraisement informations of four cities according to the four attributes in terms of SVN-set. Now, we will solved this decision making problem using the proposed operators.

Step 1: Formulate the decision matrix

The appraisement informations of four cities consider by SVN-number according to the four attributes. Re-presentation of the decision matrix shown in Eq. (16) as given by:

$$[\,A]_{4\times 4} = \begin{array}{c} \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{pmatrix} \langle 0.6, 0.4, 0.3\rangle & \langle 0.7, 0.3, 0.4\rangle & \langle 0.8, 0.4, 0.6\rangle & \langle 0.6, 0.2, 0.4\rangle \\ \langle 0.3, 0.1, 0.4\rangle & \langle 0.5, 0.2, 0.2\rangle & \langle 0.8, 0.3, 0.4\rangle & \langle 0.6, 0.3, 0.5\rangle \\ \langle 0.6, 0.3, 0.5\rangle & \langle 0.7, 0.3, 0.5\rangle & \langle 0.8, 0.3, 0.5\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.7, 0.3, 0.3\rangle & \langle 0.3, 0.4, 0.3\rangle & \langle 0.7, 0.4, 0.5\rangle & \langle 0.8, 0.3, 0.4\rangle \end{pmatrix} \end{array} \tag{16}$$

Step 2: Compute the score matrix and ϑ_{ij} value matrix

Using Eq. (1), We calculate the score value of each alternative for different attribute and represent as matrix form as:

$$[\ S]_{4\times4} = \begin{array}{c} \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{pmatrix} 0.63 & 0.67 & 0.60 & 0.67 \\ 0.60 & 0.70 & 0.70 & 0.60 \\ 0.60 & 0.63 & 0.67 & 0.63 \\ 0.70 & 0.53 & 0.60 & 0.70 \end{pmatrix} \end{array} \quad (17)$$

and calculate ϑ_{ij} value using Eq. (4), represent as follows:

$$[\vartheta]_{4\times4} = \begin{array}{c} \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{pmatrix} 1 & 0.63 & 0.42 & 0.28 \\ 1 & 0.60 & 0.42 & 0.29 \\ 1 & 0.60 & 0.38 & 0.25 \\ 1 & 0.70 & 0.37 & 0.22 \end{pmatrix} \end{array} \quad (18)$$

Step 3: Compute ϕ_k^{ij} matrix (Let parameter $k = 1$)

We calculate the values of ϕ_k^{ij} for each alternative with respects to each attributes by the Eq. (3) and represent in matrix form as:

$$[\phi]_{4\times4} = \begin{array}{c} \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{pmatrix} 0.36 & 0.25 & 0.20 & 0.18 \\ 0.34 & 0.24 & 0.21 & 0.18 \\ 0.37 & 0.25 & 0.20 & 0.18 \\ 0.37 & 0.27 & 0.20 & 0.17 \end{pmatrix} \end{array} \quad (19)$$

Step 4: Aggregate the all attribute values of alternatives

Based on the $SSVNWA$ operator, the aggregated SVN-numbers and score values of Eq. (17) are shown in Table 2 (Parameter $k = 1$ fixed).

From Table 2, we find the riskiest city is ψ_2.

Based on the $SSVNWG$ operator, the aggregated SVN-numbers and corresponding score values of Eq. (16) are shown in Table 3(Parameter $k = 1$ fixed),

From Table 3, we find the riskiest city is ψ_2.

Based on the $GSSVNWA$ operator, the aggregated SVN-numbers and corresponding score values of Eq. (16) are shown in Table 4(Parameter $\lambda = 2, 5, 10$ and Parameter $k = 1$ fixed).

From Table 4, we find the riskiest city is ψ_4 for $\lambda = 2$ and ψ_2 for $\lambda = 5, 10$.

Based on the $GSSVNWG$ operator, the aggregated SVN-numbers and corresponding score values of Eq. (16) are shown in Table 5(Parameter $\lambda = 2, 5, 10$ and Parameter $k = 1$ fixed).

From Table 5, we find the riskiest city is ψ_1 for $\lambda = 2, 5, 10$.

Step 5: Ranking order of alternatives

TABLE 2. The aggregated SVN-numbers and score values based on SSVNWA operator

Alternatives	Aggregated SVNNs	Score values
ψ_1	$\langle 0.622, 0.332, 0.398 \rangle$	0.631
ψ_2	$\langle 0.546, 0.194, 0.362 \rangle$	**0.663**
ψ_3	$\langle 0.651, 0.300, 0.424 \rangle$	0.642
ψ_4	$\langle 0.652, 0.339, 0.345 \rangle$	0.656

TABLE 3. The aggregated SVNNs and score values based on SSVNWG operator

Alternatives	Aggregated SVNNs	Score values
ψ_1	$\langle 0.633, 0.340, 0.414 \rangle$	0.636
ψ_2	$\langle 0.489, 0.204, 0.368 \rangle$	**0.669**
ψ_3	$\langle 0.614, 0.300, 0.456 \rangle$	0.619
ψ_4	$\langle 0.568, 0.351, 0.365 \rangle$	0.617

TABLE 4. The aggregated SVNNs and score values based on GSSVNWA operator

value of λ	Alternatives	Aggregated SVNNs	Score values
	ψ_1	$\langle 0.679, 0.327, 0.388 \rangle$	0.655
	ψ_2	$\langle 0.572, 0.189, 0.353 \rangle$	0.677
$\lambda = 2$	ψ_3	$\langle 0.659, 0.300, 0.414 \rangle$	0.648
	ψ_4	$\langle 0.665, 0.245, 0.343 \rangle$	**0.692**
	ψ_1	$\langle 0.689, 0.316, 0.372 \rangle$	0.667
	ψ_2	$\langle 0.632, 0.178, 0.330 \rangle$	**0.708**
$\lambda = 5$	ψ_3	$\langle 0.680, 0.300, 0.378 \rangle$	0.643
	ψ_4	$\langle 0.692, 0.337, 0.336 \rangle$	0.673
	ψ_1	$\langle 0.710, 0.294, 0.353 \rangle$	0.688
	ψ_2	$\langle 0.691, 0.163, 0.295 \rangle$	**0.744**
$\lambda = 10$	ψ_3	$\langle 0.708, 0.300, 0.320 \rangle$	0.696
	ψ_4	$\langle 0.715, 0.330, 0.326 \rangle$	0.686

TABLE 5. The aggregated SVNNs and score values based on GSSVNWG operator

value of λ	Alternatives	Aggregated SVNNs	Score values
	ψ_1	$\langle 0.656, 0.347, 0.424\rangle$	**0.628**
	ψ_2	$\langle 0.464, 0.221, 0.382\rangle$	0.620
$\lambda = 2$	ψ_3	$\langle 0.598, 0.300, 0.465\rangle$	0.611
	ψ_4	$\langle 0.536, 0.353, 0.370\rangle$	0.604
	ψ_1	$\langle 0.638, 0.364, 0.463\rangle$	**0.603**
	ψ_2	$\langle 0.413, 0.253, 0.408\rangle$	0.584
$\lambda = 5$	ψ_3	$\langle 0.548, 0.300, 0.481\rangle$	0.589
	ψ_4	$\langle 0.451, 0.361, 0.395\rangle$	0.565
	ψ_1	$\langle 0.623, 0.378, 0.513\rangle$	**0.577**
	ψ_2	$\langle 0.369, 0.273, 0.433\rangle$	0.554
$\lambda = 10$	ψ_3	$\langle 0.492, 0.300, 0.490\rangle$	0.567
	ψ_4	$\langle 0.385, 0.373, 0.430\rangle$	0.527

According to the decreasing score value of alternatives $\psi_i (i = 1, 2, 3, 4)$ and based on the Table 2, Table 3, Table 4 and Table 5, the ranking order of alternatives is presented in Table 6.

TABLE 6. Ranking order of alternatives and riskiest city for various operators

Proposed operators	Ranking order of alternatives	riskiest city
$SSVNWA$	$\psi_2 \succ \psi_4 \succ \psi_3 \succ \psi_1$	ψ_2
$SSVNWG$	$\psi_2 \succ \psi_1 \succ \psi_3 \succ \psi_4$	ψ_2
$GSSVNWA,\ \lambda = 2$	$\psi_4 \succ \psi_2 \succ \psi_1 \succ \psi_3$	ψ_4
$\lambda = 5$	$\psi_2 \succ \psi_4 \succ \psi_1 \succ \psi_3$	ψ_2
$\lambda = 10$	$\psi_2 \succ \psi_3 \succ \psi_1 \succ \psi_4$	ψ_2
$GSSVNWG,\ \lambda = 2$	$\psi_1 \succ \psi_2 \succ \psi_3 \succ \psi_4$	ψ_1
$\lambda = 5$	$\psi_1 \succ \psi_3 \succ \psi_2 \succ \psi_4$	ψ_1
$\lambda = 10$	$\psi_1 \succ \psi_3 \succ \psi_2 \succ \psi_4$	ψ_1

Step 6: The procedure of proposed algorithm end here.

In the the numerical example we analysed FD-risk assessment problem. It easy to recognize that the neutrosophic set of information is expressed by SVN-number. Here we have been examined in details for the FD-risk assessment problem. Also the process of proposed strategies are reasonable for this problem. From the numerical example, we can say that it is comfortable to use the strategy to cope with the other risk assessment problems. Therefore the proposed decision making strategy has a deep practical value.

6. Comparative Analysis

A comparative study was constructed with other existing methods to show the validity of the proposed ranking method. The proposed method is compared to the other techniques such as Wei and Wei (2018), Nancy and Garg (2016), and Rong et al. (2020) SVN environments. In Table-7, we have presented a comparative analysis. By Wei and Wei (2018) method, the best alternative is ψ_2 and the worst one is ψ_1. According to the Nancy and Garg (2016) method, the best alternative is ψ_3, and the worst one is ψ_2. Again by the Rong et al. (2020) method, the best alternative is ψ_4, and the worst one is ψ_3. By our proposed method, the best alternative is ψ_2 and the worst one is ψ_1. From Table-7, it is clear that our proposed method gives better results than the other existing method.

Table 7. Comparative studies with other existence method

Proposed operators	Ranking order of alternatives	Best Alternatives
Wei and Wei (2018)	$\psi_2 \succ \psi_4 \succ \psi_3 \succ \psi_1$	ψ_2
Nancy and Garg (2016)	$\psi_3 \succ \psi_1 \succ \psi_4 \succ \psi_2$	ψ_3
Rong et al. (2020)	$\psi_4 \succ \psi_1 \succ \psi_2 \succ \psi_3$	ψ_4
Our Method	$\psi_2 \succ \psi_4 \succ \psi_1 \succ \psi_3$	ψ_2

7. Conclusions

In recent years, aggregation operators have become a popular research topic in decision-making problems. This paper presents some new aggregation operators for solving a real MADM problem under an SVN environment. Additionally, some different aggregation operators are developed, which are the softmax SVN weighted average (SVNWA) operator, softmax SVN weighted geometric (SVNWG) operator, generalized softmax SVN weighted average (GSSVNWA) operator, and generalized softmax SVN weighted geometric (GSSVNIFWG) operator. Then, we have presented some essential properties of these operators. Moreover, using the proposed operators, we have been built a MADM strategy under an SVN environment. Finally, we have illustrated one numerical example to express the usefulness and effectiveness of the proposed MADM technique. Also, we have presented a comparative analysis with other existing methods.

In the future, we will extend the proposed operators in interval neutrosophic set Wang 2005, neutrosophic cubic set (Ali 2016), and refined neutrosophic set (Smarandache 2013) environments. Also, we will try to apply the proposed operators to different realistic decision-making problems.

Compliance with ethical standards

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