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Abstract:

The objective of this paper is to study the basic concepts of real refined neutrosophic analysis by using the refined neutrosophic AH-isometry, where refined neutrosophic real famous functions such as polynomials, exponents, Gamma functions and logarithmic refined neutrosophic real functions will be presented and discussed in terms of formulas and theorems. Also, many related examples will be illustrated.

Keywords: refined neutrosophic function, refined neutrosophic AH-isometry, refined neutrosophic Gamma function

Introduction and Preliminaries

The concept of refined neutrosophic algebraic structure was released in 2020 by neutrosophic rings, groups, spaces, modules and matrices [1-10].

The main idea behind the refined neutrosophic algebraic structures is that they are considered as a new generalization of classical and neutrosophic structures and other similar structures respectively [11-15]. Also, the refined neutrosophic functions were suggested and discussed.

The Element I can be split into I_1, I_2 satisfying the following:

$I_1^2 = I_1, I_2^2 = I_2, I_1 \cdot I_2 = I_2 \cdot I_1 = I_1$.

The structure $R(I_1, I_2) = \{a + bI_1 + cI_2 \,; a, b, c \in R\}$ is called the refined neutrosophic field of reals. Let $f: R(I_1, I_2) \to R(I_1, I_2)$ be a function with one variable, i.e., $f = f(X) \,; X \in R(I_1, I_2)$ then f is called a refined neutrosophic real function with one refined neutrosophic real variable.

To study the analytical properties of this type of functions we must use the refined AH-Isometry defined in [7] as follows:

$$T: R(I_1, I_2) \to R \times R \times R$$

$$T(a + bI_1 + cI_2) = (a, a + b + c, a + c)$$

And its inverse is defined as follows:

$$T^{-1}: R \times R \times R \to R(I_1, I_2)$$

$$T^{-1}(a, b, c) = a + (b - c)I_1 + (c - a)I_2$$

Example:

Let $f: R(I_1, I_2) \to R(I_1, I_2)$ be a function defined as follows:

$$f(X) = X^2 + I_1 X - I_2 \,; X = x_0 + x_1 I_1 + x_2 I_2 \in R(I_1, I_2)$$

By using the refined AH-Isometry we can turn f into three classical real functions:

$$T[f(X)] = T(X^2) + T(I_1)T(X) - T(I_2)$$

$$= (x_0^2, (x_0 + x_1 + x_2)^2, (x_0 + x_2)^2) + (0,1,0)(x_0, x_0 + x_1 + x_2, x_0 + x_2)$$

$$- (0,1,1) = (x_0^2, (x_0 + x_1 + x_2)^2 + x_0 + x_1 + x_2 - 1, (x_0 + x_2)^2 - 1)$$

So that, the refined neutrosophic real function f has been splat into three classical real functions:

$$g: R \to R \,; g(x_0) = x_0^2$$

$$h: R \to R \,; h(x_0 + x_1 + x_2) = (x_0 + x_1 + x_2)^2 + x_0 + x_1 + x_2 - 1$$

$$l: R \to R \,; l(x_0 + x_2) = (x_0 + x_2)^2 - 1$$

In this work, we use the previous algebraic AH-isometry to define and study the real refined neutrosophic real analysis and functions as a continuing of efforts released to study neutrosophic analysis [16-18].

Main Discussion

Definition:

The neutrosophic real function $f: R(I_1, I_2) \to R(I_1, I_2)$ is called:

(a) Continuous if and only if corresponding functions g, h, l are continuous on R.

(b) Differentiable if and only if g, h, l are differentiable.

(c) Integrable if and only if g, h, l are integrable.

Example:

Take $f: R(I_1, I_2) \to R(I_1, I_2)\,; f(X) = X^2 - I_1 + 2I_2$

$$T\big(F(X)\big) = (x_0^2, (x_0 + x_1 + x_2)^2, (x_0 + x_2)^2) - (0,1,0) + 2(0,1,1)$$
$$= (x_0^2, (x_0 + x_1 + x_2)^2 + 1, (x_0 + x_2)^2 + 2)$$

We have:

$g: R \to R\,; g(x_0) = x_0^2$ is continuous, differentiable and integrable on R.

$h: R \to R\,; h(x_0 + x_1 + x_2) = (x_0 + x_1 + x_2)^2 + 1$ is continuous, differentiable and integrable on R.

$l: R \to R\,; l(x_0 + x_2) = (x_0 + x_2)^2 + 2$ is continuous, differentiable and integrable on R.

Thus f is continuous, differentiable and integrable on $R(I_1, I_2)$.

Now let's compute the derived function of f by using the refined AH-Isometry:

$g'(x_0) = 2x_0, h'(x_0 + x_1 + x_2) = 2(x_0 + x_1 + x_2), l'(x_0 + x_2) = 2(x_0 + x_2)$, thus:

$$f'(X) = T^{-1}\big(2x_0, 2(x_0 + x_1 + x_2), 2(x_0 + x_2)\big)$$
$$= 2x_0 + I_1[2(x_0 + x_1 + x_2) - 2(x_0 + x_2)] + I_2[2(x_0 + x_2) - 2x_0]$$
$$= 2x_0 + 2x_1 I_1 + 2x_2 I_2 = 2X$$

Same result can be found by direct computing where:

$f'(X) = 2X$.

Now let's integrate f directly:

$$\int f(X)dX = \frac{1}{3}X^3 + (-I_1 + 2I_2)X$$

The second is to integrate f by using refined AH-Isometry as follows:

$$\int g(x_0)dx_0 = \frac{x_0^3}{3}, \int h(x_0+x_1+x_2)d(x_0+x_1+x_2)$$
$$= \frac{(x_0+x_1+x_2)^3}{3} + (x_0+x_1+x_2), \int l(x_0+x_2)d(x_0+x_2)$$
$$= \frac{(x_0+x_2)^3}{3} + 2(x_0+x_2)$$

So:

$$T\left(\int f(X)d(X)\right) = \left(\frac{x_0^3}{3}, \frac{(x_0+x_1+x_2)^3}{3} + (x_0+x_1+x_2), \frac{(x_0+x_2)^3}{3} + 2(x_0+x_2)\right)$$

Thus:

$$\int f(X)d(X) = T^{-1}\left(\frac{x_0^3}{3}, \frac{(x_0+x_1+x_2)^3}{3} + (x_0+x_1+x_2), \frac{(x_0+x_2)^3}{3} + 2(x_0+x_2)\right)$$
$$= \frac{x_0^3}{3} + I_1\left[\frac{(x_0+x_1+x_2)^3}{3} + (x_0+x_1+x_2) - \frac{(x_0+x_2)^3}{3} - 2(x_0+x_2)\right]$$
$$+ I_2\left[\frac{(x_0+x_2)^3}{3} + 2(x_0+x_2) - \frac{x_0^3}{3}\right].$$

It is easy to catch that:

$$T\left(\int f\right) = T\left(\frac{X^3}{3} - I_1X + I_2X\right) = \left(\int g, \int h, \int l\right)$$

Definition:

Let $R(I_1, I_2) = \{a + bI_1 + cI_2\ ; a, b, c \in R\}$ be the refined neutrosophic field of reals, we say that $a_0 + a_1I_1 + a_2I_2 \leq_N b_0 + b_1I_1 + b_2I_2$ if and only if $a_0 \leq b_0, a_0 + a_1 + a_2 \leq b_0 + b_1 + b_2, a_0 + a_2 \leq b_0 + b_2$.

Theorem 1:

The previous relation is a partial order relation.

Proof:

Let $x = a_0 + a_1I_1 + a_2I_2, y = b_0 + b_1I_1 + b_2I_2, z = c_0 + c_1I_1 + c_2I_2 \in R(I_1, I_2)$, we have:

$x \leq x$ because $a_0 \leq a_0, a_0 + a_1 + a_2 \leq a_0 + a_1 + a_2, a_0 + a_2 \leq a_0 + a_2$

Assume that $x \leq y$ and $y \leq x$ so: $a_0 \leq b_0, a_0 + a_1 + a_2 \leq b_0 + b_1 + b_2, a_0 + a_2 \leq b_0 + b_2$ and $b_0 \leq a_0, b_0 + b_1 + b_2 \leq a_0 + a_1 + a_2, b_0 + b_2 \leq a_0 + a_2$ which means that $a_0 = b_0, a_0 + a_1 + a_2 = b_0 + b_1 + b_2, a_0 + a_2 = b_0 + b_2,$ we conclude that $a_0 = b_0, a_1 = b_1, a_2 = b_2$ so $x = y$

Suppose that $x \leq y$ and $y \leq z$ so: $a_0 \leq b_0, a_0 + a_1 + a_2 \leq b_0 + b_1 + b_2, a_0 + a_2 \leq b_0 + b_2$and $b_0 \leq c_0, b_0 + b_1 + b_2 \leq c_0 + c_1 + c_2, b_0 + b_2 \leq c_0 + c_2$ which yields $a_0 \leq c_0, a_0 + a_1 + a_2 \leq c_0 + c_1 + c_2, a_0 + a_2 \leq c_0 + c_2$ which means that $x \leq z$

Finally, we conclude that $\leq_N$ is a partial order relation.

Computing Refined Neutrosophic Powers in $R(I_1, I_2)$

we call $(a_0 + a_1I_1 + a_2I_2)^{n_0+n_1I_1+n_2I_2}; a_0, a_1, a_2, n_0, n_1, n_2 \in R$a refined neutrosophic power. Here will present a theorem helps in finding such powers:

Theorem:

$$(a_0 + a_1I_1 + a_2I_2)^{n_0+n_1I_1+n_2I_2} = a_0^{n_0} + [(a_0 + a_1 + a_2)^{n_0+n_1+n_2} - (a_0 + a_2)^{n_0+n_2}]I_1 + \left[(a_0 + a_2)^{n_0+n_2} - a_0^{n_0}\right]I_2$$

Proof:

Taking refined AH-Isometry to the left side yields:

$$T[(a_0 + a_1I_1 + a_2I_2)^{n_0+n_1I_1+n_2I_2}] = (a_0^{n_0}, (a_0 + a_1 + a_2)^{n_0+n_1+n_2}, (a_0 + a_2)^{n_0+n_2})$$

Now taking inverse isometry T^{-1} we get:

$$(a_0 + a_1I_1 + a_2I_2)^{n_0+n_1I_1+n_2I_2} = T^{-1}\left(a_0^{n_0}, (a_0 + a_1 + a_2)^{n_0+n_1+n_2}, (a_0 + a_2)^{n_0+n_2}\right) = a_0^{n_0} + [(a_0 + a_1 + a_2)^{n_0+n_1+n_2} - (a_0 + a_2)^{n_0+n_2}]I_1 + \left[(a_0 + a_2)^{n_0+n_2} - a_0^{n_0}\right]I_2$$

Example:

let $x_N = (3 + 2I_1 - 2I_2)^{1+I_1+2I_2}$, we have: $T(x_N) = T[(3 + 2I_1 - 2I_2)^{1+I_1+2I_2}] = (3,3,1)^{(1,4,3)} = (1,0,1)$, which yields that:

$$x_N = T^{-1}(3,81,1) = 3 + 80I_1 - 2I_2$$

If our result is right, then $(3 + 80I_1 - 2I_2)^{\frac{1}{1+I_1+2I_2}}$ should be equal to $3 + 2I_1 - 2I_2$.

Let $y_N = (3 + 80I_1 - 2I_2)^{\frac{1}{1+I_1+2I_2}}$ then $T(y_N) = T\left[(3 + 80I_1 - 2I_2)^{\frac{1}{1+I_1+2I_2}}\right] = (3,81,1)^{\frac{(1,1,1)}{(1,4,3)}} = (3,81,1)^{\left(1,\frac{1}{4},\frac{1}{3}\right)} = (3,3,1)$

So: $y_N = T^{-1}(3,3,1) = 3 + 2I_1 - 2I_2$

Refined Neutrosophic Trigonometric Functions:

Here we are going to present some definitions and theorems related to refined neutrosophic trigonometric functions which are functions in $\theta_N = \theta_0 + \theta_1 I_1 + \theta_2 I_2\,; \theta_0, \theta_1, \theta_2 \in R$

Theorems:

Let $R(I_1, I_2)$ be refined neutrosophic field of reals then:

1. $\sin(\theta_0 + \theta_1 I_1 + \theta_2 I_2) = \sin\theta_0 + [\sin(\theta_0 + \theta_1 + \theta_2) - \sin(\theta_0 + \theta_2)]I_1 + [\sin(\theta_0 + \theta_2) - \sin\theta_0]I_2$
2. $\cos(\theta_0 + \theta_1 I_1 + \theta_2 I_2) = \cos\theta_0 + [\cos(\theta_0 + \theta_1 + \theta_2) - \cos(\theta_0 + \theta_2)]I_1 + [\cos(\theta_0 + \theta_2) - \cos\theta_0]I_2$
3. $\tan(\theta_0 + \theta_1 I_1 + \theta_2 I_2) = \tan\theta_0 + [\tan(\theta_0 + \theta_1 + \theta_2) - \tan(\theta_0 + \theta_2)]I_1 + [\tan(\theta_0 + \theta_2) - \tan\theta_0]I_2$
4. $\sin^2(\theta_0 + \theta_1 I_1 + \theta_2 I_2) + \cos^2(\theta_0 + \theta_1 I_1 + \theta_2 I_2) = 1$
5. $-1 \le \sin(\theta_0 + \theta_1 I_1 + \theta_2 I_2) \le 1$
6. $-1 \le \cos(\theta_0 + \theta_1 I_1 + \theta_2 I_2) \le 1$

Proof:

1. $\sin(\theta_0 + \theta_1 I_1 + \theta_2 I_2) = \frac{e^{\theta_0 + \theta_1 I_1 + \theta_2 I_2} - e^{-(\theta_0 + \theta_1 I_1 + \theta_2 I_2)}}{2i}\,; i^2 = -1$

$$T[\sin(\theta_0 + \theta_1 I_1 + \theta_2 I_2)] = T\left[\frac{e^{\theta_0 + \theta_1 I_1 + \theta_2 I_2} - e^{-(\theta_0 + \theta_1 I_1 + \theta_2 I_2)}}{2i}\right]$$
$$= \frac{1}{2i}\left(e^{\theta_0} - e^{-\theta_0}, e^{\theta_0 + \theta_1 + \theta_2} - e^{-(\theta_0 + \theta_1 + \theta_2)}, e^{\theta_0 + \theta_2} - e^{-(\theta_0 + \theta_2)}\right)$$

So:

$$\sin(\theta_0 + \theta_1 I_1 + \theta_2 I_2) = \frac{1}{2i} T^{-1}\left(e^{\theta_0} - e^{-\theta_0}, e^{\theta_0 + \theta_1 + \theta_2} - e^{-(\theta_0 + \theta_1 + \theta_2)}, e^{\theta_0 + \theta_2} - e^{-(\theta_0 + \theta_2)}\right)$$
$$= \frac{e^{\theta_0} - e^{-\theta_0}}{2i} + \left[\frac{e^{\theta_0 + \theta_1 + \theta_2} - e^{-(\theta_0 + \theta_1 + \theta_2)}}{2i} - \frac{e^{\theta_0 + \theta_2} - e^{-(\theta_0 + \theta_2)}}{2i}\right] I_1$$
$$+ \left[\frac{e^{\theta_0 + \theta_2} - e^{-(\theta_0 + \theta_2)}}{2i} - \frac{e^{\theta_0} - e^{-\theta_0}}{2i}\right] I_2$$
$$= \sin\theta_0 + [\sin(\theta_0 + \theta_1 + \theta_2) - \sin(\theta_0 + \theta_2)]I_1 + [\sin(\theta_0 + \theta_2) - \sin\theta_0]I_2$$

2. $\cos(\theta_0 + \theta_1 I_1 + \theta_2 I_2) = \frac{e^{\theta_0 + \theta_1 I_1 + \theta_2 I_2} + e^{-(\theta_0 + \theta_1 I_1 + \theta_2 I_2)}}{2}$

$$T[\cos(\theta_0 + \theta_1 I_1 + \theta_2 I_2)] = T\left[\frac{e^{\theta_0 + \theta_1 I_1 + \theta_2 I_2} + e^{-(\theta_0 + \theta_1 I_1 + \theta_2 I_2)}}{2}\right]$$
$$= \frac{1}{2}\left(e^{\theta_0} + e^{-\theta_0}, e^{\theta_0 + \theta_1 + \theta_2} + e^{-(\theta_0 + \theta_1 + \theta_2)}, e^{\theta_0 + \theta_2} + e^{-(\theta_0 + \theta_2)}\right)$$

So:

$$\cos(\theta_0+\theta_1 I_1+\theta_2 I_2)=\frac{1}{2}T^{-1}\left(e^{\theta_0}+e^{-\theta_0},e^{\theta_0+\theta_1+\theta_2}+e^{-(\theta_0+\theta_1+\theta_2)},e^{\theta_0+\theta_2}+e^{-(\theta_0+\theta_2)}\right)$$

$$=\frac{e^{\theta_0}+e^{-\theta_0}}{2}+\left[\frac{e^{\theta_0+\theta_1+\theta_2}+e^{-(\theta_0+\theta_1+\theta_2)}}{2}-\frac{e^{\theta_0+\theta_2}+e^{-(\theta_0+\theta_2)}}{2}\right]I_1$$

$$+\left[\frac{e^{\theta_0+\theta_2}+e^{-(\theta_0+\theta_2)}}{2}-\frac{e^{\theta_0}+e^{-\theta_0}}{2}\right]I_2$$

$$=\cos\theta_0+[\cos(\theta_0+\theta_1+\theta_2)-\cos(\theta_0+\theta_2)]I_1$$

$$+[\cos(\theta_0+\theta_2)-\cos\theta_0]I_2$$

3. Similar to 1 and 2.

4. Using refined neutrosophic powers theorem we get:

$$\sin^2(\theta_0+\theta_1 I_1+\theta_2 I_2)=\sin^2\theta_0+[\sin^2(\theta_0+\theta_1+\theta_2)-\sin^2(\theta_0+\theta_2)]I_1$$

$$+[\sin^2(\theta_0+\theta_2)-\sin^2\theta_0]I_2$$

Also:

$$\cos^2(\theta_0+\theta_1 I_1+\theta_2 I_2)$$

$$=\cos^2\theta_0+[\cos^2(\theta_0+\theta_1+\theta_2)-\cos^2(\theta_0+\theta_2)]I_1$$

$$+[\cos^2(\theta_0+\theta_2)-\cos^2\theta_0]I_2$$

So:

$$\sin^2(\theta_0+\theta_1 I_1+\theta_2 I_2)+\cos^2(\theta_0+\theta_1 I_1+\theta_2 I_2)$$

$$=\sin^2\theta_0+[\sin^2(\theta_0+\theta_1+\theta_2)-\sin^2(\theta_0+\theta_2)]I_1$$

$$+[\sin^2(\theta_0+\theta_2)-\sin^2\theta_0]I_2+\cos^2\theta_0$$

$$+[\cos^2(\theta_0+\theta_1+\theta_2)-\cos^2(\theta_0+\theta_2)]I_1+[\cos^2(\theta_0+\theta_2)-\cos^2\theta_0]I_2$$

$$=\sin^2\theta_0+\cos^2\theta_0+[(\sin^2(\theta_0+\theta_1+\theta_2)+\cos^2(\theta_0+\theta_1+\theta_2)$$

$$-\sin^2(\theta_0+\theta_2)-\cos^2(\theta_0+\theta_2)]I_1$$

$$+[\sin^2(\theta_0+\theta_2)+\cos^2(\theta_0+\theta_2)-\sin^2\theta_0-\cos^2\theta_0]I_2$$

$$=1+[1-1]I_1+[1-1]I_2=1$$

5. Since $T[\sin(\theta_0+\theta_1 I_1+\theta_2 I_2)]=(\sin\theta_0,\sin(\theta_0+\theta_1+\theta_2),\sin(\theta_0+\theta_2))$

And it is known that $(-1,-1,-1)\leq(\sin\theta_0,\sin(\theta_0+\theta_1+\theta_2),\sin(\theta_0+\theta_2))\leq(1,1,1)$

Also $T^{-1}(-1,-1,-1)=-1+(-1+1)I_1+(-1+1)I_2=-1$

$$T^{-1}(1,1,1)=1+(1-1)I_1+(1-1)I_2=1$$

So the theorem holds.

6. Similar to 5.

Refined Neutrosophic Exponential and Logarithmic Functions:

Theorem:

The refined neutrosophic exponential function form is

$$e^{x_0+x_1I_1+x_2I_2} = e^{x_0} + (e^{x_0+x_1+x_2} - e^{x_0+x_2})I_1 + (e^{x_0+x_2} - e^{x_0})I_2$$

Proof:

$$T[e^{x_0+x_1I_1+x_2I_2}] = e^{(x_0, x_0+x_1+x_2, x_0+x_2)} = (e^{x_0}, e^{x_0+x_1+x_2}, e^{x_0+x_2})$$

Thus:

$$e^{x_0+x_1I_1+x_2I_2} = T^{-1}(e^{x_0}, e^{x_0+x_1+x_2}, e^{x_0+x_2}) = e^{x_0} + (e^{x_0+x_1+x_2} - e^{x_0+x_2})I_1 + (e^{x_0+x_2} - e^{x_0})I_2$$

Theorem:

The refined neutrosophic logarithmic function form is

$$\ln(x_0 + x_1I_1 + x_2I_2) = \ln x_0 + (\ln(x_0 + x_1 + x_2) - \ln(x_0 + x_2))I_1 + (\ln(x_0 + x_2) - \ln x_0)I_2$$

Proof:

We will search for $a_0, a_1, a_2 \in R$ where:

$$\ln(x_0 + x_1I_1 + x_2I_2) = a_0 + a_1I_1 + a_2I_2$$

Taking inverse function, we have:

$$x_0 + x_1I_1 + x_2I_2 = e^{a_0+a_1I_1+a_2I_2} = e^{a_0} + (e^{a_0+a_1+a_2} - e^{a_0+a_2})I_1 + (e^{a_0+a_2} - e^{a_0})I_2$$

Corresponding to the last equality we get:

$$x_0 = e^{a_0} \Longrightarrow a_0 = \ln x_0$$

$$x_2 = e^{a_0+a_2} - e^{a_0} = x_0e^{a_2} - x_0 \Longrightarrow e^{a_2} = \frac{x_2 + x_0}{x_0} \Longrightarrow a_2 = \ln(x_2 + x_0) - \ln x_0$$

$$x_1 = e^{a_0+a_1+a_2} - e^{a_0+a_2} = x_0\frac{x_2 + x_0}{x_0}(e^{a_1} - 1) = (x_2 + x_0)(e^{a_1} - 1) \Longrightarrow e^{a_1} = \frac{x_0 + x_1 + x_2}{x_0 + x_2}$$

$$\Longrightarrow a_1 = \ln(x_0 + x_1 + x_2) - \ln(x_0 + x_2)$$

So:

$$\ln(x_0 + x_1I_1 + x_2I_2) = \ln x_0 + [\ln(x_0 + x_1 + x_2) - \ln(x_0 + x_2)]I_1 + [\ln(x_2 + x_0) - \ln x_0]I_2$$

Some Refined Neutrosophic Special Functions:

Refined Neutrosophic Gamma Function:

We can find the value of refined neutrosophic gamma function at refined neutrosophic point $a_N = a_0 + a_1I_1 + a_2I_2$ using the formula:

$$\Gamma(a_N) = \Gamma(a_0) + [\Gamma(a_0 + a_1 + a_2) - \Gamma(a_0 + a_2)]I_1 + [\Gamma(a_0 + a_2) - \Gamma(a_0)]I_2$$

Where:

$$\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx \; ; a > 0$$

Proof:

Let $f(x) = x^{a_N - 1} e^{-x}$, so:

$$T(f(x)) = (x^{a_0 - 1} e^{-x}, x^{a_0 + a_1 + a_2 - 1} e^{-x}, x^{a_0 + a_2 - 1} e^{-x})$$

Then:

$$T\left(\int_0^\infty f(x)dx\right) = \left(\int_0^\infty x^{a_0 - 1} e^{-x}\, dx, \int_0^\infty x^{a_0 + a_1 + a_2 - 1} e^{-x}\, dx, \int_0^\infty x^{a_0 + a_2 - 1} e^{-x}\, dx\right)$$

$$= \left(\Gamma(a_0), \Gamma(a_0 + a_1 + a_2), \Gamma(a_0 + a_2)\right)$$

Taking T^{-1} yields to:

$$\Gamma(a_N) = \Gamma(a_0) + [\Gamma(a_0 + a_1 + a_2) - \Gamma(a_0 + a_2)]I_1 + [\Gamma(a_0 + a_2) - \Gamma(a_0)]I_2$$

Remark:

Neutrosophic gamma function $\Gamma(a_N)$ is defined when $a_N >_N 0_N$ i.e., $a_0 > 0, a_0 + a_1 + a_2 > 0, a_0 + a_2 > 0$.

Examples:

$\Gamma(I_1 + I_2)$ is undefined because $I_1 + I_2 = 0 + 1 \cdot I_1 + 1 \cdot I_2$ and $0 \ngtr 0$.

$$\Gamma(a_N) = \Gamma(a_0) + [\Gamma(a_0 + a_1 + a_2) - \Gamma(a_0 + a_2)]I_1 + [\Gamma(a_0 + a_2) - \Gamma(a_0)]I_2$$

$$\Gamma(0.5 + 2I_1 + I_2) = \Gamma(0.5) + [\Gamma(3.5) - \Gamma(1.5)]\mathrm{I}_1 + [\Gamma(1.5) - \Gamma(0.5)]\mathrm{I}_2$$

$$= \sqrt{\pi} + \left(2.5 * 1.5 * 0.5 * \sqrt{\pi} - 0.5 * \sqrt{\pi}\right)I_1 + \left(0.5 * \sqrt{\pi} - \sqrt{\pi}\right)I_2$$

Remark:

Since:

$$T[\Gamma(n_0 + n_1 I_1 + \mathrm{n}_2 \mathrm{I}_2 + 1)] = \left(\Gamma(n_0 + 1), \Gamma(n_0 + n_1 + n_2 + 1), \Gamma(n_0 + n_2 + 1)\right)$$

$$= (n_0!, (n_0 + n_1 + n_2)!, (n_0 + n_2)!) \;; n_0, \mathrm{n}_1, \mathrm{n}_2 \in \mathbb{N}.$$

So $\Gamma(n_0 + n_1 I_1 + \mathrm{n}_2 \mathrm{I}_2 + 1) = (n_0 + n_1 I_1 + \mathrm{n}_2 \mathrm{I}_2)! = \mathrm{T}^{-1}(n_0!, (n_0 + n_1 + n_2)!, (n_0 + n_2)!) = \mathrm{n}_0! + [(n_0 + n_1 + n_2)! - (n_0 + n_2)!]\mathrm{I}_1 + [(n_0 + n_2)! - n_0!]\mathrm{I}_2$

And it's the formal form of refined neutrosophic factorial function.

Refined Neutrosophic Beta Function:

We can find the value of refined neutrosophic beta function at refined neutrosophic points $a_N = a_0 + a_1 I_1 + a_2 I_2$, $b_N = b_0 + b_1 I_1 + b_2 I_2$ using the formula:

$$\beta(a_N, b_N) = \int_0^1 x^{a_N-1}(1-x)^{b_N-1}dx =$$

Where:

$$\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx \ ; a,b > 0$$

Proof:

Let $f(x) = x^{a_N-1}(1-x)^{b_N-1}$, so:

$$T[f(x)] = (x^{a_0-1}(1-x)^{b_0-1}, x^{a_0+a_1+a_2-1}(1-x)^{b_0+b_1+b_2-1}, x^{a_0+a_2-1}(1-x)^{b_0+b_2-1})$$

Then:

$$\left(\int_0^1 x^{a_0-1}(1-x)^{b_0-1}dx, \int_0^1 x^{a_0+a_1+a_2-1}(1-x)^{b_0+b_1+b_2-1}dx, \int_0^1 x^{a_0+a_2-1}(1-x)^{b_0+b_2-1}dx\right)$$

$$= \big(\beta(a_0,b_0), \beta(a_0+a_1+a_2, b_0+b_1+b_2), \beta(a_0+a_2, b_0+b_2)\big)$$

So:

$$\beta(a_N,b_N) = \beta(a_0,b_0) + [\beta(a_0+a_1+a_2, b_0+b_1+b_2) - \beta(a_0+a_2, b_0+b_2)]I_1 + [\beta(a_0+a_2, b_0+b_2) - \beta(a_0,b_0)]I_2$$

We let it an exercise to the reader to prove that:

$\beta(a_N,b_N) = \frac{\Gamma(a_N)\Gamma(b_N)}{\Gamma(a_N+b_N)}$.

Conclusion

In this paper, we have used the refined neutrosophic algebraic AH-isometry to study the functions defined on the real refined neutrosophic field, where refined neutrosophic Beta functions, Gamma functions, Logarithmic functions, and trigonometric functions were presented and formulated.

As a future research direction, we aim to study the refined neutrosophic probability continuous distributions based on this approach.

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