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On The Algebraic Properties of 2-Cyclic Refined Neutrosophic Matrices and The Diagonalization Problem

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Abstract:

The n-cyclic refined neutrosophic algebraic structures are very diverse and rich materials. In this paper, we study the elementary algebraic properties of 2-cyclic refined neutrosophic square matrices, where we find formulas for computing determinants, eigen values, and inverses. On the other hand, we solve the diagonalization problem of these matrices, where a complete algorithm to diagonalize every diagonalizable 2-cyclic refined neutrosophic square matrix is obtained and illustrated by many related examples.

Key Words: n-cyclic refined neutrosophic ring, n –cyclic refined neutrosophic matrix, the diagonalization problem.

1.Introduction

Neutrosophic algebraic structures were defined firstly in [1], by adding an algebraic indeterminacy element I to classical algebraic structures to obtain n novel extensions. For example, we can find neutrosophic geometry, neutrosophic functions, neutrosophic rings, and neutrosophic spaces [2-7].

The concept of n-cyclic neutrosophic algebraic structure was supposed in [8], and then it has been studied widely, see [9-12].

As an important algebraic structure, neutrosophic matrices with many types were handled and studied, where we can see many results about inverses, eigen vectors, diagonalizations, and determinants were proven and established [13-24]. In the literature, we have many types of neutrosophic matrices, refined neutrosophic matrices, and n-refined neutrosophic matrices, and n-cyclic refined neutrosophic matrices [17].

The diagonalization algorithm for n-cyclic refined neutrosophic matrix has been asked as an open problem in [12], and it is still open for all values of n.

This motivates us to study the diagonalization problem for n =2, and to present an effective algorithm to diagonalize a 2-cyclic refined neutrosophic square matrix, as well as many related concepts, especially eigen values computing.

2. Preliminaries

Definition [8]

Let $(R, +, \times)$ be a ring and $I_k; 1 \leq k \leq n$ be n sub-indeterminacies. We define $R_n(I) = \{a_0 + a_1I + \dots + a_nI^n; a_i \in R\}$ to be n-cyclic refined neutrosophic ring.

Operations on $R_n(I)$ are defined as:

$$\sum_{i=0}^n x_i I_i + \sum_{i=0}^n y_i I_i = \sum_{i=0}^n (x_i + y_i) I_i, \sum_{i=0}^n x_i I_i \times \sum_{i=0}^n y_i I_i = \sum_{i,j=0}^n (x_i \times y_j) I_i I_j = \sum_{i,j=0}^n (x_i \times y_j) I_{(i+j \bmod n)}.$$

\times is the multiplication on the ring R.

In this paper, we study open problem 3, open problem 4, and open problem 5 in [12].

3. Main discussion :

Definition.

Let $M = M_0 + M_1 I_1 + M_2 I_2$ be a 2-cyclic refined neutrosophic matrix, then M is diagonalizable if and only if there exists a 2-cyclic refined neutrosophic diagonal matrix K and invertible matrix U such that $M = UKU^{-1}$.

Theorem.

Let $M = M_0 + M_1I_1 + M_2I_2$ be a 2-cyclic refined neutrosophic matrix, then M is diagonalizable if and only if: $M_0, M_0 + M_1 + M_2, M_0 - M_1 + M_2$ are diagonalizable.

Proof.

Assume that M is diagonalizable, then there exists a diagonal matrix $K = K_0 + K_1I_1 + K_2I_2$ and an invertible matrix $U = U_0 + U_1I_1 + U_2I_2$ such that $M = UKU^{-1}$.

The matrix equation $UKU^{-1} = M$ is equivalent to:

$$\begin{aligned} U_0K_0U_0^{-1} + \frac{1}{2}I_1[(U_0 + U_1 + U_2)(K_0 + K_1 + K_2)(U_0 + U_1 + U_2)^{-1} \\ - (U_0 - U_1 + U_2)(K_0 - K_1 + K_2)(U_0 - U_1 + U_2)^{-1}] \\ + \frac{1}{2}I_2[(U_0 + U_1 + U_2)(K_0 + K_1 + K_2)(U_0 + U_1 + U_2)^{-1} \\ - (U_0 - U_1 + U_2)(K_0 - K_1 + K_2)(U_0 - U_1 + U_2)^{-1} - 2U_0K_0U_0^{-1}] \\ = M_0 + M_1I_1 + M_2I_2 \end{aligned}$$

Thus:

$$\begin{cases} U_0K_0U_0^{-1} = M_0 \\ (U_0 + U_1 + U_2)(K_0 + K_1 + K_2)(U_0 + U_1 + U_2)^{-1} = M_0 + M_1 + M_2 \\ (U_0 - U_1 + U_2)(K_0 - K_1 + K_2)(U_0 - U_1 + U_2)^{-1} = M_0 - M_1 + M_2 \end{cases}$$

This implies $M_0, M_0 + M_1 + M_2, M_0 - M_1 + M_2$ are diagonalizable.

Conversely, assume that $M_0, M_0 + M_1 + M_2, M_0 - M_1 + M_2$ are diagonalizable, then there exists diagonal matrices D_0, D_1, D_2 and invertible matrices P_0, P_1, P_2 such that $P_0D_0P_0^{-1} = M_0, P_1D_1P_1^{-1} = M_0 + M_1 + M_2, P_2D_2P_2^{-1} = M_0 - M_1 + M_2$.

This implies that $M_1 = \frac{1}{2}(P_1D_1P_1^{-1} - P_2D_2P_2^{-1}), M_2 = \frac{1}{2}(P_1D_1P_1^{-1} + P_2D_2P_2^{-1} - 2P_0D_0P_0^{-1})$

We put $D = D_0 + \frac{1}{2}I_1(D_1 - D_2) + \frac{1}{2}I_2(D_1 + D_2 - 2D_0) = L_0 + \frac{1}{2}I_1L_1 +$

$$\frac{1}{2}I_2L_2; \begin{cases} L_0 = D_0 \\ L_1 = D_1 - D_2 \\ L_2 = D_1 + D_2 - 2D_0 \end{cases}.$$

$$P = P_0 + \frac{1}{2}I_1(P_1 - P_2) + \frac{1}{2}I_2(P_1 + P_2 - 2P_0) = N_0 + \frac{1}{2}I_1N_1 + \frac{1}{2}I_2N_2; \begin{cases} N_0 = P_0 \\ N_1 = P_1 - P_2 \\ N_2 = P_1 + P_2 - 2P_0 \end{cases}.$$

We have:

$$\begin{aligned} P^{-1} &= N_0^{-1} + \frac{1}{2}I_1[(N_0 + N_1 + N_2)^{-1} - (N_0 - N_1 + N_2)^{-1}] + \frac{1}{2}I_2[(N_0 + N_1 + N_2)^{-1} - \\ &(N_0 - N_1 + N_2)^{-1} - 2N_0^{-1}] = P_0^{-1} + \frac{1}{2}I_1[P_1^{-1} - P_2^{-1}] + \frac{1}{2}I_2[P_1^{-1} + P_2^{-1} - 2P_0^{-1}] \end{aligned}$$

It is easy to check that $PDP^{-1} = M_0 + M_1I_1 + M_2I_2 = M$, thus M is diagonalizable.

Example.

Consider the following 2×2 2-cyclic refined matrix:

$$X = \begin{pmatrix} 3 + \frac{1}{2}I_1 - \frac{3}{2}I_2 & \frac{1}{2}I_1 + \frac{1}{2}I_2 \\ \frac{1}{2}I_1 - \frac{1}{2}I_2 & 2 - \frac{3}{2}I_1 + \frac{1}{2}I_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{pmatrix} I_1 + \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} I_2$$

$$= X_0 + X_1 I_1 + X_2 I_2$$

We have:

$$X_0 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, X_0 + X_1 + X_2 = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, X_0 - X_1 + X_2 = \begin{pmatrix} 1 & 0 \\ -1 & 4 \end{pmatrix}$$

X_0 is diagonalizable with $X_0 = P_0^{-1} A_0 P_0$, where $P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_0 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

$X_0 + X_1 + X_2$ is diagonalizable with $X_0 + X_1 + X_2 = P_1^{-1} A_1 P_1$, where $P_1 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$X_0 - X_1 + X_2$ is diagonalizable with $X_0 - X_1 + X_2 = P_2^{-1} A_2 P_2$, where $P_2 = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

According the previous theorem, we have.

$X = P^{-1} Y P$, where:

$$Y = A_0 + \frac{1}{2} I_1 (A_1 - A_2) + \frac{1}{2} I_2 (A_1 + A_2 - 2A_0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{3}{2} \end{pmatrix} I_1 + \begin{pmatrix} -\frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} I_2$$

$$= \begin{pmatrix} 3 + \frac{1}{2}I_1 - \frac{3}{2}I_2 & 0 \\ 0 & 2 - \frac{3}{2}I_1 + \frac{1}{2}I_2 \end{pmatrix}$$

$$P = P_0^{-1} + \frac{1}{2} I_1 [P_1^{-1} - P_2^{-1}] + \frac{1}{2} I_2 [P_1^{-1} + P_2^{-1} - 2P_0^{-1}]$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & \frac{1}{2} \\ -\frac{1}{2} & -1 \end{pmatrix} I_1 + \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix} I_2 = \begin{pmatrix} 1 - I_1 + I_2 & \frac{1}{2}I_1 + \frac{1}{2}I_2 \\ -\frac{1}{2}I_1 + \frac{1}{2}I_2 & 1 - I_1 - I_2 \end{pmatrix}$$

The Eigen Values.

Definition.

Let $A = A_0 + A_1 I_1 + A_2 I_2$ be an n-cyclic refined neutrosophic matrix, we say that $T = t_0 + t_1 I_1 + t_2 I_2 \in R_2(I)$ is an eigen value if and only if $AX = tX$; $X = X_0 + X_1 I_1 + x_2 I_2$ is an n-cyclic refined neutrosophic vector, where $X_i \in R^n$.

X is called n-cyclic refined neutrosophic vector.

Theorem.

Let $A = A_0 + A_1I_1 + A_2I_2$ be an n-cyclic refined neutrosophic matrix, then $T = t_0 + t_1I_1 + t_2I_2 \in R_2(I)$ is an eigen value with $X = X_0 + X_1I_1 + X_2I_2$ as eigen vector if and only if:

t_0 is an eigen value of A_0 with X_0 as eigen vector, $t_0 + t_1 + t_2$ is an eigen value of $A_0 + A_1 + A_2$ with $X_0 + X_1 + X_2$ as eigen vector, $t_0 - t_1 + t_2$ is an eigen value of $A_0 - A_1 + A_2$ with $X_0 - X_1 + X_2$ as eigen vector.

Proof.

The equation $AX = tX$ is equivalent to:

$$\begin{aligned} A_0X_0 + \frac{1}{2}I_1[(A_0 + A_1 + A_2)(X_0 + X_1 + X_2) - (A_0 - A_1 + A_2)(X_0 - X_1 + X_2)] \\ + \frac{1}{2}I_2[(A_0 + A_1 + A_2)(X_0 + X_1 + X_2) + (A_0 - A_1 + A_2)(X_0 - X_1 + X_2) \\ - 2A_0X_0] \\ = t_0X_0 + \frac{1}{2}I_1[(t_0 + t_1 + t_2)(X_0 + X_1 + X_2) - (t_0 - t_1 + t_2)(X_0 - X_1 + X_2)] \\ + \frac{1}{2}I_2[(t_0 + t_1 + t_2)(X_0 + X_1 + X_2) + (t_0 - t_1 + t_2)(X_0 - X_1 + X_2) - 2t_0X_0] \end{aligned}$$

So that:

$$\begin{cases} t_0X_0 = A_0X_0 \dots (1) \\ (t_0 + t_1 + t_2)(X_0 + X_1 + X_2) - (t_0 - t_1 + t_2)(X_0 - X_1 + X_2) = (A_0 + A_1 + A_2)(X_0 + X_1 + X_2) - (A_0 - A_1 + A_2)(X_0 - X_1 + X_2) \\ (t_0 + t_1 + t_2)(X_0 + X_1 + X_2) + (t_0 - t_1 + t_2)(X_0 - X_1 + X_2) = (A_0 + A_1 + A_2)(X_0 + X_1 + X_2) + (A_0 - A_1 + A_2)(X_0 - X_1 + X_2) \end{cases}$$

This equivalents:

$$\begin{cases} A_0X_0 = t_0X_0 \\ (A_0 + A_1 + A_2)(X_0 + X_1 + X_2) = (t_0 + t_1 + t_2)(X_0 + X_1 + X_2) \\ (A_0 - A_1 + A_2)(X_0 - X_1 + X_2) = (t_0 - t_1 + t_2)(X_0 - X_1 + X_2) \end{cases}$$

Thus, the proof is complete.

Example.

Consider the matrix:

$$\begin{aligned} A = \begin{pmatrix} 3 + \frac{1}{2}I_1 - \frac{3}{2}I_2 & \frac{1}{2}I_1 + \frac{1}{2}I_2 \\ \frac{1}{2}I_1 - \frac{1}{2}I_2 & 2 - \frac{3}{2}I_1 + \frac{1}{2}I_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -3 \end{pmatrix} I_1 + \begin{pmatrix} -3 & 1 \\ 2 & 2 \end{pmatrix} I_2 \\ = A_0 + A_1I_1 + A_2I_2 \end{aligned}$$

The eigen values of A_0 are $\{3,2\}$.

The eigen values of $A_0 + A_1 + A_2$ are $\{2,1\}$.

The eigen values of $A_0 - A_1 + A_2$ are $\{1,4\}$.

To find the corresponding 2×2 2-cyclic refined neutrosophic matrix A , we discuss the following cases:

Case(1). If $t_0 = 3$, $t_0 + t_1 + t_2 = 2$, $t_0 - t_1 + t_2 = 1$, then:

$$t_1 = \frac{1}{2}, t_2 = \frac{-3}{2}, \text{ thus } T_1 = 3 + \frac{1}{2}I_1 - \frac{3}{2}I_2.$$

Case(2). If $t_0 = 3$, $t_0 + t_1 + t_2 = 2$, $t_0 - t_1 + t_2 = 4$, then:

$$t_1 = -1, t_2 = 0, \text{ thus } T_2 = 3 - I_1.$$

Case(3). If $t_0 = 3$, $t_0 + t_1 + t_2 = 1$, $t_0 - t_1 + t_2 = 1$, then:

$$t_1 = 0, t_2 = -2, \text{ thus } T_3 = 3 - 2I_2.$$

Case(4). If $t_0 = 3$, $t_0 + t_1 + t_2 = 1$, $t_0 - t_1 + t_2 = 4$, then:

$$t_1 = \frac{-3}{2}, t_2 = \frac{-1}{2}, \text{ thus } T_4 = 3 - \frac{3}{2}I_1 - \frac{1}{2}I_2.$$

Case(5). If $t_0 = 2$, $t_0 + t_1 + t_2 = 2$, $t_0 - t_1 + t_2 = 1$, then:

$$t_1 = \frac{1}{2}, t_2 = \frac{-1}{4}, \text{ thus } T_5 = 3 + \frac{1}{2}I_1 - \frac{1}{4}I_2.$$

Case(6). If $t_0 = 2$, $t_0 + t_1 + t_2 = 2$, $t_0 - t_1 + t_2 = 4$, then:

$$t_1 = -1, t_2 = 1, \text{ thus } T_6 = 3 - I_1 + I_2.$$

Case(7). If $t_0 = 2$, $t_0 + t_1 + t_2 = 1$, $t_0 - t_1 + t_2 = 1$, then:

$$t_1 = 0, t_2 = -1, \text{ thus } T_7 = 3 - I_2.$$

Case(8). If $t_0 = 2$, $t_0 + t_1 + t_2 = 1$, $t_0 - t_1 + t_2 = 4$, then:

$$t_1 = \frac{-3}{2}, t_2 = \frac{1}{2}, \text{ thus } T_8 = 3 - \frac{3}{2}I_1 + \frac{1}{2}I_2.$$

This implies that A has 8 eigen values.

The determinant of an n-cyclic refined neutrosophic matrix.

According to the previous discussion, we have found an algorithm to compute n-cyclic refined neutrosophic matrix.

From the point of view, we are forced to study the computing of eigen values by determinants.

Definition.

Let $A = A_0 + A_1I_1 + A_2I_2$ be an n-cyclic refined neutrosophic matrix, we define its determinant as follows:

$$\det A = \det A_0 + \frac{1}{2}I_1[\det(A_0 + A_1 + A_2) - \det(A_0 - A_1 + A_2)] + \frac{1}{2}I_2[\det(A_0 + A_1 + A_2) + \det(A_0 - A_1 + A_2) - 2 \det A_0].$$

Theorem.

Let $A = A_0 + A_1I_1 + A_2I_2, B = B_0 + B_1I_1 + B_2I_2$ be two $n \times n$ n -cyclic refined neutrosophic matrices, then:

- 1). A is invertible if and only if $\det A$ is invertible.
- 2). $\det A^T = \det A$.
- 3). $\det(A.B) = \det A . \det B$.
- 4). $T = t_0 + t_1I_1 + t_2I_2$ is an eigen of A if and only if $\det(A - TU_{n \times n}) = 0$.

Proof.

1). It is clear and easy.

2). $A^T = A_0^T + A_1^T I_1 + A_2^T I_2$, thus:

$$\begin{aligned} \det A^T &= \det A_0^T + \frac{1}{2}I_1[\det(A_0 + A_1 + A_2)^T - \det(A_0 - A_1 + A_2)^T] \\ &\quad + \frac{1}{2}I_2[\det(A_0 + A_1 + A_2)^T - \det(A_0 - A_1 + A_2)^T - 2 \det A_0^T] = \det A \end{aligned}$$

$$\begin{aligned} 3). \quad A.B &= A_0B_0 + \frac{1}{2}I_1[(A_0 + A_1 + A_2)(B_0 + B_1 + B_2) - (A_0 - A_1 + A_2)(B_0 - B_1 + B_2)] + \\ &\quad \frac{1}{2}I_2[(A_0 + A_1 + A_2)(B_0 + B_1 + B_2) + (A_0 - A_1 + A_2)(B_0 - B_1 + B_2) - 2A_0B_0] = A_0B_0 + \\ &\quad \frac{1}{2}I_1(T_1 - T_2) + \frac{1}{2}I_2(T_1 + T_2 - 2A_0B_0), \text{ where:} \end{aligned}$$

$$T_1 = (A_0 + A_1 + A_2)(B_0 + B_1 + B_2), T_2 = (A_0 - A_1 + A_2)(B_0 - B_1 + B_2)$$

$$\det(A.B) = \det A_0B_0$$

$$\begin{aligned} &+ \frac{1}{2}I_1 \left[\det \left(\frac{1}{2}T_1 - \frac{1}{2}T_2 + \frac{1}{2}T_1 + \frac{1}{2}T_2 - A_0B_0 + A_0B_0 \right) \right. \\ &\quad \left. - \det \left(A_0B_0 - \frac{1}{2}T_1 + \frac{1}{2}T_2 + \frac{1}{2}T_1 + \frac{1}{2}T_2 - A_0B_0 \right) \right] \\ &+ \frac{1}{2}I_2 \left[\det \left(\frac{1}{2}T_1 - \frac{1}{2}T_2 + \frac{1}{2}T_1 + \frac{1}{2}T_2 - A_0B_0 + A_0B_0 \right) \right. \\ &\quad \left. - \det \left(A_0B_0 - \frac{1}{2}T_1 + \frac{1}{2}T_2 + \frac{1}{2}T_1 + \frac{1}{2}T_2 - A_0B_0 \right) - 2 \det A_0B_0 \right] \\ &= \det A_0 \det B_0 + \frac{1}{2}I_1[\det T_1 - \det T_2] + \frac{1}{2}I_2[\det T_1 + \det T_2 - 2 \det A_0 \det B_0] \\ &= \det A_0 \det B_0 \\ &+ \frac{1}{2}I_1[\det(A_0 + A_1 + A_2) . \det(B_0 + B_1 + B_2) \\ &\quad + \det(A_0 - A_1 + A_2) . \det(B_0 - B_1 + B_2)] \\ &+ \frac{1}{2}I_2[\det(A_0 + A_1 + A_2) . \det(B_0 + B_1 + B_2) \\ &\quad - \det(A_0 - A_1 + A_2) . \det(B_0 - B_1 + B_2) - 2 \det A_0 \det B_0] = \det A \det B \end{aligned}$$

4). We have $A - TU_{n \times n} = (A_0 + A_1I_1 + A_2I_2) - (t_0 + t_1I_1 + t_2I_2)U_{n \times n} = (A_0 - t_0U_{n \times n}) + (A_1 - t_1U_{n \times n})I_1 + (A_2 - t_2U_{n \times n})I_2$.

$$\begin{aligned} \det(A - TU_{n \times n}) &= \det(A_0 - t_0U_{n \times n}) \\ &+ \frac{1}{2}I_1[\det(A_0 + A_1 + A_2 - (t_0 + t_1 + t_2)U_{n \times n}) \\ &- \det(A_0 - A_1 + A_2 - (t_0 - t_1 + t_2)U_{n \times n})] \\ &+ \frac{1}{2}I_2[\det(A_0 + A_1 + A_2 - (t_0 + t_1 + t_2)U_{n \times n}) \\ &+ \det(A_0 - A_1 + A_2 - (t_0 - t_1 + t_2)U_{n \times n}) - 2\det(A_0 - t_0U_{n \times n})] \end{aligned}$$

The equation $\det(A - TU_{n \times n}) = 0$ is equivalent to:

$$\begin{cases} \det(A_0 - t_0U_{n \times n}) = 0 \\ \det(A_0 + A_1 + A_2 - (t_0 + t_1 + t_2)U_{n \times n}) = 0 \\ \det(A_0 - A_1 + A_2 - (t_0 - t_1 + t_2)U_{n \times n}) = 0 \end{cases}$$

This is equivalent to:

To is eigen value of A_0 , $t_0 + t_1 + t_2$ is eigen value of $A_0 + A_1 + A_2$, $t_0 - t_1 + t_2$ is eigen value of $A_0 - A_1 + A_2$, thus T is an eigen value of A .

Theorem.

Let $A = A_0 + A_1I_1 + A_2I_2, B = B_0 + B_1I_1 + B_2I_2$ be two $n \times n$ n-cyclic refined neutrosophic matrices, then:

$$\begin{aligned} A.B &= A_0B_0 + \frac{1}{2}I_1[(A_0 + A_1 + A_2)(B_0 + B_1 + B_2) - (A_0 - A_1 + A_2)(B_0 - B_1 + B_2)] \\ &+ \frac{1}{2}I_2[(A_0 + A_1 + A_2)(B_0 + B_1 + B_2) + (A_0 - A_1 + A_2)(B_0 - B_1 + B_2) \\ &- 2A_0B_0] \end{aligned}$$

The proof is easy and clear.

Example.

Consider the following 2×2 2-cyclic refined neutrosophic matrix:

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}I_1 + \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}I_2 = \begin{pmatrix} 1 + 2I_1 + I_2 & 1 + I_1 \\ I_1 + 3I_2 & 2 + I_1 + I_2 \end{pmatrix} = A_0 + A_1I_1 + A_2I_2 \\ A_0^{-1} &= \frac{1}{2} \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix}, (A_0 + A_1 + A_2) = \begin{pmatrix} 4 & 3 \\ 4 & 4 \end{pmatrix}, (A_0 + A_1 + A_2)^{-1} = \frac{1}{4} \begin{pmatrix} 4 & -3 \\ -4 & 4 \end{pmatrix}, (A_0 - A_1 + A_2) \\ &= \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}, (A_0 - A_1 + A_2)^{-1} = -\frac{1}{2} \begin{pmatrix} 2 & -1 \\ -2 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
A^{-1} &= \frac{1}{2} \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} I_1 \left[\frac{1}{4} \begin{pmatrix} 4 & -3 \\ -4 & 4 \end{pmatrix} - \frac{-1}{2} \begin{pmatrix} 2 & -1 \\ -2 & 0 \end{pmatrix} \right] \\
&\quad + \frac{1}{2} I_2 \left[\frac{1}{4} \begin{pmatrix} 4 & -3 \\ -4 & 4 \end{pmatrix} + \frac{-1}{2} \begin{pmatrix} 2 & -1 \\ -2 & 0 \end{pmatrix} - 2 \left(\frac{1}{2} \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} \right) \right] \\
&= \begin{pmatrix} 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} + \frac{1}{2} I_1 \left[\begin{pmatrix} 1 & -\frac{3}{4} \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & 0 \end{pmatrix} \right] \\
&\quad + \frac{1}{2} I_2 \left[\begin{pmatrix} 1 & -\frac{3}{4} \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & \frac{1}{2} \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} \right] \\
&= \begin{pmatrix} 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} + \frac{1}{2} I_1 \begin{pmatrix} 2 & -\frac{5}{4} \\ -2 & 1 \end{pmatrix} + \frac{1}{2} I_2 \begin{pmatrix} -2 & \frac{7}{4} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 + I_1 - I_2 & -1 - \frac{5}{8} I_1 + \frac{7}{8} I_2 \\ -I_1 & \frac{1}{2} + \frac{1}{2} I_1 \end{pmatrix}
\end{aligned}$$

Theorem.

Let $X = X_0 + X_1 I_1 + X_2 I_2$ be a 2-cyclic refined neutrosophic matrix, then X is invertible if and only if $X_0, X_0 + X_1 + X_2, X_0 - X_1 + X_2$ are invertible, also:

$$\begin{aligned}
X^{-1} &= X_0^{-1} + \frac{1}{2} I_1 [(X_0 + X_1 + X_2)^{-1} - (X_0 - X_1 + X_2)^{-1}] \\
&\quad + \frac{1}{2} I_2 [(X_0 + X_1 + X_2)^{-1} + (X_0 - X_1 + X_2)^{-1} - 2X_0]
\end{aligned}$$

Proof.

Assume that X is invertible, the exists $Y = Y_0 + Y_1 I_1 + Y_2 I_2$ such that $X.Y = U_{n \times n}$.

$$\begin{aligned}
X.Y &= X_0 Y_0 + I_1 [X_0 Y_1 + X_1 Y_0 + X_2 Y_1 + X_1 Y_2] + I_2 [X_0 Y_2 + X_2 Y_0 + X_1 Y_1 + X_2 Y_2] \\
&= X_0 Y_0 + \frac{1}{2} I_1 [(X_0 + X_1 + X_2)(Y_0 + Y_1 + Y_2) - (X_0 - X_1 + X_2)(Y_0 - Y_1 + Y_2)] \\
&\quad + \frac{1}{2} I_2 [(X_0 + X_1 + X_2)(Y_0 + Y_1 + Y_2) + (X_0 - X_1 + X_2)(Y_0 - Y_1 + Y_2) - 2X_0 Y_0] \\
&= U_{n \times n}
\end{aligned}$$

This implies that:

$$\begin{cases} X_0 Y_0 = U_{n \times n} \\ (X_0 + X_1 + X_2)(Y_0 + Y_1 + Y_2) = (X_0 - X_1 + X_2)(Y_0 - Y_1 + Y_2) = U_{n \times n} \end{cases}$$

Hence $X_0, X_0 + X_1 + X_2, X_0 - X_1 + X_2$ are invertible.

On the other hand, we get $Y_0 = X_0^{-1}$, $Y_0 - Y_1 + Y_2 = (X_0 - X_1 + X_2)^{-1}$, $Y_0 + Y_1 + Y_2 = (X_0 + X_1 + X_2)^{-1}$, thus:

$$Y_1 = \frac{1}{2} [(X_0 + X_1 + X_2)^{-1} - (X_0 - X_1 + X_2)^{-1}]$$

$$Y_2 = \frac{1}{2}[(X_0 + X_1 + X_2)^{-1} + (X_0 - X_1 + X_2)^{-1} - 2X_0^{-1}].$$

Conclusion

In this paper, we have presented a full solution of the diagonalization problem of 2-cyclic refined neutrosophic matrices, where we have presented a novel algorithm to compute the eigen values and vectors of 2-cyclic refined neutrosophic matrices that helps in representing them as a product $A^{-1}DA$, where A is an invertible matrix, and D is diagonal matrix.

In the future, we suggest researchers to continue our efforts, and to study the possibility of diagonalization problem of 3-cyclic refined neutrosophic matrices.

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