Neutrosophic Sets and Systems

Volume 53 Article 38

6-15-2023

On Some Estimation Methods of Neutrosophic Continuous Probability Distributions Using One-Dimensional AH-Isometry

Abdulrahman Astambli

Mohamed Bisher Zein

Yasin Karmouta

Neutrosophic Sets and Systems, Vol. 53, 2023
University of New Mexico

On Some Estimation Methods of Neutrosophic Continuous Probability Distributions Using One-Dimensional AH-Isometry

Abdulrahman Astambli[1], Mohamed Bisher Zeina[2] and Yasin Karmouta[3]
[1] Faculty of Science, Dept. of Mathematical Statistics, University of Aleppo, Aleppo, Syria, e-mail: abdulrahman.est.96@gmail.com.
[2] Faculty of Science, Dept. of Mathematical Statistics, University of Aleppo, Aleppo, Syria, e-mail: bisher.zeina@gmail.com.
[3] Faculty of Science, Dept. of Mathematical Statistics, University of Aleppo, Aleppo, Syria, e-mail: yassinkarmouta@gmail.com.

Abstract: In this research, we introduce an algebraic approach to define the concept of neutrosophic maximum likelihood estimation method based on neutrosophic continuous probability distributions based on classical neutrosophic numbers of the form $N = a + bI; I^2 = I$ i.e., I is a letter not a numerical set. We prove that the neutrosophic loglikelihood function gives the same estimators given by neutrosophic likelihood function. Also, we present the concept of neutrosophic moments estimation method which produces system of neutrosophic equations to derive the neutrosophic estimators using an algebraic isomorphism. Estimators based on two mentioned methods were derived successfully for some neutrosophic continuous probability distributions. Concept of neutrosophic Fisher information is also presented. Theorems were proved using an algebraic approach depending on the one-dimensional AH-Isometry. A simulation study is also presented to show the efficiency of the presented estimators.

Keywords: AH Isometry; Neutrosophic Field of Reals; Maximum Likelihood; Moments; Probability Density Functions; Neutrosophic Fisher Information.

1. Introduction

Neutrosophic field of reals is an extension to field of reals adding new algebraic structure I satisfies $I^2 = I$ so we get $R(I) = R \cup \{I\} = \{a + bI\ ; a, b \in R\ , I^2 = I\}$ which is neutrosophic field of reals. [1]

Many mathematical studies were done based on the neutrosophic set of reals in many fields of abstract mathematics including abstract algebra, probability theory, topology, number theory, etc.[2-7].

In [8] Abobala and Hatip presented an isometry called AH-Isometry which transfers mathematical problems from $R(I)$ to $R \times R$ and an inverse isometry transfers the mathematical problem from $R \times R$ to $R(I)$. This isometry is very applicable to solve and study many types of mathematical problems including real analysis, complex analysis, algebraic structures, probability theory, operations research, etc.

Many previous studies about neutrosophic probability theory were done assuming that parameters of probability distribution functions are indeterminant, i.e. parameter θ is an interval neutrosophic number, so it can be noted by $\theta_N \in [\theta^L, \theta^U]$.[9-15]

In [16], [17] Zahid Khan, Sultan Salem et al. presented neutrosophic lognormal model and studied its critical properties then applied this model to environmental data and in lifetime data where they treated problems with interval neutrosophic numbers, in [18] Zahid Khan et al. presented neutrosophic gamma distribution and applied it to a real dataset for the purpose of dealing with inaccurate statistical data which is also described by interval neutrosophic numbers. Many other extensions were done to other types of distributions like neutrosophic exponential distribution, neutrosophic maxwell distribution, etc. and these extensions were applied successfully in may real datasets. [19-21]. Notice that all the mentioned studies are done by using neutrosophic interval numbers $N = d + I$ where $I \in [a, b]$ and not neutrosophic classical numbers of the form $N = a + bI, I^2 = I$ and this is the main difference between our study and the previous studies, so, we are going to study neutrosophic probability distribution assuming that there is uncertainty in its parameters and the random variable itself, i.e. $f(x; \theta)$ is $f(x_N; \theta_N)$ based on its algebraic structure, i.e. $x_N = x + yI, \theta_N = \theta_1 + \theta_2 I\ ; x, y, \theta_1, \theta_2 \in R\ , I^2 = I\ , 0 \cdot I = 0$ by using AH-Isometry which will transfer each neutrosophic probability density function into two crisp probability distribution functions. Based on this transformation we have successfully defined the neutrosophic log-likelihood function and studied its properties then found estimators of neutrosophic probability distributions based on maximum likelihood estimation method, also on the same algorithm we have succeed to define moments estimation method and finally the neutrosophic fisher information about the estimated parameters. One can also define many other estimation methods based on the same algorithm presented in this paper. Many examples were solved successfully and estimators of many neutrosophic probability distributions were successfully derived.

2. Preliminaries

Definition 2.1 [5] [8]

Let $R(I) = \{a + bI\ ; a, b \in R\ , I^2 = I, 0 \cdot I = 0\}$ be the neutrosophic field of reals. The one-dimensional AH-isometry is defined as follows:

$$T: R(I) \to R^2 :\ T(a + bI) = (a, a + b) \quad (1)$$

And its inverse is defined as follows:

$$T^{-1}: R^2 \to R(I) :\ T^{-1}(a, b) = a + (b - a)I \quad (2)$$

Remark:

We will call the form $a + bI$ the formal of a neutrosophic number.

Definition 2.2 [6]

Let $f: R(I) \to R(I); f = f(x_N)$ where $x_N = x + yI \in R(I)$ then f is called a neutrosophic real function with one neutrosophic variable.

Definition 2.3 [1]

A neutrosophic random variable can be defined as follows: [6] [22]

$$X_N = X + YI\ ; I^2 = I\ , 0 \cdot I = 0 \quad (3)$$

Where X, Y are crisp random variables taking values on R.

Definition 2.4 [4]

Let $R(I)$ be the neutrosophic field of reals, and let $a_N = a_1 + a_2 I, b_N = b_1 + b_2 I \in R(I)$. We say that $a_N \geq_N b_N$ iff:

$$a_1 \geq b_1 \text{ and } a_1 + a_2 \geq b_1 + b_2$$

Definition 2.5 [22]

Let $R(I)$ be the neutrosophic field of reals, the neutrosophic logarithmic function can be defined as:

$$ln(x + yI) = ln\,x + [ln(x + y) - ln(x)]I, \text{ where } x + yI >_N 0.$$

3. Results and Discussion

Definition 3.1

Suppose that $\mathbb{X}_N = X_{1N}, X_{2N}, \dots, X_{nN}$ is a sequence of neutrosophic random variables, we say that $\mathbb{X}_N$ is a neutrosophic random sample drawn from neutrosophic random variable X_N if $X_{1N}, X_{2N}, \dots, X_{nN}$ are dependent and have the same probability distribution as X_N.

Definition 3.2

Let $\mathbb{X}_N$ be a random sample drawn from X_N, we call the function:

$$L_N = L(\mathbb{X}_N; \Theta_N) = f(\mathbb{X}_N; \Theta_N) = \prod_{i=1}^{n} f(X_{iN}; \Theta_N) \quad (4)$$

The neutrosophic likelihood function where $\Theta_N = \Theta_1 + \Theta_2 I = (\theta_{1N}, \theta_{2N}, \dots, \theta_{rN})$ is a vector of unknown parameters.

Theorem 1

The formal form of neutrosophic likelihood function L_N is:

$$L_N = L(\mathbb{X}_N; \Theta_N) = L(\mathbb{X}; \Theta_1) + [L(\mathbb{X} + \mathbb{Y}; \Theta_1 + \Theta_2) - L(\mathbb{X}; \Theta_1)]I \quad (5)$$

Proof:

Using the one-dimensional AH-Isometry:

$$T\big(L(\mathbb{X}_N; \Theta_N)\big) = T\left(\prod_{i=1}^{n} f(x_i + y_i I; \Theta_1 + \Theta_2 I)\right)$$

$$= \prod_{i=1}^{n} f\big((x_i, x_i + y_i); (\Theta_1, \Theta_1 + \Theta_2)\big)$$

$$= \left(\prod_{i=1}^{n} f(x_i; \Theta_1), \prod_{i=1}^{n} f(x_i + y_i; \Theta_1 + \Theta_2)\right) \quad (6)$$

Now taking the inverse isometry T^{-1}:

$$L(\mathbb{X}_N; \Theta_N) = T^{-1}\left(\left(\prod_{i=1}^{n} f(x_i; \Theta_1), \prod_{i=1}^{n} f(x_i + y_i; \Theta_1 + \Theta_2)\right)\right)$$

$$= \prod_{i=1}^{n} f(x_i, \Theta_1) + \left[\prod_{i=1}^{n} f(x_i + y_i; \Theta_1 + \Theta_2) - \prod_{i=1}^{n} f(x_i, \Theta_1)\right] I$$

$$= L(\mathbb{X}; \Theta_1) + [L(\mathbb{X} + \mathbb{Y}; \Theta_1 + \Theta_2) - L(\mathbb{X}; \Theta_1)]I \quad (7)$$

Definition 3.3

We call $\mathcal{L}_N = ln\,L(\mathbb{X}_N; \Theta_N)$ the neutrosophic loglikelihood function.

Theorem 2

The formal form of neutrosophic loglikelihood function is:

$$\mathcal{L}_N = \mathcal{L}(\mathbb{X}; \Theta_1) + [\mathcal{L}(\mathbb{X} + \mathbb{Y}; \Theta_1 + \Theta_2) - \mathcal{L}(\mathbb{X}; \Theta_1)]I \quad (8)$$

Proof:

Similar to theorem 1.

Definition 3.4

The neutrosophic statistic $\widehat{\Theta}_N$ based on random sample that maximize the neutrosophic likelihood function is called the neutrosophic likelihood estimator.

Theorem 3

The neutrosophic statistic based on random sample that maximize the neutrosophic likelihood function is the same statistic that maximize the neutrosophic loglikelihood function.

Proof:

The neutrosophic statistic $\widehat{\Theta}_N$ that maximize the likelihood function fulfills the following conditions:

$$\frac{\partial}{\partial\Theta_N}L(\mathbb{X}_N;\Theta_N)|_{\Theta_N=\widehat{\Theta}_N}=0\ ,\frac{\partial^2}{\partial\Theta_N^2}L(\mathbb{X}_N;\Theta_N)|_{\Theta_N=\widehat{\Theta}_N}<_N 0 \qquad (9)$$

Using theorem 1 the conditions become:

$$\frac{\partial L(\mathbb{X};\Theta_1)|_{\Theta_1=\widehat{\Theta}_1}}{\partial\Theta_1}+\left[\frac{\partial L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)|_{\Theta_1+\Theta_2=\widehat{\Theta}_1+\widehat{\Theta}_2}}{\partial(\Theta_1+\Theta_2)}-\frac{\partial L(\mathbb{X};\Theta_1)|_{\Theta_1=\widehat{\Theta}_1}}{\partial\Theta_1}\right]I=0 \qquad (10)$$

Which means that:

$$\frac{\partial}{\partial\Theta_1}L(\mathbb{X};\Theta_1)|_{\Theta_1=\widehat{\Theta}_1}=0 \qquad (11)$$

$$\frac{\partial}{\partial(\Theta_1+\Theta_2)}L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)|_{\Theta_1+\Theta_2=\widehat{\Theta}_1+\widehat{\Theta}_2}=0 \qquad (12)$$

The same to the second condition which yields to:

$$\frac{\partial^2}{\partial\Theta_1^2}L(\mathbb{X};\Theta_1)|_{\Theta_1=\widehat{\Theta}_1}<0 \qquad (13)$$

$$\frac{\partial^2}{\partial(\Theta_1+\Theta_2)^2}L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)|_{\Theta_1+\Theta_2=\widehat{\Theta}_1+\widehat{\Theta}_2}<0 \qquad (14)$$

If we apply the same conditions to the neutrosophic loglikelihood function we get:

$$\frac{\partial}{\partial\Theta_N}\mathcal{L}_N=\frac{\partial\mathcal{L}(\mathbb{X};\Theta_1)|_{\Theta_1=\widehat{\Theta}_1}}{\partial\Theta_1}+\left[\frac{\partial\mathcal{L}(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)|_{\Theta_1+\Theta_2=\widehat{\Theta}_1+\widehat{\Theta}_2}}{\partial(\Theta_1+\Theta_2)}-\frac{\partial\mathcal{L}(\mathbb{X};\Theta_1)|_{\Theta_1=\widehat{\Theta}_1}}{\partial\Theta_1}\right]I=0 \qquad (15)$$

Since $\mathcal{L}_N=ln\,L_N$, we know that $\frac{\partial}{\partial\Theta_N}\mathcal{L}_N=\frac{\partial}{\partial\Theta_N}lnL_N=\frac{\frac{\partial}{\partial\Theta_N}L_N}{L_N}$

So, the first condition become:

$$\frac{\frac{\partial}{\partial\Theta_1}L(\mathbb{X};\Theta_1)}{L(\mathbb{X};\Theta_1)}|_{\Theta_1=\widehat{\Theta}_1}+\left[\frac{\frac{\partial}{\partial(\Theta_1+\Theta_2)}L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)}{L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)}|_{\Theta_1+\Theta_2=\widehat{\Theta}_1+\widehat{\Theta}_2}-\frac{\frac{\partial}{\partial\Theta_1}L(\mathbb{X};\Theta_1)}{L(\mathbb{X};\Theta_1)}|_{\Theta_1=\widehat{\Theta}_1}\right]I=0 \quad (16)$$

Which means that both following equations hold:

$$\frac{\frac{\partial}{\partial\Theta_1}L(\mathbb{X};\Theta_1)}{L(\mathbb{X};\Theta_1)}|_{\Theta_1=\widehat{\Theta}_1}=0 \qquad (17)$$

$$\frac{\frac{\partial}{\partial(\Theta_1+\Theta_2)}L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)}{L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)}|_{\Theta_1+\Theta_2=\widehat{\Theta}_1+\widehat{\Theta}_2}=0 \qquad (18)$$

And this yields to:

$$\frac{\partial}{\partial\Theta_1}L(\mathbb{X};\Theta_1)|_{\Theta_1=\widehat{\Theta}_1}=0 \qquad (19)$$

$$\frac{\partial}{\partial(\Theta_1+\Theta_2)}L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)|_{\Theta_1+\Theta_2=\widehat{\Theta}_1+\widehat{\Theta}_2}=0 \qquad (20)$$

And these are the same equations as (11), (12).

Same proof can be applied to the second condition.

Example 1:

Let $X_{1N},X_{2N},\dots,X_{nN}$ be a neutrosophic random sample drawn from the density of neutrosophic power distribution:

$$f(x_N;\theta_N)=\theta_N x_N^{\theta_N-1};0\le_N x_N\le_N 1$$

Let's take AH-Isometry to $f(x_N;\theta_N)$:

$$\begin{aligned}T[f(x_N;\theta_N)]&=T\left[\theta_N x_N^{\theta_N-1}\right]=T\left[(\theta_1+\theta_2I)(x+yI)^{(\theta_1+\theta_2I)-1}\right]\\&=T[(\theta_1+\theta_2I)]T\left[(x+yI)^{(\theta_1+\theta_2I)-1}\right]\\&=T[(\theta_1+\theta_2I)]T[(x+yI)]^{T[(\theta_1+\theta_2I)-1]}\\&=(\theta_1,\theta_1+\theta_2)(x,x+y)^{(\theta_1,\theta_1+\theta_2)-(1,1)}\\&=\left(\theta_1x^{\theta_1-1},(\theta_1+\theta_2)(x+y)^{(\theta_1+\theta_2)-1}\right)\\&=\left(f(x;\theta_1),f(x+y;\theta_1+\theta_2)\right)\end{aligned}$$

So, by applying equation (15) considering properties of probability density functions we get:

$$T\left[\frac{\partial}{\partial\theta_N} ln\, L(\mathbb{X}_N;\theta_N)\right] = \left(\sum_{i=1}^{n}\frac{\partial}{\partial\theta_1} ln\, f(x_i;\theta_1), \sum_{i=1}^{n}\frac{\partial}{\partial(\theta_1+\theta_2)} ln\, f(x_i+y_i;\theta_1+\theta_2)\right)$$

$$= \left(\sum_{i=1}^{n}\frac{\partial}{\partial\theta_1} ln\left(\theta_1 x_i^{\theta_1-1}\right), \sum_{i=1}^{n}\frac{\partial}{\partial(\theta_1+\theta_2)} ln\left((\theta_1+\theta_2)(x_i+y_i)^{(\theta_1+\theta_2)-1}\right)\right)$$

$$= \left(\sum_{i=1}^{n}\frac{\partial\, ln\,\theta_1}{\partial\theta_1}+\frac{\partial(\theta_1-1)\, ln\, x_i}{\partial\theta_1}, \sum_{i=1}^{n}\frac{\partial\, ln(\theta_1+\theta_2)}{\partial(\theta_1+\theta_2)}+\frac{\partial(\theta_1+\theta_2-1)\, ln(x_i+y_i)}{\partial(\theta_1+\theta_2)}\right)$$

$$= \left(\sum_{i=1}^{n}\frac{1}{\theta_1}+ln\, x_i, \sum_{i=1}^{n}\frac{1}{\theta_1+\theta_2}+ln(x_i+y_i)\right) = \left(\frac{n}{\theta_1}+\sum_{i=1}^{n} ln\, x_i, \frac{n}{\theta_1+\theta_2}+\sum_{i=1}^{n} ln(x_i+y_i)\right)$$

$$T\left[\frac{\partial}{\partial\theta_N} ln\, L(\mathbb{X}_N;\theta_N)\right] = T[0]$$

$$\left(\frac{n}{\hat\theta_1}+\sum_{i=1}^{n} ln\, x_i, \frac{n}{\hat\theta_1+\hat\theta_2}+\sum_{i=1}^{n} ln(x_i+y_i)\right) = (0,0)$$

$$\left(\hat\theta_1, \hat\theta_1+\hat\theta_2\right) = \left(-\frac{n}{\sum_{i=1}^{n} ln\, x_i}, -\frac{n}{\sum_{i=1}^{n} ln(x_i+y_i)}\right)$$

$$T^{-1}\left(\hat\theta_1, \hat\theta_1+\hat\theta_2\right) = T^{-1}\left(-\frac{n}{\sum_{i=1}^{n} ln\, x_i}, -\frac{n}{\sum_{i=1}^{n} ln(x_i+y_i)}\right)$$

$$\Rightarrow \hat\theta_N = -\frac{n}{\sum_{i=1}^{n} ln\, x_i}+\left[-\frac{n}{\sum_{i=1}^{n} ln(x_i+y_i)}+\frac{n}{\sum_{i=1}^{n} ln\, x_i}\right]I$$

Example 2:

Let $X_{1N}, X_{2N}, \dots, X_{nN}$ be a neutrosophic random sample drawn from the density of neutrosophic Maxwell distribution:

$$f(x_N;\theta_N) = \sqrt{\frac{2}{\pi}}\,\theta_N^{\frac{3}{2}} x_N^2 e^{-\frac{1}{2}\theta_N x_N^2}; x_N >_N 0$$

So:

$$\left(\sum_{i=1}^{n}\frac{\partial}{\partial\theta_1} ln\, f(x_i;\theta_1), \sum_{i=1}^{n}\frac{\partial}{\partial(\theta_1+\theta_2)} ln\, f(x_i+y_i;\theta_1+\theta_2)\right) = (0,0)$$

$$\left(\sum_{i=1}^{n}\frac{\partial}{\partial\theta_1} ln\left(\sqrt{\frac{2}{\pi}}\,\theta_1^{\frac{3}{2}} x^2 e^{-\frac{1}{2}\theta_1 x^2}\right), \sum_{i=1}^{n}\frac{\partial}{\partial(\theta_1+\theta_2)} ln\left(\sqrt{\frac{2}{\pi}}\,(\theta_1+\theta_2)^{\frac{3}{2}}(x+y)^2 e^{-\frac{1}{2}(\theta_1+\theta_2)(x+y)^2}\right)\right) = (0,0)$$

$$\left(\frac{3n}{2\hat\theta_1}-\frac{1}{2}\sum_{i=1}^{n} x_i^2, \frac{3n}{2\left(\hat\theta_1+\hat\theta_2\right)}-\sum_{i=1}^{n}(x_i+y_i)^2\right) = (0,0)$$

$$\left(\hat\theta_1, \hat\theta_1+\hat\theta_2\right) = \left(\frac{3n}{\sum_{i=1}^{n} x_i^2}, \frac{3n}{\sum_{i=1}^{n}(x_i+y_i)^2}\right)$$

Taking T^{-1}:

$$T^{-1}\left(\hat\theta_1, \hat\theta_1+\hat\theta_2\right) = T^{-1}\left(\frac{3n}{\sum_{i=1}^{n} x_i^2}, \frac{3n}{\sum_{i=1}^{n}(x_i+y_i)^2}\right)$$

$$\hat\theta_N = \frac{3n}{\sum_{i=1}^{n} x_i^2}+\left[\frac{3n}{\sum_{i=1}^{n}(x_i+y_i)^2}-\frac{3n}{\sum_{i=1}^{n} x_i^2}\right]I$$

Example 3:

Let $X_{1N}, X_{2N}, \dots, X_{nN}$ be a neutrosophic random sample drawn from the density of neutrosophic exponential distribution:

$$f(x_N;\theta_N) = \frac{1}{\theta_N} e^{-\frac{x_N}{\theta_N}} \quad ; \quad x_N >_N 0$$

So:

$$\left(\sum_{i=1}^{n}\frac{\partial}{\partial\theta_1} ln\, f(x_i;\theta_1), \sum_{i=1}^{n}\frac{\partial}{\partial(\theta_1+\theta_2)} ln\, f(x_i+y_i;\theta_1+\theta_2)\right) = (0,0)$$

$$\left(\sum_{i=1}^{n}\frac{\partial}{\partial\theta_1}ln\frac{1}{\theta_1}e^{-\frac{x}{\theta_1}},\sum_{i=1}^{n}\frac{\partial}{\partial(\theta_1+\theta_2)}ln\frac{1}{\theta_1+\theta_2}e^{-\frac{(x+y)}{(\theta_1+\theta_2)}}\right)=(0,0)$$

$$\left(-\frac{n}{\hat{\theta}_1}+\frac{n\bar{X}}{\theta_1^2},-\frac{n}{\hat{\theta}_1+\hat{\theta}_2}+\frac{n\bar{X}+n\bar{Y}}{\left(\hat{\theta}_1+\hat{\theta}_2\right)^2}\right)=(0,0)$$

$$\left(\hat{\theta}_1,\hat{\theta}_1+\hat{\theta}_2\right)=(\bar{X},\bar{X}+\bar{Y})$$

$$T^{-1}\left(\hat{\theta}_1,\hat{\theta}_1+\hat{\theta}_2\right)=T^{-1}(\bar{X},\bar{X}+\bar{Y})$$

$$\Rightarrow\hat{\theta}_N=\bar{X}+\bar{Y}I$$

Definition 3.5

Let X_N be a neutrosophic random variable, we call $\alpha_{kN}=E(X_N^k)$ the k[th] moment of the neutrosophic random variable X_N.

Definition 3.6

Let $X_{1N},X_{2N},\ldots,X_{nN}$ be a neutrosophic random sample drawn from the neutrosophic random variable X_N, we call $A_{kN}(X)=\frac{1}{n}\sum_{i=1}^{n}X_{iN}^k$ the sample moment of order k.

Definition 3.7

The parameter that satisfies the following system of equations:

$$\alpha_{kN}=A_{kN}(X) \qquad (21)$$

Is called the moments estimator where k is the number of unknown parameters.

Theorem 4

Equations (21) can be written in R^2 in the following form:

$$\left(\int_{-\infty}^{+\infty}x^kf(x;\theta_1)\,dx,\int_{-\infty}^{+\infty}(x+y)^kf(x+y;\theta_1+\theta_2)\,d(x+y)\right)=\left(\frac{1}{n}\sum_{i=1}^{n}x_i^k,\frac{1}{n}\sum_{i=1}^{n}(x_i+y_i)^k\right) \qquad (22)$$

Proof:

$$\alpha_{kN}=\alpha_{kN}(\theta_N)=E(X_N^k)=\int_{-\infty}^{+\infty}x_N^kf(x_N;\theta_N)\,dx_N \qquad (23)$$

Taking AH-Isometry:

$$T[\alpha_{kN}]=T[E(X_N^k)]=T\left[\int_{-\infty}^{+\infty}x_N^kf(x_N;\theta_N)\,dx_N\right]$$

$$=\left(\int_{-\infty}^{+\infty}x^kf(x;\theta_1)\,dx,\int_{-\infty}^{+\infty}(x+y)^kf(x+y;\theta_1+\theta_2)\,d(x+y)\right) \qquad (24)$$

Also:

$$A_{kN}(X)=\frac{1}{n}\sum_{i=1}^{n}X_{iN}^k \qquad (25)$$

And taking the AH-Isometry:

$$T[A_{kN}(X)]=T\left[\frac{1}{n}\sum_{i=1}^{n}X_{iN}^k\right]=\left(\frac{1}{n}\sum_{i=1}^{n}x_i^k,\frac{1}{n}\sum_{i=1}^{n}(x_i+y_i)^k\right) \qquad (26)$$

Equations (24) and (26) proves the theorem.

e.g., for one parameter, we substitute $k=1$:

$$\left(\int_{-\infty}^{+\infty}xf(x;\theta_1)\,dx,\int_{-\infty}^{+\infty}(x+y)f(x+y;\theta_1+\theta_2)\,d(x+y)\right)=(\bar{X},\bar{X}+\bar{Y}) \qquad (27)$$

for two parameters, we substitute $k=2$:

$$\left(\int_{-\infty}^{+\infty}x^2f(x;\theta_1)\,dx,\int_{-\infty}^{+\infty}(x+y)^2f(x+y;\theta_1+\theta_2)\,d(x+y)\right)=\left(\frac{1}{n}\sum_{i=1}^{n}x_i^2,\frac{1}{n}\sum_{i=1}^{n}(x_i+y_i)^2\right) \qquad (28)$$

And so on.

Example 4:

Let $X_{1N},X_{2N},\ldots,X_{nN}$ be a neutrosophic random sample drawn from the density given in example 1, then to find the moments estimator we have to solve the equation:

$$\alpha_{1N}=A_{1N}$$

$$\left(\int_{-\infty}^{+\infty}xf(x;\theta_1)\,dx,\int_{-\infty}^{+\infty}(x+y)f(x+y;\theta_1+\theta_2)\,d(x+y)\right)=(\bar{X},\bar{X}+\bar{Y})$$

$$\left(\int_0^1 x\theta_1 x^{\theta_1-1}\,dx, \int_0^1 (x+y)(\theta_1+\theta_2)(x+y)^{\theta_1+\theta_2-1}\,d(x+y)\right) = (\bar{X}, \bar{X}+\bar{Y})$$

$$\left(\frac{\hat{\theta}_1}{\hat{\theta}_1+1}, \frac{\hat{\theta}_1+\hat{\theta}_2}{\hat{\theta}_1+\hat{\theta}_2+1}\right) = (\bar{X}, \bar{X}+\bar{Y})$$

$$\left(\hat{\theta}_1, \hat{\theta}_1+\hat{\theta}_2\right) = \left(\frac{\bar{X}}{1-\bar{X}}, \frac{\bar{X}+\bar{Y}}{1-(\bar{X}+\bar{Y})}\right)$$

$$T^{-1}\left(\hat{\theta}_1, \hat{\theta}_1+\hat{\theta}_2\right) = T^{-1}\left(\frac{\bar{X}}{1-\bar{X}}, \frac{\bar{X}+\bar{Y}}{1-(\bar{X}+\bar{Y})}\right)$$

$$\hat{\theta}_N = \hat{\theta}_1 + \hat{\theta}_2 I = \frac{\bar{X}}{1-\bar{X}} + \left[\frac{\bar{X}+\bar{Y}}{1-(\bar{X}+\bar{Y})} - \frac{\bar{X}}{1-\bar{X}}\right] I$$

Example 5:

Let $X_{1N}, X_{2N}, \dots, X_{nN}$ be a neutrosophic random sample drawn from the density given in example 2, then to find the moments estimator we have to solve the equation:

Using equation (27):

$$\left(\int_0^{+\infty} x\sqrt{\frac{2}{\pi}}\theta_1^{\frac{3}{2}} x^2 e^{-\frac{1}{2}\theta_1 x^2}\,dx, \int_0^{+\infty} (x+y)\sqrt{\frac{2}{\pi}}(\theta_1+\theta_2)^{\frac{3}{2}}(x+y)^2 e^{-\frac{1}{2}(\theta_1+\theta_2)(x+y)^2}\,d(x+y)\right) = (\bar{X}, \bar{X}+\bar{Y})$$

$$\left(\sqrt{\frac{8}{\pi\hat{\theta}_1}}, \sqrt{\frac{8}{\pi\left(\hat{\theta}_1+\hat{\theta}_2\right)}}\right) = (\bar{X}, \bar{X}+\bar{Y})$$

$$\left(\hat{\theta}_1, \hat{\theta}_1+\hat{\theta}_2\right) = \left(\frac{8}{\pi\bar{X}^2}, \frac{8}{\pi(\bar{X}+\bar{Y})^2}\right)$$

$$\hat{\theta}_N = T^{-1}\left(\frac{8}{\pi\bar{X}^2}, \frac{8}{\pi(\bar{X}+\bar{Y})^2}\right) = \frac{8}{\pi\bar{X}^2} + \left[\frac{8}{\pi(\bar{X}+\bar{Y})^2} - \frac{8}{\pi\bar{X}^2}\right] I$$

Example 6:

Let $X_{1N}, X_{2N}, \dots, X_{nN}$ be a neutrosophic random sample drawn from the density given in example 3, then to find the moments estimator we have to solve the equation:

Using equation (27):

$$\left(\int_0^{+\infty} x\frac{1}{\theta_1}e^{-\frac{x}{\theta_1}}\,dx, \int_0^{+\infty} (x+y)\frac{1}{\theta_1+\theta_2}e^{-\frac{(x+y)}{(\theta_1+\theta_2)}}\,d(x+y)\right) = (\bar{X}, \bar{X}+\bar{Y})$$

$$\left(\hat{\theta}_1, \hat{\theta}_1+\hat{\theta}_2\right) = (\bar{X}, \bar{X}+\bar{Y})$$

$$T^{-1}\left(\hat{\theta}_1, \hat{\theta}_1+\hat{\theta}_2\right) = T^{-1}(\bar{X}, \bar{X}+\bar{Y})$$

$$\hat{\theta}_N = \bar{X} + \bar{Y} I$$

Definition 3.8

We call the partial derivative of neutrosophic log-likelihood function the neutrosophic score function and we denote it by:

$$U(\mathbb{X}_N; \theta_N) = \frac{\partial}{\partial\Theta_N}\mathcal{L}_N \qquad (29)$$

Remark:

Notice that equation (29) is a neutrosophic random sample since it is a function of $\mathbb{X}_N$.

Theorem 5

Expected value of neutrosophic score function is equal to zero.

Proof:

$$T\left[\int_{-\infty}^{+\infty} L(\mathbb{x}_N; \Theta_N)\,d\mathbb{x}_N\right] = T[1] \qquad (30)$$

Where $\int_{-\infty}^{+\infty} L(\mathbb{x}_N; \Theta_N)\,d\mathbb{x}_N = 1$ because $L(\mathbb{x}_N; \Theta_N)$ is a neutrosophic probability density function.

$$T\left[\frac{\partial}{\partial\Theta_N}\int_{-\infty}^{+\infty} L(\mathbb{x}_N; \Theta_N)\,d\mathbb{x}_N\right] = T\left[\frac{\partial}{\partial\Theta_N}1\right] \qquad (31)$$

$$\left(\frac{\partial}{\partial\Theta_1}\int_{-\infty}^{+\infty} L(\mathbb{x};\Theta_1)\,d\mathbb{x}, \frac{\partial}{\partial(\Theta_1+\Theta_2)}\int_{-\infty}^{+\infty} L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)\,d(\mathbb{x}+\mathbb{y})\right) = (0,0) \quad (32)$$

$$\left(\int_{-\infty}^{+\infty}\frac{\partial}{\partial\Theta_1} ln\,L(\mathbb{x};\Theta_1)\,L(\mathbb{x};\Theta_1)\,dx_1, \int_{-\infty}^{+\infty}\frac{\partial}{\partial(\Theta_1+\Theta_2)} ln\,L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)\,L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)\,d(\mathbb{x}+\mathbb{y})\right) = (0,0) \quad (33)$$

$$\left(E\left[\frac{\partial}{\partial\Theta_1} ln\,L(\mathbb{X};\Theta_1)\right], E\left[\frac{\partial}{\partial(\Theta_1+\Theta_2)} ln\,L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)\right]\right) = (0,0) \quad (34)$$

$$\left(E[U(\mathbb{X};\Theta_1)], E[U(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)]\right) = (0,0) \quad (35)$$

Taking T^{-1} get:

$$E\big(U(\mathbb{X}_N;\theta_N)\big) = 0 \quad (36)$$

Definition 3.9

We will call variance of neutrosophic score function the neutrosophic Fisher information about the neutrosophic parameter Θ_N $(NFI_n(\Theta_N))$ i.e.:

$$NFI_n(\Theta_N) = Var\big(U(\mathbb{X}_N;\theta_N)\big) = E\big(U^2(\mathbb{X}_N;\theta_N)\big) \quad (37)$$

Theorem 6

$$Var\big(U(\mathbb{X}_N;\theta_N)\big) = E([U(\mathbb{X}_N;\theta_N)]^2) = -nE\left(\frac{\partial^2 ln\,f(x_N;\theta_N)}{\partial\theta_N^2}\right) \quad (38)$$

Proof:

$$\frac{\partial}{\partial\Theta_N}E\big(U(\mathbb{X}_N;\theta_N)\big) = 0 \quad (39)$$

$$T\left[\frac{\partial}{\partial\Theta_N}E\big(U(\mathbb{X}_N;\theta_N)\big)\right] = T[0] \quad (40)$$

$$\left(\frac{\partial}{\partial\Theta_1}\int_{-\infty}^{+\infty}\frac{\partial}{\partial\Theta_1} ln\,L(\mathbb{x};\Theta_1)\,L(\mathbb{x};\Theta_1)\,d\mathbb{x}, \frac{\partial}{\partial(\Theta_1+\Theta_2)}\int_{-\infty}^{+\infty}\frac{\partial}{\partial(\Theta_1+\Theta_2)} ln\,L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)\,L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)\,d(\mathbb{x}+\mathbb{y})\right) \quad (41)$$

$$\left(\int_{-\infty}^{+\infty}\left(\frac{\partial^2 ln\,L(\mathbb{x};\Theta_1)L(\mathbb{x};\Theta_1)}{\partial\Theta_1^2} + \frac{\partial L(\mathbb{x};\Theta_1)}{\partial\Theta_1}\frac{\partial\,ln\,L(\mathbb{x};\Theta_1)}{\partial\Theta_1}\right)d\mathbb{x}, \left(\int_{-\infty}^{+\infty}\frac{\partial^2 ln\,L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)}{\partial(\Theta_1+\Theta_2)^2} + \frac{\partial L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)}{\partial(\Theta_1+\Theta_2)}\frac{\partial\,ln\,L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)}{\partial(\Theta_1+\Theta_2)}\right)d(\mathbb{x}+\mathbb{y})\right) = (0,0) \quad (42)$$

$$\left(\int_{-\infty}^{+\infty}\left(\frac{\partial^2 ln\,L(\mathbb{x};\Theta_1)L(\mathbb{x};\Theta_1)}{\partial\Theta_1^2} + \left(\frac{\partial\,ln\,L(\mathbb{x};\Theta_1)}{\partial\Theta_1}\right)^2 L(\mathbb{x};\Theta_1)\right)d\mathbb{x}, \int_{-\infty}^{+\infty}\left(\frac{\partial^2 ln\,L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)}{\partial(\Theta_1+\Theta_2)^2} + \left(\frac{\partial\,ln\,L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)}{\partial(\Theta_1+\Theta_2)}\right)^2 L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)\right)d(\mathbb{x}+\mathbb{y})\right) = (0,0) \quad (43)$$

$$\left(E\left(\frac{\partial^2 ln\,L(\mathbb{X};\Theta_1)}{\partial\Theta_1^2}\right) + E\left(\frac{\partial\,ln\,L(\mathbb{X};\Theta_1)}{\partial\Theta_1}\right)^2, E\left(\frac{\partial^2 ln\,L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)}{\partial(\Theta_1+\Theta_2)^2}\right) + E\left(\frac{\partial\,ln\,L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)}{\partial(\Theta_1+\Theta_2)}\right)^2\right) = (0,0) \quad (44)$$

$$E\left[\frac{\partial^2 ln\,L(\mathbb{X};\Theta_1)}{\partial\Theta_1^2}\right] + E\left[\left(\frac{\partial\,ln\,L(\mathbb{X};\Theta_1)}{\partial\Theta_1}\right)^2\right] + \left[E\left[\left(\frac{\partial^2 ln\,L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)}{\partial(\Theta_1+\Theta_2)^2}\right)^2\right] + E\left[\left(\frac{\partial\,ln\,L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)}{\partial(\Theta_1+\Theta_2)}\right)^2\right] - \left(E\left[\frac{\partial^2 ln\,L(\mathbb{X};\Theta_1)}{\partial\Theta_1^2}\right] + E\left[\left(\frac{\partial\,ln\,L(\mathbb{X};\Theta_1)}{\partial\Theta_1}\right)^2\right]\right)\right]I = 0 \quad (45)$$

$$E\left[\frac{\partial^2}{\partial\Theta_N^2} ln\,L(\mathbb{X}_N;\Theta_N)\right] + E\left[\left(\frac{\partial}{\partial\Theta_N} ln\,L(\mathbb{X}_N;\Theta_N)\right)^2\right] = 0 \quad (46)$$

$$E([U(\mathbb{X}_N;\theta_N)]^2) = -nE\left(\frac{\partial^2 ln\,f(x_N;\theta_N)}{\partial\theta_N^2}\right) \quad (47)$$

Theorem 7

Neutrosophic Fisher information can be written in the following form:

$$NFI_n(\theta_N) = FI_n(\theta_1) + [FI_n(\theta_1+\theta_2) - FI_n(\theta_1)]I \quad (48)$$

Where:

$$FI_n(\theta) = nE\frac{\partial^2}{\partial\theta^2} ln\,f(x;\theta) \quad (49)$$

Proof:

Using equations (47) and properties of AH-Isometry we get:

$$NFI_n(\theta_N) = E([U(\mathbb{X}_N;\theta_N)]^2) = -nE\left(\frac{\partial^2 \ln f(x_N;\theta_N)}{\partial\theta_N^2}\right) \quad (50)$$

$$T\big(NFI_n(\theta_N)\big) = T\left(-nE\left(\frac{\partial^2}{\partial(\theta_1+\theta_2 I)^2}\ln f(x+yI;\theta_1+\theta_2 I)\right)\right)$$

$$= \left(-nE\frac{\partial^2}{\partial\theta_1^2}\ln f(x;\theta_1), -nE\frac{\partial^2}{\partial(\theta_1+\theta_2)^2}\ln f(x+y;\theta_1+\theta_1)\right) \quad (51)$$

Taking T^{-1}get:

$$NFI_n(\theta_N) = T^{-1}\left(-nE\frac{\partial^2}{\partial\theta_1^2}\ln f(x;\theta_1), -nE\frac{\partial^2}{\partial(\theta_1+\theta_2)^2}\ln f(x+y;\theta_1+\theta_1)\right)$$

$$= -nE\frac{\partial^2}{\partial\theta_1^2}\ln f(x;\theta_1) + \left[-nE\frac{\partial^2}{\partial(\theta_1+\theta_2)^2}\ln f(x+y;\theta_1+\theta_2) + nE\frac{\partial^2}{\partial\theta_1^2}\ln f(x;\theta_1)\right]I$$

$$= FI_n(\theta_1) + [FI_n(\theta_1+\theta_2) - FI_n(\theta_1)]I \quad (52)$$

Example 7:

Let $\mathbb{X}_N$ be a neutrosophic random sample of distribution given in example 3, then:

$$T\big(f(x_N;\theta_N)\big) = \left(\frac{1}{\theta_1}e^{-\frac{x}{\theta_1}}, \frac{1}{(\theta_1+\theta_2)}e^{-\frac{(x+y)}{(\theta_1+\theta_2)}}\right)$$

$$T(\ln f(x_N;\theta_N)) = \left(-\ln\theta_1 - \frac{x}{\theta_1}, -\ln(\theta_1+\theta_2) - \frac{(x+y)}{(\theta_1+\theta_2)}\right)$$

$$T\left(\frac{\partial}{\partial\theta_N}\ln f(x_N;\theta_N)\right) = \left(-\frac{1}{\theta_1}+\frac{x}{\theta_1^2}, -\frac{1}{(\theta_1+\theta_2)}+\frac{(x+y)}{(\theta_1+\theta_2)^2}\right)$$

$$T\left(\frac{\partial^2}{\partial\theta_N^2}\ln f(x_N;\theta_N)\right) = \left(\frac{1}{\theta_1^2}-\frac{2x}{\theta_1^3}, \frac{1}{(\theta_1+\theta_2)^2}-\frac{2(x+y)}{(\theta_1+\theta_2)^3}\right)$$

$$T\left(-nE\left(\frac{\partial^2}{\partial\theta_N^2}\ln f(x_N;\theta_N)\right)\right) = \left(-nE\left(\frac{1}{\theta_1^2}-\frac{2x}{\theta_1^3}\right), -nE\left(\frac{1}{(\theta_1+\theta_2)^2}-\frac{2(x+y)}{(\theta_1+\theta_2)^3}\right)\right)$$

$$T\left(-nE\left(\frac{\partial^2}{\partial\theta_N^2}\ln f(x_N;\theta_N)\right)\right) = \left(\frac{n}{\theta_1^2}, \frac{n}{(\theta_1+\theta_2)^2}\right)$$

$$-nE\left(\frac{\partial^2}{\partial\theta_N^2}\ln f(x_N;\theta_N)\right) = T^{-1}\left(\left(\frac{n}{\theta_1^2}, \frac{n}{(\theta_1+\theta_2)^2}\right)\right) = \frac{n}{\theta_1^2} + \left[\frac{n}{(\theta_1+\theta_2)^2} - \frac{n}{\theta_1^2}\right]I = NIF_n(\theta_N)$$

Simulation Analysis:

In this part, performance of two estimation methods was evaluated based on Monte Carlo simulation to the three studied neutrosophic probability distributions using R software with various sample sizes and with total replication of $N = 10000$ times with sample sizes of 5,15,30,50 and 100 and with fixed parameter $\theta_N = 2 + I$. Goodness of estimation was assessed depending on average bias and root mean square error defined below: [18]

$$AB = \frac{\sum_{i=1}^{N}(\hat{\theta}_{Ni} - \theta_N)}{N}$$

$$RMSE = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\theta}_{Ni} - \theta_N)^2}{N}}$$

Table (1) shows results of simulation analysis for neutrosophic power distribution and compares the two proposed estimation methods, notice that average bias of moments estimator is decreasing faster than maximum likelihood's average bias, which proves by simulation that moments estimator is asymptotically unbiased.

Table 1: Simulation performance of Neutrosophic Power Distribution.

n	Maximum Likelihood			Moments		
	$RMSE$	AB	Average $\hat{\theta}_N$	$RMSE$	AB	Average $\hat{\theta}_N$
5	$1.48 + 0.76I$	$0.4992 + 0.2356I$	$2.50 + 1.24I$	$1.45 + 0.75I$	$0.3877 + 0.2213I$	$2.39 + 1.22I$

15	$0.61 + 0.29I$	$0.1443 + 0.0638I$	$2.14 + 1.06I$	$0.63 + 0.28I$	$0.1101 + 0.0559I$	$2.11 + 1.06I$
30	$0.39 + 0.22I$	$0.0678 + 0.0422I$	$2.07 + 1.04I$	$0.41 + 0.21I$	$0.0505 + 0.0406I$	$2.05 + 1.04I$
50	$0.29 + 0.16I$	$0.0386 + 0.0204I$	$2.04 + 1.02I$	$0.31 + 0.15I$	$0.0282 + 0.0193I$	$2.03 + 1.02I$
100	$0.21 + 0.10I$	$0.0192 + 0.0126I$	$2.02 + 1.01I$	$0.22 + 0.10I$	$0.0147 + 0.0104I$	$2.01 + 1.01I$

Table (2) shows results of simulation analysis for neutrosophic Exponential distribution and compares the two proposed estimation methods and we see that both methods give the same estimators.

Table 2: Simulation performance of Neutrosophic Exponential Distribution.

n	Maximum Likelihood			Moments		
	$RMSE$	AB	Average $\hat{\theta}_N$	$RMSE$	AB	Average $\hat{\theta}_N$
5	$0.89 + 0.46I$	$-0.0046 + 0.0097I$	$2.00 + 1.01I$	$0.89 + 0.46I$	$-0.0046 + 0.0097I$	$2.00 + 1.01I$
15	$0.61 + 0.29I$	$0.0067 - 0.0178I$	$2.01 + 0.98I$	$0.61 + 0.29I$	$0.0067 - 0.0178I$	$2.01 + 0.98I$
30	$0.39 + 0.22I$	$-0.0024 + 0.0121I$	$2.00 + 1.01I$	$0.39 + 0.22I$	$-0.0024 + 0.0121I$	$2.00 + 1.01I$
50	$0.29 + 0.16I$	$-0.0024 - 0.0013I$	$2.00 + 1.00I$	$0.29 + 0.16I$	$-0.0024 - 0.0013I$	$2.00 + 1.00I$
100	$0.21 + 0.10I$	$-0.0008 + 0.0033I$	$2.00 + 1.00I$	$0.21 + 0.10I$	$-0.0008 + 0.0033I$	$2.00 + 1.00I$

Table (3) shows results of simulation analysis for neutrosophic Maxwell distribution and compares the two proposed estimation methods, notice that average bias of moments estimator is decreasing faster than maximum likelihood's average bias, which proves by simulation that moments estimator is asymptotically unbiased.

Table 3: Simulation performance of Neutrosophic Maxwell Distribution.

n	Maximum Likelihood			Moments		
	$RMSE$	AB	Average $\hat{\theta}_N$	$RMSE$	AB	Average $\hat{\theta}_N$
5	$1.04 + 0.49I$	$0.3119 + 0.1552I$	$2.31 + 1.16I$	$1.02 + 0.48I$	$0.2470 + 0.1260I$	$2.25 + 1.13I$
15	$0.47 + 0.23I$	$0.0882 + 0.0568I$	$2.09 + 1.06I$	$0.47 + 0.24I$	$0.0687 + 0.0488I$	$2.07 + 1.05I$
30	$0.31 + 0.16I$	$0.0429 + 0.0185I$	$2.04 + 1.02I$	$0.32 + 0.16I$	$0.0351 + 0.0113I$	$2.04 + 1.01I$
50	$0.24 + 0.12I$	$0.0274 + 0.0105I$	$2.03 + 1.01I$	$0.24 + 0.12I$	$0.0231 + 0.0073I$	$2.02 + 1.01I$
100	$0.17 + 0.08I$	$0.0160 + 0.0046I$	$2.02 + 1.00I$	$0.17 + 0.08I$	$0.0135 + 0.0032I$	$2.01 + 1.00I$

6. Conclusions and future research directions

In this paper we have introduced the concept of neutrosophic likelihood estimation method and neutrosophic moments estimation method and studied its properties based on AH-Isometry. We also presented theorems on these two estimation methods. We see that two estimation methods yields to different estimators. We also presented the concept of neutrosophic fisher information and presented some theorems related to it. In future work we are looking forward to study the properties of estimators like biasness, consistency and sufficiency. This paper opens the way to study the theory of neutrosophic statistical inference which is using neutrosophic classical numbers $N = a + bI; I^2 = I$ (not interval neutrosophic numbers).

Funding: "This research received no external funding"

Conflicts of Interest: "The authors declare no conflict of interest."

Acknowledgments: Authors are very thankful to editors-in-chief and to reviewers for their valuable help and recommendations.

References

[1] M. Ali, F. Smarandache, M. Shabir and L. Vladareanu, "Generalization of Neutrosophic Rings and Neutrosophic Fields," *Neutrosophic Sets and Systems,* vol. 5, 2014.

[2] M. Abobala, "Neutrosophic Real Inner Product Spaces," *Neutrosophic Sets and Systems,* vol. 43, pp. 225-246, 2021.

[3] M. Abobala, "Semi Homomorphisms and Algebraic Relations Between Strong Refined Neutrosophic Modules and Strong Neutrosophic Modules," *Neutrosophic Sets and Systems,* vol. 39, 2021.

[4] M. Abobala and M. Ibrahim, "An Introduction to Refined Neutrosophic Number Theory," *Neutrosophic Sets and Systems,* vol. 45, pp. 40-53, 2021.

[5] W. B. V. Kandasamy and F. Smarandache, Neutrosophic Rings, (USA) : Hexis, Phoenix, Arizona, 2006.

[6] M. B. Zeina and A. Hatip, "Neutrosophic Random Variables," *Neutrosophic Sets and Systems,* vol. 39, pp. 44-52, 2021.

[7] F. Smarandache, Symbolic Neutrosophic Theory, Belgium: EuropaNova, 2015.

[8] M. Abobala and A. Hatip, "An Algebraic Approach to Neutrosophic Euclidean Geometry," *Neutrosophic Sets and Systems,* vol. 43, pp. 114-123, 2021.

[9] Z. Khan and M. Gulistan, "Neutrosophic Design of the Exponential Model with Applications," *Neutrosophic Sets and Systems,* vol. 48, 2022.

[10] W.-Q. Duan, Z. Khan, M. Gulistan and A. Khurshid, "Neutrosophic Exponential Distribution: Modeling and Applications for Complex Data Analysis," *Hindawi,* vol. 2021, p. 8, 2021.

[11] M. F. Alaswad, "A Study of the Integration of Neutrosophic Thick Function," *International Journal of Neutrosophic Science,* vol. 6, 2020.

[12] M. F. Alaswad, "A Study of Neutrosophic Differential Equation by Using a Neutrosophic Thick Function," *Neutrosophic Knowledge,,* vol. 1, 2020.

[13] H. Rashad and M. Mohamed, "Neutrosophic Theory and Its Application in Various Queueing Models: Case Studies," *Neutrosophic Sets and Systems,* vol. 42, pp. 117-135, 2021.

[14] M. B. Zeina, "Neutrosophic Event-Based Queueing Model," *International Journal of Neutrosophic Science,* vol. 6, 2020.

[15] M. B. Zeina, "Erlang Service Queueing Model with Neutrosophic Parameters," *International Journal of Neutrosophic Science,* vol. 6, no. 2, pp. 106-112, 2020.

[16] Z. Khan, A. Amin, S. A. Khan and M. Gulistan, "Statistical Development of the Neutrosophic Lognormal Model with Application to Environmental Data," *Neutrosophic Sets and Systems,* vol. 47, pp. 1-11, 2021.

[17] S. Salem, Z. Khan, H. Ayed, A. Brahmia and A. Amin, "The Neutrosophic Lognormal Model in Lifetime Data Analysis: Properties and Applications," *Journal of Function Spaces,* p. 9, 2021.

[18] Z. Khan, A. Al-Bossly, M. M. A. Almazah and F. S. Alduais, "On Statistical Development of Neutrosophic Gamma Distribution with Applications to Complex Data Analysis," *Complexity,* p. 8, 2021.

[19] W.-Q. Duan, Z. Khan, M. Gulistan and A. Khurshid, "Neutrosophic Exponential Distribution: Modeling and Applications for Complex Data Analysis," *Complexity,* p. 8, 2021.

[20] F. Shah, M. Aslam, Z. Khan, M. M. A. Almazah and F. S. Alduais, "On Neutrosophic Extension of the Maxwell Model: Properties and Applications," *Journal of Function Spaces,* p. 9, 2022.

[21] Z. Khan and M. Gulistan, "Neutrosophic Design of the Exponential Model with Applications," *Neutrosophic Sets and Systems,* vol. 48, pp. 291-305, 2022.

[22] M. B. Zeina and M. Abobala, "A Novel Approach of Neutrosophic Continuous Probability Distributions using AH-Isometry used in Medical Applications," in *Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics*, Elsevier, 2023.

Received: Sep 13, 2022. Accepted: Dec 20, 2022