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Estimating the State Dependent Utility Function

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I. INTRODUCTION

Basic to the other papers in this symposium which concern valuing environmental assets and environmental risk is the question of how individuals make valuation decisions under conditions of uncertainty; of corresponding interest is the question of how researchers might empirically estimate these values. Underlying these empirical inquiries is a body of theory, generally referred to as "expected utility theory," from which axioms are derived which serve to establish testable hypotheses. Those hypotheses, in turn, provide the research design for data collection efforts.

This paper reports on results from recent research concerning the implications of received theory for efforts to estimate individual valuations of public projects, or public goods, which involve uncertain benefits. Such projects or goods are the substance of topics considered in this symposium. This topic is necessarily esoteric and details will be of interest to a limited number of readers. For this reason, a brief, "homey" sketch of research results is given in what immediately follows. In section II, more technical arguments relevant for research results are provided. Given the limited interest in these arguments, they will be mercifully brief. Readers interested in greater detail are invited to contact the author.

The bottom line from the author's theoretical research can be briefly described as follows. Research designed to estimate benefits attributable to uncertain environmental assets (including, in some cases, environmental risk) is generally concerned with measures of consumer "surplus" (derived benefits in excess of costs) for various quantities (or levels) of the environmental asset and, in some instances, an "option price"—that is, what an individual would be willing to pay to preserve his/her options for future consumption (use) of the environmental asset. Three parameters are of empirical interest for estimated measures of consumer surplus and/or option prices. Denoted a, b and β, these parameters reflect the manner in which individuals value environmental assets and their reaction or responses to risk and uncertainty.

Consider three types of experiments which might be used to derive data to be used in estimating a, b and β. In experiment I, individual

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values are derived under conditions where the individual understands the environmental commodity and relevant uncertainties (as reflected in probability measures). In experiment II, individuals make value choices before receiving full information on the nature of the environmental asset, but choices do not affect known probabilities associated with the asset. In experiment III, individuals make choices which affect uncertainties (probabilities) associated with the environmental asset. The author's research demonstrates the following, which is relevant for the design of experiments aimed at providing data for estimating benefits attributable to environmental assets. First, type I experiments can provide information relevant for estimating only *one* of the three parameters of interest: the parameter β. Second, type II experiments allow for the estimation of but two of the three parameters: a and β. Third, type III experiments are required for deriving estimates for the parameter b. Thus, type II *and* type III experiments are needed if researchers are to be able to provide comprehensive, predictive estimates of individual willingness to pay for environmental assets involving uncertainty.

II. TECHNICAL PRESENTATION

Consider the consumer who will experience state i with probability π^i, $i = 1, 2, \ldots, n$ and whose preferences for state contingent consumption, $(x^1, x^2, \ldots, x^n)$, $x^i \epsilon R^m_+$, can be represented by the von Neumann utility function[1]

$$U(x,\pi;\beta,a,b) \equiv \sum_{i=1}^{n} \pi^i[a^i u(x^i;\beta^i) + b^i]$$

where $\beta \equiv (\beta^1,\beta^2, \ldots,\beta^n)\epsilon R^{ns}$, $a \equiv (a^1,a^2, \ldots a^n)\ \epsilon R^n_{++}$ and $b \equiv (b^1, b^2, \ldots,b^n)\epsilon R^n$ are unknown parameters to be estimated by the analyst. Suppose further that data from three types of experiments are potentially available to the analyst. In *type I experiments,* the consumer is allowed to make choices *after* the state is revealed. In *type II experiments,* the consumer is also allowed to make choices which do not affect the probabilities of the states *before* the state is revealed. In *type III experiments,* the scope of choice is further expanded to include choices which affect the probabilities of the states. An obvious question in this context concerns what can be learned from the different types of data. The not so obvious answer is that type I experiments provide information about β only. Type II experiments provide information about β and a, and type III experiments are required to obtain information about b. The converse also holds: predicting the results of type I experiments requires knowledge of only β. Predicting type II results requires knowledge of β and a, and predicting the results of type III experiments further requires knowledge of b. These results are related to the problem of estimating willingness to pay for public projects with uncertain benefits, by showing that "surplus" is type I data while "option price" and the "willingness to pay locus" are type II data. Finally, a relatively simple type II experiment is proposed for estimating a.

Type I Experiments

Consider first what can be learned from type I experiments. We may regard these experiments as specifying a set of alternatives for each state, $A^i \subset R^m_+$, from which the consumer must choose. Let x^i denote the result of this choice. Then

$$x^i \in \arg\max_x a^i u(x;\beta^i) + b^i \qquad i = 1,2, \ldots, n.$$

Since $a^i > 0$, this is true if and only if

$$x^i \in \arg\max_x u\,(x;\beta^i) \qquad i = 1,2, \ldots, n.$$

This means that the results of type I experiments are completely independent of the values of both a and b. Thus these experiments provide no information about either a or b.[2]

Proposition 1: The results of type I experiments are invariant with respect to a and b. (Equivalently, hypotheses regarding a and b are untestable and, therefore, meaningless in the context of type I experiments). Knowledge of β is sufficient to predict the results of type I experiments.

To see how type I data could be used to estimate β, let

$$A^i = \{x \in R^m_+ \mid p^i x \leq y^i\} \qquad i = 1,2, \ldots, n$$

1. It is assumed that $u(x,\beta)$ is increasing in x and continuously differentiable with respect to x and β.

2. The type I choice context can be expanded to include "lotteries" by supposing that $A^i \quad P$ where

$$P \equiv \{(x,x',\rho,1-\rho) \mid x,x' \in R^m_+, \rho \in R_+\}$$

Here we note that

$$(x,x',\rho,1-\rho) \in \arg\max \rho[a^i u(z;\beta^i) + b^i] + (1-\rho)[a^i u(x';\beta^i) + b^i]$$
$$\text{s.t.} \quad (x,x',\rho,1-\rho) \in A^i$$

if and only if

$$(x,x',\rho,1-\rho) \in \arg\max \rho u(x;\beta^i) + (1-\rho)u(x';\beta^i)$$
$$\text{s.t.} \quad (x,x',\rho,1-\rho) \in A^i$$

This reflects the fact that the von Neumann utility function $u(x;\beta^i)$ is unique only up to a positive affine transformation:

$$a^i u(x;\beta^i) + b^i, a^i > 0.$$

Here A_i is the budget set of the consumer who has income y^i and must pay prices p^i in state i. Further let

$$v(p^i,y^i;\beta^i) \equiv \max u(x;\beta^i) \quad \text{s.t.} \quad p^i x \leqq y^i$$

be the indirect utility function of the consumer. Then an obvious application of Roy's identity implies that the quantity demanded of the jth good in the ith state is[3]

$$x_j^i = x_j(p^i,y^i;\beta^i) = -\frac{\partial v}{\partial p_j}(p^i,y^i;\beta^i)/\frac{\partial v}{\partial y}(p^i,y^i;\beta^i)$$

Data on "state dependent" qualities demanded could thus be employed to estimate β^i by well known methods.

Consumer surplus can easily be shown to be type I data which depends only upon β. Let s^i denote the "surplus" (price compensating variation) associated with allowing the individual to trade at prices p^{-i} in state i rather than at prices p^i. Then

$$a^i v(p^{-i},y^i - s^i;\beta^i) + b^i = a^i v(p^i,y^i;\beta^i) + b^i$$

or since $a^i > 0$,

$$(1) \quad v(p^{-i},y^i - s^i;\beta^i) = v(p^i,y^i;\beta^i) \qquad i = 1,2, \ldots,n$$

Thus the surplus point $(s^1,s^2 \ldots,s^n)$, and the expected value of surplus, $\Sigma \pi^i s^i$, depend only upon β. Alternatively, predicting the value of expected surplus requires only knowledge of β.

Type II Experiments

Suppose now that we expand the opportunity for choice by allowing the consumer to choose $z \epsilon B$ *before* the state is revealed. After the state is revealed the consumer may choose $x^i \epsilon A^i(z)$. Thus the choice made prior to discovery of the state affects the alternatives that will be available after discovery. The consumer's problem then is to

$$\max_{z,x} \sum_{i=1}^{n} \pi^i[a^i u(x^i;\beta^i) + b^i]$$

$$\text{s.t. } z \epsilon B \text{ and } x^i \epsilon A^i(z) \qquad i = 1,2, \ldots,n$$

3. Roy's Identity is discussed in a number of texts, *see, e.g.*, H. R. VARIAN, MICROECONOMIC ANALYSIS 93 (1978). *See also* Diewert, *Applications of Duality Theory*, in 2 FRONTIERS OF QUANTITATIVE ECONOMICS (M. Intriligator & D. Kendrick eds. 1974) for an excellent discussion of demand estimation.

But z solves this problem if and only if

$$z \in \arg\max_{\text{s.t. } z\in B} \sum_{i=1}^{n} \pi^i a^i u(x^i(z);\beta^i) + \sum_{i=1}^{n} \pi^i b^i$$

or

$$z \in \arg\max_{\text{s.t. } z\in B} \sum_{i=1}^{n} \pi^i a^i u(x^i(z);\beta^i)$$

where

$$x^i(z) \in \arg\max u(x;\beta^i) \quad \text{s.t. } x\in A^i(z) \qquad i = 1,2, \ldots,n$$

The results of these type II experiments are thus completely independent of b. Data from such experiments can, therefore, provide information only about β and a.

Proposition 2: The results of type II experiments are invariant with respect to b. (Equivalently, hypotheses about b are untestable and, therefore, meaningless in the context of type II experiments). Knowledge of β and a is sufficient to predict the results of type II experiments.

To see how type II experiments could be used to obtain information about a, suppose, as before, that the consumer has income y^i and must pay prices p^i in state i. Suppose further that the consumer can buy (or sell) before the state is revealed, and contracts for delivery of some good, say the first, after the state is revealed. Let q denote the current price in this "futures" market and let z denote the consumer's demand (+) or supply (−) of these contracts. The budget constraint of the consumer in state i will then be

$$p^i x \leqq y^i + (p^i_1 - q)z \qquad i = 1,2, \ldots,n$$

Employing the indirect utility function the consumer's problem then is to

$$\max_{z} \sum_{i=1}^{n} \pi^i[a^i v(p^i, y^i + (p^i_1 - q)z;\beta^i) + b^i]$$

$$\text{s.t. } y^i + (p^i_1 - q)z \geqq 0 \qquad i = 1,2, \ldots,n$$

This problem has the first order necessary condition for an interior (not bankrupt in any state) solution that:[4]

$$\sum_{i=1}^{n} \pi^i a^i \frac{\partial v}{\partial y}(p^i, y^i + (p^i_1 - q)z;\beta^i)(p^i_1 - q) = 0$$

If we suppose that β has already been estimated (by type I experiments) and that we have r observations on the quantity of contracts demanded, z_j, at futures prices q_j, $j = 1,2, \ldots, r$ then we may use the first order condition to estimate a. Let

$$c_{ij} \equiv \pi^i \frac{\partial v}{\partial y}(p^i, m^i + (p^i_1 - q_j)z_j;\beta^i)(p^i_1 - q_j)$$

Then the first order condition for z_j to be optimal at price q_j is

$$\sum_{i=1}^{n} a^i c_{ij} = 0 \qquad j = 1,2, \ldots, r$$

where the c_{ij}'s are known and the a^i's are to be determined. Since a will be determined only up to a positive scalar multiple (a positive affine transformation of the utility function), we may choose a convenient nomalization, namely $a^1 = 1$, and express linear restrictions on a in the obvious matrix notation as

$$c = -\bar{a}C$$

where $c = (c_{11}, c_{12}, \ldots, c_{1r})$ and $\bar{a} = (a^2, a^3, \ldots, a^n)$.
Clearly $\bar{a}$ can be learned if $r = n - 1$ and C^{-1} exists:

$$\bar{a} = -cC^{-1}$$

It should be emphasized that this process for estimating a requires only that a futures market for *some* commodity exist. This is a reflection of the fact that the a^i are "state specific" parameters and not "commodity specific" parameters. This is an important consideration in public project evaluation since the futures market (or other type II experiment) need not involve the commodity to be publicly provided.

The "willingness to pay locus" can be defined as the set of n-tuples of contingent payments $(\gamma^1, \gamma^2, \ldots, \gamma^n)$ that the consumer would be willing to make to be able to trade at prices p^{-i} rather than prices p^i:[5]

4. For this condition to be meaningful it must be the case that $q \in (\underline{p}_{-1}, \bar{p}_1)$ where $\underline{p}_{-1} \equiv \min_i \{p^i_1\}$ and $\bar{p}_1 \equiv \max_i \{p^i_1\}$. Hence there must be some variation in the price of the first good across states.

5. For a more detailed discussion of "willingness to pay," "option price," and related concepts, *see* Graham, *Cost-Benefit Analysis Under Uncertainty,* 71 AM. ECON. REV. 715 (1981), and the references cited therein.

$$\sum_{i=1}^{n} \pi^i[a^i v(p^{-i},y^i-\gamma^i;\beta^i) + b^i] \equiv \sum_{i=1}^{n} \pi^i[a^i v(p^i,y^i;\beta^i) + b^i]$$

or, equivalently,

$$(2) \quad \sum_{i=1}^{n} \pi^i a^i[v(p^{-i},y^i-\gamma^i;\beta^i) - v(p^i,y^i;\beta^i)] \equiv 0$$

Thus γ is completely independent of b. From equation (1) we note that all of the bracketed terms in (2) vanish when $\gamma^i = s^i$. This implies that the willingness to pay locus passes through the surplus point (which was itself shown to be independent of a). The gradient of the locus (the vector orthogonal to the surface) at γ, however, has as its ith component

$$-a^i\pi^i \frac{\partial v}{\partial y}(p^i,y^i-\gamma^i;\beta^i)$$

Thus the slope of the locus depends upon a.

Another point along the locus that deserves consideration corresponds to "option price," defined as the largest sure payment, OP, that the individual would made to trade at prices p^{-i} rather than p^i:

$$(3) \quad \sum_{i=1}^{n} \pi^i a^i[v(p^{-i},y^i-OP;\beta^i) - v(p^i,y^i;\beta^i)] \equiv 0$$

Since surplus is generally presumed to vary across states ($s^i \neq s^j$), the surplus point will, in general, be distinct from the option price point. Accordingly let

$$I \equiv \{i \mid s^i \neq OP\} \neq \phi$$

Then equation (3) can be rewritten as

$$(4) \quad \sum_{i \in I} \pi^i a^i[v(p^{-i},y^i-OP;\beta^i) - v(p^i,y^i;\beta^i)] = 0$$

where *none* of the bracketed terms in (4) vanish. Clearly OP must depend upon a^i, $i \in I$. This reflects the fact that option price is also type II data which depends upon a as well as β.

A question of lingering interest is whether knowledge of the surplus point (or the expected value of surplus) can be used to provide bounds for option price. The motivation is obvious: estimates of expected surplus may be more easily obtained and yet option price may be the information required to resolve the policy issue. While it is now generally acknowledged that no theoretical basis exists for supposing that option price exceeds or falls short of expected surplus, it remains tempting to conjecture that under "reasonable circumstances" an estimate of expected surplus could be used to provide an upper or lower bound for option price. Unfortunately the implication of propositions 1 and 2 is that such "reasonable circumstances" necessarily represent hypotheses about a which are untestable in the context of the type I data that has permitted estimation of expected surplus. There is simply no way around the fact that surplus provides information about β only and that option price requires knowledge of a also—knowledge that can be obtained and tested only in the context of type II experiments.

Type III Experiments

The general dependence of type III results upon b remains to be demonstrated. Let Δ^n denote the unit simplex in R^n, i.e.

$$\Delta^n \equiv \{\pi \epsilon R^n_+ \mid \sum_{i=1}^{n} \pi^i = 1\}$$

and define

$$\Pi(u \equiv \{\pi \epsilon \Delta^n \mid \sum_{i=1}^{n} \pi^i[a^i v(p^i,y^i;\beta^i) + b^i] = u \}$$

For the consumer who has income y^i and must pay prices p^i in state i, $\Pi(u)$ is the set of all probability distributions over states that yield utility u. The gradient of this indifference "curve" at π has its ith component:

$$a^i v(p^i,y^i j\beta^i) + b^i \qquad i = 1,2, \ldots,n$$

Thus the "slope" of the indifference surface for probability distributions depends upon b and choices among such distributions (type III experiments) will therefore depend upon b.

CONCLUSION

The nature of the implications of this analysis can, perhaps, be best conveyed by a simple example: the "surplus" that an individual derives from having a passive restraint system in his automobile in the states "wreck" and "no wreck" provides information about β but no information about "option price"—the price the individual would actually pay to obtain the system. Knowledge of the option price would provide information about a but no information about the value of reducing the probability of a wreck.