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Topological structures via MBJ-neutrosophic sets

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Abstract. In this paper, we introduce the concept of quasi-coincidences in MBJ-neutrosophic sets and obtain some of its properties. Also by defining the pre-image and the image of an MBJ-neutrosophic set under a mapping, we confirm that their properties are naturally extensions of the classical case. Next, we define two types of MBJ-neutrosophic neighborhood system and discuss their various properties and introduce the notion of $\circ$-[resp. $*$-]MBJ-neutrosophic bases and deal with some of their properties. Finally, we define $\circ$-C_I, $*$-C_I, $\circ$-C_{II} and $*$-C_{II} corresponding to the first countability and the second countability in classical topological spaces, and we obtain the relationships between them.

Keywords:MBJ-neutrosophic set; MBJ-neutrosophic topology; MBJ-neutrosophic neighborhood; MBJ-neutrosophic base and subbase; MBJ-neutrosophic local base.

1. **Introduction**

In the real world, we are faced with a complex system that includes various types of uncertainty to solve epidemics such as covid, conflicts between countries, and international energy and food problems. As a tool to solve such complex systems, Zadeh [1] first proposed the concept of fuzzy sets which generalises crisp sets. Smarandache [2, 3] introduced the notion of neutrosophic sets that is a triple $\langle T, I, F\rangle$ of three fuzzy sets (T, I and F are called the *truth*, the *indeterminate* and *false membership functions*) and can extend the concepts of classical sets, fuzzy sets, interval-valued fuzzy sets [4] and intuitionistic fuzzy sets [5]. We can see that neutrosophic set is applied to a variety of fields (See the site http://fs.gallup.unm.edu/neutrosophy.htm). Recently, Takallo et al. [6] defined an MBJ-neutrosophic set that generalized the neutrosophic set by replacing the interval-valued fuzzy set with the fuzzy set I of a neutrosophic set, and applied it to BCK/BCI-algebra. After that time, it can be seen from the literature that several researchers [7–10] have applied the MBJ-netrosophic set to BCK/BCI-algebras. In particular, Khalid et al. [11] studied MBJ-neutrosophic T-ideals on B-algebras. Manivasan and Kalidass [12] discussed MBJ-neutrosophic ideals on KU-algebras.

Topology has been intended in a natural way as background for geometry and modern analysis. It is not only a nice and powerful tool in many branches in Mathematics but also has had a beauty of its own. However, topology studies based on MBJ-neutrosophic sets could not be found in the literature. Then it is our aim to study basic properties for topology via MBJ-neutrosophic sets. Thus, before conducting our study, we would like to review topology studies based on fuzzy sets, intuitionistic fuzzy sets, interval-valued fuzzy sets and neutrosophic sets respectively. Chang [13] first applied fuzzy sets to topologies (See [14–20] for further researches). Coker [21] introduced the concept of intuitionistic fuzzy topologies and studied its some properties. After then, El-Latif and Khalaf [22], Singh and Srivastava [23], and Saleh [24] discussed connectedness and separation axioms in intuitionistic fuzzy topological spaces. Mondal and Samanta [25] defined an interval-valued fuzzy topology and dealt with some of its properties. After that time, Hongmei and Xuehai [26], and Kandil et al. [27] investigated separation axioms and in interval-valued fuzzy topological spaces. Smarandache [28], Lupia'n$\widetilde{e}$z [29, 30], and Salama and Alblowi [31] studied basic properties of neutrosophic topologies respectively. Kim et al. [32] defined an ordinary single valued topology by considering the single valued neutrosophic degree of openness for ordinary subsets and dealt with some of its properties. Recently, Lee et al. [33] studied topological structures based on cubic sets introduced by Jun et al. [34].

So to do this, we proceed with our research in the following order. First, we recall the concepts of fuzzy sets, interval-valued fuzzy sets and neutrosophic sets needed in the next sections. Second, we defined the inclusion, the union, the intersection and the compliment for MBJ-neutrosophic sets and obtain some of their properties. Moreover, we introduce the notion of MBJ-neutrosophic quasi-coincidences and discuss some of its properties. Third, We define two types of topologies and neighborhoods based on MBJ-neutrosophic sets and study their respective properties. Finally, by MBJ-neutrosophic set, the concepts of two types of base, subbase and local base are introduced and their respective properties are studied. Also we extend the concepts of first coutabilities and second countabilities in classical topology to the MBJ-neutrosophic sets and find the relationships between them. Furthermore, we present an example which the converse of a proposition does not hold.

Throughout this paper, let $I = [0, 1]$ and let J denote a index set.

2. Preliminaries

We list the notions of fuzzy sets, interval-valued fuzzy sets and neutrosophic sets needed in the next sections.

For a nonempty set X, a mapping $A : X \to I$ is called a *fuzzy set* in X. The special fuzzy set $\mathbf{0}$ [resp. 1] defined by:

$$\mathbf{0}(x) = 0 \text{ [resp. } \mathbf{1}(x) = 1]$$

is called the *fuzzy empty set* [resp. the *fuzzy whole set*] in X (See [1]). We denote the collection of all fuzzy sets in X by I^X.

For a nonempty set X, a mapping $\bar{A} = (A^{\in}, A^{\notin}) : X \to I \times I$ satisfying the following condition: for each $x \in X$,

$$0 \leq A^{\in}(x) + A^{\notin}(x) \leq 1$$

is called an *intuitionistic fuzzy set* in X (See [5]). The intuitionistic fuzzy sets $\bar{\mathbf{0}}$ and $\bar{\mathbf{1}}$ defined as follows: for each $x \in X$,

$$\bar{\mathbf{0}}(x) = (0, 1) \text{ and } \bar{\mathbf{1}}(x) = (1, 0)$$

are called the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in X. $IFS(X)$ denotes the set of all intuitionistic fuzzy sets.

Let $[I]$ be the set of all closed subintervals of I and members of $[I]$ are called *interval numbers* and are denoted by $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, etc., where $\widetilde{a} = [a^-, a^+]$ and $0 \leq a^- \leq a^+ \leq 1$ (See [34] for the definitions of the order between two interval-valued numbers, the infimum and the supremum of an arbitrary interval-valued numbers).

For a nonempty set X, a mapping $\widetilde{A} = [A^-, A^+] : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, IVFS) in X. The special interval-valued fuzzy set $\widetilde{\mathbf{0}}$ [resp. $\widetilde{\mathbf{1}}$] defined by: for each $x \in X$,

$$\widetilde{\mathbf{0}}(x) = [0, 0] \text{ [resp. } \widetilde{\mathbf{1}}(x) = [1, 1]]$$

is called the *interval-valued fuzzy empty set* [resp. the *interval-valued fuzzy whole set* in X (See [4]). $IVFS(X)$ denotes the set of all IVFSs in X.

For a nonempty set X, the form $A = \langle A^T, A^I, A^F \rangle$ is called a *neutrosophic set* in X, where $A^T : X \to I$ represents a *truth membership function*, $A^I : X \to I$ represents an *indeterminate membership* and $A^F : X \to I$ represents a *false membership function* (See [2]). We will denote the set of all neutrosophic sets in X as $NS(X)$.

3. Basic properties of MBJ-neutrosophic sets

By modifying some concepts related to fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets and neutrosophic fuzzy sets, we give some basic definitions based on MBJ-neutrosophic sets.

Definition 3.1 ([6])**.** Let X be a nonempty set. Then the form

$$\mathcal{A} = \left\langle M_A, \widetilde{B}_A, J_A \right\rangle$$

is called an *MBJ-neutrosophic set* in X, where $M_A,\ J_A \in I^X$ which are called a *truth membership function* and a *false membership function* respectively, and $\widetilde{B}_A \in IVFS(X)$ which is called an *indeterminate interval-valued membership function.*

It is clear that for any subset A of a nonempty set X, $\langle \chi_A, [\chi_A, \chi_A], \chi_{A^c} \rangle \in MBJN(X)$, where χ_A denotes the characteristic function of A. Then we can consider an MBJ-neutrosophic set as a generalization of classical sets.

We can consider special MBJ-neutrosophic sets:

$$\ddot{\emptyset} = \left\langle \mathbf{0}, \widetilde{\mathbf{0}}, \mathbf{1} \right\rangle,\ \dot{\emptyset} = \left\langle \mathbf{0}, \widetilde{\mathbf{1}}, \mathbf{1} \right\rangle,\ \breve{\emptyset} = \left\langle \mathbf{0}, \widetilde{\mathbf{1}}, \mathbf{0} \right\rangle,\ \hat{\emptyset} = \left\langle \mathbf{0}, \widetilde{\mathbf{0}}, \mathbf{0} \right\rangle,$$

$$\ddot{X} = \left\langle \mathbf{1}, \widetilde{\mathbf{1}}, \mathbf{0} \right\rangle,\ \dot{X} = \left\langle \mathbf{1}, \widetilde{\mathbf{0}}, \mathbf{0} \right\rangle,\ \breve{X} = \left\langle \mathbf{1}, \widetilde{\mathbf{0}}, \mathbf{1} \right\rangle,\ \hat{X} = \left\langle \mathbf{1}, \widetilde{\mathbf{1}}, \mathbf{1} \right\rangle.$$

We will denote the set of all MBJ-neutrosophic sets in X as $MBJNS(X)$.

Definition 3.2. Let X be a nonempty set and let $\mathcal{A} \in MBJNS(X)$. Then the *complement* of $\mathcal{A}$, denoted by $\mathcal{A}^{c,1}$ resp. $\mathcal{A}^{c,2}$ and $\mathcal{A}^{c,3}$], is an MBJ-neutrosophic set in X defined as follows:

$$\mathcal{A}^{c,1} = \left\langle M_A^c, \widetilde{B}_A^c, J_A^c \right\rangle \text{ [resp. } \mathcal{A}^{c,2} = \left\langle J_A, \widetilde{B}_A, M_A \right\rangle \text{ and } \mathcal{A}^{c,3} = \left\langle J_A, \widetilde{B}_A^c, M_A \right\rangle].$$

Definition 3.3. Let X be a nonempty set and let $\mathcal{A},\ \mathcal{B} \in MBJNS(X)$. Then two type's inclusion relations between $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \sqsubset \mathcal{B}$ (called the $\circ$-*inclusion*) and $\mathcal{A} \Subset \mathcal{B}$ ($*$-*inclusion*), are defined as follows: for each $x \in X$,

(i) $\mathcal{A} \sqsubset \mathcal{B}$ if and only if $M_A(x) \leq M_A(x)$, $\widetilde{B}_A(x) \leq \widetilde{B}_B(x)$, $J_A(x) \geq J_A(x)$,

(ii) $\mathcal{A} \Subset \mathcal{B}$ if and only if $M_A(x) \leq M_A(x)$, $\widetilde{B}_A(x) \geq \widetilde{B}_B(x)$, $J_A(x) \geq J_A(x)$.

The following is an immediate consequence of Definitions 3.1 and 3.3.

Proposition 3.4. *Let X be a nonempty set and let $\mathcal{A} \in MBJNS(X)$. Then the followings are hold:*

(1) $\ddot{\emptyset} \sqsubset \mathcal{A} \sqsubset \ddot{X}$, $\ddot{\emptyset} \sqsubset \dot{\emptyset} \sqsubset \breve{\emptyset} \sqsubset \ddot{X}$, $\breve{X} \sqsubset \dot{X} \sqsubset \ddot{X}$, $\hat{X} \sqsubset \ddot{X}$,

(2) $\dot{\emptyset} \Subset \mathcal{A} \Subset \dot{X}$, $\dot{\emptyset} \Subset \hat{X} \Subset \ddot{X} \Subset \dot{X}$, $\dot{\emptyset} \Subset \ddot{\emptyset} \Subset \hat{\emptyset}$, $\dot{\emptyset} \Subset \breve{\emptyset} \Subset \hat{\emptyset}$.

Definition 3.5. Let X be a nonempty set and let $\mathcal{A},\ \mathcal{B} \in MBJNS(X)$.

(i) The *intersection* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \sqcap \mathcal{B}$ (called the $\circ$-*intersection*) and $\mathcal{A} \Cap \mathcal{B}$ (called the $*$-*intersection*), is a MBJ-neutrosophic set in X defined as follows: for each $x \in X$,

$$(\mathcal{A} \sqcap \mathcal{B})(x) = \left\langle M_A(x) \wedge M_B(x), \widetilde{B}_A(x) \wedge \widetilde{B}_B(x), J_A(x) \vee J_B(x) \right\rangle,$$

$$(\mathcal{A} \Cap \mathcal{B})(x) = \left\langle M_A(x) \wedge M_B(x), \widetilde{B}_A(x) \vee \widetilde{B}_B(x), J_A(x) \vee J_B(x) \right\rangle.$$

(ii) The *union* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \sqcup \mathcal{B}$ (called the $\circ$-*union*) and $\mathcal{A} \Cup \mathcal{B}$ (called the $*$-*union*), is an MBJ-neutrosophic set in X defined as follows: for each $x \in X$,

$$(\mathcal{A} \sqcup \mathcal{B})(x) = \left\langle M_A(x) \vee M_B(x), \widetilde{B}_A(x) \vee \widetilde{B}_B(x), J_A(x) \wedge J_B(x) \right\rangle,$$

$$(\mathcal{A} \Cup \mathcal{B})(x) = \left\langle M_A(x) \vee M_B(x), \widetilde{B}_A(x) \wedge \widetilde{B}_B(x), J_A(x) \wedge J_B(x) \right\rangle.$$

Definition 3.6. Let X be a nonempty set and let $(\mathcal{A}_j)_{j \in J} \subset MBJNS(X)$.

(i) The *intersection* of $(\mathcal{A}_j)_{j \in J}$, denoted by $\sqcap_{j \in J}\mathcal{A}_j$ and $\Cap_{j \in J}\mathcal{A}_j$, is a MBJ-neutrosophic set in X defined as follows: for each $x \in X$,

$$(\sqcap j \in J\mathcal{A}_j)(x) = \left\langle \bigwedge_{j \in J} M_{A_j}(x), \bigwedge_{j \in J} \widetilde{B}_{A_j}(x), \bigvee_{j \in J} J_{A_j}(x) \right\rangle,$$

$$(\Cap_{j \in J}\mathcal{A}_j)(x) = \left\langle \bigwedge_{j \in J} M_{A_j}(x), \bigvee_{j \in J} \widetilde{B}_{A_j}(x), \bigvee_{j \in J} J_{A_j}(x) \right\rangle.$$

(ii) The *union* of $(\mathcal{A}_j)_{j \in J}$, denoted by $\sqcup_{j \in J}\mathcal{A}_j$ and $\Cup_{j \in J}\mathcal{A}_j$, is a MBJ-neutrosophic set in X defined as follows: for each $x \in X$,

$$(\sqcup j \in J\mathcal{A}_j)(x) = \left\langle \bigvee_{j \in J} M_{A_j}(x), \bigvee_{j \in J} \widetilde{B}_{A_j}(x), \bigwedge_{j \in J} J_{A_j}(x) \right\rangle,$$

$$(\Cup_{j \in J}\mathcal{A}_j)(x) = \left\langle \bigvee_{j \in J} M_{A_j}(x), \bigwedge_{j \in J} \widetilde{B}_{A_j}(x), \bigwedge_{j \in J} J_{A_j}(x) \right\rangle.$$

From Definitions 3.1, 3.2, 3.3, 3.5 and 3.6, we obtain a similar consequence of (Theorem 1, [25]), (Corollary 2.8, [21]) and (Theorem 1.2, [35]).

Proposition 3.7. *Let X be a nonempty set, let $\mathcal{A},\ \mathcal{B},\ \mathcal{C} \in MBJNS(X)$ and let $(\mathcal{A}_j)_{j \in J} \subset MBJNS(X)$.*

(1) $\mathcal{A} \sqsubset \mathcal{A} \sqcap \mathcal{B},\ \mathcal{B} \sqsubset \mathcal{A} \sqcap \mathcal{B},\ \mathcal{A} \Subset \mathcal{A} \Cap \mathcal{B},\ \mathcal{B} \Subset \mathcal{A} \Cap \mathcal{B}$.

(2) $\mathcal{A} \sqcup \mathcal{B} \sqsubset \mathcal{A},\ \mathcal{A} \sqcup \mathcal{B} \sqsubset \mathcal{B},\ \mathcal{A} \Cup \mathcal{B} \Subset \mathcal{A},\ \mathcal{A} \Cup \mathcal{B} \Subset \mathcal{B}$.

(3) *If* $\mathcal{A} \sqsubset \mathcal{B}$, *then* $\mathcal{A} \sqcap \mathcal{C} \sqsubset \mathcal{B} \sqcap \mathcal{C},\ \mathcal{A} \sqcup \mathcal{C} \sqsubset \mathcal{B} \sqcup \mathcal{C}$.

(4) *If* $\mathcal{A} \Subset \mathcal{B}$, *then* $\mathcal{A} \Cap \mathcal{C} \Subset \mathcal{B} \Cap \mathcal{C},\ \mathcal{A} \Cup \mathcal{C} \Subset \mathcal{B} \Cup \mathcal{C}$.

(5) *If* $\mathcal{A} \sqsubset \mathcal{B}$ *and* $\mathcal{B} \sqsubset \mathcal{C}$, *then* $\mathcal{A} \sqsubset \mathcal{C}$.

(6) *If* $\mathcal{A} \Subset \mathcal{B}$ *and* $\mathcal{B} \Subset \mathcal{C}$, *then* $\mathcal{A} \Subset \mathcal{C}$.

(7) $\mathcal{A} \sqcup \mathcal{A} = \mathcal{A},\ \mathcal{A} \sqcap \mathcal{A} = \mathcal{A},\ \mathcal{A} \Cup \mathcal{A} = \mathcal{A},\ \mathcal{A} \Cap \mathcal{A} = \mathcal{A}.$

(8) $\mathcal{A} \sqcup \mathcal{B} = \mathcal{B} \sqcup \mathcal{A},\ \mathcal{A} \sqcap \mathcal{B} = \mathcal{B} \sqcap \mathcal{A},\ \mathcal{A} \Cup \mathcal{B} = \mathcal{B} \Cup \mathcal{A},\ \mathcal{A} \Cap \mathcal{B} = \mathcal{B} \Cap \mathcal{A}.$

(9) $\mathcal{A} \sqcup (\mathcal{B} \sqcup \mathcal{C}) = (\mathcal{A} \sqcup \mathcal{B}) \sqcup \mathcal{C},\ \mathcal{A} \sqcap (\mathcal{B} \sqcap \mathcal{C}) = (\mathcal{A} \sqcap \mathcal{B}) \sqcap \mathcal{C},$
$\mathcal{A} \Cup (\mathcal{B} \Cup \mathcal{C}) = (\mathcal{A} \Cup \mathcal{B}) \Cup \mathcal{C},\ \mathcal{A} \Cap (\mathcal{B} \Cap \mathcal{C}) = (\mathcal{A} \Cap \mathcal{B}) \Cap \mathcal{C}.$

(10) $\mathcal{A} \sqcup (\mathcal{B} \sqcap \mathcal{C}) = (\mathcal{A} \sqcup \mathcal{B}) \sqcap (\mathcal{A} \sqcup \mathcal{C}),\ \mathcal{A} \sqcap (\mathcal{B} \sqcup \mathcal{C}) = (\mathcal{A} \sqcap \mathcal{B}) \sqcup (\mathcal{A} \sqcap \mathcal{C}),$
$\mathcal{A} \Cup (\mathcal{B} \Cap \mathcal{C}) = (\mathcal{A} \Cup \mathcal{B}) \Cap (\mathcal{A} \Cup \mathcal{C}),\ \mathcal{A} \Cap (\mathcal{B} \Cup \mathcal{C}) = (\mathcal{A} \Cap \mathcal{B}) \Cup (\mathcal{A} \Cap \mathcal{C}).$

$(10)'$ $\mathcal{A} \sqcup (\sqcap_{j\in J}\mathcal{A}_j) = \sqcap_{j\in J}(\mathcal{A} \sqcup \mathcal{A}_j),\ \mathcal{A} \sqcap (\sqcup_{j\in J}\mathcal{A}_j) = \sqcup_{j\in J}(\mathcal{A} \sqcap \mathcal{A}_j),$
$\mathcal{A} \Cup (\Cap_{j\in J}\mathcal{A}_j) = \Cap_{j\in J}(\mathcal{A} \Cup \mathcal{A}_j),\ \mathcal{A} \Cap (\Cup_{j\in J}\mathcal{A}_j) = \Cup_{j\in J}(\mathcal{A} \Cap \mathcal{A}_j).$

(11) $(\mathcal{A} \sqcup \mathcal{B})^{c,2} = \mathcal{A}^{c,2} \Cap \mathcal{B}^{c,2},\ (\mathcal{A} \sqcup \mathcal{B})^{c,i} = \mathcal{A}^{c,i} \sqcap \mathcal{B}^{c,i},$
$(\mathcal{A} \sqcap \mathcal{B})^{c,2} = \mathcal{A}^{c,2} \Cup \mathcal{B}^{c,2},\ (\mathcal{A} \sqcap \mathcal{B})^{c,i} = \mathcal{A}^{c,i} \sqcup \mathcal{B}^{c,i},$
$(\mathcal{A} \Cup \mathcal{B})^{c,2} = \mathcal{A}^{c,2} \sqcap \mathcal{B}^{c,2},\ (\mathcal{A} \Cup \mathcal{B})^{c,i} = \mathcal{A}^{c,i} \Cap \mathcal{B}^{c,i},$
$(\mathcal{A} \Cap \mathcal{B})^{c,2} = \mathcal{A}^{c,2} \sqcup \mathcal{B}^{c,2},\ (\mathcal{A} \Cap \mathcal{B})^{c,i} = \mathcal{A}^{c,i} \Cup \mathcal{B}^{c,i}$ *for* $i = 1,\ 3.$

(12) $\ddot{\emptyset}^{c,2} = \dot{X},\ \ddot{X}^{c,2} = \dot{\emptyset},\ \ddot{\emptyset}^{c,i} = \ddot{X},\ \ddot{X}^{c,i} = \ddot{\emptyset},$
$\dot{\emptyset}^{c,2} = \ddot{X},\ \dot{X}^{c,2} = \dot{\emptyset},\ \dot{\emptyset}^{c,i} = \dot{X},\ \dot{X}^{c,i} = \dot{\emptyset}$ *for* $i = 1,\ 3,$
$\breve{\emptyset}^{c,1} = \breve{X},\ \breve{X}^{c,1} = \breve{\emptyset},\ \breve{\emptyset}^{c,2} = \breve{\emptyset},\ \breve{X}^{c,2} = \breve{X},\ \breve{\emptyset}^{c,3} = \hat{\emptyset},\ \breve{X}^{c,3} = \hat{X},$
$\hat{\emptyset}^{c,1} = \hat{X},\ \hat{X}^{c,1} = \hat{\emptyset},\ \hat{\emptyset}^{c,2} = \hat{\emptyset},\ \hat{X}^{c,2} = \hat{X},\ \hat{\emptyset}^{c,3} = \breve{\emptyset},\ \hat{X}^{c,3} = \breve{X}.$

(13) $\mathcal{A} \sqcap \mathcal{A}^{c,i} \neq \ddot{\emptyset},\ \mathcal{A} \sqcup \mathcal{A}^{c,i} \neq \ddot{X}$ *and* $\mathcal{A} \Cap \mathcal{A}^{c,i} \neq \dot{\emptyset},\ \mathcal{A} \Cup \mathcal{A}^{c,i} \neq \dot{X}$ *in general for* $i = 1,\ 3$ (See Example 3.8).

Example 3.8. Let $X = \{a, b, c\}$ and let $\mathcal{A}$ be the MBJ-neutrosophic set in X given by: for each $x \in X$,

$$\mathcal{A} = \langle 0.5, [0.5, 0.5], 0, 5 \rangle .$$

Then we can easily check that $\mathcal{A} \sqcap \mathcal{A}^{c,i} \neq \ddot{\emptyset},\ \mathcal{A} \sqcup \mathcal{A}^{c,i} \neq \ddot{X}$ and $\mathcal{A} \Cap \mathcal{A}^{c,i} \neq \dot{\emptyset},\ \mathcal{A} \Cup \mathcal{A}^{c,i} \neq \dot{X}$.

Remark 3.9. From Propositions 3.4 and 3.7, we can see that

$$(MBJNS(X), \sqcup, \sqcap, ^{c,i}, \ddot{\emptyset}, \ddot{X}) \text{ and } (MBJNS(X), \Cup, \Cap, ^{c,i}, \dot{\emptyset}, \dot{X})$$

form Boolean algebras except the condition (13) of Proposition 3.7.

Let $M_a,\ J_a \in I$ and let $\widetilde{B}_a \in [I]$. Then the form

$$\widetilde{\overline{a}} = \left\langle M_a, \widetilde{B}_a, J_a \right\rangle = \langle a, \widetilde{a}, \overline{a} \rangle$$

is called an *MBJ-neutrosophic number* (briefly, MBJNN). We can consider the following special MBJNNs:

$$\ddot{0} = \left\langle 0, \widetilde{0}, 1 \right\rangle,\ \dot{0} = \left\langle 0, \widetilde{1}, 1 \right\rangle,\ \breve{0} = \left\langle 0, \widetilde{1}, 0 \right\rangle,\ \hat{0} = \left\langle 0, \widetilde{0}, 0 \right\rangle,$$

$$\ddot{1} = \left\langle 1, \widetilde{1}, 0 \right\rangle,\ \dot{1} = \left\langle 1, \widetilde{0}, 0 \right\rangle,\ \breve{1} = \left\langle 1, \widetilde{0}, 1 \right\rangle,\ \hat{1} = \left\langle 1, \widetilde{1}, 1 \right\rangle.$$

We will denote the set of all MBJNNs as $I \times [I] \times I$.

Definition 3.10. Let $\widetilde{\overline{a}}$, $\widetilde{\overline{b}}$ be two MBJNNs and let $(\widetilde{\overline{a}}_j)_{j\in J}$ be a family of MBJNNs.

(i) The *order* between $\widetilde{\overline{a}}$ and $\widetilde{\overline{b}}$, denoted by $\widetilde{\overline{a}} \leq^{\circ} \widetilde{\overline{b}}$ [resp. $\widetilde{\overline{a}} \leq^{*} \widetilde{\overline{b}}$], is defined as follows:

$$\widetilde{\overline{a}} \leq^{\circ} \widetilde{\overline{b}} \iff a \leq b,\ \widetilde{a} \leq \widetilde{a},\ \overline{a} \geq \overline{b} \text{ [resp. } \widetilde{\overline{a}} \leq^{*} \widetilde{\overline{b}} \iff a \leq b,\ \widetilde{a} \geq \widetilde{a},\ \overline{a} \geq \overline{b}].$$

(ii) The *equality* of $\widetilde{\overline{a}}$ and $\widetilde{\overline{b}}$, denoted by $\widetilde{\overline{a}} = \widetilde{\overline{b}}$, is defined as follows:

$$\widetilde{\overline{a}} = \widetilde{\overline{b}} \iff \widetilde{\overline{a}} \leq^{\circ} \widetilde{\overline{b}},\ \widetilde{\overline{b}} \leq^{\circ} \widetilde{\overline{a}} \text{ or } \widetilde{\overline{a}} \leq^{*} \widetilde{\overline{b}},\ \widetilde{\overline{b}} \leq^{*} \widetilde{\overline{a}}.$$

(iii) The *infimum* of $(\widetilde{\overline{a}}_j)_{j\in J}$, denoted by $\bigwedge^{\circ}{}_{j\in J}\widetilde{\overline{a}}_j$ [resp. $\bigwedge^{*}_{j\in J}\widetilde{\overline{a}}_j$], is an MBJ-neutrosophic number defined as follows:

$$\overset{\circ}{\bigwedge}_{j\in J}\widetilde{\overline{a}}_j = \left\langle \bigwedge_{j\in J} a_j, \bigwedge_{j\in J} \widetilde{a}_j, \bigvee_{j\in J} \overline{a}_j \right\rangle \text{ [resp. } \overset{*}{\bigwedge_{j\in J}} \widetilde{\overline{a}}_j = \left\langle \bigwedge_{j\in J} a_j, \bigvee_{j\in J} \widetilde{a}_j, \bigvee_{j\in J} \overline{a}_j \right\rangle].$$

and

(iv) The *supremum* of $(\widetilde{\overline{a}}_j)_{j\in J}$, denoted by $\bigvee^{\circ}{}_{j\in J}\widetilde{\overline{a}}_j$ [resp. $\bigvee^{*}_{j\in J}\widetilde{\overline{a}}_j$], is an MBJ-neutrosophic number defined as follows:

$$\overset{\circ}{\bigvee}_{j\in J}\widetilde{\overline{a}}_j = \left\langle \bigvee_{j\in J} a_j, \bigvee_{j\in J} \widetilde{a}_j, \bigwedge_{j\in J} \overline{a}_j \right\rangle \text{ [resp. } \overset{*}{\bigvee_{j\in J}} \widetilde{\overline{a}}_j = \left\langle \bigvee_{j\in J} a_j, \bigwedge_{j\in J} \widetilde{a}_j, \bigwedge_{j\in J} \overline{a}_j \right\rangle].$$

(v) The *complement* of $\widetilde{\overline{a}}$, denoted by $\widetilde{\overline{a}}^{c,1}$ [resp. $\widetilde{\overline{a}}^{c,2}$ and $\widetilde{\overline{a}}^{c,3}$], is an MBJ-neutrosophic number defined as follows:

$$\widetilde{\overline{a}}^{c,1} = \langle 1-a, \widetilde{a}^{c}, 1-\overline{a}\rangle \text{ [resp. } \widetilde{\overline{a}}^{c,2} = \langle \overline{a}, \widetilde{a}, a\rangle \text{ and } \widetilde{\overline{a}}^{c,2} = \langle \overline{a}, \widetilde{a}^{c}, a\rangle].$$

Remark 3.11. (1) Definitions 3.3, 3.5 and 3.6 are redefined by Definition 3.10 as follows. Let X be a nonempty set, $\mathcal{A}$, $\mathcal{B} \in MBJNS(X)$ and let $(\mathcal{A}_j) \subset MBJN(X)$. Then

- $\mathcal{A} \sqsubset \mathcal{B}$ if and only if $\mathcal{A}(x) \leq^{\circ} \mathcal{B}(x)$ for each $x \in X$,
- $\mathcal{A} \Subset \mathcal{B}$ if and only if $\mathcal{A}(x) \leq^{*} \mathcal{B}(x)$ for each $x \in X$,
- $(\mathcal{A} \sqcap \mathcal{B})(x) = \mathcal{A}(x) \wedge^{\circ} \mathcal{B}(x)$, $(\mathcal{A} \sqcup \mathcal{B})(x) = \mathcal{A}(x) \vee^{\circ} \mathcal{B}(x)$ for each $x \in X$,
- $(\mathcal{A} \Cap \mathcal{B})(x) = \mathcal{A}(x) \wedge^{*} \mathcal{B}(x)$, $(\mathcal{A} \Cup \mathcal{B})(x) = \mathcal{A}(x) \vee^{*} \mathcal{B}(x)$ for each $x \in X$,
- $(\sqcap_{j\in J}\mathcal{A}_j)(x) = \bigwedge^{\circ} \mathcal{A}_j(x)$, $(\sqcup_{j\in J}\mathcal{A}_j)(x) = \bigvee^{\circ} \mathcal{A}_j(x)$ for each $x \in X$,
- $(\Cap_{j\in J}\mathcal{A}_j)(x) = \bigwedge^{*} \mathcal{A}_j(x)$, $(\Cup_{j\in J}\mathcal{A}_j)(x) = \bigvee^{*} \mathcal{A}_j(x)$ for each $x \in X$.

(2) We can easily see that MBJNNs have similar properties to Proposition 3.7, and then

$$(I \times [I] \times I, \wedge^{\circ}, \vee^{\circ}, {}^{c,i}, \ddot{0}, \ddot{1}) \text{ and } (I \times [I] \times I, \wedge^{*}, \vee^{*}, {}^{c,i}, \dot{0}, \dot{1})$$

form Boolean algebras except the property corresponding the condition (13) of Proposition 3.7.

Definition 3.12. Let X be a nonempty set, let $\widetilde{\overline{a}} \in I \times [I] \times I$ and let $\mathcal{A} \in MBJNS(X)$. Then we define two type's MBJ-neutrosophic points as followings"

(i) A is called a $\circ$-*MBJ-neutrosophic point* (briefly, $\circ$-MBJNP) with the support $x \in X$ and the value $\widetilde{\overline{a}}$ with $a > 0,\ a^- > 0,\ \overline{a} < 1$, denoted by $A = x^{\circ}_{\widetilde{\overline{a}}}$, if for each $y \in X$,

$$x^{\circ}_{\widetilde{\overline{a}}} = \begin{cases} \widetilde{\overline{a}} & \text{if } y = x \\ \ddot{0} & \text{otherwise,} \end{cases}$$

(ii) A is called a $*$- *MBJ-neutrosophic point* (briefly, $*$-MBJNP) with the support $x \in X$ and the value $\widetilde{\overline{a}}$ with $a > 0,\ a^+ < 1,\ \overline{a} < 1$, denoted by $A = x^{\circ}_{\widetilde{\overline{a}}}$, if for each $y \in X$,

$$x^{*}_{\widetilde{\overline{a}}} = \begin{cases} \widetilde{\overline{a}} & \text{if } y = x \\ \dot{0} & \text{otherwise.} \end{cases}$$

We denote the set of all fuzzy points in X by $MBJN_P(X)$.

For a nonempty set X, let x_a [resp. $x_{\widetilde{a}}$] denotes the fuzzy point [resp. interval-valued fuzzy point] in X with the support $x \in X$ and the value $a \in I$ [resp. $\widetilde{a} \in [I]$] (See [16] [resp. [25]]). We denote the set of all fuzzy points [resp. interval-valued fuzzy points] in X as $F_P(X)$ [resp. $IVF_P(X)$]. It is well-known that $A = \bigcup_{x_a \in A} x_a$ for each $A \in I^X$ (See [18]) and $\widetilde{A} = \bigcup_{x_{\widetilde{a}} \in \widetilde{A}} x_{\widetilde{a}}$ for each $\widetilde{A} \in IVFS(X)$ (See [25]).

Definition 3.13. Let X be a nonempty set, let $x^{\circ}_{\widetilde{\overline{a}}},\ x^{*}_{\widetilde{\overline{a}}} \in MBJN_P(X)$ and let $\mathcal{A} \in MBJNS(X)$. Then

(i) $x^{\circ}_{\widetilde{\overline{a}}}$ is said to *belong to* $\mathcal{A}$, denoted by $x^{\circ}_{\widetilde{\overline{a}}} \in \mathcal{A}$, if $\widetilde{\overline{a}} \leq^{\circ} \mathcal{A}(x)$,

(ii) $x^{*}_{\widetilde{\overline{a}}}$ is said to *belong to* $\mathcal{A}$, denoted by $x^{*}_{\widetilde{\overline{a}}} \in \mathcal{A}$, if $\widetilde{\overline{a}} \leq^{*} \mathcal{A}(x)$,

(iii) $x^{\circ}_{\widetilde{\overline{a}}}$ is said to $\circ$-*quasi-coincident with* $\mathcal{A}$, denoted by $x^{\circ}_{\widetilde{\overline{a}}} q^i \mathcal{A}$, if $\mathcal{A}(x) >^{\circ} \mathcal{A}^i(x)$ $(i = 1,\ 2,\ 3)$,

(iv) $x^{*}_{\widetilde{\overline{a}}}$ is said to $*$-*quasi-coincident with* $\mathcal{A}$, denoted by $x^{*}_{\widetilde{\overline{a}}} q^i \mathcal{A}$, if $\mathcal{A}(x) >^{*} \mathcal{A}^{c,i}(x)$ $(i = 1,\ 2,\ 3)$,

(v) $\mathcal{A}$ is said to be $\circ$-*quasi-coincident with* $\mathcal{B}$, denoted by $\mathcal{A} q^{\circ,i} \mathcal{B}$, if there is $x \in X$ such that $\mathcal{A}(x) >^{\circ} \mathcal{B}^i(x)$ $(i = 1,\ 2,\ 3)$,

(vi) $\mathcal{A}$ is said to be $*$-*quasi-coincident with* $\mathcal{B}$, denoted by $\mathcal{A} q^{*.i} \mathcal{B}$, if there is $x \in X$ such that $\mathcal{A}(x) >^{*} \mathcal{B}^i(x)$ $(i = 1,\ 2,\ 3)$.

It is obvious that $\mathcal{A} = \sqcup_{x^{\circ}_{\widetilde{\overline{a}}} \in \mathcal{A}} x^{\circ}_{\widetilde{\overline{a}}}$ and $\mathcal{A} = \Cup_{x^{*}_{\widetilde{\overline{a}}} \in \mathcal{A}} x^{*}_{\widetilde{\overline{a}}}$ for each $\mathcal{A} \in MBJN(X)$.

For a fuzzy point x_a and two fuzzy sets $A,\ B$, $x_a q A$ [resp. AqB] means that x_a is quasi-coincident with A [resp. A is quasi-coincident with B] (See [16]). Also, an interval-valued fuzzy point $x_{\widetilde{a}}$ and two interval-valued fuzzy sets $\widetilde{A},\ \widetilde{B}$, $x_{\widetilde{a}} q \widetilde{A}$ [resp. $\widetilde{A} q \widetilde{B}$] means that $x_{\widetilde{a}}$ is quasi-coincident with $\widetilde{A}$ [resp. $\widetilde{A}$ is quasi-coincident with $\widetilde{B}$] (See [33]).

Throughout this paper, for any fuzzy set A [resp. interval-valued fuzzy set $\widetilde{A}$] in X, if $a \geq A(x)$ [resp. $\widetilde{a} \geq \widetilde{A}(x)$], then we say that x_a $\circ$-*belongs to* A [resp. $x_{\widetilde{a}}$ $*$-*belongs to* $\widetilde{A}$] and denoted by $x^{\circ}_a \in A$ [resp. $x^{*}_{\widetilde{a}} \in \widetilde{A}$]. Moreover, if $a < A^c(x)$ [resp. $\widetilde{a} < \widetilde{A}^c(x)$], then we say that

x_a *is* $\circ$-*quasi-coincident with* A [resp. $x_{\tilde{a}}$ *is* $*$-*quasi-coincident with* $\widetilde{A}$] and denoted by $x_a q^\circ A$ [resp. $x_{\tilde{a}} q^* \widetilde{A}$].

Remark 3.14. From Definition 3.13, we can easily see that the followings hold.

(1) $x^\circ_{\bar{\tilde{a}}} \in \mathcal{A}$ if and only if $x_a \in M_A$, $x_{\tilde{a}} \in \widetilde{B}_A$, $x^\circ_{\bar{a}} \in J_A$.

(2) $x^*_{\bar{\tilde{a}}} \in \mathcal{A}$ if and only if $x_a \in M_A$, $x^*_{\tilde{a}} \in \widetilde{B}_A$, $x^\circ_{\bar{a}} \in J_A$.

(3) $x^\circ_{\bar{\tilde{a}}} q^1 \mathcal{A}$ if and only if $x_a q M_A$, $x_{\tilde{a}} q \widetilde{B}_A$, $x_{\bar{a}} q^\circ J_A$,
$x^\circ_{\bar{\tilde{a}}} q^2 \mathcal{A}$ if and only if $a > J_A(x)$, $\tilde{a} > \widetilde{B}_A(x)$, $\bar{a} < M_A(x)$,
$x^\circ_{\bar{\tilde{a}}} q^3 \mathcal{A}$ if and only if $a > J_A(x)$, $x_{\tilde{a}} q \widetilde{B}_A$, $\bar{a} < M_A(x)$.

(4) $x^*_{\bar{\tilde{a}}} q^1 \mathcal{A}$ if and only if $x_a q M_A$, $x_{\tilde{a}} q^* \widetilde{B}_A$, $x_{\bar{a}} q^\circ J_A$,
$x^*_{\bar{\tilde{a}}} q^2 \mathcal{A}$ if and only if $a > J_A(x)$, $\tilde{a} < \widetilde{B}_A(x)$, $\bar{a} < M_A(x)$,
$x^*_{\bar{\tilde{a}}} q^3 \mathcal{A}$ if and only if $a > J_A(x)$, $\tilde{a} < \widetilde{B}^c_A(x)$, $\bar{a} < M_A(x)$.

From now on, we will use only $\mathcal{A}^{c,1}$ as the complement of an MBJNS $\mathcal{A}$ in X and write $\mathcal{A}^{c,1} = \mathcal{A}^c$. Also, we use $x^\circ_{\bar{\tilde{a}}} q\mathcal{A}$, $x^*_{\bar{\tilde{a}}} q\mathcal{A}$, $\mathcal{A}q^\circ\mathcal{B}$ and $\mathcal{A}q^*\mathcal{B}$ instead of $x^\circ_{\bar{\tilde{a}}} q^1\mathcal{A}$, $x^*_{\bar{\tilde{a}}} q^1\mathcal{A}$, $\mathcal{A}q^{\circ,1}\mathcal{B}$ and $\mathcal{A}q^{*,1}\mathcal{B}$ respectively.

If there is $x \in X$ such that $\mathcal{A}(x) >^\circ \mathcal{B}^c(x)$ [resp. $\mathcal{A}(x) >^* \mathcal{B}^c(x)$], then we say that $\mathcal{A}$ and $\mathcal{B}$ are $\circ$- [resp. $*$-]*quasi-coincident* (*with each other*) *at* x. We say that $\mathcal{A}$ *is not* $\circ$-[resp. $*$-]*quasi-coincident with* $\mathcal{B}$, denoted by $\mathcal{A}\neg q^\circ\mathcal{B}$ [resp. $\mathcal{A}\neg q^*\mathcal{B}$], if the following conditions hold:

$$M_A \neg q M_B,\ \widetilde{B}_A \neg q \widetilde{B}_B,\ J_A \neg q^\circ J_B \text{ [resp. } M_A \neg q M_B,\ \widetilde{B}_A \neg q^* \widetilde{B}_B,\ J_A \neg q^\circ J_B],\ \text{i.e.,}$$

$$M_A(x) \leq M^c_B(x),\ \widetilde{B}_A(x) \leq \widetilde{B}^c_B(x),\ J_A(x) \geq J^c_B(x),$$

$$M_A(x) \leq M^c_B(x),\ \widetilde{B}_A(x) \geq \widetilde{B}^c_B(x),\ J_A(x) \geq J^c_B(x)$$

for each $x \in X$.

Definition 3.15. Let $\mathcal{A},\ \mathcal{B} \in MBJNS(X)$. Then

(i) $\mathcal{A}$ and $\mathcal{B}$ are said to be $\circ$-*intersecting*, if there is $x \in X$ such that

$$(M_A \cap M_B)(x) \neq 0,\ (\widetilde{B}_A \cap \widetilde{B}_B)(x) \neq [0,0],\ (J_A \cup J_B)(x) \neq 1,$$

(ii) $\mathcal{A}$ and $\mathcal{B}$ are said to be $*$-*intersecting*, if there is $x \in X$ such that

$$(M_A \cap M_B)(x) \neq 0,\ (\widetilde{B}_A \cup \widetilde{B}_B)(x) \neq [1,1],\ (J_A \cup J_B)(x) \neq 1.$$

In either case, we say that $\mathcal{A}$ and $\mathcal{B}$ $\circ$-*intersect at* x [resp. $*$-*intersect at* x].

It is obvious that if $\mathcal{A}$ and $\mathcal{B}$ are $\circ$- [resp. $*$-]quasi-coincident at x, then they are $\circ$- [resp. $*$-]intersect at x.

The following is an immediate consequence of Definition 3.13 and Remark 3.14.

Lemma 3.16. *Let* $x^{\circ}_{\tilde{\tilde{a}}},\ x^{*}_{\tilde{\tilde{a}}} \in MBJN_P(X)$ *and let* $\mathcal{A} \in MBJNS(X)$*. Then*

(1) $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{A}$ *if and only if* $x^{\circ}_{\tilde{\tilde{a}}} \neg q \mathcal{A}^c$,

(2) $x^{*}_{\tilde{\tilde{a}}} \in \mathcal{A}$ *if and only if* $x^{*}_{\tilde{\tilde{a}}} \neg q \mathcal{A}^c$.

The following is an immediate consequence of Definitions 3.3 and 3.13, and Lemma 3.5 (1).

Lemma 3.17. *Let* $\mathcal{A},\ \mathcal{B} \in MBJNS(X)$*. Then the followings are equivalent:*

(1) $\mathcal{A} \sqsubset \mathcal{B}$,

(2) $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B}$ *for each* $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{A}$,

(3) $\mathcal{A} \neg q \mathcal{B}^c$.

Also, the following is an immediate consequence of Definitions 3.3 and 3.13, and Lemma 3.5 (2).

Lemma 3.18. *Let* $\mathcal{A},\ \mathcal{B} \in MBJNS(X)$*. Then the followings are equivalent:*

(1) $\mathcal{A} \Subset \mathcal{B}$,

(2) $x^{*}_{\tilde{\tilde{a}}} \in \mathcal{B}$ *for each* $x^{*}_{\tilde{\tilde{a}}} \in \mathcal{A}$,

(3) $\mathcal{A} \neg q \mathcal{B}^c$.

Lemma 3.19. *Let* $\mathcal{A},\ \mathcal{B} \in MBJNS(X)$*, let* $(\mathcal{A})_{j \in J} \subset MBJNS(X)$ *and let* $x^{\circ}_{\tilde{\tilde{a}}} \in MBJN_P(X)$*.*

(1) $x^{\circ}_{\tilde{\tilde{a}}} q(\sqcup_{j \in J} \mathcal{A}_j)$ *if and only if there is* $j_0 \in J$ *such that* $x^{\circ}_{\tilde{\tilde{a}}} q \mathcal{A}_{j_0}$.

(2) $x^{\circ}_{\tilde{\tilde{a}}} q(\mathcal{A} \sqcap \mathcal{B})$ *if and only if* $x^{\circ}_{\tilde{\tilde{a}}} q \mathcal{A}$ *and* $x^{\circ}_{\tilde{\tilde{a}}} q \mathcal{B}$.

Proof. The proof is easy. □

Lemma 3.20. *Let* $\mathcal{A},\ \mathcal{B} \in MBJNS(X)$*, let* $(\mathcal{A})_{j \in J} \subset MBJNS(X)$ *and let* $x^{*}_{\tilde{\tilde{a}}} \in MBJN_P(X)$*.*

(1) $x^{*}_{\tilde{\tilde{a}}} q(\Cup_{j \in J} \mathcal{A}_j)$ *if and only if there is* $j_0 \in J$ *such that* $x^{*}_{\tilde{\tilde{a}}} q \mathcal{A}_{j_0}$.

(2) $x^{*}_{\tilde{\tilde{a}}} q(\mathcal{A} \Cap \mathcal{B})$ *if and only if* $x^{*}_{\tilde{\tilde{a}}} q \mathcal{A}$ *and* $x^{*}_{\tilde{\tilde{a}}} q \mathcal{B}$.

Proof. The proof is similar to Proposition 3.19. □

Lemma 3.21. *Let* X *be a nonempty set and let* $A \in I^X$ *such that* $A(x) \neq 1$ *for each* $x \in X$*. Then* $x_{a^c} q^{\circ} A$ *for each* $a \in I$ *such that* $A(x) < a < 1$*.*

Proof. The proof is straightforward. □

Lemma 3.22. *Let* X *be a nonempty set and let* $\widetilde{A} \in IVFS(X)$ *such that* $\widetilde{A}(x) \neq [1,1]$ *for each* $x \in X$*. Then* $x_{\widetilde{a}^c} q^{*} \widetilde{A}$ *for each* $\widetilde{a} \in [I]$ *such that* $\widetilde{A}(x) < \widetilde{a} < [1,1]$*.*

Proof. The proof is straightforward. □

Definition 3.23. Let X, Y be nonempty sets and let $f : X \to Y$ and let $\mathcal{A} \in MBJNS(X)$, $\mathcal{B} \in MBJNS(Y)$.

(i) The *pre-image* of $\mathcal{B}$ under f, denoted by denoted by

$$f^{-1}(\mathcal{B}) = \left\langle f^{-1}(M_B), f^{-1}(\widetilde{B}_B), f^{-1}(J_B) \right\rangle,$$

is an MBJ-neutrosophic set in X defined as follows: for each $x \in X$,

$$f^{-1}(M_B)(x) = M_B(f(x)),\ f^{-1}(\widetilde{B}_B)(x) = \widetilde{B}_B(f(x)),\ f^{-1}(J_B)(x) = J_B(f(x)).$$

(ii) The $\circ$-*image* and the $*$-*image* of $\mathcal{A}$ under f, denoted by $f^\circ(\mathcal{A})$ and $f^*(\mathcal{A})$, are cubic sets in Y respectively defined as follows: for each $y \in Y$,

$$f^\circ(\mathcal{A})(y) = \begin{cases} \left\langle \bigvee_{x\in f^{-1}(y)} M_A(x), \bigvee_{x\in f^{-1}(y)} \widetilde{B}_A(x), \bigwedge_{x\in f^{-1}(y)} J_A(x) \right\rangle & \text{if } f^{-1}(y) \neq \emptyset \\ \ddot{0} & \text{otherwise,} \end{cases}$$

$$f^*(\mathcal{A})(y) = \begin{cases} \left\langle \bigvee_{x\in f^{-1}(y)} M_A(x), \bigwedge_{x\in f^{-1}(y)} \widetilde{B}_A(x), \bigwedge_{x\in f^{-1}(y)} J_A(x) \right\rangle & \text{if } f^{-1}(y) \neq \emptyset \\ \dot{0} & \text{otherwise.} \end{cases}$$

Remark 3.24. Let us denote $\bigwedge_{x\in f^{-1}(y)} J_A(x)$ and $\bigwedge_{x\in f^{-1}(y)} \widetilde{B}_A(x)$ as $f^\circ(J_A)$ and $f^*(\widetilde{B}_A)$ respectively. Then we can see that

$$f^\circ(\mathcal{A}) = \left\langle f(M_A, f(\widetilde{B}_A), f^\circ(J_A) \right\rangle,\ f^*(\mathcal{A}) = \left\langle f(M_A, f^*(\widetilde{B}_A), f^\circ(J_A) \right\rangle.$$

We have a similar consequence of (Lemma 1.1, [17]), (Theorem 2, [25]), (Corollary 2.10, [21]) and (Theorem 1.10, [35]).

Proposition 3.25. *Let X, Y be nonempty sets, let $f : X \to Y$ be a mapping, let $\mathcal{A}$, $\mathcal{A}_1, \mathcal{A}_2 \in MBJNS(X)$, $\mathcal{B}$, $\mathcal{B}_1, \mathcal{B}_2 \in MBJNS(Y)$, let $(\mathcal{A}_j)_{j\in J} \subset MBJNS(X)$ and let $(\mathcal{B}_j)_{j\in J} \subset MBJNS(Y)$.*

(1) $f^{-1}(\mathcal{B}^c) = [f^{-1}(\mathcal{B})]^c$.

(2) $f^{-1}(\ddot{\emptyset}) = \ddot{\emptyset},\ f^{-1}(\ddot{Y}) = \ddot{Y}, f^{-1}(\dot{\emptyset}) = \dot{\emptyset},\ f^{-1}(\dot{Y}) = \dot{Y}$.

(3) $f^\circ(\mathcal{A}^c) \sqsupset [f^\circ(\mathcal{A})]^c$ *and* $f^*(\mathcal{A}^c) \Supset [f^*(\mathcal{A})]^c$, *if f is injective, then* $f^\circ(\mathcal{A}^c) = [f^\circ(\mathcal{A})]^c$ *and* $f^*(\mathcal{A}^c) = [f^*(\mathcal{A})]^c$.

(4) *If* $\mathcal{B}_1 \sqsubset \mathcal{B}_2$, *then* $f^{-1}(\mathcal{B}_1) \sqsubset f^{-1}(\mathcal{B}_2)$.

(5) *If* $\mathcal{B}_1 \Subset \mathcal{B}_2$, *then* $f^{-1}(\mathcal{B}_1) \Subset f^{-1}(\mathcal{B}_2)$.

(6) *If* $\mathcal{A}_1 \sqsubset \mathcal{A}_2$, *then* $f^\circ(\mathcal{A}_1) \sqsubset f^\circ(\mathcal{A}_2)$.

(7) *If* $\mathcal{A}_1 \Subset \mathcal{A}_2$, *then* $f^*(\mathcal{A}_1) \Subset f^*(\mathcal{A}_2)$.

(8) $f^\circ(f^{-1}(\mathcal{B})) \sqsubset \mathcal{B}$. *In particular, if f is surjective, then* $f^\circ(f^{-1}(\mathcal{B})) = \mathcal{B}$.

(9) $\mathcal{A} \sqsubset f^{-1}(f^\circ(\mathcal{A}))$. *In particular, if f is injective, then* $\mathcal{A} = f^{-1}(f^\circ(\mathcal{A}))$.

(10) $f^*(f^{-1}(\mathcal{B})) \Subset \mathcal{B}$. *In particular, if f is surjective, then* $f^*(f^{-1}(\mathcal{B})) = \mathcal{B}$.

(11) $\mathcal{A} \Subset f^{-1}(f^*(\mathcal{A}))$. *In particular, if f is injective, then* $\mathcal{A} = f^{-1}(f^*(\mathcal{A}))$.

(12) *If* $f^\circ(\mathcal{A}) \sqsubset \mathcal{B}$, *then* $\mathcal{A} \sqsubset f^{-1}(\mathcal{B})$.

(13) *If* $f^*(\mathcal{A}) \Subset \mathcal{B}$, *then* $\mathcal{A} \Subset f^{-1}(\mathcal{B})$.

(14) *For each* $x^\circ_{\tilde{\tilde{a}}} \in MBJN_P(X)$, $f^\circ(x^\circ_{\tilde{\tilde{a}}}) \in MBJN_P(Y)$ *and* $f^\circ(x^\circ_{\tilde{\tilde{a}}}) = [f(x)]^\circ_{\tilde{\tilde{a}}}$.

(15) *For each* $x^\circ_{\tilde{\tilde{a}}} \in MBJN_P(X)$, *if* $x^\circ_{\tilde{\tilde{a}}} q \mathcal{A}$, *then* $f^\circ(x^\circ_{\tilde{\tilde{a}}}) q f^\circ(\mathcal{A})$.

(16) *For each* $x^*_{\tilde{\tilde{a}}} \in MBJN_P(X)$, $f^*(x^*_{\tilde{\tilde{a}}}) \in MBJN_P(Y)$ *and* $f^*(x^*_{\tilde{\tilde{a}}}) = [f(x)]^*_{\tilde{\tilde{a}}}$.

(17) *For each* $x^*_{\tilde{\tilde{a}}} \in MBJN_P(X)$, *if* $x^*_{\tilde{\tilde{a}}} q \mathcal{A}$, *then* $f^*(x^*_{\tilde{\tilde{a}}}) q f^*(\mathcal{A})$.

(18) $f^\circ(\sqcup_{j \in J} \mathcal{A}_j) = \sqcup_{j \in J} f^\circ(\mathcal{A}_j)$.

(19) $f^*(\Cup_{j \in J} \mathcal{A}_j) = \Cup_{j \in J} f^*(\mathcal{A}_j)$.

(20) $f^{-1}(\sqcup_{j \in J} \mathcal{B}_j) = \sqcup_{j \in J} f^{-1}(\mathcal{B}_j)$ *and* $f^{-1}(\Cup_{j \in J} \mathcal{B}_j) = \Cup_{j \in J} f^{-1}(\mathcal{B}_j)$.

(21) $f^{-1}(\sqcap_{j \in J} \mathcal{B}_j) = \sqcap_{j \in J} f^{-1}(\mathcal{B}_j)$ *and* $f^{-1}(\Cap_{j \in J} \mathcal{B}_j) = \Cap_{j \in J} f^{-1}(\mathcal{B}_j)$.

(22) *If* $g : Y \to Z$ *is a mapping, then* $(g \circ f)^{-1}(\mathcal{C}) = f^{-1}(g^{-1}(\mathcal{C}))$ *for each* $\mathcal{C} \in MBJNS(Z)$, *where* $g \circ f$ *denotes the composition of f and g.*

Definition 3.26. Let X be a nonempty set and let $\mathcal{A} \in MBJNS(X)$. Then

(i) the $\circ$-[resp. $*$-]*support* of $\mathcal{A}$, denoted by $supp^\circ(\mathcal{A})$ [resp. $supp^*(\mathcal{A})$], is a subset of X defined as follows:

$$supp^\circ(\mathcal{A}) = \{x \in X : \mathcal{A}(x) >^\circ \ddot{0}\} \text{ [resp. } supp^*(\mathcal{A}) = \{x \in X : \mathcal{A}(x) >^* \dot{0}\}].$$

(ii) $\mathcal{A}$ is said to be $\circ$-[resp. $*$-] *finite*, if $supp^\circ(\mathcal{A})$ [resp. $supp^*(\mathcal{A})$] is finite.

Proposition 3.27. *Let X be a nonempty set and let* $\mathcal{A}$, $\mathcal{B} \in MBJNS(X)$. *Then*

(1) *the* $supp^\circ((\mathcal{A} \sqcup \mathcal{B})^c) = supp^\circ(\mathcal{A}^c) \sqcap supp^*(\mathcal{B}_c)$,

(2) *the* $supp^*((\mathcal{A} \sqcup \mathcal{B})^c) = supp^*(\mathcal{A}^c) \sqcap supp^*(\mathcal{B}_c)$.

Proof. The proof is straightforward. □

4. **MBJ-neutrosophic neighborhoods**

We define a $\circ$-[resp. $*$]-MBJ-neutrosophic neighborhood of a $\circ$-[resp. a $*$]-MBJ-neutrosophic point with respect to a $\circ$-[resp. $*$]-MBJ-neutrosophic topology and obtain its various properties.

Definition 4.1. Let X be a nonempty set and let $\mathcal{A} \in MBJNS(X)$. Then $\mathcal{A}$ is called a *constant MBJ-neutrosophic set* in X, denoted by $\mathcal{A} = C_{\tilde{\tilde{a}}}$, if there is $\tilde{\tilde{a}} \in I \times [I] \times I$ such that $\mathcal{A}(x) = \tilde{\tilde{a}}$ for each $x \in X$.

Definition 4.2. Let τ be a family of cubic sets in a nonempty set X. Consider the following conditions:

(MBJNO$_0$) $C_{\widetilde{\widetilde{a}}} \in \tau$ for each $\widetilde{\widetilde{a}} \in I \times [I] \times I$,

($\circ$-MBJNO$_1$) $\ddot{\emptyset},\ \ddot{X} \in \tau$,

($\circ$-MBJNO$_2$) $\mathcal{A} \sqcap \mathcal{B} \in \tau$ for any $\mathcal{A},\ \mathcal{B} \in \tau$,

($\circ$-MBJNO$_3$) $\sqcup_{j\in J}\mathcal{A}_j \in \tau$ for each $(\mathcal{A}_j)_{j\in J} \subset \tau$,

($*$-MBJNO$_1$) $\dot{\emptyset},\ \dot{X} \in \tau$,

($*$-MBJNO$_2$) $\mathcal{A} \Cap \mathcal{B} \in \tau$ for any $\mathcal{A},\ \mathcal{B} \in \tau$,

($*$-MBJNO$_3$) $\Cup_{j\in J}\mathcal{A}_j \in \tau$ for each $(\mathcal{A}_j)_{j\in J} \subset \tau$.

(i) τ is called a $\circ$-*MBJ-neutrosophic topology* (briefly, $\circ$-MBJNT) on X in Chang's sense, if it satisfies the conditions ($\circ$-MBJNO$_1$), ($\circ$-MBJNO$_2$) and ($\circ$-MBJNO$_3$).

(ii) τ is called a $*$-*MBJ-neutrosophic topology* (briefly, $*$-MBJNT) on X in Chang's sense, if it satisfies the conditions ($*$-MBJNO$_1$), ($*$-MBJNO$_2$) and ($*$-MBJNO$_3$).

(iii) τ is called a $\circ$-*MBJ-neutrosophic topology* (briefly, $*$-MBJNT) on X in Lowen's sense, if it satisfies the conditions (MBJNO$_0$), ($\circ$-MBJNO$_2$) and ($\circ$-MBJNO$_3$).

(iv) τ is called a $*$-*cubic topology* on X in Lowen's sense, if it satisfies the conditions (MBJNO$_0$), ($*$-MBJNO$_2$) and ($*$-MBJNO$_3$).

In either case, the pair (X, τ) is called a $\circ$-*MBJ-neutrosophic topological space* [resp. $*$-*MBJ-neutrosophic topological space*] and each member of τ is called a $\circ$-*MBJ-neutrosophic open set* (briefly, $\circ$-MBJNOS) [resp. $*$-*MBJ-neutrosophic open set* (briefly, $*$-MBJNOS)]. We will denote the set of all $\circ$-MBJNTs in Chang's sense [resp. Lowen's sense] on X as $MBJNT^\circ(X)$ [resp. $MBJNT^\circ_L(X)$]. Also, we will denote the set of all $*$-MBJNTs in Chang's sense [resp. Lowen's sense] on X as $MBJNT^*(X)$ [resp. $MBJNT^*_L(X)$]. An MBJ-neutrosophic set $\mathcal{A}$ is called a $\circ$-*MBJ-neutrosophic closed set* (briefly, $\circ$-MBJNCS) [resp. $*$-*MBJ-neutrosophic closed set* (briefly, $*$-MBJNCS)] in X, if $\mathcal{A}^c \in \tau$. For a $\circ$-MBJ-neutrosophic topological space X, we denote the set of all $\circ$-MBJNOs [resp. $\circ$-MBJNCSs] in X as $MBJNO^\circ(X)$ [resp. $MBJNC^\circ(X)$]. Also, for a $*$-MBJ-neutrosophic topological space X, we denote the set of all $*$-MBJNOSs [resp. $*$-M BJNCSs] in X as $MBJNO^*(X)$ [resp. $MBJNC^*(X)$].

Example 4.3. (1) Let $X = \{x, y\}$ and let $\mathcal{A}_j \in MBJNS(X)$ $(j = 1,\ 2,\ 3,\ 4,\ 5,\ 6)$ defined as follows:

$$\mathcal{A}_1(x) = \langle 0.4, [0.6, 0.8], 0.8\rangle\ ,\ \mathcal{A}_1(y) = \langle 0.6, [0.5, 0.9], 0.7\rangle\ ,$$

$$\mathcal{A}_2(x) = \langle 0.5, [0.4, 0.7], 0.4\rangle\ ,\ \mathcal{A}_2(y) = \langle 0.3, [0.7, 0.8], 0.9\rangle\ ,$$

$$\mathcal{A}_3(x) = \langle 0.5, [0.6, 0.8], 0.4\rangle\ ,\ \mathcal{A}_3(y) = \langle 0.6, [0.7, 0.9], 0.7\rangle\ ,$$

$$\mathcal{A}_4(x) = \langle 0.4, [0.4, 0.7], 0.8\rangle\ ,\ \mathcal{A}_4(y) = \langle 0.3, [0.5, 0.8], 0.9\rangle\ ,$$

$$\mathcal{A}_5(x) = \langle 0.5, [0.4, 0.7], 0.4\rangle\ ,\ \mathcal{A}_5(y) = \langle 0.6, [0.5, 0.8], 0.7\rangle\ ,$$

$$\mathcal{A}_6(x) = \langle 0.4, [0.6, 0.8], 0.8 \rangle\,,\ \mathcal{A}_6(y) = \langle 0.3, [0.7, 0.9], 0.9 \rangle\,.$$

Let us consider the following two families:

$$\tau = \{\ddot{\emptyset}, \ddot{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\},\ \eta = \{\dot{\emptyset}, \dot{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_5, \mathcal{A}_6\}.$$

Then we can easily check that $\tau \in MBJNT^{\circ}(X)$ and $\eta \in MBJNT^{*}(X)$. Furthermore, we can easily see that

$$\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\} \bigcup \{C_{\widetilde{\widetilde{a}}} : \widetilde{\widetilde{a}} \in I \times [I] \times I\} \in MBJNT^{\circ}_{L}(X)$$

and

$$\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_5, \mathcal{A}_6\} \bigcup \{C_{\widetilde{\widetilde{a}}} : \widetilde{\widetilde{a}} \in I \times [I] \times I\} \in MBJNT^{*}_{L}(X).$$

(2) Let X be a nonempty set and let τ be the family of MBJ-neutrosophic sets in X defined as follows:

$$\tau = \{\mathcal{A} \in MBJNS(X) : \mathcal{A} = \ddot{X} \text{ or } supp^{\circ}(\mathcal{A}^c) \text{ is } \circ\text{-finite}\}$$

$$[\text{resp. } \tau = \{\mathcal{A} \in MBJNS(X) : \mathcal{A} = \dot{X} \text{ or } supp^{*}(\mathcal{A}^c) \text{ is } *\text{-finite}\}].$$

Then by Proposition 3.27, it is obvious that $\tau \in MBJNT^{\circ}(X)$ [resp. $\tau \in MBJNT^{*}(X)$]. In this case, we will call τ as *MBJ-neutrosophic* $\circ$-[resp. $*$-]*cofinite topology* on X.

Remark 4.4. (1) From Definition 4.2, it is obvious that $\{\ddot{\emptyset},\ \ddot{X}\} \in MBJNT^{\circ}(X)$, $\{\dot{\emptyset},\ \dot{X}\} \in MBJNT^{*}(X)$ and $MBJNS(X)$ are both $\circ$-MBJNT and $*$-MBJNT on X. In this case, $\{\ddot{\emptyset},\ \ddot{X}\}$ [resp. $\{\dot{\emptyset},\ \dot{X}\}$ and $MBJNS(X)$] is called the $\circ$-*MBJ-neutrosophic indiscrete topology* [resp. $*$-*MBJ-neutrosophic indiscrete topology* and *MBJ-neutrosophic discrete topology*] on X and will be denoted by $\mathcal{I}^{\circ}$ [resp. $\mathcal{I}^{*}$ and $\mathcal{D}$]. The pair $(X, \mathcal{I}^{\circ})$ [resp. $(X, \mathcal{I}^{*})$ and $(X, \mathcal{D})$] is called a $\circ$-*MBJ-neutrosophic indiscrete space* [resp. $*$-*MBJ-neutrosophic indiscrete space* and *MBJ-neutrosophic discrete space*]. It is clear that that $\mathcal{I}^{\circ} \subset \tau \subset \mathcal{D}$ for each $\tau \in MBJNT^{\circ}(X)$ and $\mathcal{I}^{*} \subset \tau \subset \mathcal{D}$ for each $\tau \in MBJNT^{*}(X)$. Moreover, by Proposition 3.4, we can easily check that for each $\tau \in MBJNT^{\circ}(X)$ [resp. $\tau \in MBJNT^{*}(X)$], τ have the least element $\ddot{\emptyset}$ [resp. $\dot{\emptyset}$] and greatest element $\ddot{X}$ [resp. $\dot{X}$].

(2) Let T be a classical topology on a nonempty set X. Then clearly,

$$\chi^{\circ}_{\mathcal{T}} = \{\langle \chi_{A}, [\chi_{A}, \chi_{A}], \chi_{A^c} \rangle \in MBJN(X) : A \in T\} \in MBJNT^{\circ}(X),$$

$$\chi^{*}_{\mathcal{T}} = \{\langle \chi_{A}, [\chi_{A^c}, \chi_{A^c}], \chi_{A^c} \rangle \in MBJN(X) : A \in T\} \in MBJNT^{*}(X).$$

(3) We denote the set of all fuzzy topologies (See [13, 20]) on a nonempty set X as FT(X) and let $\tau^c = \{A^c \in I^X : A \in \tau\}$ for each $\tau \in FT(X)$. Then it is obvious that for each $\tau \in FT(X)$,

$$\{\langle A, [A, A], A^c \rangle \in MBJN(X) : A \in \tau\} \in MBJNT^{\circ}(X),$$

$$\{\langle A, [A^c, A^c], A^c \rangle \in MBJN(X) : A \in \tau\} \in MBJNT^{*}(X).$$

(4) Let us denote the set of all interval-valued fuzzy topologies (See [25]) on a nonempty set X as $IVFT(X)$. Then we can easily check that for each $\tau \in IVFT(X)$,

$$\{\left\langle A^-, \widetilde{A}, A^{-c}\right\rangle \in MBJN(X) : \widetilde{A} \in \tau\} \in MBJNT^{\circ}(X),$$

$$\{\left\langle A^-, \widetilde{A}, A^{+c}\right\rangle \in MBJN(X) : \widetilde{A} \in \tau\} \in MBJNT^{\circ}(X),$$

$$\{\left\langle A^+, \widetilde{A}, A^{-c}\right\rangle \in MBJN(X) : \widetilde{A} \in \tau\} \in MBJNT^{\circ}(X),$$

$$\{\left\langle A^+, \widetilde{A}, A^{+c}\right\rangle \in MBJN(X) : \widetilde{A} \in \tau\} \in MBJNT^{\circ}(X),$$

$$\{\left\langle A^-, \widetilde{A}^c, A^{-c}\right\rangle \in MBJN(X) : \widetilde{A} \in \tau\} \in MBJNT^{*}(X),$$

$$\{\left\langle A^-, \widetilde{A}^c, A^{+c}\right\rangle \in MBJN(X) : \widetilde{A} \in \tau\} \in MBJNT^{*}(X),$$

$$\{\left\langle A^+, \widetilde{A}^c, A^{-c}\right\rangle \in MBJN(X) : \widetilde{A} \in \tau\} \in MBJNT^{*}(X),$$

$$\{\left\langle A^+, \widetilde{A}^c, A^{+c}\right\rangle \in MBJN(X) : \widetilde{A} \in \tau\} \in MBJNT^{*}(X).$$

(5) We denote the set of all interval-valued fuzzy topologies [resp. cotopologies] (See [25]) and all fuzzy topologies [resp. cotopologies] (See [13, 20]) on a set X as $IVFT(X)$ [resp. $IVFCT(X)$] and $FT(X)$ [resp. $FCT(X)$] respectively, where the term "cotoplogy" means the dual of "topology". For each $\tau \in MBJNT^{\circ}(X)$ [resp. $\tau \in MBJNT^{*}(X)$], let us consider the following families:

$$\tau_{_M} = \{M_A \in I^X : \mathcal{A} \in \tau\},\ \tau_{_{\widetilde{B}}} = \{\widetilde{B}_A \in IVFSX : \mathcal{A} \in \tau\},\ \tau_{_J} = \{J_A \in I^X : \mathcal{A} \in \tau\}.$$

Then we can easily see that the followings hold:

$$\tau \in MBJNT^{\circ}(X) \Longleftrightarrow \tau_{_M} \in FT(X),\ \tau_{_{\widetilde{B}}} \in IVFT(X),\ \tau_{_J} \in FCT(X)$$

and

$$\tau \in MBJNT^{*}(X) \Longleftrightarrow \tau_{_M} \in FT(X),\ \tau_{_{\widetilde{B}}} \in IVFCT(X),\ \tau_{_J} \in FCT(X).$$

(6) Let (X, τ) be a neutrosophic topological space proposed by Salama and Alblowi [31] and consider two families τ° and τ^{*} defined by:

$$\tau^{\circ} = \{\left\langle A^T, [A^I, A^I], A^F\right\rangle \in MBJNS(X) : \mathcal{A} = \left\langle A^T, A^I, A^F\right\rangle \in \tau\}$$

and

$$\tau^{*} = \{\left\langle A^T, [A^{I,c}, A^{I,c}], A^F\right\rangle \in MBJNS(X) : \mathcal{A} \in \tau\},$$

where $A^{I,c}$ denotes the complement of the fuzzy set A^I. Then clearly $\tau^{\circ} \in MBJNT^{\circ}(X)$ and $\tau^{*} \in MBJNT^{\circ}(X)$. Moreover, it is well-known (Example 4.1, [31]) that every fuzzy topology is a neutrosophic topolog.

(7) Let (X, τ) be an intuitionistic fuzzy topological space introduced by Coker [21] and consider the family $\tau_{_{I,N}}$ of neutrosophic sets in X defined by:

$$\tau_{_{I,N}} = \{\left\langle A^{\in}, [A^{\in}, A^{\in}], A^{\notin}\right\rangle \in NS(X) : \bar{A} = (A^{\in}, A^{\notin}) \in \tau\}.$$

Then clearly $\tau_{I,N}$ is a neutrosophic topology on X. On the other hand, let $\tau_{I,IV}$ be the family of interval-valued fuzzy sets in X given by:

$$\tau_{I,IV} = \{[A^{\in}, A^{\notin,c}] \in IVFS(X) : \bar{A} \in \tau\},$$

where $A^{\notin,c}$ denotes the complement of the fuzzy set $A^{\notin}$. Then we can easily check that $\tau_{I,IV}$ is an interval-valued fuzzy topology on X.

(8) Let (X, τ) be an interval-valued fuzzy topological space and consider the family $\tau_{IV,N}$ defined by:

$$\tau_{IV,N} = \{\left\langle A^{-}, \widetilde{A}, A^{-,c} \right\rangle \in NS(X) : \bar{A} = (A^{\in}, A^{\notin}) \in \tau\},$$

where $A^{-,c}$ denotes the complement of the fuzzy set A^{-}. Then clearly $\tau_{IV,N}$ is a neutrosophic topology on X. On the other hand, let $\tau_{IV,I}$ be the family of intuitionistic fuzzy sets in X given by:

$$\tau_{IV,I} = \{(A^{-}, A^{+,c}) \in IFS(X) : \widetilde{A} \in \tau\}.$$

Then it is clear that $\tau_{I,IV}$ is an intuitionistic fuzzy topology on X.

Let T [resp. FT, IF, $IVFT$, NT and $MBJNT$] be a classical [resp. a fuzzy, an intuitionistic fuzzy, an interval-valued fuzzy, a neutrosophic and an MBJ-neutrosophic] topology on a set X. Then from (2)–(8) and Proposition 4.5, we have the following among T, FT, IFT, $IVFT$, NT and MBJNT:

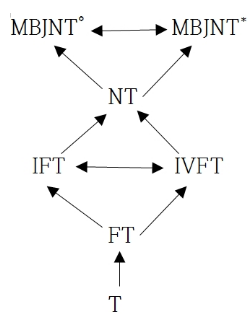

FIGURE 1. Implications among the above topologies

The following is an immediate consequence of Definition 4.2 and Proposition 3.7 (12).

Proposition 4.5. *Let X be a nonempty set. If $\tau \in MBJNT^{\circ}(X)$, then the family*

$$\tau^{*} = \{\left\langle M_A, \widetilde{B}_A^c, J_A \right\rangle \in MBJNS(X) : \mathcal{A} \in \tau\} \in MBJNT^{*}(X).$$

Also the converse holds.

The following is an immediate consequence of Definition 4.2.

Proposition 4.6. (1) *Let* $\tau \in MBJNT^{\circ}(X)$ *or* $\tau \in MBJNT^{\circ}_{L}(X)$ *and let* $\tau^c = \{\mathcal{A}^c : \mathcal{A} \in \tau\}$. *Then* τ^c *satisfies the following conditions:*

(MBJNC$_0$) *for each* $\widetilde{\widetilde{a}} \in I \times [I] \times I$, $C_{\widetilde{\widetilde{a}}} \in \tau^c$,

($\circ$-MBJNC$_1$) $\ddot{\emptyset}$, $\ddot{X}$,

($\circ$-MBJNC$_2$) $\mathcal{A} \sqcup \mathcal{B} \in \tau^c$ *for any* $\mathcal{A}$, $\mathcal{B} \in \tau^c$,

($\circ$-MBJNC$_3$) $\sqcap_{j \in J} \mathcal{A}_j \in \tau^c$ *for each* $(\mathcal{A}_j)_{j \in J} \subset \tau^c$.

(2) *Let* $\tau \in MBJNT^{*}(X)$ *or* $\tau \in MBJNT^{*}_{L}(X)$ *and let* $\tau^c = \{\mathcal{A}^c : \mathcal{A} \in \tau\}$. *Then* τ^c *satisfies the following conditions:*

(MBJNC$_0$) *for each* $\widetilde{\widetilde{a}} \in I \times [I] \times I$, $C_{\widetilde{\widetilde{a}}} \in \tau^c$,

($*$-MBJNC$_1$) $\dot{\emptyset}$, $\dot{X}$,

($*$-MBJNC$_2$) $\mathcal{A} \Cup \mathcal{B} \in \tau^c$ *for any* $\mathcal{A}$, $\mathcal{B} \in \tau^c$,

($*$-MBJNC$_3$) $\Cap_{j \in J} \mathcal{A}_j \in \tau^c$ *for each* $(\mathcal{A}_j)_{j \in J} \subset \tau^c$.

In this case, τ^c will be called a $\circ$-*MBJ-neutrosophic cotopology* [resp. $*$-*MBJ-neutrosophic cotopology*] on X.

Now we will deal with neighborhood structures based on MBJ-neutrosophic sets.

Definition 4.7. Let (X, τ) be a $\circ$-MBJ-neutrosophic topological space or a $*$-MBJ-neutrosophic topological space, let $\mathcal{A} \in MBJN(X)$ and let $x^{\circ}_{\widetilde{\widetilde{a}}}$, $x^{*}_{\widetilde{\widetilde{a}}} \in MBJN_P(X)$.

(i) $\mathcal{A}$ is called a $\circ$-*MBJ-neutrosophic neighborhood* (briefly, $\circ$-MBJNN) of $x^{\circ}_{\widetilde{\widetilde{a}}}$, if there is $\mathcal{B} \in \tau$ such that $x^{\circ}_{\widetilde{\widetilde{a}}} \in \mathcal{B} \sqsubset \mathcal{A}$. A $\circ$-MBJNN $\mathcal{A}$ is said to be $\circ$-*MBJ-neutrosophic open*, if $\mathcal{A} \in \tau$. The collection of all $\circ$-MBJNNs of $x^{\circ}_{\widetilde{\widetilde{a}}}$ is called the *system of* $\circ$-*MBJ-neutrosophic neighborhoods of* $x^{\circ}_{\widetilde{\widetilde{a}}}$ and will be denoted by $\mathcal{N}(x^{\circ}_{\widetilde{\widetilde{a}}})$.

(ii) $\mathcal{A}$ is called a $\circ$-*MBJ-neutrosophic Q-neighborhood* (briefly, $\circ$-MBJNQN) of $x^{\circ}_{\widetilde{\widetilde{a}}}$, if there is $\mathcal{B} \in \tau$ such that $x^{\circ}_{\widetilde{\widetilde{a}}} q \mathcal{B} \sqsubset \mathcal{A}$. The family of all $\circ$-MBJNQNs of $x^{\circ}_{\widetilde{\widetilde{a}}}$ is called the *system of* $\circ$-*MBJ-neutrosophic Q-neighborhoods of* $x^{\circ}_{\widetilde{\widetilde{a}}}$ and will be denoted by $\mathcal{N}_Q(x^{\circ}_{\widetilde{\widetilde{a}}})$.

(iii) $\mathcal{A}$ is called a $*$-*MBJ-neutrosophic neighborhood* (briefly, $*$-MBJNN) of $x^{*}_{\widetilde{\widetilde{a}}}$, if there is $\mathcal{B} \in \tau$ such that $x^{*}_{\widetilde{\widetilde{a}}} \in \mathcal{B} \Subset \mathcal{A}$. A $*$-MBJNN $\mathcal{A}$ is said to be $*$-*MBJ-neutrosophic open*, if $\mathcal{A} \in \tau$. The collection of all $*$-MBJNNs of $x^{*}_{\widetilde{\widetilde{a}}}$ is called the *system of* $*$-*MBJ-neutrosophic neighborhoods of* $x^{*}_{\widetilde{\widetilde{a}}}$ and will be denoted by $\mathcal{N}(x^{*}_{\widetilde{\widetilde{a}}})$.

(iv) $\mathcal{A}$ is called a $*$-*MBJ-neutrosophic Q-neighborhood* (briefly, $*$-MBJNQN) of $x^{*}_{\widetilde{\widetilde{a}}}$, if there is $\mathcal{B} \in \tau$ such that $x^{*}_{\widetilde{\widetilde{a}}} q \mathcal{B} \Subset \mathcal{A}$. The family of all $*$-MBJNQNs of $x^{*}_{\widetilde{\widetilde{a}}}$ is called the *system of* $*$-*MBJ-neutrosophic Q-neighborhoods of* $x^{*}_{\widetilde{\widetilde{a}}}$ and will be denoted by $\mathcal{N}_Q(x^{*}_{\widetilde{\widetilde{a}}})$.

Example 4.8. Let (X, τ) and (X, η) be the $\circ$-MBJ-neutrosophic topological space and $*$-MBJ-neutrosophic topological space given Example 4.3. Consider four MBJ-neutrosophic points and four MBJ-neutrosophic sets in X given by:

$$x^{\circ}_{\langle 0.3,[0.4,0.6],0.5\rangle},\ x^{*}_{\langle 0.4,[0.5,0.8],0.6\rangle},\ C_{\langle 0.4,[0.5,0.7],0.3\rangle},\ C_{\langle 0.6,[0.6,0.9],0.7\rangle}$$

and

$$x^{\circ}_{\langle 0.7,[0.9,0.9],0.5\rangle},\ x^{*}_{\langle 0.6,[0.2,0.4],0.3\rangle},\ C_{\langle 0.6,[0.3,0.5],0.7\rangle},\ C_{\langle 0.6,[0.3,0.7],0.5\rangle}.$$

Then we can easily check that

$$C_{\langle 0.4,[0.5,0.7],0.3\rangle} \in \mathcal{N}(x^{\circ}_{\langle 0.3,[0.4,0.6],0.5\rangle}),\ C_{\langle 0.6,[0.6,0.9],0.7\rangle} \in \mathcal{N}(x^{*}_{\langle 0.4,[0.5,0.8],0.6\rangle})$$

and

$$C_{\langle 0.6,[0.3,0.5],0.7\rangle} \in \mathcal{N}_Q(x^{\circ}_{\langle 0.7,[0.9,0.9],0.5\rangle}),\ C_{\langle 0.6,[0.3,0.7],0.5\rangle} \in \mathcal{N}_Q(x^{*}_{\langle 0.6,[0.2,0.4],0.3\rangle}).$$

Let $\widetilde{N}(x_{\tilde{a}})$ [resp. $\widetilde{N}_Q(x_{\tilde{a}})$] be the set of all interval-valued fuzzy neighborhoods [resp. Q-neighborhoods] of an interval-valued fuzzy point $x_{\tilde{a}}$ (See [25] [resp. [33]) and let $N(x_a)$ [resp. $N_Q(x_a)$] denote the set of all fuzzy neighborhoods [resp. Q-neighborhoods] of a fuzzy point x_a (See [16]).

Remark 4.9. From Remarks 3.14 and 4.4 (5), Definitions 3.13 and 4.7, we can easily check that the following holds:

$$\mathcal{A} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}}) \Longleftrightarrow M_A \in N_{\tau_M}(x_a),\ \widetilde{B}_A \in \widetilde{N}_{\tau_{\widetilde{B}}}(x_{\tilde{a}}),\ J_A \in N^{\circ}_{\tau_J}(x^{\circ}_{\bar{a}}),$$

$$\mathcal{A} \in \mathcal{N}(x^{*}_{\tilde{\tilde{a}}}) \Longleftrightarrow M_A \in N_{\tau_M}(x_a),\ \widetilde{B}_A \in \widetilde{N}^{*}_{\tau_{\widetilde{B}}}(x^{*}_{\tilde{a}}),\ J_A \in N^{\circ}_{\tau_J}(x^{\circ}_{\bar{a}}),$$

$$\mathcal{A} \in \mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}}) \Longleftrightarrow M_A \in N_{\tau_M,Q}(x_a),\ \widetilde{B}_A \in \widetilde{N}_{\tau_{\widetilde{B}},Q}(x_{\tilde{a}}),\ J_A \in N^{\circ}_{\tau_J,Q}(x^{\circ}_{\bar{a}}),$$

$$\mathcal{A} \in \mathcal{N}_Q(x^{*}_{\tilde{\tilde{a}}}) \Longleftrightarrow M_A \in N_{\tau_M,Q}(x_a),\ \widetilde{B}_A \in \widetilde{N}^{*}_{\tau_{\widetilde{B}},Q}(x^{*}_{\tilde{a}}),\ J_A \in N^{\circ}_{\tau_J,Q}(x^{\circ}_{\bar{a}}),$$

where $J_A \in N^{\circ}_{\tau_J}(x^{\circ}_{\bar{a}})$ if and only if there is $J_B \in \tau_J$ such that $x^{\circ}_{\bar{a}} \in J_B \supset J_A$, $\widetilde{B}_A \in \widetilde{N}^{*}_{\tau_{\widetilde{B}}}(x^{*}_{\tilde{a}})$ if and only if there is $\widetilde{B}_B \in \tau_{\widetilde{B}}$ such that $x^{*}_{\tilde{a}} \in \widetilde{B}_B \supset \widetilde{B}_A$, $J_A \in N^{\circ}_{\tau_J,Q}(x^{\circ}_{\bar{a}})$ if and only if there is $J_B \in \tau_J$ such that $x_{\bar{a}} q^{\circ} J_B \supset J_A$ and $\widetilde{B}_A \in \widetilde{N}^{*}_{\tau_{\widetilde{B}},Q}(x^{*}_{\tilde{a}})$ if and only if there is $\widetilde{B}_B \in \tau_{\widetilde{B}}$ such that $x_{\tilde{a}} q^{*} \widetilde{B}_B \supset \widetilde{B}_A$.

Theorem 4.10. *Let (X, τ) be a $\circ$-MBJ-neutrosophic topological space or a $*$-MBJ-neutrosophic topological space and let $\mathcal{A} \in MBJN(X)$.*

(1) *$\mathcal{A} \in \tau$ if and only if $\mathcal{A} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$ for each $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{A}$.*

(2) *$\mathcal{A} \in \tau$ if and only if $\mathcal{A} \in \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$ for each $x^{*}_{\tilde{\tilde{a}}} \in \mathcal{A}$.*

Proof. (1) By Remark 3.14 (1), it is clear that $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{A}$ if and only if $x_a \in M_A$, $x_{\tilde{a}} \in \widetilde{B}_A$, $x^{\circ}_{\bar{a}} \in J_A$. From Proposition 1.8 in [18] and Theorem 7 in [25], we have

$$M_A \in \tau_M \iff M_A \in N_{\tau_M}(x_a) \text{ for each } x_a \in M_A$$

and

$$\widetilde{B}_A \in \tau_{\widetilde{B}} \iff \widetilde{B}_A \in N_{\tau_{\widetilde{B}}}(x_{\tilde{a}}) \text{ for each } x_{\tilde{a}} \in \widetilde{B}_A.$$

It is sufficient to prove that $J_A \in \tau_J \iff J_A \in N^{\circ}_{\tau_J}(x^{\circ}_{\bar{a}})$ for each $x^{\circ}_{\bar{a}} \in J_A$.

Suppose $J_A \in \tau_J$ and let $x^{\circ}_{\bar{a}} \in J_A$. Then clearly, $J_A \in N^{\circ}_{\tau_J}(x^{\circ}_{\bar{a}})$. Conversely, suppose the necessary condition holds. Then there is $(J_B)_{x^{\circ}_{\bar{a}}} \in \tau_J$ such that

$$x^{\circ}_{\bar{a}} \in (J_B)_{x^{\circ}_{\bar{a}}} \in \tau_J \supset J_A.$$

Thus $J_A = \bigcap_{x^{\circ}_{\bar{a}} \in J_A} (J_B)_{x^{\circ}_{\bar{a}}}$. By Remark 4.4 (5), $(J_B)_{x^{\circ}_{\bar{a}}} \in \tau_J \in FCT(X)$ for each $x^{\circ}_{\bar{a}} \in J_A$. So $J_A \in \tau_J$. Hence the result holds.

(2) From the procedure of the proof of (1), we get

$$M_A \in \tau_M \iff M_A \in N_{\tau_M}(x_a) \text{ for each } x_a \in M_A$$

and

$$J_A \in \tau_J \iff J_A \in N^{\circ}_{\tau_J}(x^{\circ}_{\bar{a}}) \text{ for each } x^{\circ}_{\bar{a}} \in J_A.$$

It is sufficient to prove that $\widetilde{B}_A \in \tau_{\widetilde{B}} \iff \widetilde{B}_A \in N_{\tau^*_{\widetilde{B}}}(x^*_{\tilde{a}})$ for each $x^*_{\tilde{a}} \in \widetilde{B}_A$.

Suppose $\widetilde{B}_A \in \tau_{\widetilde{B}}$ and let $x^*_{\tilde{a}} \in \widetilde{B}_A$. Then clearly, $\widetilde{B}_A \in N^*_{\tau_{\widetilde{B}}}(x^*_{\tilde{a}})$. Conversely, suppose the necessary condition holds. Then there is $(\widetilde{B}_B)_{x^*_{\tilde{a}}} \in \tau_{\widetilde{B}}$ such that

$$x^*_{\tilde{a}} \in (\widetilde{B}_B)_{x^*_{\tilde{a}}} \supset \widetilde{B}_A.$$

Thus $\widetilde{B}_A = \bigcap_{x^*_{\tilde{a}} \in \widetilde{B}_A} (\widetilde{B}_B)_{x^*_{\tilde{a}}}$. By Remark 4.4 (5), $(\widetilde{B}_B)_{x^*_{\tilde{a}}} \in \tau_{\widetilde{B}} \in IVFCT(X)$ for each $x^*_{\tilde{a}} \in \widetilde{B}_A$. So $\widetilde{B}_A \in \tau_{\widetilde{B}}$. Hence the result holds. □

Theorem 4.11. *Let (X, τ) be a ∘-MBJ-neutrosophic topological space or a ∗-MBJ-neutrosophic topological space and let $\mathcal{A} \in MBJNS(X)$.*

(1) $\mathcal{A} \in \tau$ if and only if $\mathcal{A} \in \mathcal{N}_Q(x^{\circ}_{\tilde{\bar{a}}})$ for each $x^{\circ}_{\tilde{\bar{a}}} \in MBJN_P(X)$ such that $0 < a < M_A(x)$, $[0,0] < \tilde{a} < \widetilde{B}_A(x)$, $J_A(x) < \bar{a} < 1$ and $x^{\circ}_{\tilde{\bar{a}}} q \mathcal{A}$.

(2) $\mathcal{A} \in \tau$ if and only if $\mathcal{A} \in \mathcal{N}_Q(x^*_{\tilde{\bar{a}}})$ for each $x^{\circ}_{\tilde{\bar{a}}} \in MBJN_P(X)$ such that $0 < a < M_A(x)$, $\widetilde{B}_A(x) < \tilde{a} < [1,1]$, $J_A(x) < \bar{a} < 1$ and $x^*_{\tilde{\bar{a}}} q \mathcal{A}$.

Proof. (1) From Remark 4.9, we have

$$\mathcal{A} \in \mathcal{N}_Q(x^{\circ}_{\tilde{\bar{a}}}) \iff M_A \in N_{\tau_M, Q}(x_a),\ \widetilde{B}_A \in \widetilde{N}_{\tau_{\widetilde{B}}, Q}(x_{\tilde{a}}),\ J_A \in N^{\circ}_{\tau_J, Q}(x_{\bar{a}}).$$

From Theorem 3.2 [19] and Lemma 4.12 [33], it is obvious that

$$M_A \in \tau_M \iff M_A \in N_{\tau_M, Q}(x_a)$$

for each $x_a \in F_P(X)$ such $0 < a < M_A(x)$ and $x_a q M_A$ and

$$\widetilde{B}_A \in \tau_{\widetilde{B}} \iff \widetilde{B}_A \in \widetilde{N}_{\tau_{\widetilde{B}}, Q}(x_a)$$

for each $x_{\tilde{a}} \in IVF_P(X)$ such $[0,0] < \tilde{a} < \widetilde{M}_A(x)$ and $x_{\tilde{a}} q \widetilde{M}_A$. It is sufficient to show that

$$J_A \in \tau_J \iff J_A \in N^{\circ}_{\tau_J, Q}(x_{\bar{a}}) \quad (1)$$

for each $x_{\bar{a}} \in F_P(X)$ such that $J_A(x) < \bar{a} < 1$ and $x_{\bar{a}} q^\circ J_A$.

Suppose $J_A \in \tau_J$ and let $x_{\bar{a}} \in F_P(X)$ such that $J_A(x) < \bar{a} < 1$ and $x_{\bar{a}} q^\circ J_A$. Then clearly, $J_A \in N^\circ_{\tau_J,Q}(x_{\bar{a}})$. Conversely, suppose the necessary condition holds. Since $J_A(x) < \bar{a} < 1$, by Lemma 3.21, $x_{\bar{a}^c} q^\circ J_A$. Then by the hypothesis, $J_A \in N^\circ_{\tau_J,Q}(x_{\bar{a}^c})$. Thus there is $U_{x_{\bar{a}^c}} \in \tau_J$ such that $x_{\bar{a}^c} q^\circ U_{x_{\bar{a}^c}} \supset J_A$. Since $x_{\bar{a}} q^\circ J_A$, $x_{\bar{a}^c} \in J_A$. So $J_A = \bigcap_{x_{\bar{a}^c} \in J_A} U_{x_{\bar{a}^c}}$. Since τ_J is a fuzzy cotopology on X and $U_{x_{\bar{a}^c}} \in \tau_J$, $J_A \in \tau_J$. Hence the result holds.

(2) From Remark 4.9, we get

$$\mathcal{A} \in \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}}) \Longleftrightarrow M_A \in N_{\tau_M,Q}(x_a),\ \widetilde{B}_A \in \widetilde{N}^*_{\tau_{\widetilde{B}},Q}(x_{\tilde{a}}),\ J_A \in N^\circ_{\tau_J,Q}(x_{\bar{a}}).$$

From (1), it is clear that

$$M_A \in \tau_M \Longleftrightarrow M_A \in N_{\tau_M,Q}(x_a)$$

for each $x_a \in F_P(X)$ such $0 < a < M_A(x)$ and $x_a q M_A$ and

$$J_A \in \tau_J \Longleftrightarrow J_A \in N^\circ_{\tau_J,Q}(x_{\bar{a}})$$

for each $x_{\bar{a}} \in F_P(X)$ such that $J_A(x) < \bar{a} < 1$ and $x_{\bar{a}} q^\circ J_A$. It is sufficient to show that

$$\widetilde{B}_A \in \tau_{\widetilde{B}} \Longleftrightarrow \widetilde{B}_A \in \widetilde{N}^*_{\tau_{\widetilde{B}},Q}(x_{\tilde{a}}) \tag{2}$$

for each $x_{\tilde{a}} \in IVF_P(X)$ such that $\widetilde{B}_A(x) < \tilde{a} < [1,1]$ and $x_{\tilde{a}} q^\circ \widetilde{B}_A$.

Suppose $\widetilde{B}_A \in \tau_{\widetilde{B}}$ and let $x_{\tilde{a}} \in IVF_P(X)$ such that $\widetilde{B}_A(x) < \tilde{a} < [1,1]$ and $x_{\tilde{a}} q^* \widetilde{B}_A$. Then clearly, $\widetilde{B}_A \in N^*_{\tau_{\widetilde{B}},Q}(x_{\tilde{a}})$. Conversely, suppose the necessary condition holds. Since $\widetilde{B}_A(x) < \tilde{a} < [1,1]$, by Lemma 3.22, $x_{\tilde{a}^c} q^* \widetilde{B}_A$. Then by the hypothesis, $\widetilde{B}_A \in N^*_{\tau_{\widetilde{B}},Q}(x_{\tilde{a}^c})$. Thus there is $\widetilde{U}_{x_{\tilde{a}^c}} \in \tau_{\widetilde{B}}$ such that $x_{\tilde{a}^c} q^* \widetilde{U}_{x_{\tilde{a}^c}} \supset \widetilde{B}_A$. Since $x_{\tilde{a}} q^* \widetilde{B}_A$, $x_{\tilde{a}^c} \in \widetilde{B}_A$. So $\widetilde{B}_A = \bigcap_{x_{\tilde{a}^c} \in \widetilde{B}_A} \widetilde{U}_{x_{\tilde{a}^c}}$. Since $\tau_{\widetilde{B}}$ is an interval-valued fuzzy cotopology on X and $\widetilde{U}_{x_{\tilde{a}^c}} \in \tau_{\widetilde{B}}$, $\widetilde{B}_A \in \tau_{\widetilde{B}}$. Hence the result holds. □

Lemma 4.12. *Let (X, τ) be an interval-valued fuzzy topological space and let $x_{\tilde{a}} \in IVF_P(X)$.*

(1) If $\widetilde{A} \in \widetilde{N}(x_{\tilde{a}})$, then $x_{\tilde{a}} \in \widetilde{A}$.

(2) If $\widetilde{A},\ \widetilde{B} \in \widetilde{N}(x_{\tilde{a}})$, then $\widetilde{A} \cap \widetilde{B} \in \widetilde{N}(x_{\tilde{a}})$.

(3) If $\widetilde{A} \in \widetilde{N}(x_{\tilde{a}})$ and $\widetilde{A} \subset \widetilde{B}$, then $\widetilde{B} \in \widetilde{N}(x_{\tilde{a}})$.

(4) If $\widetilde{A} \in \widetilde{N}(x_{\tilde{a}})$, then there is $\widetilde{B} \in \widetilde{N}(x_{\tilde{a}})$ such that $\widetilde{B} \subset \widetilde{A}$ and $\widetilde{B} \in \widetilde{N}(y_{\tilde{b}})$ for $y_{\tilde{b}} \in \widetilde{B}$.

Conversely, if for each $x_{\tilde{a}} \in IVF_P(X)$, $\widetilde{N}_{x_{\tilde{a}}}$ satisfies the conditions (1), (2) and (3), then the family τ of IVFSs in X given by:

$$\tau = \{\widetilde{A} \in IVFS(X) : \widetilde{A} \in \widetilde{N}_{x_{\tilde{a}}} \text{ for each } x_{\tilde{a}} \in \widetilde{A}\}$$

is an interval-valued fuzzy topology on X. Furthermore, if $\widetilde{N}_{x_{\tilde{a}}}$ satisfies the condition (4), then $\widetilde{N}_{x_{\tilde{a}}}$ is exactly the system of interval-valued fuzzy neidhborhood of $x_{\tilde{a}}$ with respect to τ, i.e., $\widetilde{N}_{x_{\tilde{a}}} = \widetilde{N}(x_{\tilde{a}})$.

Proof. The proof is almost similar to a classical case (See [36]). □

From Remark 4.9, Proposition 2.2 [16] and Lemma 4.12, we have the followings.

Theorem 4.13. *Let (X,τ) be a $\circ$-MBJ-neutrosophic topological space and let $x^{\circ}_{\tilde{\tilde{a}}} \in MBJN_P(X)$.*

(1) *If $\mathcal{A} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$, then $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{A}$.*

(2) *If $\mathcal{A},\ \mathcal{B} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$, then $\mathcal{A} \sqcap \mathcal{B} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$.*

(3) *If $\mathcal{A} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$ and $\mathcal{A} \sqsubset \mathcal{B}$, then $\mathcal{B} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$.*

(4) *If $\mathcal{A} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$, then there is $\mathcal{B} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$ such that $\mathcal{B} \sqsubset \mathcal{A}$ and $\mathcal{B} \in \mathcal{N}(y^{\circ}_{\tilde{\tilde{a}}})$ for $y^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B}$.*

Conversely, if for each $x^{\circ}_{\tilde{\tilde{a}}} \in MBJN_P(X)$, $\mathcal{N}_{x^{\circ}_{\tilde{\tilde{a}}}}$ satisfies the conditions (1), (2) and (3), then the family τ of cubic sets in X given by:

$$\tau = \{\mathcal{A} \in MBJN(X) : \mathcal{A} \in \mathcal{N}_{x^{\circ}_{\tilde{\tilde{a}}}} \text{ for each } x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{A}\}$$

is a $\circ$-MBJ-neutrosophic topology on X. Furthermore, if $\mathcal{N}^{\circ}_{x_{\tilde{\tilde{a}}}}$ satisfies the condition (4), then $\mathcal{N}_{x^{\circ}_{\tilde{\tilde{a}}}}$ is exactly the system of $\circ$-MBJ-neutrosophic neidhborhood of $x_{\tilde{\tilde{a}}^{\circ}}$ with respect to τ, i.e., $\mathcal{N}_{x^{\circ}_{\tilde{\tilde{a}}}} = \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$.

Theorem 4.14. *Let (X,τ) be a $*$-MBJ-neutrosophic topological space and let $x^{*}_{\tilde{\tilde{a}}} \in MBJN_P(X)$.*

(1) *If $\mathcal{A} \in \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$, then $x^{*}_{\tilde{\tilde{a}}} \in \mathcal{A}$.*

(2) *If $\mathcal{A},\ \mathcal{B} \in \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$, then $\mathcal{A} \Cap \mathcal{B} \in \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$.*

(3) *If $\mathcal{A} \in \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$ and $\mathcal{A} \Subset \mathcal{B}$, then $\mathcal{B} \in \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$.*

(4) *If $\mathcal{A} \in \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$, then there is $\mathcal{B} \in \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$ such that $\mathcal{B} \Subset \mathcal{A}$ and $\mathcal{B} \in \mathcal{N}(y^{*}_{\tilde{\tilde{a}}})$ for $y^{*}_{\tilde{\tilde{a}}} \in \mathcal{B}$.*

Conversely, if for each $x^{*}_{\tilde{\tilde{a}}} \in MBJN_P(X)$, $\mathcal{N}_{x^{*}_{\tilde{\tilde{a}}}}$ satisfies the conditions (1), (2) and (3), then the family τ of cubic sets in X given by:

$$\tau = \{\mathcal{A} \in MBJN(X) : \mathcal{A} \in \mathcal{N}_{x^{*}_{\tilde{\tilde{a}}}} \text{ for each } x^{*}_{\tilde{\tilde{a}}} \in \mathcal{A}\}$$

is a $*$-MBJ-neutrosophic topology on X. Furthermore, if $\mathcal{N}^{*}_{x_{\tilde{\tilde{a}}}}$ satisfies the condition (4), then $\mathcal{N}_{x^{*}_{\tilde{\tilde{a}}}}$ is exactly the system of $*$-MBJ-neutrosophic neidhborhood of $x_{\tilde{\tilde{a}}^{*}}$ with respect to τ, i.e., $\mathcal{N}_{x^{*}_{\tilde{\tilde{a}}}} = \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$.

Lemma 4.15. *Let (X,τ) be a fuzzy cotopological space, let $x_a \in F_P(X)$ and let $N^{\circ}_{\tau,Q}(x_a)$ be the family of fuzzy sets in X defined as follows: for each $A \in I^X$,*

$$A \in N^{\circ}_{\tau,Q}(x_a) \textit{ if and only if there is } B \in \tau \textit{ such that } x_a q^{\circ} B \supset A.$$

(1) *If $A \in N^{\circ}_{\tau,Q}(x_a)$, then $x_a q^{\circ} A$.*

(2) *If $A,\ B \in N^{\circ}_{\tau,Q}(x_a)$, then $A \cup B \in N^{\circ}_{\tau,Q}(x_a)$.*

(3) *If $A \in N^{\circ}_{\tau,Q}(x_a)$ and $B \subset A$, then $B \in N^{\circ}_{\tau,Q}(x_a)$.*

(4) *If* $A \in N^{\circ}_{\tau,Q}(x_a)$*, then there is* $B \in N^{\circ}_{\tau,Q}(x_a)$ *such that* $A \subset B$ *and* $B \in N^{\circ}_{\tau,Q}(y_b)$ *for* $y_b q^{\circ} B$*.*

Conversely, if for each $x_a \in F_P(X)$*,* N°_{Q,x_a} *satisfies the conditions (1), (2) and (3), then the family* τ *of fuzzy sets in* X *given by:*

$$\tau = \{A \in I^X : A \in N^{\circ}_{Q,x_a} \text{ for each } x_a q^{\circ} A\}$$

is a fuzzy cotopology on X*. Furthermore, if* N°_{Q,x_a} *satisfies the condition (4), then* N°_{Q,x_a} *is exactly the system of fuzzy* $\circ$*-Q-neighborhoods of* x_a *with respect to* τ*, i.e.,* $N^{\circ}_{Q,x_a} = N^{\circ}_{\tau,Q}(x_a)$*.*

Proof. (1) Suppose $A \in N^{\circ}_{\tau,Q}(x_a)$. Then there is $U \in \tau$ such that $x_a q^{\circ} U \supset A$. Thus $a < U^c(x) \leq A^c(x)$. So $x_a q^{\circ} A$.

(2) Suppose $A,\ B \in N^{\circ}_{\tau,Q}(x_a)$. Then there are $U,\ V \in \tau$ such that $x_a q^{\circ} U \supset A$ and $x_a q^{\circ} V \supset B$. Thus $a < U^c(x) \leq A^c(x)$ and $a < V^c(x) \leq B^c(x)$. So we get

$$a < U^c(x) \wedge V^c(x) \leq A^c(x) \wedge B^c(x) = (U \cup V)^c(x) \leq (A \cup B)^c(x).$$

Hence $x_a q^{\circ} U \cup V \supset A \cup B$ and $U \cup V \in \tau$. Therefore $A \cup B \in N^{\circ}_{\tau,Q}(x_a)$.

(3) $A \in N^{\circ}_{\tau,Q}(x_a)$ and $B \subset A$. Then there is $U \in \tau$ such that $x_a q^{\circ} U \supset A$. Thus $a < U^c(x) \leq A^c(x) \leq B^c(x)$. So $x_a q^{\circ} U \supset B$. Hence $B \in N^{\circ}_{\tau,Q}(x_a)$.

(4) Suppose $A \in N^{\circ}_{\tau,Q}(x_a)$. Then there is $B \in \tau$ such that $x_a q^{\circ} B \supset A$. Since $B \supset B$, $B \in N^{\circ}_{\tau,Q}(x_a)$ and moreover, $B \in N^{\circ}_{\tau,Q}(y_b)$ for each $y_b q^{\circ} B$.

Conversely, suppose N_{Q,x_a} satisfies the conditions (1), (2) and (3) for each $x_a \in F_P(X)$. From the definition of τ, it is clear that $\mathbf{0},\ \mathbf{1} \in \tau$. Now let $A,\ B \in \tau$ and let $x_a q^{\circ}(A \cup B)$. Then by Lemma 3.19 (2), $x_a q^{\circ} A$ and $x_a q_{\circ} B$. So by the definition of τ, $A \in N^{\circ}_{Q,x_a}$ and $B \in N^{\circ}_{Q,x_a}$. By the condition (2), $A \cup B \in N_{Q,x_a}$. Hence $A \cup B \in \tau$. Finally, let $(A_j)_{j \in J} \subset \tau$, let $A = \bigcap_{j \in J} A_j$ and let $x_a q^{\circ} A$. By Lemma 3.19 (1), there is $j \in J$ such that $x_a q^{\circ} A_j$. Since $A_j \in \tau$, $A_j \in N^{\circ}_{Q,x_a}$. Since $A_j \supset A$, by the condition (3), $A \in \tau$, i.e., $\bigcap_{j \in J} A_j \in \tau$. Therefore τ is a fuzzy cotopology on X.

Now suppose N°_{Q,x_a} satisfies the conditions (4). Then we can easily prove similarly to a classical case that $N^{\circ}_{Q,x_a} = N^{\circ}_{Q}(x_a)$. □

From Remarks 4.4 (5) and 4.9, Proposition 2.2 [16], Lemma 4.18 [33] and Lemma 4.15, we obtain the following.

Theorem 4.16. *Let* (X, τ) *be a* $\circ$*-MBJ-neutrosophic topological space and let* $x^{\circ}_{\tilde{\tilde{a}}} \in MBJN_P(X)$*.*

(1) *If* $\mathcal{A} \in \mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$*, then* $x^{\circ}_{\tilde{\tilde{a}}} q \mathcal{A}$*.*

(2) *If* $\mathcal{A},\ \mathcal{B} \in \mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$*, then* $\mathcal{A} \sqcap \mathcal{B} \in \mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$*.*

(3) *If* $\mathcal{A} \in \mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$ *and* $\mathcal{A} \sqsubset \mathcal{B}$*, then* $\mathcal{B} \in \mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$*.*

(4) *If* $\mathcal{A} \in \mathcal{N}_Q(x^\circ_{\tilde{\tilde{a}}})$, *then there is* $\mathcal{B} \in \mathcal{N}_Q(x^\circ_{\tilde{\tilde{a}}})$ *such that* $\mathcal{B} \sqsubset \mathcal{A}$ *and* $\mathcal{B} \in \mathcal{N}_Q(y^\circ_{\tilde{\tilde{a}}})$ *for* $y^\circ_{\tilde{\tilde{a}}} q\mathcal{B}$.

Conversely, if for each $x^\circ_{\tilde{\tilde{a}}} \in MBJN_P(X)$, $\mathcal{N}_{Q,x^\circ_{\tilde{\tilde{a}}}}$ *satisfies the conditions (1), (2) and (3), then the family* τ *of MBJ-neutrosophic sets in* X *given by:*

$$\tau = \{\mathcal{A} \in MBJN(X) : \mathcal{A} \in \mathcal{N}_{Q,x^\circ_{\tilde{\tilde{a}}}} \text{ for each } x^\circ_{\tilde{\tilde{a}}} \in \mathcal{A}\}$$

is a $\circ$*-MBJ-neutrosophic topology on* X*. Furthermore, if* $\mathcal{N}^\circ_{Q,x_{\tilde{\tilde{a}}}}$ *satisfies the condition (4), then* $\mathcal{N}_{Q,x^\circ_{\tilde{\tilde{a}}}}$ *is exactly the system of* $\circ$*-MBJ-neutrosophic neidhborhood of* $x^\circ_{\tilde{\tilde{a}}}$ *with respect to* τ*, i.e.,* $\mathcal{N}_{Q,x^\circ_{\tilde{\tilde{a}}}} = \mathcal{N}_Q(x^\circ_{\tilde{\tilde{a}}})$.

Lemma 4.17. *Let* (X, τ) *be an interval-valued fuzzy cotopological space, let* $x_{\tilde{a}} \in IVF_P(X)$ *and let* $\widetilde{N}^*_{\tau,Q}(x_{\tilde{a}})$ *be the family of interval-valued fuzzy sets in* X *defined as follows: for each* $\widetilde{A} \in IVFS(X)$,

$$\widetilde{A} \in \widetilde{N}^*_{\tau,Q}(x_{\tilde{a}}) \text{ if and only if there is } \widetilde{B} \in \tau \text{ such that } x_{\tilde{a}} q^* \widetilde{B} \supset \widetilde{A}.$$

(1) *If* $\widetilde{A} \in \widetilde{N}^*_{\tau,Q}(x_{\tilde{a}})$, *then* $x_{\tilde{a}} q^* \widetilde{A}$.

(2) *If* $\widetilde{A},\ \widetilde{B} \in \widetilde{N}^*_{\tau,Q}(x_{\tilde{a}})$, *then* $\widetilde{A} \cup \widetilde{B} \in \widetilde{N}^*_{\tau,Q}(x_{\tilde{a}})$.

(3) *If* $\widetilde{A} \in \widetilde{N}^*_{\tau,Q}(x_{\tilde{a}})$ *and* $\widetilde{B} \subset \widetilde{A}$, *then* $\widetilde{B} \in \widetilde{N}^*_{\tau,Q}(x_{\tilde{a}})$.

(4) *If* $\widetilde{A} \in \widetilde{N}^*_{\tau,Q}(x_{\tilde{a}})$, *then there is* $\widetilde{B} \in \widetilde{N}^*_{\tau,Q}(x_{\tilde{a}})$ *such that* $\widetilde{A} \subset \widetilde{B}$ *and* $\widetilde{B} \in \widetilde{N}^*_{\tau,Q}(y_{\tilde{b}})$ *for* $y_{\tilde{b}} q^* \widetilde{B}$.

Conversely, if for each $x_{\tilde{a}} \in IVF_P(X)$, $\widetilde{N}^*_{Q,x_{\tilde{a}}}$ *satisfies the conditions (1), (2) and (3), then the family* τ *of interval-valued fuzzy sets in* X *given by:*

$$\tau = \{\widetilde{A} \in IVFS(X) : \widetilde{A} \in \widetilde{N}^*_{Q,x_{\tilde{a}}} \text{ for each } x_{\tilde{a}} q^* \widetilde{A}\}$$

is an interval-valued fuzzy cotopology on X*. Furthermore, if* $\widetilde{N}^*_{Q,x_{\tilde{a}}}$ *satisfies the condition (4), then* $\widetilde{N}^*_{Q,x_{\tilde{a}}}$ *is exactly the system of interval-valued fuzzy* $*$*-Q-neighborhoods of* $x_{\tilde{a}}$ *with respect to* τ*, i.e.,* $\widetilde{N}^*_{Q,x_{\tilde{a}}} = \widetilde{N}^*_{\tau,Q}(x_{\tilde{a}})$.

Proof. The proof is similar to Lemma 3.15. □

From Remarks 4.4 (5) and 4.9, Proposition 2.2 [16], Lemmas 4.15 and Lemma 4.17, we get the following.

Theorem 4.18. *Let* (X, τ) *be a* $*$*-MBJ-neutrosophic topological space and let* $x^*_{\tilde{\tilde{a}}} \in MBJN_P(X)$.

(1) *If* $\mathcal{A} \in \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$, *then* $x^*_{\tilde{\tilde{a}}} q\mathcal{A}$.

(2) *If* $\mathcal{A},\ \mathcal{B} \in \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$, *then* $\mathcal{A} \Cap \mathcal{B} \in \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$.

(3) *If* $\mathcal{A} \in \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$ *and* $\mathcal{A} \Subset \mathcal{B}$, *then* $\mathcal{B} \in \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$.

(4) *If* $\mathcal{A} \in \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$, *then there is* $\mathcal{B} \in \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$ *such that* $\mathcal{B} \Subset \mathcal{A}$ *and* $\mathcal{B} \in \mathcal{N}_Q(y^*_{\tilde{\tilde{a}}})$ *for* $y^*_{\tilde{\tilde{a}}} q\mathcal{B}$.

Conversely, if for each $x^*_{\tilde{\tilde{a}}} \in MBJN_P(X)$, $\mathcal{N}_{Q,x^*_{\tilde{\tilde{a}}}}$ *satisfies the conditions (1), (2) and (3), then the family* τ *of MBJ-neutrosophic sets in* X *given by:*

$$\tau = \{\mathcal{A} \in MBJN(X) : \mathcal{A} \in \mathcal{N}_{Q,x^*_{\tilde{\tilde{a}}}} \text{ for each } x^*_{\tilde{\tilde{a}}} \in \mathcal{A}\}$$

is a $*$*-MBJ-neutrosophic topology on* X*. Furthermore, if* $\mathcal{N}^*_{Q,x_{\tilde{\tilde{a}}}}$ *satisfies the condition (4), then* $\mathcal{N}_{Q,x^*_{\tilde{\tilde{a}}}}$ *is exactly the system of* $*$*-MBJ-neutrosophic neidhborhood of* $x^*_{\tilde{\tilde{a}}}$ *with respect to* τ*, i.e.,* $\mathcal{N}_{Q,x^*_{\tilde{\tilde{a}}}} = \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$.

5. MBJ-neutrosophic bases and MBJ-neutrosophic local bases

We introduce the concept of $\circ$-[resp. $*$-]MBJ-neutrosophic bases and $\circ$-[resp. $*$-]MBJ-neutrosophic local bases, and discuss some of their properties. Also we define $\circ$-C_I, $*$-C_I, $\circ$-C_{II} and $*$-C_{II}, and we obtain the relationships between them. Moreover, we give an Example that the converse of Proposition 5.23 does not hold.

Definition 5.1. Let (X, τ) be a $\circ$-[resp. $*$-]MBJ-neutrosophic topological space and let $\beta \subset \tau$, $\sigma \subset \tau$.

(i) β is called a $\circ$-*MBJ-neutrosophic base* (briefly, $\circ$-MBJNB) [resp. $*$-*MBJ-neutrosophic base* (briefly, $*$-MBJNB)] for τ, if for each $\mathcal{A} \in \tau, \mathcal{A} = \ddot{\emptyset}$ [resp. $\mathcal{A} = \dot{\emptyset}$] or there is $\beta' \subset \beta$ such that $\mathcal{A} = \sqcup \beta'$ [resp. $\mathcal{A} = \Cup \beta'$].

(ii) σ is called a $\circ$-*MBJ-neutrosophic subbase* (briefly, $\circ$-MBJNSB) [resp. $*$-*MBJ-neutrosophic subbase* (briefly, $*$-MBJNSB)] for τ, if the family $\beta = \{\sqcap \eta : \eta \text{ is a finite subset of } \sigma\}$ [resp. $\beta = \{\Cap \eta : \eta \text{ is a finite subset of } \sigma\}$] is a $\circ$-MBJNB [resp. $*$-MBJNB] for τ.

Now we will introduce the concepts of bases and subbases for a fuzzy cotopology and an interval-valued fuzzy cotopology.

Definition 5.2. Let τ be a fuzzy cotopology on a nonempty set X and let $\beta,\ \sigma \subset \tau$.

(i) β is called a $\circ$-*fuzzy base for* τ, if for each $A \in \tau$, $A = \mathbf{1}$ or there is $\beta' \subset \beta$ such that $A = \bigcap \beta'$.

(ii) σ is called a $\circ$-*fuzzy subbase for* τ, if the family $\beta = \{\bigcup \eta : \eta \text{ is a finite subset of } \sigma\}$ is a $\circ$-fuzzy base for τ.

Definition 5.3. Let τ be an interval-valued fuzzy cotopology on a nonempty set X and let $\beta,\ \sigma \subset \tau$.

(i) β is called a $*$-*interval-valued fuzzy base for* τ, if for each $\widetilde{A} \in \tau$, $\widetilde{A} = \widetilde{\mathbf{1}}$ or there is $\beta' \subset \beta$ such that $\widetilde{A} = \bigcap \beta'$.

(ii) σ is called a $*$-*interval-valued fuzzy subbase for* τ, if the family $\beta = \{\bigcup \eta : \eta \text{ is a finite subset of } \sigma\}$ is a $*$-fuzzy base for τ.

Example 5.4. (1) Let $X = \{x, y\}$ and consider the fuzzy sets A_i $(i = 1, 2, \cdots, 9)$ in X given by:

$$A_1(x) = 0.3,\ A_1(y) = 0.7,\ A_2(x) = 0.6,\ A_2(y) = 0.4,\ A_3(x) = 0.5,\ A_3(y) = 0.6,$$

$$A_4(x) = 0.6,\ A_4(y) = 0.7,\ A_5(x) = 0.5,\ A_5(y) = 0.7,\ A_6(x) = 0.6,\ A_6(y) = 0.6,$$

$$A_7(x) = 0.3,\ A_7(y) = 0.4,\ A_8(x) = 0.3,\ A_8(y) = 0.6,\ A_9(x) = 0.5,\ A_9(y) = 0.4.$$

Then we can easily see that the family $\tau = \{\mathbf{0}, \mathbf{1}, A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9\}$ is a fuzzy cotopology on X. Now let us two subfamilies β and σ of τ given by:

$$\beta = \{\mathbf{0}, A_1, A_2, A_3, A_4, A_5, A_6\},\ \sigma = \{\mathbf{0}, A_1, A_2, A_3\}.$$

Then we can easily check that β and σ are $\circ$-fuzzy base and $\circ$-fuzzy subbase for τ respectively.

(2) Let $X = \{x, y\}$ and consider the interval-valued fuzzy sets $\widetilde{A}_i$ $(i = 1, 2, \cdots, 9)$ in X given by:

$$\widetilde{A}_1(x) = [0.3, 0.5],\ A_1(y) = [0.7, 0.8],\ \widetilde{A}_2(x) = [0.6, 0.7],\ \widetilde{A}_2(y) = [0.4, 0.6],$$

$$\widetilde{A}_3(x) = [0.5, 0.6],\ \widetilde{A}_3(y) = [0.6, 0.7], \widetilde{A}_4(x) = [0.6, 0.7],\ \widetilde{A}_4(y) = [0.7, 0.8],$$

$$\widetilde{A}_5(x) = [0.5, 0.6],\ \widetilde{A}_5(y) = [0.7, 0.8],\ \widetilde{A}_6(x) = [0.6, 0.7],\ \widetilde{A}_6(y) = [0.6, 0.7],$$

$$\widetilde{A}_7(x) = [0.3, 0.5],\ \widetilde{A}_7(y) = [0.4, 0.6],\ \widetilde{A}_8(x) = [0.3, 0.5],\ \widetilde{A}_8(y) = [0.6, 0.7],$$

$$\widetilde{A}_9(x) = [0.5, 0.6],\ \widetilde{A}_9(y) = [0.4, 0.6].$$

Then we can easily see that the family $\tau = \{\widetilde{\mathbf{0}}, \widetilde{\mathbf{1}}, \widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_5, \widetilde{A}_6, \widetilde{A}_7, \widetilde{A}_8, \widetilde{A}_9\}$ is a fuzzy cotopology on X. Now let us two subfamilies β and σ of τ given by:

$$\beta = \{\widetilde{\mathbf{0}}, \widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_5, \widetilde{A}_6\},\ \sigma = \{\widetilde{\mathbf{0}}, \widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3\}.$$

Then we can easily check that β and σ are $*$-interval-valued fuzzy base and $*$-interval-valued fuzzy subbase for τ respectively.

(3) Let $X = \{x, y\}$ and consider the MBJ-neutrosophic sets $\mathcal{A}_i$ $(i = 1, 2, \cdots, 9)$ in X given by:

$$\mathcal{A}_1(x) = \langle 0.5, [0.5, 0.6], 0.3 \rangle,\ \mathcal{A}_1(y) = \langle 0.6, [0.6, 0.7], 0.7 \rangle,$$

$$\mathcal{A}_2(x) = \langle 0.6, [0.6, 0.8], 0.6 \rangle,\ \mathcal{A}_2(y) = \langle 0.3, [0.3, 0.5], 0.4 \rangle,$$

$$\mathcal{A}_3(x) = \langle 0.8, [0.8, 0.9], 0.5 \rangle,\ \mathcal{A}_3(y) = \langle 0.2, [0.2, 0.4], 0.6 \rangle,$$

$$\mathcal{A}_4(x) =< \langle 0.5, [0.5, 0.6], 0.5 \rangle,\ \mathcal{A}_4(y) = \langle 0.3, [0.3, 0.5], 0.7 \rangle,$$

$$\mathcal{A}_5(x) = \langle 0.5, [0.5, 0.6], 0.5 \rangle,\ \mathcal{A}_5(y) = \langle 0.2, [0.2, 0.4], 0.7 \rangle,$$

$$\mathcal{A}_6(x) = \langle 0.6, [0.6, 0.8], 0.6 \rangle,\ \mathcal{A}_6(y) = \langle 0.2, [0.2, 0.4], 0.6 \rangle,$$

$$\mathcal{A}_7(x) = \langle 0.6, [0.6, 0.8], 0.3 \rangle,\ \mathcal{A}_7(y) = \langle 0.6, [0.6, 0.7], 0.4 \rangle,$$

$$\mathcal{A}_8(x) = \langle 0.8, [0.8, 0.9], 0.3 \rangle,\ \mathcal{A}_8(y) = \langle 0.6, [0.6, 0.7], 0.6 \rangle,$$

$$\mathcal{A}_9(x) = \langle 0.8, [0.8, 0.9], 0.5\rangle\,,\ \mathcal{A}_9(y) = \langle 0.3, [0.3, 0.5], 0.4\rangle\,.$$

Then we can easily see that the family $\tau = \{\ddot{\emptyset}, \ddot{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7, \widetilde{A}_8, \mathcal{A}_9\}$ is a $\circ$-MBJ-neutrosophic topology on X. Now let us two subfamilies β and σ of τ given by:

$$\beta = \{\ddot{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6\},\ \sigma = \{\ddot{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}.$$

Then we can easily check that β and σ are $\circ$-MBJNB and $*$-MBJNSB for τ respectively.

(4) Let $X = \{x, y\}$ and consider the MBJ-neutrosophic sets $\mathcal{A}_i$ $(i = 1, 2, \cdots, 9)$ in X given by:

$$\mathcal{A}_1(x) = \langle 0.5, [0.3, 0.5], 0.3\rangle\,,\ \mathcal{A}_1(y) = \langle 0.6, [0.7, 0.8], 0.7\rangle\,,$$

$$\mathcal{A}_2(x) = \langle 0.6, [0.6, 0.7], 0.6\rangle\,,\ \mathcal{A}_2(y) = \langle 0.3, [0.4, 0.6], 0.4\rangle\,,$$

$$\mathcal{A}_3(x) = \langle 0.8, [0.5, 0.6], 0.5\rangle\,,\ \mathcal{A}_3(y) = \langle 0.2, [0.6, 0.7], 0.6\rangle\,,$$

$$\mathcal{A}_4(x) =< \langle 0.5, [0.6, 0.7], 0.5\rangle\,,\ \mathcal{A}_4(y) = \langle 0.3, [0.7, 0.8], 0.7\rangle\,,$$

$$\mathcal{A}_5(x) = \langle 0.5, [0.5, 0.6], 0.5\rangle\,,\ \mathcal{A}_5(y) = \langle 0.2, [0.7, 0.8], 0.7\rangle\,,$$

$$\mathcal{A}_6(x) = \langle 0.6, [0.6, 0.7], 0.6\rangle\,,\ \mathcal{A}_6(y) = \langle 0.2, [0.6, 0.7], 0.6\rangle\,,$$

$$\mathcal{A}_7(x) = \langle 0.6, [0.3, 0.5], 0.3\rangle\,,\ \mathcal{A}_7(y) = \langle 0.6, [0.4, 0.6], 0.4\rangle\,,$$

$$\mathcal{A}_8(x) = \langle 0.8, [0.3, 0.5], 0.3\rangle\,,\ \mathcal{A}_8(y) = \langle 0.6, [0.6, 0.7], 0.6\rangle\,,$$

$$\mathcal{A}_9(x) = \langle 0.8, [0.5, 0.6], 0.5\rangle\,,\ \mathcal{A}_9(y) = \langle 0.3, [0.4, 0.6], 0.4\rangle\,.$$

Then we can easily see that the family $\tau = \{\dot{\emptyset}, \dot{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7, \widetilde{A}_8, \mathcal{A}_9\}$ is a $*$-MBJ-neutrosophic topology on X. Now let us two subfamilies β and σ of τ given by:

$$\beta = \{\dot{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6\},\ \sigma = \{\dot{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}.$$

Then we can easily check that β and σ are $*$-MBJNB and $*$-MBJNSB for τ respectively.

Remark 5.5. (1) Let (X, τ) be a $\circ$-MBJ-neutrosophic topological space or $*$-MBJ-neutrosophic topological space and let $\beta \subset \tau$. Consider the following families:

$$\beta_M = \{M_A \in I^X : \mathcal{A} \in \beta\},\ \beta_{\widetilde{B}} = \{\widetilde{B}_A \in IVFS(X) : \mathcal{A} \in \beta\},\ \beta_J = \{J_A \in I^X : \mathcal{A} \in \beta\}.$$

Then from Remark 4.4 (5), Definitions 5.1 and 5.2, we can easily check that

(a) β is a $\circ$-MBJNB for τ if and only if β_M is a fuzzy base for $\tau_{_M}$, $\beta_{\widetilde{B}}$ is an interval-valued fuzzy base for $\tau_{_{\widetilde{B}}}$ and β_J is a $\circ$-fuzzy base for $\tau_{_J}$,

(b) β is a $*$-MBJNB for τ if and only if β_M is a fuzzy base for $\tau_{_M}$, $\beta_{\widetilde{B}}$ is a $*$-interval-valued fuzzy base for $\tau_{_{\widetilde{B}}}$ and β_J is a $\circ$-fuzzy base for $\tau_{_J}$.

(2) Let (X, τ) be a $\circ$-MBJ-neutrosophic topological space or $*$-MBJ-neutrosophic topological space and let $\sigma \subset \tau$. Consider the following families:

$$\sigma_M = \{M_A \in I^X : \mathcal{A} \in \sigma\},\ \sigma_{\widetilde{B}} = \{\widetilde{B}_A \in IVFS(X) : \mathcal{A} \in \sigma\},\ \sigma_J = \{J_A \in I^X : \mathcal{A} \in \sigma\}.$$

Then from Remark 4.4 (5), Definitions 5.1 and 5.2, we can easily see that

(a) σ is a $\circ$-MBJNSB for τ if and only if σ_M is a fuzzy subbase for τ_M, $\sigma_{\widetilde{B}}$ is an interval-valued fuzzy subbase for $\tau_{\widetilde{B}}$ and σ_J is a $\circ$-fuzzy subbase for τ_J,

(b) σ is a $*$-MBJNSB for τ if and only if σ_M is a fuzzy subbase for τ_M, $\sigma_{\widetilde{B}}$ is a $*$-interval-valued fuzzy subbase for $\tau_{\widetilde{B}}$ and σ_J is a $\circ$-fuzzy subbase for τ_J.

Lemma 5.6. *Let (X, τ) be a fuzzy cotopological space and let $\beta \subset \tau$. Then β is a $\circ$-fuzzy base for τ if and only if for each $x_a \in F_P(X)$ and for each fuzzy closed $\circ$-Q-neighborhood A of x_a, there is $B \in \beta$ such that $x_a q^\circ B \supset A$.*

Proof. ($\Rightarrow$) The proof is straightforward from the definition of a $\circ$-fuzzy base and the necessary condition of Lemma 3.20 (1).

($\Leftarrow$) Suppose the necessary condition holds. Assume that β is not a $\circ$-fuzzy base for τ. Then there is $A \in \tau$ such that $U = \bigcap\{B \in \beta : B \supset A\} \neq A$. Thus there is $x \in X$ such that $U(x) > A(x)$. Let $a = U^c(x)$. Then clearly, $A(x) + a < U(x) + a < 1$. Thus $x_a q^\circ A$. On the other hand, let $B \in \beta$ such that $B \supset A$. Then clearly, $B \supset U$. Thus $B(x) + a \geq U(x) + a = 1$. So $x_a \neg q^\circ \widetilde{B}$. This contradicts the hypothesis. □

From Proposition 2.4 in [16], Lemma 7 in [33], Lemmas 3.19 (1) and 5.6, we have the following.

Theorem 5.7. *Let (X, τ) be a $\circ$-MBJ-neutrosophic topological space and let $\beta \subset \tau$. Then β is a $\circ$-MBJNB for τ if and only if for each $x^\circ_{\tilde{\tilde{a}}} \in MBJN_P(X)$ and for each $\circ$-MBJ-neutrosophic open Q-neighborhood $\mathcal{A}$ of $x^\circ_{\tilde{\tilde{a}}}$, there is $\mathcal{B} \in \beta$ such that $x^\circ_{\tilde{\tilde{a}}} q^\circ \mathcal{B} \sqsubset \mathcal{A}$.*

Lemma 5.8. *Let (X, τ) be an interval-valued fuzzy cotopological space and let $\beta \subset \tau$. Then β is a $*$-interval-valued fuzzy base for τ if and only if for each $x_{\tilde{a}} \in IVF_P(X)$ and for each fuzzy closed $*$-Q-neighborhood $\widetilde{A}$ of $x_{\tilde{a}}$, there is $\widetilde{B} \in \beta$ such that $x_{\tilde{a}} q^* B \supset A$.*

Proof. The proof is similar to Lemma 5.6. □

Theorem 5.9. *Let (X, τ) be a $*$-MBJ-neutrosophic topological space and let $\beta \subset \tau$. Then β is a $*$-MBJNB for τ if and only if for each $x^*_{\tilde{\tilde{a}}} \in MBJN_P(X)$ and for each $*$-MBJ-neutrosophic open Q-neighborhood $\mathcal{A}$ of $x^*_{\tilde{\tilde{a}}}$, there is $\mathcal{B} \in \beta$ such that $x^*_{\tilde{\tilde{a}}} q^* \mathcal{B} \Subset \mathcal{A}$.*

Proof. The proof is straightforward from Proposition 2.4 in [16], Lemmas 5.8 and 5.6. □

The following gives a necessary and sufficient condition for a subset of $MBJNS(X)$ to be a $\circ$-MBJNB for a $\circ$-MBJ-neutrosophic topology on a set X.

Theorem 5.10. *Let X be a set and let $\beta \subset MBJNS(X)$. Then β is a $\circ$-MBJNB for some $\circ$-MBJ-neutrosophic topology τ if and only if the followings hold:*

(1) $\ddot{X} = \sqcup\beta$,

(2) *if* $\mathcal{B}_1,\ \widetilde{B}_2 \in \beta$ *and* $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B}_1 \sqcup \mathcal{B}_2$, *then there is* $\mathcal{B} \in \beta$ *such that*

$$x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B} \sqsubset \mathcal{B}_1 \sqcap \mathcal{B}_2.$$

In this case, τ is called the *$\circ$-MBJ-neutrosophic topology on X generated by β.*

Proof. ($\Rightarrow$) Suppose β is a $\circ$-MBJNB for a $\circ$-MBJ-neutrosophic topology τ. Since $\ddot{X} \in \tau$, $\ddot{X} = \sqcap\beta$. Then the condition (1) holds. Now suppose $\mathcal{B}_1,\ \widetilde{B}_2 \in \beta$ and $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B}_1 \sqcap \mathcal{B}_2$. Since $\beta \subset \tau$, $\mathcal{B}_1,\ \widetilde{B}_2 \in \tau$. Then $\mathcal{B}_1 \sqcap \mathcal{B}_2 \in \tau$. Since $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B}_1 \sqcap \mathcal{B}_2$, $\mathcal{B}_1 \sqcap \mathcal{B}_2 \neq \ddot{\emptyset}$. By the definition of a $\circ$-MBJNB, there is $\beta^{'} \subset \beta$ such that $\mathcal{B}_1 \sqcap \mathcal{B}_2 = \sqcup\beta^{'}$. Thus there is $\mathcal{B} \in \beta$ such that $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B} \sqsubset \mathcal{B}_1 \sqcap \mathcal{B}_2$. So the condition (2) holds.

($\Leftarrow$) Suppose the conditions (1) and (2) hold and let

$$\tau = \{\mathcal{U} \in MBJNS(X) : \mathcal{U} = \ddot{\emptyset} \text{ or there is } \beta^{'} \subset \beta \text{ such that } \mathcal{U} = \sqcup\beta^{'}\}.$$

Then clearly, $\ddot{\emptyset},\ \ddot{X} \in \tau$. Thus the condition ($\circ$-MBJNO$_1$) holds. Now suppose $\mathcal{U}_1,\ \mathcal{U}_2 \in \tau$ and $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{U}_1 \sqcap \mathcal{U}_2$. Then there are $\mathcal{B}_1,\ \mathcal{B}_2 \in \beta$ such that $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B}_1 \sqsubset \mathcal{U}_1$ and $x_{\tilde{\tilde{a}}} \in \mathcal{B}_2 \sqsubset \mathcal{U}_2$. Thus $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B}_1 \sqcap \mathcal{B}_2 \sqsubset \mathcal{U}_1 \sqcap \mathcal{U}_2$. By the condition (2), there is $\mathcal{B} \in \beta$ such that $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B} \sqsubset \mathcal{U}_1 \sqcap \mathcal{U}_2$. So $\mathcal{U}_1 \sqcap \mathcal{U}_2 \in \tau$. Hence the condition ($\circ$-MBJNO$_2$) holds. Since τ consists of all $\circ$-MBJ unions of members of β, the $\circ$-MBJ union of any family of members of τ is also a member of τ. Then ($\circ$-MBJNO$_3$) holds. This completes the proof. □

Also, we have a necessary and sufficient condition for a subset of $MBJNS(X)$ to be a $*$-MBJNB for a $*$-MBJ-neutrosophic topology on a set X.

Theorem 5.11. *Let X be a set and let $\beta \subset MBJNS(X)$. Then β is a $*$-MBJNB for some $*$-MBJ-neutrosophic topology τ if and only if the followings hold:*

(1) $\ddot{X} = \Cup\beta$,

(2) *if* $\mathcal{B}_1,\ \widetilde{B}_2 \in \beta$ *and* $x^{*}_{\tilde{\tilde{a}}} \in \mathcal{B}_1 \Cap \mathcal{B}_2$, *then there is* $\mathcal{B} \in \beta$ *such that*

$$x^{*}_{\tilde{\tilde{a}}} \in \mathcal{B} \Subset \mathcal{B}_1 \Cap \mathcal{B}_2.$$

In this case, τ is called the *$*$-MBJ-neutrosophic topology on X generated by β.*

Proof. The proof is similar to Theorem 5.10. □

The following provides a sufficient condition for a subset of $MBJNS(X)$ to be a $\circ$-MBJNB for a $\circ$-MBJ-neutrosophic topology on a set X.

Proposition 5.12. *Let X be a set and let $\sigma \subset MBJNS(X)$ such that $\ddot{X} = \sqcup\sigma$. Then there is a unique $\circ$-MBJ-neutrosophc topology τ on X such that σ is a $\circ$-MBJNSB for τ.* In this case, τ is called the *$\circ$-MBJ-neutrosophc topology on X generated by σ.*

Proof. Let $\beta = \{\sqcap\eta : \eta$ is a finite subset of $\sigma\}$ and let

$$\tau = \{\mathcal{U} \in MBJNS(X) : \mathcal{U} = \ddot{\emptyset} \text{ or there is } \beta' \subset \beta \text{ such that } \mathcal{U} = \sqcup\beta'\}.$$

Then clearly, $\ddot{X}$, $\ddot{\emptyset} \in \tau$ by the definition of τ. Thus τ satisfies the condition ($\circ$-MBJNO$_1$). Let $\mathcal{U}_j \in \tau$ for each $j \in J$. Then there is $\beta_j \subset \beta$ such that $\mathcal{U}_j = \sqcup\{\mathcal{B} \in MBJNS(X) : \mathcal{B} \in \beta_j\}$. Thus $\sqcup_{j\in J}\mathcal{U}_j = \sqcup_{j\in J}(\sqcup_{\mathcal{B}\in\beta_j}\mathcal{B}$. So $\sqcup_{j\in J}\mathcal{U}_j \in \tau$. Hence the condition ($\circ$-MBJNO$_3$) holds. Finally, suppose $\mathcal{U}_1$, $\mathcal{U}_2 \in \tau$ and $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{U}_1 \sqcap \mathcal{U}_2$. Then by Theorem 5.10, there are $\mathcal{B}_1$, $\mathcal{B}_2 \in \beta$ such that $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B}_1 \sqcap \mathcal{B}_2$, $\mathcal{B}_1 \sqsubset \mathcal{U}_1$ and $\mathcal{B}_2 \sqsubset \mathcal{U}_2$. Since each of $\mathcal{B}_1$ and $\mathcal{B}_2$ is the $\circ$-intersection of a finite number of members of σ, $\mathcal{B}_1 \sqcap \mathcal{B}_2 \in \beta$. So there is $\beta' \subset \beta$ such that $\mathcal{U}_1 \sqcap \mathcal{U}_2 = \sqcup_{\mathcal{B}\in\beta'}\mathcal{B}$. Hence $\mathcal{U}_1 \sqcap \mathcal{U}_2 \in \tau$, i.e., the condition ($\circ$-MBJNO$_2$) holds. Therefore $\tau \in PCT(X)$. It is obvious that τ is the unique $\circ$-MBJ-neutrosophic topology on X having σ as a $\circ$-MBJNSB. □

Also, we get a sufficient condition for a subset of $MBJNS(X)$ to be a $\circ$-MBJNB for a $\circ$-MBJ-neutrosophic topology on a set X.

Proposition 5.13. *Let X be a set and let $\sigma \subset MBJNS(X)$ such that $\dot{X} = \Cup\sigma$. Then there is a unique $*$-MBJ-neutrosophc topology τ on X such that σ is a $*$-MBJNSB for τ.* In this case, τ is called the *$*$-MBJ-neutrosophc topology on X generated by σ.*

Proof. The proof is similar to Proposition 5.12. □

Definition 5.14. Let (X, τ) be a $\circ$-MBJ-neutrosophc topological space or $*$-MBJ-neutrosophc topological space, let $x^{\circ}_{\tilde{\tilde{a}}}$, $x^{*}_{\tilde{\tilde{a}}} \in MBJN_P(X)$ and let $\beta(x^{\circ}_{\tilde{\tilde{a}}}) \subset \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$, $\beta(x^{*}_{\tilde{\tilde{a}}}) \subset \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$.

(i) $\beta(x^{\circ}_{\tilde{\tilde{a}}})$ is called a *$\circ$-MBJ-neutrosophic neighborhood base* (briefly, $\circ$-MBJNNB) for $\mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$, if for each $\mathcal{A} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$, there is $\mathcal{A} \in \beta(x^{\circ}_{\tilde{\tilde{a}}})$ such that $\mathcal{B} \sqsubset \mathcal{A}$.

(ii) $\beta(x^{*}_{\tilde{\tilde{a}}})$ is called a *$*$-MBJ-neutrosophic neighborhood base* (briefly, $*$-MBJNNB) for $\mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$, if for each $\mathcal{A} \in \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$, there is $\mathcal{A} \in \beta(x^{*}_{\tilde{\tilde{a}}})$ such that $\mathcal{B} \Subset \mathcal{A}$.

(iii) (X, τ) is said to *satisfy the $\circ$-first axiom of countability* or to be $\circ$-C_I, if each $x^{\circ}_{\tilde{\tilde{a}}} \in MBJN_P(X)$ has a countable $\circ$-MBJNNB.

(iv) (X, τ) is said to *satisfy the $*$-first axiom of countability* or to be $*$-C_I, if each $x^{*}_{\tilde{\tilde{a}}} \in MBJN_P(X)$ has a countable $*$-MBJNNB.

From Remark 4.9 and 4.4 (5), we can rewrite Definition 5.14 as followings.

Remark 5.15. Let $\beta(x_a) \subset N_{\tau_M}(x_a)$, $\beta(x_{\tilde{a}}) \subset \widetilde{N}_{\tau_{\widetilde{B}}}(x_{\tilde{a}})$, $\beta(x^*_{\tilde{a}}) \subset \widetilde{N}^*_{\tau_{\widetilde{B}}}(x^*_{\tilde{a}})$, $\beta(x^\circ_{\bar{a}}) \subset N^\circ_{\tau_J}(x^\circ_{\bar{a}})$. Then we have

(1) $\beta(x^\circ_{\tilde{\tilde{a}}})$ is a $\circ$-MBJNNB for $\mathcal{N}(x^\circ_{\tilde{\tilde{a}}}) \Longleftrightarrow$

(i) $\beta(x_a)$ is a fuzzy neighborhood base for $N_{\tau_M}(x_a)$,

(ii) $\beta(x_{\tilde{a}})$ is an interval-valued fuzzy neighborhood base for $\widetilde{N}_{\tau_{\widetilde{B}}}(x_{\tilde{a}})$,

(iii) $\beta(x^\circ_{\bar{a}})$ is a $\circ$-fuzzy neighborhood base for $N^\circ_{\tau_J}(x^\circ_{\bar{a}})$.

(2) $\beta(x^*_{\tilde{\tilde{a}}})$ is a $*$-MBJNNB for $\mathcal{N}(x^*_{\tilde{\tilde{a}}}) \Longleftrightarrow$

(i) $\beta(x_a)$ is a fuzzy neighborhood base for $N_{\tau_M}(x_a)$,

(ii) $\beta(x^*_{\tilde{a}})$ is an $*$-interval-valued fuzzy neighborhood base for $\widetilde{N}^*_{\tau_{\widetilde{B}}}(x^*_{\tilde{a}})$,

(iii) $\beta(x^\circ_{\bar{a}})$ is a $\circ$-fuzzy neighborhood base for $N^\circ_{\tau_J}(x^\circ_{\bar{a}})$.

(3) (X, τ) is a $\circ$-C_I $\Longleftrightarrow$

(i) (X, τ_M) is a fuzzy C_I (See [?]),

(ii) $(X, \tau_{\widetilde{B}})$ is an interval-valued fuzzy C_I, i.e., each $x_{\tilde{a}} \in IVF_P(X)$ has a countable interval-valued fuzzy neighborhood base for $\widetilde{N}_{\tau_{\widetilde{B}}}(x_{\tilde{a}})$,

(iii) (X, τ_J) is a fuzzy $\circ$-C_I, i.e., each $x^\circ_{\bar{a}} \in F_P(X)$ has a countable $\circ$-fuzzy neighborhood base for $N^\circ_{\tau_J}(x^\circ_{\bar{a}})$.

(4) (X, τ) is a $*$-C_I $\Longleftrightarrow$

(i) (X, τ_M) is a fuzzy C_I,

(ii) $(X, \tau_{\widetilde{B}})$ is an interval-valued fuzzy $*$-C_I, i.e., each $x_{\tilde{a}} \in IVF_P(X)$ has a countable $*$-interval-valued fuzzy neighborhood base for $\widetilde{N}^*_{\tau_{\widetilde{B}}}(x^*_{\tilde{a}})$,

(iii) (X, τ_J) is a fuzzy $\circ$-C_I.

Definition 5.16. Let (X, τ) be a $\circ$-MBJ-neutrosophc topological space or $*$-MBJ-neutrosophc topological space, let $x^\circ_{\tilde{\tilde{a}}},\ x^*_{\tilde{\tilde{a}}} \in MBJN_P(X)$ and let $\beta_Q(x^\circ_{\tilde{\tilde{a}}}) \subset \mathcal{N}_Q(x^\circ_{\tilde{\tilde{a}}})$, $\beta_Q(x^*_{\tilde{\tilde{a}}}) \subset \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$.

(i) $\beta_Q(x^\circ_{\tilde{\tilde{a}}})$ is called a $\circ$-*MBJ-neutrosophic Q- neighborhood base* (briefly, $\circ$-MBJNQNB) for $\mathcal{N}_Q(x^\circ_{\tilde{\tilde{a}}})$, if for each $\mathcal{A} \in \mathcal{N}_Q(x^\circ_{\tilde{\tilde{a}}})$, there is $\mathcal{B} \in \beta_Q(x^\circ_{\tilde{\tilde{a}}})$ such that $\mathcal{B} \sqsubset \mathcal{A}$.

(ii) $\beta_Q(x^*_{\tilde{\tilde{a}}})$ is called a $*$-*MBJ-neutrosophic Q-neighborhood base* (briefly, $*$-MBJNQNB) for $\mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$, if for each $\mathcal{A} \in \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$, there is $\mathcal{B} \in \beta_Q(x^*_{\tilde{\tilde{a}}})$ such that $\mathcal{B} \Subset \mathcal{A}$.

(iii) (X, τ) is said to *satisfy the $\circ$-Q-first axiom of countability* or to be $\circ$-Q-C_I, if each $x^\circ_{\tilde{\tilde{a}}} \in MBJN_P(X)$ has a countable $\circ$-MBJNQNB.

(iv) (X, τ) is said to *satisfy the $*$-first axiom of countability* or to be $*$-Q-C_I, if each $x^*_{\tilde{\tilde{a}}} \in MBJN_P(X)$ has a countable $*$-MBJNQNB.

From Remark 4.9, we can rewrite Definition 5.16 as followings.

Remark 5.17. Let $\beta(x_a) \subset N_{\tau_M}(x_a)$, $\beta(x_{\tilde{a}}) \subset \widetilde{N}_{\tau_{\widetilde{B}}}(x_{\tilde{a}})$, $\beta(x^*_{\tilde{a}}) \subset \widetilde{N}^*_{\tau_{\widetilde{B}}}(x^*_{\tilde{a}})$, $\beta(x^\circ_{\bar{a}}) \subset N^\circ_{\tau_J}(x^\circ_{\bar{a}})$. Then we have

(1) $\beta_Q(x^{\circ}_{\tilde{\tilde{a}}})$ is a $\circ$-MBJNQNB for $\mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}}) \Longleftrightarrow$

(i) $\beta_Q(x_a)$ is a fuzzy Q-neighborhood base for $N_{\tau_M,Q}(x_a)$,

(ii) $\beta_Q(x_{\tilde{a}})$ is an interval-valued fuzzy Q-neighborhood base for $\widetilde{N}_{\tau_{\widetilde{B}},Q}(x_{\tilde{a}})$,

(iii) $\beta_Q(x^{\circ}_{\bar{a}})$ is a $\circ$-fuzzy Q-neighborhood base for $N^{\circ}_{\tau_J,Q}(x^{\circ}_{\bar{a}})$.

(2) $\beta_Q(x^{*}_{\tilde{\tilde{a}}})$ is a $*$-MBJNQNB for $\mathcal{N}_Q(x^{*}_{\tilde{\tilde{a}}}) \Longleftrightarrow$

(i) $\beta_Q(x_a)$ is a fuzzy Q-neighborhood base for $N_{\tau_M,Q}(x_a)$,

(ii) $\beta_Q(x^{*}_{\tilde{a}})$ is an $*$-interval-valued fuzzy Q-neighborhood base for $\widetilde{N}^{*}_{\tau_{\widetilde{B}},Q}(x^{*}_{\tilde{a}})$,

(iii) $\beta_Q(x^{\circ}_{\bar{a}})$ is a $\circ$-fuzzy Q-neighborhood base for $N^{\circ}_{\tau_J,Q}(x^{\circ}_{\bar{a}})$.

(3) (X,τ) is a $\circ$-Q-$C_I \Longleftrightarrow$

(i) (X,τ_M) is a fuzzy Q-C_I (See [16]),

(ii) $(X,\tau_{\widetilde{B}})$ is an interval-valued fuzzy Q-C_I, i.e., each $x_{\tilde{a}} \in IVF_P(X)$ has a countable interval-valued fuzzy Q-neighborhood base for $\widetilde{N}_{\tau_{\widetilde{B}},Q}(x_{\tilde{a}})$,

(iii) (X,τ_J) is a fuzzy $\circ$-Q-C_I, i.e., each $x^{\circ}_{\bar{a}} \in F_P(X)$ has a countable $\circ$-Q-fuzzy neighborhood base for $N^{\circ}_{\tau_J,Q}(x^{\circ}_{\bar{a}})$.

(4) (X,τ) is a $*$-Q-$C_I \Longleftrightarrow$

(i) (X,τ_M) is a fuzzy Q-C_I,

(ii) $(X,\tau_{\widetilde{B}})$ is an interval-valued fuzzy $*$-Q-C_I, i.e., each $x_{\tilde{a}} \in IVF_P(X)$ has a countable $*$-interval-valued fuzzy Q-neighborhood base for $\widetilde{N}^{*}_{\tau_{\widetilde{B}},Q}(x^{*}_{\tilde{a}})$,

(iii) (X,τ_J) is a fuzzy $\circ$-Q-C_I.

Example 5.18. (1) Let $X=\{x,y\}$ and let $\tau=\{\ddot{\emptyset},\ddot{X},\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\mathcal{A}_4,\mathcal{A}_5,\mathcal{A}_6,\mathcal{A}_7,\widetilde{A}_8,\mathcal{A}_9\}$ be the $\circ$-MBJ-neutrosophic topology on X given in Example 5.4 (3). Consider two families $\mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$ and $\mathcal{N}_Q(y^{\circ}_{\tilde{\tilde{b}}})$of MBJ-neutrosophic sets in X defined by:

$$\mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})=\{\mathcal{A}\in MBJNS(X):\mathcal{A}_1\sqsubset\mathcal{A}\}$$

and

$$\mathcal{N}_Q(y^{\circ}_{\tilde{\tilde{b}}})=\{\mathcal{A}\in MBJNS(X):\mathcal{A}_1\sqsubset\mathcal{A}\},$$

where $\tilde{\tilde{a}} >^{\circ} \mathcal{A}^c_1(x)=\langle 0.5,[0.4,0.5],0.7\rangle$, i.e., $a>0.5$, $\tilde{a}>[0.4,0.5]$, $\bar{a}<0.7$ and $\tilde{\tilde{b}} >^{\circ} \mathcal{A}^c_1(y)=\langle 0.4,[0.3,0.4],0.3\rangle$, i.e., $a>0.4$, $\tilde{a}>[0.3,0.4]$, $\bar{a}<0.3$. Then we can easily see that $\mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$ and $\mathcal{N}_Q(y^{\circ}_{\tilde{\tilde{b}}})$ are MBJ-neutrosophic neighborhood system of $x^{\circ}_{\tilde{\tilde{a}}}$ and $y^{\circ}_{\tilde{\tilde{b}}}$ with respect to τ respectively. Now consider the subfamily $\beta_Q(x^{\circ}_{\tilde{\tilde{a}}})$ [resp. $\beta_Q(y^{\circ}_{\tilde{\tilde{b}}})$] of $\mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$ [resp. $\mathcal{N}_Q(y^{\circ}_{\tilde{\tilde{b}}})$] given by: for each $n\in\mathbb{N}$,

$$\beta_Q(x^{\circ}_{\tilde{\tilde{a}}})=\{\mathcal{A}\in\mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}}):\tilde{\tilde{a}}=\left\langle 0.5+\frac{1}{n},[0.4+\frac{1}{n},0.5+\frac{1}{n}],0.7-\frac{1}{n}\right\rangle\}$$

$$[\text{resp. } \beta_Q(y^{\circ}_{\tilde{\tilde{b}}})=\{\mathcal{A}\in\mathcal{N}_Q(y^{\circ}_{\tilde{\tilde{b}}}):\tilde{\tilde{b}}=\left\langle 0.4+\frac{1}{n},[0.3+\frac{1}{n},0.4+\frac{1}{n}],0.3-\frac{1}{n}\right\rangle\}].$$

Then it is obvious that $\beta_Q(x^{\circ}_{\tilde{\tilde{a}}})$ [resp. $\beta_Q(y^{\circ}_{\tilde{\tilde{b}}})$] is a $\circ$-MBJNQNB for $\mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$ [resp. $\mathcal{N}_Q(y^{\circ}_{\tilde{\tilde{b}}})$]. Moreover, $\beta_Q(x^{\circ}_{\tilde{\tilde{a}}})$ [resp. $\beta_Q(y^{\circ}_{\tilde{\tilde{b}}})$] is a countable $\circ$-MBJNQNB. Note that if there is no condition for MBJNN $\tilde{\tilde{a}}$, then $\mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}}) = \{\ddot{X}\}$ for each $x_{\tilde{\tilde{a}}} \in MBJN_P(X)$. Thus (X,τ) is $\circ$-Q-C_I. Note that by placing constraints on MBJNN $\tilde{\tilde{a}}$, we can make $\mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$ have more members of τ.

(2) Let $X = \{x, y\}$ and let $\tau = \{\ddot{\emptyset}, \ddot{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7, \widetilde{A}_8, \mathcal{A}_9\}$ be the $*$-MBJ-neutrosophic topology on X given in Example 5.4 (4). Consider the family $\mathcal{N}_Q(x^{*}_{\tilde{\tilde{a}}})$ of MBJ-neutrosophic sets in X defined by:

$$\mathcal{N}_Q(x^{*}_{\tilde{\tilde{a}}}) = \{\mathcal{A} \in MBJNS(X) : \mathcal{A}_2 \Subset \mathcal{A}\}$$

and

$$\mathcal{N}_Q(y^{*}_{\tilde{\tilde{b}}}) = \{\mathcal{A} \in MBJNS(X) : \mathcal{A}_2 \Subset \mathcal{A}\},$$

where $\tilde{\tilde{a}} >^{*} \mathcal{A}^c_2(x) = \langle 0.4, [0.3, 0.4], 0.4\rangle$, i.e., $a > 0.4$, $\tilde{a} < [0.3, 0.4]$, $\bar{a} < 0.4$ and $\tilde{\tilde{b}} >^{*} \mathcal{A}^c_2(y) = \langle 0.7, [0.4, 0.6], 0.6\rangle$, i.e., $a > 0.7$, $\tilde{a} < [0.4, 0.6]$, $\bar{a} < 0.6$. Then we can easily check that $\mathcal{N}_Q(x^{*}_{\tilde{\tilde{a}}})$ and $\mathcal{N}_Q(y^{*}_{\tilde{\tilde{b}}})$ are MBJ-neutrosophic neighborhood system of $x^{*}_{\tilde{\tilde{a}}}$ and $y^{*}_{\tilde{\tilde{b}}}$ with respect to τ respectively. Now consider the subfamily $\beta_Q(x^{*}_{\tilde{\tilde{a}}})$ [resp. $\beta_Q(y^{*}_{\tilde{\tilde{b}}})$] of $\mathcal{N}_Q(x^{*}_{\tilde{\tilde{a}}})$ [resp. $\mathcal{N}_Q(y^{*}_{\tilde{\tilde{b}}})$] given by: for each $n \in \mathbb{N}$,

$$\beta_Q(x^{*}_{\tilde{\tilde{a}}}) = \{\mathcal{A} \in \mathcal{N}_Q(x^{*}_{\tilde{\tilde{a}}}) : \tilde{\tilde{a}} = \left\langle 0.4 + \frac{1}{n}, [0.3 - \frac{1}{n}, 0.4 - \frac{1}{n}], 0.4 - \frac{1}{n} \right\rangle\}$$

$$\text{[resp. } \beta_Q(y^{*}_{\tilde{\tilde{b}}}) = \{\mathcal{A} \in \mathcal{N}_Q(y^{*}_{\tilde{\tilde{b}}}) : \tilde{\tilde{b}} = \left\langle 0.7 + \frac{1}{n}, [0.4 - \frac{1}{n}, 0.6 - \frac{1}{n}], 0.6 - \frac{1}{n} \right\rangle\}.$$

Then it is clear that $\beta_Q(x^{*}_{\tilde{\tilde{a}}})$ [resp. $\beta_Q(y^{*}_{\tilde{\tilde{b}}})$] is a $*$-MBJNQNB for $\mathcal{N}_Q(x^{*}_{\tilde{\tilde{a}}})$ [resp. $\mathcal{N}_Q(y^{*}_{\tilde{\tilde{b}}})$]. Furthermore, $\beta_Q(x^{*}_{\tilde{\tilde{a}}})$ [resp. $\beta_Q(y^{*}_{\tilde{\tilde{b}}})$] is a countable $*$-MBJNQNB. Also, note that if there is no condition for MBJNN $\tilde{\tilde{a}}$, then $\mathcal{N}_Q(x^{*}_{\tilde{\tilde{a}}}) = \{\dot{X}\}$ for each $x_{\tilde{\tilde{a}}} \in MBJN_P(X)$. Thus (X,τ) is $*$-Q-C_I. Also, note that by placing constraints on MBJNN $\tilde{\tilde{a}}$, we can make $\mathcal{N}_Q(x^{*}_{\tilde{\tilde{a}}})$ have more members of τ.

(3) Let X be an infinite set and let τ be the MBJ-neutrosophic $\circ$-[resp. $*$-]cofinite topology on X. Assume that $\beta(x^{\circ}_{\tilde{\tilde{a}}}) = \{\mathcal{B}_n : n \in \mathbb{N}\}$ is a $\circ$-MBJNNB for $x^{\circ}_{\tilde{\tilde{a}}}$. Let $y^{\circ}_{\tilde{\tilde{b}}} \in MBJN_P(X)$ such that $x \neq y$. Then clearly, $y^{\circ}_{\tilde{\tilde{b}}} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$. Thus there is $n \in \mathbb{N}$ such that $y^{\circ}_{\tilde{\tilde{b}}} \notin \mathcal{B}_n$. So $\sqcap_{n\in\mathbb{N}}\mathcal{B}_n = \{x^{\circ}_{\tilde{\tilde{a}}}\}$. On the other hand, we have

$$x^{\circ,c}_{\tilde{\tilde{a}}} = [\sqcap_{n\in\mathbb{N}}\mathcal{B}_n]^c = \sqcup_{n\in\mathbb{N}}\mathcal{B}^c_n.$$

Since $\mathcal{B}^c_n$ is $\circ$-finite for each $n \in \mathbb{N}$, $\sqcup_{n\in\mathbb{N}}\mathcal{B}^c_n$ is countable. So $x^{\circ,c}_{\tilde{\tilde{a}}}$ is countable. This is a contradiction. Hence (X,τ) is not $\circ$-C_I. Similarly, we can check that (X,τ) is not $*$-C_I.

Proposition 5.19. *Let (X,τ) be a $\circ$-MBJ-neutrosophc topological space. If (X,τ) is $\circ$-C_I, then it is $\circ$-Q-C_I.*

Proof. Suppose (X,τ) is $\circ$-C_I. It is well-known (Proposition 3.1, [16]) that if (X,τ_M) is fuzzy C_I, then it is fuzzy Q-C_I.

Let $x_{\widetilde{a}} \in IVF_P(X)$. Consider a sequence $\{\widetilde{a}_n = [a_n^-, a_n^+]\}_{n\in\mathbb{N}}$ of interval numbers converging to $\widetilde{a}^c = [1-a^+, 1-a^-]$ and interval-valued fuzzy points $x_{\widetilde{a}_n}$ in X, where $a_n^- \in (1-a^+, 1]$ and $a_n^+ \in (1-a^-, 1]$. Since (X,τ) is $\circ$-C_I, by Remark 5.15 (3), $(X,\tau_{\widetilde{B}})$ is interval-valued fuzzy C_I. Then for each $n \in \mathbb{N}$, there is a countable interval-valued open neighborhood base $\beta_n(x_{\widetilde{a}_n})$. Thus for each $\widetilde{B} \in \beta_n(x_{\widetilde{a}_n})$, $\widetilde{B}(x) \geq \widetilde{a}_n > \widetilde{a}^c$. So $\widetilde{B} \in \widetilde{N}_{\tau_{\widetilde{B}},Q}(x_{\widetilde{a}})$. Let $\beta(x_{\widetilde{a}})$ be the collection of all the members of all $\beta_n(x_{\widetilde{a}_n})$. It is clear that $\beta(x_{\widetilde{a}})$ is a family of interval-valued fuzzy open Q-neighborhoods of $x_{\widetilde{a}}$. Let $\widetilde{A} \in \widetilde{N}_{\tau_{\widetilde{B}},Q}(x_{\widetilde{a}})$. Then clearly, $\widetilde{A}(x) > \widetilde{a}^c$. Since $\{\widetilde{a}_n\}_{n\in\mathbb{N}}$ is convergent to $\widetilde{a}^c$, there is $m \in \mathbb{N}$ such that $\widetilde{A}(x) \geq \widetilde{a}_m > \widetilde{a}^c$, i.e., $x_{\widetilde{a}_m} \in \widetilde{A}$ and $\widetilde{A}$ is an interval-valued fuzzy open neighborhood of $x_{\widetilde{a}_m}$. Thus there is $\widetilde{B} \in \beta_n(x_{\widetilde{a}_n}) \subset \beta(x_{\widetilde{a}})$ such that $\widetilde{B} \subset \widetilde{A}$ and $\widetilde{B}(x) \geq \widetilde{a}_m > \widetilde{a}^c$. So $\beta(x_{\widetilde{a}})$ is a countable interval-valued fuzzy Q-neighborhood of $x_{\widetilde{a}}$. Hence $(X,\tau_{\widetilde{B}})$ is interval-valued fuzzy Q-C_I.

Now let $x^{\circ}_{\bar{a}} \in F_P(X)$. Consider a sequence $\{\bar{a}_n\}_{n\in\mathbb{N}}$ in $[0,\bar{a}^c)$ converging to $\bar{a}^c$ and a $\circ$-fuzzy points $x^{\circ}_{\bar{a}_n}$ in X. Since (X,τ) is $\circ$-C_I, by Remark 5.15 (3), (X,τ_J) is $\circ$-fuzzy C_I. Then for each $n \in \mathbb{N}$, there is a countable $\circ$-fuzzy closed neighborhood base $\beta_n(x^{\circ}_{\bar{a}_n})$. Thus for each $B \in \beta_n(x^{\circ}_{\bar{a}_n})$, $B(x) \leq \bar{a}_n < \bar{a}^c$. So $B \in N^{\circ}_{\tau_J,Q}(x^{\circ}_{\bar{a}})$. Let $\beta(x_{\bar{a}})$ be the collection of all the members of all $\beta_n(x_{\bar{a}_n})$. It is clear that $\beta(x^{\circ}_{\bar{a}})$ is a family of $\circ$-fuzzy closed Q-neighborhoods of $x_{\bar{a}}^{\circ}$. Let $A \in N_{\tau_J,Q}(x^{\circ}_{\bar{a}})$. Then clearly, $A(x) < \bar{a}^c$. Since $\{\bar{a}_n\}_{n\in\mathbb{N}}$ is convergent to $\bar{a}^c$, there is $m \in \mathbb{N}$ such that $A(x) \leq \bar{a}_m < \bar{a}^c$, i.e., $x^{\circ}_{\bar{a}_m} \in A$ and A is a $\circ$-fuzzy closed neighborhood of $x^{\circ}_{\bar{a}_m}$. Thus there is $B \in \beta_n(x^{\circ}_{\bar{a}_n}) \subset \beta(x^{\circ}_{\bar{a}})$ such that $B \supset A$ and $B(x) \leq \bar{a}_m < \bar{a}^c$. So $\beta(x^{\circ}_{\bar{a}})$ is a countable $\circ$-fuzzy Q-neighborhood of $x_{\bar{a}}^{\circ}$. Hence (X,τ_J) is $\circ$-fuzzy Q-C_I. Therefore by Remark 5.15, (X,τ) is $\circ$-Q-C_I. □

Proposition 5.20. *Let (X,τ) be a $*$-MBJ-neutrosophc topological space. If (X,τ) is $*$-C_I, then it is $*$-Q-C_I.*

Proof. Suppose (X,τ) is a $*$-C_I. Then from the proof of Proposition 5.19, the conditions (i) and (iii) of Remark 5.15. It is sufficient to show that (ii) of Remark 5.15 holds. Let $x^*_{\widetilde{a}} \in IVF_P(X)$. Consider a sequence $\{\widetilde{a}_n\}_{n\in\mathbb{N}}$ of interval numbers converging to $\widetilde{a}^c$ and a $*$-interval-valued fuzzy points $x^*_{\widetilde{a}_n}$ in X, where $a_n^- \in [0, 1-a_n^+)$ and $a_n^+ \in [0, 1-a_n^-)$. Since (X,τ) is $*$-C_I, by Remark 5.15 (4), $(X,\tau_{\widetilde{B}})$ is a $*$-interval-valued fuzzy C_I. Then for each $n \in \mathbb{N}$, there is a countable $*$-interval-valued fuzzy closed neighborhood base $\beta_n(x^*_{\widetilde{a}_n})$. Thus for each $\widetilde{B} \in \beta_n(x^*_{\widetilde{a}_n})$, $\widetilde{B}(x) \leq \widetilde{a}_n < \widetilde{a}^c$. So $\widetilde{B} \in N^{*c}_{\tau_{\widetilde{B}},Q}(x^*_{\widetilde{a}})$. Let $\beta(x^*_{\widetilde{a}})$ be the collection of all the members of all $\beta_n(x^*_{\widetilde{a}_n})$. It is clear that $\beta(x^*_{\widetilde{a}})$ is a family of $*$-interval-valued fuzzy closed Q-neighborhoods of $x^*_{\widetilde{a}}$. Let $\widetilde{A} \in \widetilde{N}^*_{\tau_{\widetilde{B}},Q}(x^*_{\widetilde{a}})$. Then clearly, $\widetilde{A}(x) < \widetilde{a}^c$. Since $\{\widetilde{a}_n\}_{n\in\mathbb{N}}$ is convergent to $\widetilde{a}^c$, there is $m \in \mathbb{N}$ such that $\widetilde{A}(x) \leq \widetilde{a}_m < \widetilde{a}^c$, i.e., $x^*_{\widetilde{a}_m} \in \widetilde{A}$ and $\widetilde{A}$ is a

$*$-interval-valued fuzzy closed neighborhood of $x^*_{\tilde{a}_m}$. Thus there is $\widetilde{B} \in \beta_n(x^*_{\tilde{a}_n}) \subset \beta(x^*_{\tilde{a}})$ such that $\widetilde{B} \supset widetildeA$ and $\widetilde{B}(x) \leq \tilde{a}_m < \tilde{a}^c$. So $\beta(x^*_{\tilde{a}})$ is a countable $*$-interval-valued fuzzy Q-neighborhood of $x_{\tilde{a}}^*$. Hence (X, τ_J) is $*$-interval-valued fuzzy Q-C_I. Therefore by Remark 5.15, (X, τ) is $*$-Q-C_I. □

Definition 5.21. Let (X, τ) be a $\circ$-MBJ-neutrosophc topological space or $*$-MBJ-neutrosophc topological space.

(i) (X, τ) is said to *satisfy the $\circ$-second axiom of countability* or to be $\circ$-C_{II}, if there is a countable $\circ$-base β for τ.

(ii) (X, τ) is said to *satisfy the $*$-second axiom of countability* or to be $*$-C_{II}, if there is a countable $*$-base β for τ.

From Remark 4.4 (5), and Definitions 5.2 and 5.3, we can rewrite Definition 5.1 (i) as followings.

Remark 5.22. Let (X, τ) be a $\circ$-MBJ-neutrosophc topological space or $*$-MBJ-neutrosophc topological space.

(1) (X, τ) is $\circ$-C_{II} $\Longleftrightarrow$

(i) (X, τ_M) is fuzzy C_{II}, i.e., there is a countable fuzzy base β_M for τ_M,

(ii) $(X, \widetilde{B})$ is interval-valued fuzzy C_{II}, i.e., there is a countable interval-valued fuzzy base $\beta_{\widetilde{B}}$ for $\tau_{\widetilde{B}}$,

(iii) (X, τ_J) is $\circ$-fuzzy C_{II}, i.e., there is a countable $\circ$-fuzzy base β_J for τ_J.

(2) (X, τ) is $*$-C_{II} $\Longleftrightarrow$

(i) (X, τ_M) is fuzzy C_{II}, i.e., there is a countable fuzzy base β_M for τ_M,

(ii) $(X, \widetilde{B})$ is $*$-interval-valued fuzzy C_{II}, i.e., there is a countable $*$-interval-valued fuzzy base $\beta_{\widetilde{B}}$ for $\tau_{\widetilde{B}}$,

(iii) (X, τ_J) is $\circ$-fuzzy C_{II}, i.e., there is a countable $\circ$-fuzzy base β_J for τ_J.

Proposition 5.23. *Let (X, τ) be a $\circ$-MBJ-neutrosophc topological space or a $*$-MBJ-neutrosophc topological space.*

(1) If (X, τ) is $\circ$-C_{II}, then it is $\circ$-C_I.

(2) If (X, τ) is $*$-C_{II}, then it is $*$-C_I.

Proof. (1) Suppose (X, τ) is $\circ$-C_{II}. It is well-known (Theorem 3.3, [14]) that if (X, τ_M) is fuzzy C_{II}, then it is fuzzy C_I. Thus (X, τ_M) satisfies the condition (i) of Remark 5.15 (3).

Let $x_{\tilde{a}} \in IVF_P(X)$. Then by the hypothesis and Remark 5.22 (1), there is a countable interval-valued fuzzy base $\beta_{\widetilde{B}}$ for $\tau_{\widetilde{B}}$. Let $\beta_{\widetilde{B}, x_{\tilde{a}}}$ be the subfamily of $\beta_{\widetilde{B}}$ given by $\beta_{\widetilde{B}, x_{\tilde{a}}} = \{\widetilde{B} : x_{\tilde{a}} \in \widetilde{B} \in \beta_{\widetilde{B}}\}$. Then clearly, $\beta_{\widetilde{B}, x_{\tilde{a}}}$ is countable. Let $\widetilde{A} \in \tau_{\widetilde{B}}$ such that $x_{\tilde{a}} \in \widetilde{A}$. Since $\beta_{\widetilde{B}}$ is an interval-valued fuzzy base for $\tau_{\widetilde{B}}$, by Theorem 5.10, there is $\widetilde{B} \in \beta_{\widetilde{B}}$ such that $x_{\widetilde{a}} \in \widetilde{B} \subset \widetilde{A}$. Then by the definition of $\beta_{\widetilde{B},x_{\widetilde{a}}}$, $\widetilde{B} \in \beta_{\widetilde{B},x_{\widetilde{a}}}$. Thus $(X, \tau_{\widetilde{B}}$ is interval-valued fuzzy C_I. So $(X, \tau_{\widetilde{B}}$ satisfies the condition (ii) of Remark 5.15 (3).

Now let $x^{\circ}_{\overline{a}} \in F_P(X)$. Then by the hypothesis and Remark 5.22 (1), there is a countable $\circ$-fuzzy base β_J for τ_J. Let $\beta_{J,x^{\circ}_{\overline{a}}}$ be the subfamily of β_J defined by $\beta_{J,x^{\circ}_{\overline{a}}} = \{B : x^{\circ}_{\overline{a}} \in B \in \beta_J\}$. Then clearly, $\beta_{J,x^{circ}_{\overline{a}}}$ is countable. Let $A \in \tau_J$ such that $x^{\circ}_{\overline{a}} \in A$. Since β_J is a $\circ$-fuzzy base for τ_J, by Theorem 5.10, there is $B \in \beta_J$ such that $x^{\circ}_{\overline{a}} \in B \supset A$. Then by the definition of $\beta_{J,x^{\circ}_{\overline{a}}}$, $B \in \beta_{J,x^{\circ}_{\overline{a}}}$. Thus $(X, \tau_J$ is $\circ$-fuzzy C_I. So $(X, \tau_J$ satisfies the condition (iii) of Remark 5.15 (3). Hence (X, τ) is $\circ$-C_I.

(2) Suppose (X, τ) is $*$-C_{II}. From (1), it is obvious that the conditions (i) and (iii) of Remark 5.15 (4). It is sufficient to prove that (ii) of Remark 5.15 (4) holds. □

The following is an immediate consequence of Propositions 5.19, 5.20 and 5.23.

Corollary 5.24. *Let (X, τ) be a $\circ$-MBJ-neutrosophc topological space or a $*$-MBJ-neutrosophc topological space.*

(1) *If (X, τ) is $\circ$-C_{II}, then it is $\circ$-Q-C_I.*

(2) *If (X, τ) is $*$-C_{II}, then it is $*$-Q-C_I.*

The converse of Proposition 5.23 does not hold in general (See Example 5.33).

Definition 5.25 ([37]). Let X be a classical topological space and let $A : X \to I$ be a mapping.

(i) A is said to be *lower semi-continuous* [resp. *upper semi-continuous*] at $a \in X$, if for each $h < A(a)$ [resp. $k > A(a)$], there is a neighborhood V of a such that $h < A(x)$ [resp. $k > A(x)$] for each $x \in V$.

(ii) A is said to be *lower semi-continuous* [resp. *upper semi-continuous*] on X, if it is lower semi-continuous [resp. upper semi-continuous] at each $a \in X$.

It is well-known (6.2, [37]) that A is continuous on X if and only if A is both upper and lower semi-continuous on X. Moreover, A is lower semi-continuous on X if and only if $1 - A$ is upper semi-continuous on X.

For a fuzzy set A in a set X and any $a \in I$, the *weak* [resp. *strong*] *a-cut* or *a-level set* of A, denoted by $[A]_a$ [resp. $[A]^a$], is a subset of X defined as follows:

$$[A]_a = \{x \in X : A(x) \geq a\} \text{ [resp.} [A]^a = \{x \in X : A(x) > a\} \text{ (See [14]).}$$

Definition 5.26 ([14]). Let (X, T) be a classical topological space. Then the *induced fuzzy topology* on X, denoted by $F(T)$, is a family of fuzzy sets in X given as follows:

$$F(T) = \{A \in I^X : A \text{ is lower semi-continuous}\}.$$

It is well-known (Proposition 3.3, [15]) that for a classical topological space (X, T) and $A \in I^X$, A is fuzzy open [resp. closed] in $(X, F(T))$ if and only if for each $a \in I$, $[A]^a \in T$ [resp. $[A]_a \in T^c$]. Then from Definition 5.25 and the above fact,

$$F(T) = \{A \in I^X : [A]^a \in T,\ a \in I\} \text{ and } CF(T) = \{A \in I^X : [A]_a \in T^c,\ a \in I\},$$

where $CF(T)$ will be called the *induced fuzzy cotopology* on X. Furthermore, we can easily see that the family $\{[A, A] \in IVFS(X) : A \in F(T)\}$ is an interval-valued fuzzy topology on X, and it will be called the *induced interval-valued fuzzy topology* on X and denoted by $IVF(T)$. Also, $CIVF(T) = \{[A, A] \in IVFS(X) : A \in CF(T)\}$ is an interval-valued fuzzy cotopology on X.

Remark 5.27. Let (X, T) be a classical topological space and consider the families τ and η of MBJ-neutrosophic sets in X defined as follows:

$$\tau = \{\mathcal{A} \in MBJNS(X) : M_A \in F(T),\ \widetilde{B}_A \in IVF(T),\ J_A \in CF(T)\},$$

$$\eta = \{\mathcal{A} \in MBJNS(X) : M_A \in F(T),\ \widetilde{B}_A \in CIVF(T),\ J_A \in CF(T)\}.$$

Then clearly, $\tau \in MBJNT^\circ(X)$ and $\eta \in MBJNT^*(X)$. In this case, we will call τ and η as the *induced* $\circ$*-MBJ-neutrosophic topology* and the *induced* $*$*-MBJ-neutrosophic topology* on X.

Result 5.28 (See Lemma 3.1, [16]). Let (X, T) be a classical complete regular topological space. Then for each $B \in F(T)$, there is a family $\beta_F \subset I^X$ each member of which is continuous with respect to T, such that $B = \bigcup_{A \in \beta_F} A$. In other words, the family

$$\beta_F = \{A \in I^X : A \text{ is continuous on } I\}$$

forms a fuzzy base for $F(T)$.

Remark 5.29. Let (X, T) be a classical complete regular topological space and consider the following families:

$$\beta_{IVF} = \{[A, A] \in IVF(X) : A \in \beta_F\},$$

$$\beta_F^\circ = \{A \in I^X : A^c \text{ is continuous on } I\},$$

$$\beta_{IVF}^* = \{[A, A] \in IVF(X) : A \in \beta_F^\circ\}.$$

Then from Result 5.28, we can easily see that β_{IVF}, β_F° and β_{IVF}^* form an interval-valued fuzzy base for $IVF(T)$, a $\circ$-fuzzy base for $CF(T)$ and a $*$-interval-valued fuzzy base for $CIVF(T)$ respectively.

From Remarks 5.5 (1), 5.27 and 5.29, and Result 5.28, we have the following.

Lemma 5.30. *Let (X, T) be a classical complete regular topological space and consider the following families:*

$$\beta = \{\mathcal{A} \in MBJNS(X) : M_A \in \beta_F,\ \widetilde{B}_A \in \beta_{IVF},\ J_A \in \beta_F^\circ\},$$

$$\beta^* = \{\mathcal{A} \in MBJNS(X) : M_A \in \beta_F,\ \widetilde{B}_A \in \beta_{IVF}^*,\ J_A \in \beta_F^\circ\}.$$

(1) *β forms a $\circ$-MBJNB for τ.*

(2) *β^* forms a $*$-MBJNB for η.*

Result 5.31 (See Theorem 3.1, [16])**.** Let (X, T) be the subspace I of the real axis and let $F(T)$ be the induced fuzzy topology for T. Then $(X, F(T)$ is fuzzy C_{II} but not fuzzy C_I.

Lemma 5.32 (See Theorem 3.1, [16])**.** *Let (X, T) be the subspace I of the real axis.*

(1) *$(X, IVF(T))$ is interval-valued fuzzy C_{II} but not interval-valued fuzzy C_I.*

(2) *$(X, CF(T))$ is $\circ$-fuzzy C_{II} but not $\circ$-fuzzy C_I.*

(3) *$(X, CIVF(T))$ is $*$-interval-valued fuzzy C_{II} but not $*$-interval-valued fuzzy C_I.*

Proof. The proofs are similar to Result 5.31. □

From Remark 5.26, Lemmas 5.30 and 5.32, and Result 5.31, we can give as an example which the converse of Proposition 5.23 does not hold.

Example 5.33. Let (X, T) be the subspace I of the real axis.

(1) (X, τ) is $\circ$-C_{II} but not $\circ$-C_I.

(2) (X, η) is $*$-C_{II} but not $*$-C_I.

6. Conclusions

Through the study, we obtained several results as follows:

(1) $(MBJNS(X), \sqcup, \sqcap, ^{c,i}, \ddot{\emptyset}, \ddot{X})$ and $(MBJNS(X), \Cup, \Cap, ^{c,i}, \dot{\emptyset}, \dot{X})$ form Boolean algebras except the condition (13) of Proposition 3.7.

(2) An MBJ-neutrosophic neighborhood system generates an MBJ-neutrosophic topology (See Theorem 4.16 and 4.18).

(3) The characterization of MBJ-neutrosophic base (See Theorems 5.6 and 5.8).

(4) A necessary and sufficient condition for a set of MB J-neutrosophic sets to be an MBJ-neutrosophic topology (See Theorems 5.9 and 5.10).

(5) The relationships among $\circ$-C_I, $*$-C_I, $\circ$-C_{II} and $*$-C_{II}.

Before conducting our research, we came across an interesting paper written by Al-shami [38] during a literature search. By defining soft separation axioms in soft topological spaces, he proposed an algorithm for decision-making problems. In the future, we try to study separation axioms based on MBJ-neutrosophic sets and to apply them to decision-making problems. Moreover, we expect that one can apply MBJ-neutrosophic sets to a category theory, a graph theory, a group theory, etc.

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