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Neutrosophic $\mathcal{N}-$structures on Sheffer stroke BCH-algebras

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Abstract. The aim of the study is to introduce a neutrosophic $\mathcal{N}-$subalgebra and neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCH-algebras. We prove that the level-set of a neutrosophic $\mathcal{N}-$subalgebra (neutrosophic $\mathcal{N}-$ideal) of a Sheffer stroke BCH-algebra is its subalgebra (ideal) and vice versa. Then it is shown that the family of all neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke BCH-algebra forms a complete distributive modular lattice. Also, we state that every neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCH-algebra is its neutrosophic $\mathcal{N}-$subalgebra but the inverse is generally not true. We examine relationships between neutrosophic $\mathcal{N}-$ideals of Sheffer stroke BCH-algebras by means of a surjective homomorphism between these algebras. Finally, certain subsets of a Sheffer stroke BCH-algebra are defined by means of $\mathcal{N}-$functions on this algebraic structure and some properties are investigated.

Keywords: Sheffer stroke BCH-algebra; subalgebra; neutrosophic $\mathcal{N}-$ subalgebra; neutrosophic $\mathcal{N}-$ideal.

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1. **Introduction**

Sheffer stroke (or Sheffer operation) introduced by H. M. Sheffer is one of the two operators that can be used by itself, without any other logical operators to build a logical formal system [22]. Since it provides new, basic and easily applicable axiom systems for many algebraic structures, this operation has many applications in algebraic structures such as orthoimplication algebras [1], ortholattices [3], Boolean algebras [15], strong Sheffer stroke non-associative MV-algebras [4] and their neutrosophic N-structures [19], Sheffer Stroke Hilbert algebras [16] and their neutrosophic N-structures [17]. Besides, the concepts of BCK-algebras and BCI-algebras were introduced by Y. Imai and K. Iséki ([10], [11]) and BCK-algebras are proper subclasses of BCI-algebras. Also, a new class of these algebras so-called BCH-algebras is introduced and studied by Hu and Li ([8], [9]) and the new class contains BCK-algebras and BCI-algebras. Some properties of these new algebraic structures have been investigated by Chaudhry ([5], [6]), Dudek and Thomys [7]. Recently, BCH-algebras with Sheffer stroke, subalgebras, minimal and medial elements and BCA-parts of these algebras are studied by Oner et al [18].

On the other side, Zadeh introduced the fuzzy set theory [26] which is a generalization of ordinary sets, has the truth (t) (membership) function and positive meaning of information. Hence, scientists have been interested in negative meaning of information, and so, Atanassov introduced the intuitionistic fuzzy set theory [2] which is a generalization of fuzzy sets, has truth (t) (membership) and the falsehood (f) (nonmembership) functions. Besides, Smarandache introduced the neutrosophic set theory which is a generalization of the intuitionistic fuzzy set theory and has the indeteminacy/neutrality (i) function with membership and nonmembership functions [23, 24]. Thus, neutrosophic sets are defined on three components (t, i, f) [27]. In recent times, neutrospohic sets are applied to the algebraic structures such as BCK/BCI-algebras and BE-algebras [12–14, 20, 25].

We give basic definitions and notions on Sheffer stroke BCH-algebras, neutrosophic $\mathcal{N}-$functions and neutrosophic $\mathcal{N}-$structures. Also, neutrosophic $\mathcal{N}-$subalgebra, a neutrosophic $\mathcal{N}-$ideal and a level set on neutrosophic $\mathcal{N}-$structures are introduced on Sheffer stroke BCH-algebras. Then we prove that the level set of a neutrosophic $\mathcal{N}-$subalgebra of a Sheffer stroke BCH-algebra is its subalgebra and the inverse always is true, and that the family of all neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke BCH-algebra forms a complete distributive modular lattice. Moreover, it is shown that every neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCH-algebra is its neutrosophic $\mathcal{N}-$subalgebra but the inverse does not mostly hold. Finally, we define special subsets of a Sheffer stroke BCH-algebra by means of the $\mathcal{N}-$functions T_N, I_N and F_N and its any elements a_t, a_i, a_f and show that these subsets are ideals of this algebraic structure if a neutrosophic $\mathcal{N}-$structure on this algebraic structure is the neutrosophic $\mathcal{N}-$ideal.

2. **Preliminaries**

In this section, we give basic definitions and notions about Sheffer stroke BCH-algebras and neutrosophic $\mathcal{N}-$structures.

Definition 2.1. [3] Let $\mathcal{A} = \langle A, | \rangle$ be a groupoid. The operation $\circ$ on S is said to be a *Sheffer operation (or Sheffer stroke)* if it satisfies the following conditions for all $a, b, c \in A$:
$(S1)$ $a|b = b|a$,
$(S2)$ $(a|a)|(a|b) = a$,
$(S3)$ $a|((b|c)|(b|c)) = ((a|b)|(a|b))|c$,
$(S4)$ $(a|((a|a)|(b|b)))|(a|((a|a)|(b|b))) = a$.

Definition 2.2. [18] A Sheffer stroke BCH-algebra is an algebra $(A, |, 0)$ of type $(2, 0)$ such that 0 is the constant in A the following axioms are satisfied:
$(sBCH.1)$ $(a|(a|a))|(a|(a|a)) = 0$,

$(sBCH.2)$ $(a|(b|b))|(a|(b|b)) = (b|(a|a))|(b|(a|a)) = 0$ imply $a = b$,
$(sBCH.3)$ $((a|(b|b))|(a|(b|b)))|(c|c) = ((a|(c|c))|(a|(c|c)))|(b|b)$,
for all $a, b, c \in A$.

Definition 2.3. [18] Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra. Then a relation $\leqslant$ on A defined by

$$a \leqslant b \text{ if and only if } (a|(b|b))|(a|(b|b)) = 0,$$

is a partial order on A.

Lemma 2.4. *[18] Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra. Then the following features hold for all $a, b, c \in A$:*

(1) $(a|(a|a))|(a|a) = a$,
(2) $a|(((a|(b|b))|(b|b))|((a|(b|b))|(b|b))) = 0|0$,
(3) $(0|0)|(a|a) = a$,
(4) $(a|(0|0))|(a|(0|0)) = a$,
(5) $a|((b|(c|c))|(b|(c|c))) = b|((a|(c|c))|(a|(c|c)))$,
(6) $((a|(a|(b|b)))|(a|(a|(b|b))))|(b|b) = 0|0$,
(7) $((a|(b|b))|(a|(b|b)))|(a|a) = 0|(b|b)$,
(8) $0|(a|(b|b)) = ((0|(a|a))|(0|(a|a)))|(0|(b|b))$,
(9) $a \leqslant b$ *implies* $0|(a|a) = 0|(b|b)$.

Definition 2.5. [18] Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra. Then a nonempty subset S of A is called a subalgebra of A, if $(a|(b|b))|(a|(b|b)) \in S$, for all $a, b \in S$.

Definition 2.6. [18] A nonempty subset I of a Sheffer stroke BCH-algebra $(A, |, 0)$ is called an ideal of A if it satisfies
(I1) $0 \in I$,
(I2) $(a|(b|b))|(a|(b|b)) \in I$ and $a_2 \in I$ imply $a_1 \in I$,
for all $a, b \in A$.

Definition 2.7. [21] A modular lattice is any lattice which satisfies $a \leqslant b \longrightarrow a \vee (b \wedge c) = b \wedge (a \vee c)$.

Theorem 2.8. *[21] Every distributive lattice is a modular lattice.*

Definition 2.9. [12] $\mathcal{F}(A, [-1, 0])$ denotes the collection of functions from a set A to $[-1, 0]$ and a element of $\mathcal{F}(A, [-1, 0])$ is called a negative-valued function from A to $[-1, 0]$ (briefly, $\mathcal{N}-$function on A). An $\mathcal{N}-$structure refers to an ordered pair (A, f) of A and $\mathcal{N}-$function f on A.

Definition 2.10. [14] A neutrosophic $\mathcal{N}-$structure over a nonempty universe A is defined by

$$A_N := \frac{A}{(T_N, I_N, F_N)} = \{\frac{A}{(T_N(a), I_N(a), F_N(a))} : a \in A\}$$

where T_N, I_N and F_N are $\mathcal{N}-$function on A, called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function, respectively.

Every neutrosophic $\mathcal{N}-$structure A_N over A satisfies the condition

$$(\forall a \in A)(-3 \leqslant T_N(a) + I_N(a) + F_N(a) \leqslant 0).$$

3. Neutrosophic $\mathcal{N}-$structures

In this section, neutrosophic $\mathcal{N}-$subalgebras and neutrosophic $\mathcal{N}-$ideals of Sheffer stroke BCH-algebras. Unless indicated otherwise, A denotes a Sheffer stroke BCH-algebra.

Definition 3.1. A neutrosophic $\mathcal{N}-$subalgebra A_N of a Sheffer stroke BCH-algebra A is a neutrosophic $\mathcal{N}-$structure on A satisfying the condition

$$\begin{array}{c} T_N((a|(b|b))|(a|(b|b))) \leqslant \max\{T_N(a), T_N(b)\}, \\ \min\{I_N(a), I_N(b)\} \leqslant I_N((a|(b|b))|(a|(b|b))) \\ and \\ \min\{F_N(a), F_N(b)\} \leqslant F_N((a|(b|b))|(a|(b|b))), \end{array} \tag{1}$$

for all $a, b \in A$.

Example 3.2. Consider the Sheffer stroke BCH-algebra A where $A = \{0, x, y, 1\}$ and Sheffer stroke $|$ on A has Cayley table in Table 1 [18]:

Table 1. Cayley table of Sheffer stroke $|$ on A

$\circ$	0	x	y	1
0	1	1	1	1
x	1	y	1	y
y	1	1	x	x
1	1	y	x	0

Then a neutrosophic $\mathcal{N}-$structure

$$A_N = \{\frac{a}{(-0.63, -0.3, -0.08)} : \ a \in A - \{1\}\} \cup \{\frac{1}{(0, -0.98, -0.84)}\}$$

on A is a neutrosophic $\mathcal{N}-$subalgebra of A.

Definition 3.3. Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCH-algebra A and α, β, γ be any elements of $[-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. For the sets

$$T_N^{\alpha} := \{a \in A : T_N(a) \leqslant \alpha\},$$

$$I_N^{\beta} := \{a \in A : \beta \leqslant I_N(a)\}$$

and

$$F_N^{\gamma} := \{a \in A : \gamma \leqslant F_N(a)\},$$

the set $A_N(\alpha, \beta, \gamma) := \{a \in A : T_N(a) \leqslant \alpha, \beta \leqslant I_N(a) \text{ and } \gamma \leqslant F_N(x)\}$ is called the $(\alpha, \beta, \gamma)-$level set of A_N. Moreover, $A_N(\alpha, \beta, \gamma) = T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$.

Theorem 3.4. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCH-algebra A and α, β, γ be any elements of $[-1, 0]$ with $-3 \leqslant \alpha + \beta + \gamma \leqslant 0$. If A_N is a neutrosophic $\mathcal{N}-$subalgebra of A, then the nonempty level set $A_N(\alpha, \beta, \gamma)$ of A_N is a subalgebra of A.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$subalgebra of A and a, b be any elements of $A_N(\alpha, \beta, \gamma)$, for $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leqslant \alpha + \beta + \gamma \leqslant 0$. Then $T_N(a), T_N(b) \leqslant \alpha; \beta \leqslant I_N(a), I_N(b)$ and $\gamma \leqslant F_N(a), F_N(b)$. Since

$$T_N((a|(b|b))|(a|(b|b))) \leqslant \max\{T_N(a), T_N(b)\} \leqslant \alpha,$$

$$\beta \leqslant \min\{I_N(a), I_N(b)\} \leqslant I_N((a|(b|b))|(a|(b|b)))$$

and

$$\gamma \leqslant \min\{F_N(a), F_N(b)\} \leqslant F_N((a|(b|b))|(a|(b|b))),$$

for all $a, b \in A$, it follows that $(a|(b|b))|(a|(b|b)) \in T_N^{\alpha}, I_N^{\beta}, F_N^{\gamma}$. Then

$$(a|(b|b))|(a|(b|b)) \in T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma} = A_N(\alpha, \beta, \gamma).$$

Thus, $A_N(\alpha, \beta, \gamma)$ is a subalgebra of A. □

Theorem 3.5. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCH-algebra A and $T_N^{\alpha}, I_N^{\beta}$ and F_N^{γ} be subalgebras of A, for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leqslant \alpha + \beta + \gamma \leqslant 0$. Then A_N is a neutrosophic $\mathcal{N}-$subalgebra of A.*

Proof. Let $T_N^{\alpha}, I_N^{\beta}$ and F_N^{γ} be subalgebras of A, for all $\alpha, \beta, \gamma \in [-1, 0]$ with

$$-3 \leqslant \alpha + \beta + \gamma \leqslant 0.$$

Assume that

$$\alpha_1 = \max\{T_N(a), T_N(b)\} < T_N((a|(b|b))|(a|(b|b))) = \alpha_2,$$

$$\beta_1 = I_N((a|(b|b))|(a|(b|b))) < \min\{I_N(a), I_N(b)\} = \beta_2$$

and

$$\gamma_1 = F_N((a|(b|b))|(a|(b|b))) < \min\{F_N(a), F_N(b)\} = \gamma_2.$$

If $\alpha = \frac{1}{2}(\alpha_1 + \alpha_2), \beta = \frac{1}{2}(\beta_1 + \beta_2), \gamma = \frac{1}{2}(\gamma_1 + \gamma_2) \in [-1, 0)$, then $\alpha_1 < \alpha < \alpha_2$, $\beta_1 < \beta < \beta_2$ and $\gamma_1 < \gamma < \gamma_2$. Hence, $a, b \in T_N^\alpha, I_N^\beta, F_N^\gamma$ but $(a|(b|b))|(a|(b|b)) \notin T_N^\alpha, I_N^\beta, F_N^\gamma$ which is a contradiction. Thus,

$$T_N((a|(b|b))|(a|(b|b))) \leqslant \max\{T_N(a), T_N(b)\},$$

$$\min\{I_N(a), I_N(b)\} \leqslant I_N((a|(b|b))|(a|(b|b)))$$

and

$$\min\{F_N(a), F_N(b)\} \leqslant F_N((a|(b|b))|(a|(b|b))),$$

for all $a, b \in A$. Thereby, A_N is a neutrosophic $\mathcal{N}-$subalgebra of A. □

Theorem 3.6. *Let $\{A_{N_i} : i \in \mathbb{N}\}$ be a family of all neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke BCH-algebra A. Then $\{A_{N_i} : i \in \mathbb{N}\}$ forms a complete distributive modular lattice.*

Proof. Let S be a nonempty subset of $\{A_{N_i} : i \in \mathbb{N}\}$. Since every A_{N_i} is a neutrosophic $\mathcal{N}-$subalgebra of A, for all $i \in \mathbb{N}$, it satisfies the condition (1), for all $a, b \in A$. Then $\bigcap S$ satisfies the condition (1), and so, $\bigcap S$ is a neutrosophic $\mathcal{N}-$subalgebra of A. Let B be a family of all neutrosophic $\mathcal{N}-$subalgebras of A containing $\bigcup\{A_{N_i} : i \in \mathbb{N}\}$. Thus, $\bigcap B$ is a neutrosophic $\mathcal{N}-$subalgebra of A. If $\bigwedge_{i\in\mathbb{N}} A_{N_i} = \bigcap_{i\in\mathbb{N}} A_{N_i}$ and $\bigvee_{i\in\mathbb{N}} A_{N_i} = \bigcap B$, then $(\{A_{N_i} : i \in \mathbb{N}\}, \bigvee, \bigwedge)$ forms a complete lattice. Also, this lattice is distibutive by the definitions of $\bigvee$ and $\bigwedge$, and so, it is modular from Theorem 2.8. □

Lemma 3.7. *Let A_N be a neutrosophic $\mathcal{N}-$subalgebra of a Sheffer stroke BCH-algebra A. Then*

$$T_N(0) \leqslant T_N(a), I_N(a) \leqslant I_N(0) \text{ and } F_N(a) \leqslant F_N(0), \tag{2}$$

for all $a \in A$.

Proof. Let A_N be a neutrosophic $\mathcal{N}-$subalgebra of A. Then it is obtained from (sBCH.1) that

$$T_N(0) = T_N((a|(a|a))|(a|(a|a))) \leqslant \max\{T_N(a), T_N(a)\} = T_N(a),$$

$$I_N(a) = \min\{I_N(a), I_N(a)\} \leqslant I_N((a|(a|a))|(a|(a|a))) = I_N(0)$$

and

$$F_N(a) = \min\{F_N(a), F_N(a)\} \leqslant F_N((a|(a|a))|(a|(a|a))) = F_N(0),$$

for all $a \in A$. □

The inverse of Lemma 3.7 is not true in general.

Example 3.8. Consider the Sheffer stroke BCH-algebra A in Example 3.2. Then a neutrosophic $\mathcal{N}-$structure

$$A_N = \{\frac{x}{(-0.05, -0.3, -0.29)}\} \cup \{\frac{a}{(-1, -0.03, -0.08)} : a \in A - \{x\}\}$$

on A satisfies the condition (2) but it is not a neutrosophic $\mathcal{N}-$subalgebra of A since $I_N((x|(0|0))|(x|(0|0))) = I_N(x) = -0.3 < -0.03 = \min\{I_N(x), I_N(0)\}$.

Lemma 3.9. *A neutrosophic $\mathcal{N}-$subalgebra A_N of a Sheffer stroke BCH-algebra A satisfies*

$$\begin{gathered} T_N((a|(b|b))|(a|(b|b))) \leqslant T_N(b), \\ I_N(b) \leqslant I_N((a|(b|b))|(a|(b|b))) \\ and \\ F_N(b) \leqslant F_N((a|(b|b))|(a|(b|b))), \end{gathered} \tag{3}$$

for all $a, b \in A$ if and only if T_N, I_N and F_N are constant.

Proof. Let A_N be a a neutrosophic $\mathcal{N}-$subalgebra of A satisfying the condition (3). Since $T_N(a) = T_N((a|(0|0))|(a|(0|0))) \leqslant T_N(0)$, $I_N(0) \leqslant I_N((a|(0|0))|(a|(0|0))) = I_N(a)$ and $F_N(0) \leqslant F_N((a|(0|0))|(a|(0|0))) = F_N(a)$ from Lemma 2.4 (4), it is obtained from Lemma 3.7 that $T_N(a) = T_N(0)$, $I_N(a) = I_N(0)$ and $F_N(a) = F_N(0)$, for all $a \in A$. Therefore, T_N, I_N and F_N are constant. Conversely, it is clear since T_N, I_N and F_N are constant. □

Definition 3.10. A neutrosophic $\mathcal{N}-$structure A_N on a Sheffer stroke BCH-algebra A is called a neutrosophic $\mathcal{N}-$ideal of A if

$$\begin{gathered} T_N(0) \leqslant T_N(a) \leqslant \max\{T_N((a|(b|b))|(a|(b|b))), T_N(b)\}, \\ \min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\} \leqslant I_N(a) \leqslant I_N(0) \\ and \\ \min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\} \leqslant F_N(a) \leqslant F_N(0), \end{gathered} \tag{4}$$

for all $a, b \in A$.

Example 3.11. Consider the Sheffer stroke BCH-algebra A in Example 3.2. Then a neutrosophic $\mathcal{N}-$structure

$$A_N = \{\frac{a}{(-0.71, -0.11, -0.07)} : x = 0, x\} \cup \{\frac{a}{(-0.48, -0.35, -1)} : a = y, 1\}$$

on A is a neutrosophic $\mathcal{N}-$ideal of A.

Lemma 3.12. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCH-algebra A. Then A_N is a neutrosophic $\mathcal{N}-$ideal of A if and only if*

(1) *$a \leqslant b$ implies $T_N(a) \leqslant T_N(b)$, $I_N(b) \leqslant I_N(a)$ and $F_N(b) \leqslant F_N(a)$,*
(2) *$T_N((a|a)|(b|b)) \leqslant \max\{T_N(a), T_N(b)\}$, $\min\{I_N(a), I_N(b)\} \leqslant I_N((a|a)|(b|b))$ and $\min\{F_N(a), F_N(b)\} \leqslant F_N((a|a)|(b|b))$,*

for all $a, b \in A$.

Proof. Let A_N be a neutrosophic $\mathcal{N}-$ideal of A.

(1) Suppose that $a \leqslant b$. Then $(a|(b|b))|(a|(b|b)) = 0$. Thus, we have from Lemma 3.7 that

$$T_N(a) \leqslant \max\{T_N((a|(b|b))|(a|(b|b))), T_N(b)\} = \max\{T_N(0), T_N(b)\} = T_N(b),$$

$$I_N(b) = \min\{I_N(0), I_N(b)\} = \min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\} \leqslant I_N(a)$$

and

$$F_N(b) = \min\{F_N(0), F_N(b)\} = \min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\} \leqslant F_N(a),$$

for all $a, b \in A$.

(2) Since $(((((a|a)|(b|b))|(b|b))|(((a|a)|(b|b))|(b|b)))|(a|a))|(((((a|a)|(b|b))|(b|b))|(((a|a)|(b|b))|(b|b)))|(a|a)) = (((a|a)|(b|b))|(((a|a)|(b|b))|((a|a)|(b|b))))|(((a|a)|(b|b))|(((a|a)|(b|b))|((a|a)|(b|b)))) = 0$ from (S1), (S3) and (sBCH.1), we obtain from Definition 2.3 and (1) that

$$\begin{aligned} T_N((a|a)|(b|b)) &\leqslant \max\{T_N((((a|a)|(b|b))|(b|b))|(((a|a)|(b|b))|(b|b))), T_N(b)\} \\ &\leqslant \max\{T_N(a), T_N(b)\}, \end{aligned}$$

$$\begin{aligned} \min\{I_N(a), I_N(b)\} &\leqslant \min\{I_N((((a|a)|(b|b))|(b|b))|(((a|a)|(b|b))|(b|b))), I_N(b)\} \\ &\leqslant I_N((a|a)|(b|b)) \end{aligned}$$

and

$$\begin{aligned} \min\{F_N(a), F_N(b)\} &\leqslant \min\{F_N((((a|a)|(b|b))|(b|b))|(((a|a)|(b|b))|(b|b))), F_N(b)\} \\ &\leqslant F_N((a|a)|(b|b)), \end{aligned}$$

for all $a, b \in A$.

Conversely, let A_N be a neutrosophic $\mathcal{N}-$structure on A satisfying (1) and (2). Since $(0|(a|a))|(0|(a|a)) = (((0|0)|(0|0))|((0|0)|a))|(((0|0)|(0|0))|((0|0)|a)) = 0$ from (S1)-(S2) and Lemma 2.4 (4), we get that $0 \leqslant a$, for all $a \in A$. Then it follows from (1) that $T_N(0) \leqslant T_N(a)$, $I_N(a) \leqslant I_N(0)$ and $F_N(a) \leqslant F_N(0)$, for all $a \in A$. Since $(a|(((a|(b|b))|(b|b))|((a|(b|b))|(b|b))))|(a|(((a|(b|b))|(b|b))|((a|(b|b))|(b|b)))) = 0$ from Lemma 2.4 (2) and (S2), we have from Definition 2.3 that $a \leqslant (a|(b|b))|(b|b)$, for all $a, b \in A$. Hence, it is obtained (1), (2) and (S2) that

$$\begin{aligned} T_N(a) &\leqslant T_N((a|(b|b))|(b|b)) \\ &= T_N((((a|(b|b))|(a|(b|b)))|((a|(b|b))|(a|(b|b))))(b|b)) \\ &\leqslant \max\{T_N((a|(b|b))|(a|(b|b))), T_N(b)\}, \end{aligned}$$

$$\begin{aligned} \min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\} &\leqslant I_N((((a|(b|b))|(a|(b|b)))|((a|(b|b))|(a|(b|b))))(b|b)) \\ &= I_N((a|(b|b))|(b|b)) \\ &\leqslant I_N(a) \end{aligned}$$

and

$$\begin{aligned}\min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\} &\leqslant F_N((((a|(b|b))|(a|(b|b)))|((a|(b|b))|(a|(b|b))))|(b|b)) \\ &= F_N((a|(b|b))|(b|b)) \\ &\leqslant F_N(a),\end{aligned}$$

for all $a, b \in A$. Thus, A_N is a neutrosophic $\mathcal{N}-$ideal of A. □

Lemma 3.13. *Let A_N be a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCH-algebra A. Then*

(1) $T_N(b) \leqslant T_N(a|(b|b))$, $I_N(a|(b|b)) \leqslant I_N(b)$ *and* $F_N(a|(b|b)) \leqslant F_N(b)$,

(2) $T_N((a|(b|b))|(a|(b|b))) \leqslant \max\{T_N(a), T_N(b)\}$, $\min\{I_N(a), I_N(b)\} \leqslant I_N((a|(b|b))|(a|(b|b)))$ *and* $\min\{F_N(a), F_N(b)\} \leqslant F_N((a|(b|b))|(a|(b|b)))$,

(3) $T_N(a) \leqslant T_N((a|(b|b))|(b|b))$, $I_N((a|(b|b))|(b|b)) \leqslant I_N(a)$ *and* $F_N((a|(b|b))|(b|b)) \leqslant F_N(a)$.

for all $a, b, c \in A$.

Proof. Let A_N be a neutrosophic $\mathcal{N}-$ideal of A. Then

(1) Since

$$\begin{aligned}&(b|((a|(b|b))|(a|(b|b))))|(b|((a|(b|b))|(a|(b|b)))) \\ &= (a|((b|(b|b))|(b|(b|b))))|(a|((b|(b|b))|(b|(b|b)))) \\ &= (a|0)|(a|0) \\ &= (((0|0)|(0|0))|((0|0)|(a|a)))|(((0|0)|(0|0))|((0|0)|(a|a))) \\ &= 0\end{aligned}$$

from (sBCH.1), (S1)-(S2), Lemma 2.4 (3) and (5), we obtain that $b \leqslant a|(b|b)$, for all $a, b \in A$. Thus, it follows from Lemma 3.12 (1) that

$$T_N(b) \leqslant T_N(a|(b|b)), I_N(a|(b|b)) \leqslant I_N(b) \text{ and } F_N(a|(b|b)) \leqslant F_N(b),$$

for all $a, b \in A$.

(2) Since

$$\begin{aligned}&(((a|(b|b))|(a|(b|b)))|(a|a))|(((a|(b|b))|(a|(b|b)))|(a|a)) \\ &= ((b|b)|((a|(a|a))|(a|(a|a))))|((b|b)|((a|(a|a))|(a|(a|a)))) \\ &= ((b|b)|0)|((b|b)|0) \\ &= (((0|0)|(0|0))|((0|0)|b))|(((0|0)|(0|0))|((0|0)|b)) \\ &= 0\end{aligned}$$

from (S1)-(S3), Lemma 2.4 (3), it is obtained that $(a|(b|b))|(a|(b|b)) \leqslant a$, for all $a, b \in A$. Hence, we have from Lemma 3.12 (1) that

$$T_N((a|(b|b))|(a|(b|b))) \leqslant T_N(a) \leqslant \max\{T_N(a), T_N(b)\},$$

$$\min\{I_N(a), I_N(b)\} \leqslant I_N(a) \leqslant I_N((a|(b|b))|(a|(b|b)))$$

and

$$\min\{F_N(a), F_N(b)\} \leqslant F_N(a) \leqslant F_N((a|(b|b))|(a|(b|b))),$$

for all $a, b \in A$.

(3) Since

$$\begin{aligned}&(a|(((a|(b|b))|(b|b))|((a|(b|b))|(b|b))))|(a|(((a|(b|b))|(b|b))|((a|(b|b))|(b|b))))\\&= ((a|(b|b))|((a|(b|b))|(a|(b|b))))|((a|(b|b))|((a|(b|b))|(a|(b|b))))\\&= 0\end{aligned}$$

from Lemma 2.4 (5) and (sBCH.1), it follows from Lemma 3.12 (1) that

$$T_N(a) \leqslant T_N((a|(b|b))|(b|b)),\ I_N((a|(b|b))|(b|b)) \leqslant I_N(a) \text{ and } F_N((a|(b|b))|(b|b)) \leqslant F_N(a),$$

for all $a, b \in A$. □

Theorem 3.14. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCH-algebra A. Then A_N is a neutrosophic $\mathcal{N}-$ideal of A if and only if*

$$\begin{aligned}&((b|(c|c))|(b|(c|c)))|(a|a) = 0|0 \ \textit{ implies } \ T_N(b) \leqslant \max\{T_N(a), T_N(c)\},\\&\min\{I_N(a), I_N(c)\} \leqslant I_N(b) \ \textit{ and } \ \min\{F_N(a), F_N(c)\} \leqslant F_N(b),\end{aligned} \tag{5}$$

for all $a, b, c \in A$.

Proof. Let A_N be a neutrosophic $\mathcal{N}-$ideal of A and $((b|(c|c))|(b|(c|c)))|(a|a) = 0|0$. Since $(((b|(c|c))|(b|(c|c)))|(a|a))|(((b|(c|c))|(b|(c|c)))|(a|a)) = 0$ from (S2), it follows that $(b|(c|c))|(b|(c|c)) \leqslant a$. Then it is obtained from Lemma 3.12 (1) that

$$T_N(b) \leqslant \max\{T_N((b|(c|c))|(b|(c|c))), T_N(c)\} \leqslant \max\{T_N(a), T_N(c)\},$$

$$\min\{I_N(a), I_N(c)\} \leqslant \min\{I_N((b|(c|c))|(b|(c|c))), I_N(c)\} \leqslant I_N(b)$$

and

$$\min\{F_N(a), F_N(c)\} \leqslant \min\{F_N((b|(c|c))|(b|(c|c))), F_N(c)\} \leqslant F_N(b),$$

for all $a, b, c \in A$.

Conversely, let A_N be a neutrosophic $\mathcal{N}-$structure on A satisfying the condition (5). Since

$$\begin{aligned}((0|(a|a))|(0|(a|a)))|(a|a) &= 0|(a|a)\\&= ((0|0)|(0|0))|((0|0)|a)\\&= 0|0\end{aligned}$$

from (S2), (S3) and Lemma 2.4 (3), we have from the condition (5) that

$$T_N(0) \leqslant \max\{T_N(a), T_N(a)\} = T_N(a),$$

$$I_N(a) = \min\{I_N(a), I_N(a)\} \leqslant I_N(0)$$

and

$$F_N(a) = \min\{F_N(a), F_N(a)\} \leqslant F_N(0),$$

for all $a \in A$. Since

$$\begin{aligned}&((a|(b|b))|(a|(b|b)))|(((a|(b|b))|(a|(b|b)))|((a|(b|b))|(a|(b|b))))\\&=(a|(b|b))|((a|(b|b))|(a|(b|b)))\\&=0|0\end{aligned}$$

from (S1), (S2) and (sBCH.1), it follows from the condition (5) that

$$T_N(a) \leqslant \max\{T_N((a|(b|b))|(a|(b|b))), T_N(b)\},$$

$$\min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\} \leqslant I_N(a)$$

and

$$\min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\} \leqslant F_N(a),$$

for all $a, b \in A$. Therefore, A_N is a neutrosophic $\mathcal{N}-$ideal of A. □

Theorem 3.15. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCH-algebra A and α, β, γ be any elements of $[-1, 0]$ with $-3 \leqslant \alpha + \beta + \gamma \leqslant 0$. If A_N is a neutrosophic $\mathcal{N}-$ideal of A, then the nonempty (α, β, γ)-level set $A_N(\alpha, \beta, \gamma)$ of A_N is an ideal of A.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$ideal of A and $A_N(\alpha, \beta, \gamma) \neq \emptyset$, for $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leqslant \alpha + \beta + \gamma \leqslant 0$. Since $T_N(0) \leqslant T_N(a) \leqslant \alpha$, $\beta \leqslant I_N(a) \leqslant I_N(0)$ and $\gamma \leqslant F_N(a) \leqslant F_N(0)$, for all $a \in A$, it is obtained that $0 \in T_N(\alpha, \beta, \gamma)$. Let $(a|(b|b))|(a|(b|b)), b \in A_N(\alpha, \beta, \gamma)$. Since

$$T_N((a|(b|b))|(a|(b|b))), T_N(b) \leqslant \alpha,$$

$$\beta \leqslant I_N((a|(b|b))|(a|(b|b))), I_N(b)$$

and

$$\gamma \leqslant F_N((a|(b|b))|(a|(b|b))), F_N(b),$$

it follows that

$$T_N(a) \leqslant \max\{T_N((a|(b|b))|(a|(b|b))), T_N(b)\} \leqslant \alpha,$$

$$\beta \leqslant \min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\} \leqslant I_N(a)$$

and

$$\gamma \leqslant \min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\} \leqslant F_N(a),$$

for all $a, b \in A$, which imply that $a \in A_N(\alpha, \beta, \gamma)$. Thus, $A_N(\alpha, \beta, \gamma)$ is an ideal of A. □

Theorem 3.16. *Let A_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke BCH-algebra A and $T_N^\alpha, I_N^\beta, F_N^\gamma$ be ideals of A, for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leqslant \alpha + \beta + \gamma \leqslant 0$. Then A_N is a neutrosophic $\mathcal{N}-$ideal of A.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$structure on A and $T_N^{\alpha}, I_N^{\beta}, F_N^{\gamma}$ be ideals of A, for all $\alpha,\beta,\gamma \in [-1,0]$ with $-3 \leqslant \alpha+\beta+\gamma \leqslant 0$. Assume that $T_N(a) < T_N(0)$, $I_N(0) < I_N(a)$ and $F_N(0) < F_N(a)$, for some $a \in A$. If $\alpha = \frac{1}{2}(T_N(0)+T_N(a))$, $\beta = \frac{1}{2}(I_N(0)+I_N(a))$ and $\gamma = \frac{1}{2}(F_N(0)+F_N(a))$ in $[-1,0)$, then $T_N(a) < \alpha < T_N(0)$, $I_N(0) < \beta < I_N(a)$ and $F_N(x0) < \gamma < F_N(a)$. Thus, $0 \notin T_N^{\alpha}, I_N^{\beta}, F_N^{\gamma}$ which is a contradiction with (I1). Hence, $T_N(0) \leqslant T_N(a)$, $I_N(a) \leqslant I_N(0)$ and $F_N(a) \leqslant F_N(0)$, for all $a \in A$. Suppose that

$$\alpha_1 = \max\{T_N((a|(b|b))|(a|(b|b))), T_N(b)\} < T_N(a) = \alpha_2,$$

$$\beta_1 = I_N(a) < \min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\} = \beta_2,$$

and

$$\gamma_1 = F_N(a) < \min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\} = \gamma_2.$$

If $\alpha^* = \frac{1}{2}(\alpha_1+\alpha_2)$, $\beta^* = \frac{1}{2}(\beta_1+\beta_2)$ and $\gamma^* = \frac{1}{2}(\gamma_1+\gamma_2)$ in $[-1,0)$, then $\alpha_1 < \alpha^* < \alpha_2$, $\beta_1 < \beta^* < \beta_2$ and $\gamma_1 < \gamma^* < \gamma_2$. Thus, $(a|(b|b))|(a|(b|b))$, $b \in T_N^{\alpha^*}, I_N^{\beta^*}, F_N^{\gamma^*}$ but $a \notin T_N^{\alpha^*}, I_N^{\beta^*}, F_N^{\gamma^*}$, which is a contradiction with (I2). Thus,

$$T_N(a) \leqslant \max\{T_N((a|(b|b))|(a|(b|b)), T_N(b)\},$$

$$\min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\} \leqslant I_N(a)$$

and

$$\min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\} \leqslant I_N(a),$$

for all $a,b \in A$. Hence, A_N is a neutrosophic $\mathcal{N}-$ideal of A. □

Definition 3.17. Let $(A, |_A, 0_A)$ and $(B, |_B, 0_B)$ be Sheffer stroke BCH-algebras. Then a mapping $f : A \longrightarrow B$ is called a homomorphism if $f(a|_A b) = f(a)|_B f(b)$, for all $a,b \in A$ and $f(0_A) = 0_B$.

Theorem 3.18. *Let $(A, |_A, 0_A)$ and $(B, |_B, 0_B)$ be Sheffer stroke BCH-algebras, $f : A \longrightarrow B$ be a surjective homomorphism and $B_N = \frac{B}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}-$structure on B. Then B_N is a neutrosophic $\mathcal{N}-$ideal of B if and only if $B_N^f = \frac{A}{(T_N^f, I_N^f, F_N^f)}$ is a neutrosophic $\mathcal{N}-$ideal of A where the $\mathcal{N}-$functions $T_N^f, I_N^f, F_N^f : A \longrightarrow [-1,0]$ on A are defined by $T_N^f(a) = T_N(f(a))$, $I_N^f(a) = I_N(f(a))$ and $F_N^f(a) = F_N(f(a))$, for all $a \in A$, respectively.*

Proof. Let $(A, |, 0)$ and $(B, |, 0)$ be Sheffer stroke BCH-algebras, $f : A \longrightarrow B$ be a surjective homomorphism and $B_N = \frac{B}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}-$ideal of B. Then $T_N^f(0_A) = T_N(f(0_A)) = T_N(0_B)) \leqslant T_N(x) = T_N(f(a)) = T_N^f(a)$, $I_N^f(a) = I_N(f(a)) = I_N(x) \leqslant I_N(0_B) =$

$I_N(f(0_A)) = I_N^f = (0_A)$ and $F_N^f(a) = F_N(f(a)) = F_N(x) \leqslant F_N(0_B) = F_N(f(0_A)) = F_N^f = (0_A)$, for all $a \in A$. Moreover,

$$\begin{aligned} T_N^f(a) &= T_N(f(a)) \\ &\leqslant \max\{T_N((f(a)|_B(f(b)|_Bf(b)))|_B(f(a)|_B(f(b)|_Bf(b)))), T_N(f(b))\} \\ &= \max\{T_N(f((a|_A(b|_Ab))|_A(a|_A(b|_Ab)))), T_N(f(b))\} \\ &= \max\{T_N^f((a|_A(b|_Ab))|_A(a|_A(b|_Ab))), T_N^f(b)\}, \end{aligned}$$

$$\begin{aligned} &\min\{I_N^f((a|_A(b|_Ab))|_A(a|_A(b|_Ab))), I_N^f(b)\} \\ &= \min\{I_N(f((a|_A(b|_Ab))|_A(a|_A(b|_Ab)))), I_N(f(b))\} \\ &= \min\{I_N((f(a)|_B(f(b)|_Bf(b)))|_B(f(a)|_B(f(b)|_Bf(b)))), I_N(f(b))\} \\ &\leqslant I_N(f(a)) \\ &= I_N^f(a) \end{aligned}$$

and

$$\begin{aligned} &\min\{F_N^f((a|_A(b|_Ab))|_A(a|_A(b|_Ab))), F_N^f(b)\} \\ &= \min\{F_N(f((a|_A(b|_Ab))|_A(a|_A(b|_Ab)))), F_N(f(b))\} \\ &= \min\{F_N((f(a)|_B(f(b)|_Bf(b)))|_B(f(a)|_B(f(b)|_Bf(b)))), F_N(f(b))\} \\ &\leqslant F_N(f(a)) \\ &= F_N^f(a), \end{aligned}$$

for all $a, b \in A$. Hence, $B_N^f = \dfrac{A}{(T_N^f, I_N^f, F_N^f)}$ is a neutrosophic $\mathcal{N}-$ideal of A.

Conversely, let B_N^f be a neutrosophic $\mathcal{N}-$ideal of A. Thus,

$$T_N(0_B) = T_N(f(0_A)) = T_N^f(0_A) \leqslant T_N^f(a) = T_N(f(a)) = T_N(x),$$

$$I_N(x) = I_N(f(a)) = I_N^f(a) \leqslant I_N^f(0_A) = I_N(f(0_A)) = I_N(0_B)$$

and

$$F_N(x) = F_N(f(a)) = F_N^f(a) \leqslant F_N^f(0_A) = F_N(f(0_A)) = F_N(0_B),$$

for all $x \in B$. Also,

$$\begin{aligned} T_N(x) &= T_N(f(a)) \\ &= T_N^f(a) \\ &\leqslant \max\{T_N^f((a|_A(b|_Ab))|_A(a|_A(b|_Ab))), T_N^f(b)\} \\ &= \max\{T_N(f((a|_A(b|_Ab))|_A(a|_A(b|_Ab)))), T_N(f(b))\} \\ &= \max\{T_N((f(a)|_B(f(b)|_Bf(b)))|_B(f(a)|_B(f(b)|_Bf(b)))), T_N(f(b))\} \\ &= \max\{T_N((x|_B(y|_By))|_B(x|_B(y|_By))), T_N(y)\}, \end{aligned}$$

$$\begin{aligned}
&\min\{I_N((x|_B(y|_By))|_B(x|_B(y|_By))), I_N(y)\}\\
&= \min\{I_N((f(a)|_B(f(b)|_Bf(b)))|_B(f(a)|_B(f(b)|_Bf(b)))), I_N(f(b))\}\\
&= \min\{I_N(f((a|_A(b|_Ab))|_A(a|_A(b|_Ab)))), I_N(f(b))\}\\
&= \min\{I_N^f((a|_A(b|_Ab))|_A(a|_A(b|_Ab))), I_N^f(b)\}\\
&\leqslant I_N^f(a)\\
&= I_N(f(a))\\
&= I_N(x)
\end{aligned}$$

and

$$\begin{aligned}
&\min\{F_N((x|_B(y|_By))|_B(x|_B(y|_By))), F_N(y)\}\\
&= \min\{F_N((f(a)|_B(f(b)|_Bf(b)))|_B(f(a)|_B(f(b)|_Bf(b)))), F_N(f(b))\}\\
&= \min\{F_N(f((a|_A(b|_Ab))|_A(a|_A(b|_Ab)))), F_N(f(b))\}\\
&= \min\{F_N^f((a|_A(b|_Ab))|_A(a|_A(b|_Ab))), F_N^f(b)\}\\
&\leqslant F_N^f(a)\\
&= F_N(f(a))\\
&= F_N(x)
\end{aligned}$$

for all $x, y \in B$. Therefore, $B_N = \dfrac{B}{(T_N, I_N, F_N)}$ is a neutrosophic $\mathcal{N}-$ideal of B. □

Theorem 3.19. *Every neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCH-algebra A is a neutrosophic $\mathcal{N}-$subalgebra of A.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$ideal of A. Since

$$\begin{aligned}
&(((a|(b|b))|(a|(b|b)))|(a|a))|(((a|(b|b))|(a|(b|b)))|(a|a))\\
&= (0|(b|b))|(0|(b|b))\\
&= (((0|0)|(0|0))|((0|0)|b))|(((0|0)|(0|0))|((0|0)|b))\\
&= 0
\end{aligned}$$

from (S2), Lemma 2.4 (3) and (7), it follows that $(a|(b|b))|(a|(b|b)) \leqslant a$, for all $a, b \in A$. Then it is obtained from Lemma 3.12 (1) that

$$T_N((a|(b|b))|(a|(b|b))) \leqslant T_N(a) \leqslant \max\{T_N(a), T_N(b)\},$$

$$\min\{I_N(a), I_N(b)\} \leqslant I_N(a) \leqslant I_N((a|(b|b))|(a|(b|b)))$$

and

$$\min\{F_N(a), F_N(b)\} \leqslant F_N(a) \leqslant F_N((a|(b|b))|(a|(b|b))),$$

for all $a, b \in A$. Thereby, A_N is a neutrosophic $\mathcal{N}-$subalgebra of A. □

The inverse of Theorem 3.19 does notusually hold.

Example 3.20. Consider the Sheffer stroke BCH-algebra S in Example 3.2. Then a neutrosophic $\mathcal{N}-$structure

$$A_N = \{\frac{0}{(-0.82, -0.49, -0.17)}, \frac{1}{(-0.1, -0.91, -0.5)}\} \cup \{\frac{a}{(-0.61, -0.54, -0.3)} : a = x, y\}$$

on A is a neutrosophic $\mathcal{N}-$subalgebra of A but it is not a neutrosophic $\mathcal{N}-$ideal of A since $I_N(1) = -0.91 < -0.54 = \min\{I_N((1|(y|y))|(1|(y|y))), I_N(y)\}$.

Lemma 3.21. *Let A_N be a neutrosophic $\mathcal{N}-$subalgebra of a Sheffer stroke BCH-algebra A satisfying*

$$\begin{array}{c} T_N(a|(b|b)) \leqslant \max\{T_N((a|((b|(c|c))|(b|(c|c))))|(a|((b|(c|c))|(b|(c|c))))), T_N(a|(c|c))\} \\ \min\{I_N((a|((b|(c|c))|(b|(c|c))))|(a|((b|(c|c))|(b|(c|c))))), I_N(a|(c|c))\} \leqslant I_N(a|(b|b)) \\ \text{and} \\ \min\{F_N((a|((b|(c|c))|(b|(c|c))))|(a|((b|(c|c))|(b|(c|c))))), F_N(a|(c|c))\} \leqslant F_N(a|(b|b)), \end{array} \tag{6}$$

for all $a, b, c \in A$. Then A_N is a neutrosophic $\mathcal{N}-$ideal of A.

Proof. Let S_N be a neutrosophic $\mathcal{N}-$subalgebra of A satisfying the condition (6). By Lemma 3.7, $T_N(0) \leqslant T_N(a), I_N(a) \leqslant I_N(0)$ *and* $F_N(a) \leqslant F_N(0)$, for all $a \in A$. By substituting $[a := 0|0]$, $[b := a]$ and $[c := b]$ in the condition (6), simultaneously, it follows from Lemma 2.4 (3) that

$$\begin{array}{rcl} T_N(a) & = & T_N((0|0)|(a|a)) \\ & \leqslant & \max\{T_N(((0|0)|((a|(b|b))|(a|(b|b))))|((0|0)|((a|(b|b))|(a|(c|c))))), T_N((0|0)|(b|b))\} \\ & = & \max\{T_N((a|(b|b))|(a|(b|b))), T_N(b)\}, \end{array}$$

$$\begin{array}{rcl} \min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\} & = & \min\{I_N(((0|0)|((a|(b|b))|(a|(b|b))))|((0| \\ & & 0)|((a|(b|b))|(a|(b|b))))), I_N((0|0)|(b|b))\} \\ & \leqslant & I_N((0|0)|(a|a)) \\ & = & I_N(a) \end{array}$$

and

$$\begin{array}{rcl} \min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\} & = & \min\{F_N(((0|0)|((a|(b|b))|(a|(b|b))))|((0| \\ & & 0)|((a|(b|b))|(a|(b|b))))), F_N((0|0)|(b|b))\} \\ & \leqslant & F_N((0|0)|(a|a)) \\ & = & F_N(a), \end{array}$$

for all $a, b \in A$. Thus, A_N is a neutrosophic $\mathcal{N}-$ideal of A. □

Lemma 3.22. *Let A_N be a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCH-algebra A. Then the subsets $A_{T_N} = \{a \in A : T_N(a) = T_N(0)\}$, $A_{I_N} = \{a \in A : I_N(a) = I_N(0)\}$ and $A_{F_N} = \{a \in A : F_N(a) = F_N(0)\}$ of A are ideals of A.*

Proof. Let A_N be a neutrosophic $\mathcal{N}-$ideal of A. Then it is clear that $0 \in A_{T_N}, A_{I_N}, A_{F_N}$. Suppose that $(a|(b|b))|(a|(b|b)), b \in A_{T_N}, A_{I_N}, A_{F_N}$. Since

$$T_N(b) = T_N(0) = T_N((a|(b|b))|(a|(b|b))),$$

$$I_N(b) = I_N(0) = I_N((a|(b|b))|(a|(b|b)))$$

and

$$F_N(b) = F_N(0) = F_N((a|(b|b))|(a|(b|b))),$$

it follows that

$$T_N(a) = T_N(a) \leqslant \max\{T_N((a|(b|b))|(a|(b|b))), T_N(b)\} = \max\{T_N(0), T_N(0)\} = T_N(0),$$

$$I_N(0) = \min\{I_N(0), I_N(0)\} = \min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\} \leqslant I_N(a)$$

and

$$F_N(0) = \min\{F_N(0), F_N(0)\} = \min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\} \leqslant F_N(a).$$

Thus, $T_N(a) = T_N(0)$, $I_N(a) = I_N(0)$ and $F_N(a) = F_N(0)$, and so, $a \in A_{T_N}, A_{I_N}, A_{F_N}$. Hence, A_{T_N}, A_{I_N} and A_{F_N} are ideals of A. □

Definition 3.23. Let A be a Sheffer stroke BCH-algebra. Define the subsets

$$A_N^{a_t} := \{a \in A : T_N(a) \leqslant T_N(a_t)\},$$

$$A_N^{a_i} := \{a \in A : I_N(a_i) \leqslant I_N(a)\}$$

and

$$A_N^{a_f} := \{a \in A : F_N(a_f) \leqslant F_N(a)\}$$

of S, for all $a_t, a_i, a_f \in A$. Also, it is obvious that $a_t \in A_N^{a_t}, a_i \in A_N^{a_i}$ and $a_f \in A_N^{a_f}$.

Example 3.24. Consider the Sheffer stroke BCH-algebra A in Example 3.2. Let

$$T_N(a) = \begin{cases} 0, & \text{if } a = 0, 1 \\ -0.46, & \text{if } a = x \\ -0.23, & a = y, \end{cases} \qquad I_N(a) = \begin{cases} -0.17, & \text{if } a = 1 \\ 0, & \text{otherwise}, \end{cases}$$

$$F_N(a) = \begin{cases} -1, & \text{if } a = 0 \\ -0.4, & \text{otherwise}, \end{cases} \qquad a_t = x, a_i = 0 \text{ and } a_f = 1.$$

Then

$$A_N^{a_t} = \{a \in A : T_N(a) \leqslant T_N(x)\} = \{x\},$$

$$A_N^{a_i} = \{a \in A : I_N(0) \leqslant I_N(a)\} = \{0, x, y\}$$

and

$$A_N^{a_f} = \{a \in A : F_N(1) \leqslant F_N(a)\} = \{x, y, 1\}.$$

Theorem 3.25. *Let a_t, a_i and a_f be any elements of a Sheffer stroke BCH-algebra A. If A_N is a neutrosophic $\mathcal{N}-$ideal of A, then $A_N^{a_t}, A_N^{a_i}$ and $A_N^{a_f}$ are ideals of A.*

Proof. Let a_t, a_i and a_f be any elements of A and A_N be a neutrosophic $\mathcal{N}-$ideal of A. since $T_N(0) \leqslant T_N(a_t)\}$, $I_N(a_i) \leqslant I_N(0)$ and $F_N(a_f) \leqslant F_N(0)$, for all $a_t, a_i, a_f \in A$, it is obtained that $0 \in A_N^{a_t}, A_N^{a_i}, A_N^{a_f}$. Suppose that $(a|(b|b))|(a|(b|b)), b \in A_N^{a_t}, A_N^{a_i}, A_N^{a_f}$. Since

$$T_N((a|(b|b))|(a|(b|b))), T_N(b) \leqslant T_N(a_t),$$

$$I_N(a_i) \leqslant I_N((a|(b|b))|(a|(b|b))), I_N(b)$$

and

$$F_N(a_f) \leqslant F_N((a|(b|b))|(a|(b|b))), F_N(b),$$

it follows that

$$T_N(a) \leqslant \max\{T_N((a|(b|b))|(a|(b|b))), T_N(b)\} \leqslant T_N(a_t),$$

$$I_N(a_i) \leqslant \min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\} \leqslant I_N(a)$$

and

$$F_N(a_f) \leqslant \min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\} \leqslant F_N(a),$$

which imply that $a \in A_N^{a_t}, A_N^{a_i}, A_N^{a_f}$. Hence, $A_N^{a_t}, A_N^{a_i}$ and $A_N^{a_f}$ are ideals of A. □

Example 3.26. Consider the Sheffer stroke BCH-algebra A in Example 3.2. For a neutrosophic $\mathcal{N}-$ideal

$$A_N = \{\frac{a}{(-1, -0.47, -0.81)} : \ a = 0, x\} \cup \{\frac{a}{(-0.34, -0.69, -0.95)} : \ a = y, 1\}$$

of A and $a_t = 0, a_i = x, a_f = y \in S$, the subsets

$$A_N^{a_t} = \{a \in A : T_N(a) \leqslant T_N(0)\} = \{0, x\},$$

$$A_N^{a_i} = \{a \in A : I_N(x) \leqslant I_N(a)\} = \{0, x\}$$

and

$$A_N^{a_f} = \{a \in A : F_N(y) \leqslant F_N(a)\} = A$$

of A are ideals of A.

Theorem 3.27. *Let a_t, a_i and a_f be any elements of a Sheffer stroke BCH-algebra A and A_N be a neutrosophic $\mathcal{N}-$structure on A.*

(1) *If* $A_N^{a_t}, A_N^{a_i}$ *and* $A_N^{a_f}$ *are ideals of* A*, then*

$$\max\{T_N((b|(c|c))|(b|(c|c))), T_N(c)\} \leqslant T_N(a) \Rightarrow T_N(b) \leqslant T_N(a),$$

$$I_N(a) \leqslant \min\{I_N((b|(c|c))|(b|(c|c))), I_N(c)\} \Rightarrow I_N(a) \leqslant I_N(b) \quad \textit{and} \tag{7}$$

$$F_N(a) \leqslant \min\{F_N((b|(c|c))|(b|(c|c))), F_N(c)\} \Rightarrow F_N(a) \leqslant F_N(b),$$

for all $a, b, c \in A$.

(2) *If* A_N *satisfies the condition (7) and*

$$T_N(0) \leqslant T_N(a), \quad I_N(a) \leqslant I_N(0) \quad \textit{and} \quad F_N(a) \leqslant F_N(0), \tag{8}$$

for all $a \in A$*, then* $A_N^{a_t}, A_N^{a_i}$ *and* $A_N^{a_f}$ *are ideals of* A*, for all* $a_t \in T_N^{-1}$*,* $a_i \in I_N^{-1}$ *and* $a_f \in F_N^{-1}$.

Proof. Let a_t, a_i and a_f be any elements of A and A_N be a neutrosophic $\mathcal{N}-$structure on A.

(1) Suppose that $A_N^{a_t}, A_N^{a_i}$ and $A_N^{a_f}$ are ideals of A and

$$\max\{T_N((b|(c|c))|(b|(c|c))), T_N(c)\} \leqslant T_N(a),$$

$$I_N(a) \leqslant \min\{I_N((b|(c|c))|(b|(c|c))), I_N(c)\}$$

and

$$F_N(a) \leqslant \min\{F_N((b|(c|c))|(b|(c|c))), F_N(c)\}.$$

Since $(b|(c|c))|(b|(c|c)), c \in A_N^{a_t}, A_N^{a_i}, A_N^{a_f}$ where $a_t = a_i = a_f = a$, it is obtained that $b \in A_N^{a_t}, A_N^{a_i}, A_N^{a_f}$ where $a_t = a_i = a_f = a$. Thus, $T_N(b) \leqslant T_N(a)$, $I_N(a) \leqslant I_N(b)$ and $F_N(a) \leqslant F_N(b)$, for all $a, b, c \in A$.

(2) Let A_N be a neutrosophic $\mathcal{N}-$structure on A satisfying the conditions (7) and (8), for any $a_t \in T_N^{-1}$, $a_i \in I_N^{-1}$ and $a_f \in F_N^{-1}$. Then it follows from the condition (8) that $0 \in A_N^{a_t}, A_N^{a_i}, A_N^{a_f}$. Assume that $(a|(b|b))|(a|(b|b)), b \in A_N^{a_t}, A_N^{a_i}, A_N^{a_f}$. Thus,

$$T_N((a|(b|b))|(a|(b|b))), T_N(b) \leqslant T_N(a_t),$$

$$I_N(a_i) \leqslant I_N((a|(b|b))|(a|(b|b))), I_N(b)$$

and

$$F_N(a_f) \leqslant F_N((a|(b|b))|(a|(b|b))), F_N(b).$$

Since

$$\max\{T_N((a|(b|b))|(a|(b|b))), T_N(b)\} \leqslant T_N(a_t),$$

$$I_N(a_i) \leqslant \min\{I_N((a|(b|b))|(a|(b|b))), I_N(b)\}$$

and

$$F_N(a_f) \leqslant \min\{F_N((a|(b|b))|(a|(b|b))), F_N(b)\},$$

we have from the condition (7) that

$$T_N(a) \leqslant T_N(a_t), I_N(a_i) \leqslant I_N(a) \text{ and } F_N(a_f) \leqslant F_N(a),$$

which imply that $a \in A_N^{a_t}, A_N^{a_i}, A_N^{a_f}$. Hence, $A_N^{a_t}, A_N^{a_i}$ and $A_N^{a_f}$ are ideals of A. □

Example 3.28. Consider the Sheffer stroke BCH-algebra A in Example 3.2. Let

$$T_N(a) = \begin{cases} -1, & \text{if } a = 0, y \\ -0.05, & \text{otherwise,} \end{cases} \qquad I_N(a) = \begin{cases} -0.79, & \text{if } a = x, 1 \\ 0, & \text{otherwise,} \end{cases}$$

$$F_N(a) = \begin{cases} -0.02, & \text{if } a = 0 \\ -0.72, & \text{otherwise,} \end{cases} \qquad \text{and} \quad a_t = 0, a_i = 1 \ a_f = y \in A.$$

Then the ideals

$$A_N^{a_t} = \{0, y\}, A_N^{a_i} = A \text{ and } A_N^{a_f} = A$$

of A satisfy the condition (7).

Let

$$A_N = \{\frac{a}{(-0.8, -0.32, 0)} : \ a = 0, y\} \cup \{\frac{a}{(-0.27, -0.45, -0.51)} : \ a = x, 1\}$$

be a neutrosophic $\mathcal{N}-$structure on A satisfying the conditions (7) and (8). Then the subsets $A_N^{a_t} = A, A_N^{a_i} = \{0, y\}$ and $A_N^{a_f} = \{0, y\}$ of A are ideals of A, where $a_t = x, a_i = y$ and $a_f = 0$.

4. **Conclusion**

In this study, we introduce a neutrosophic $\mathcal{N}-$subalgebra, a neutrosophic $\mathcal{N}-$ideal and a level-set of neutrosophic $\mathcal{N}-$structures on Sheffer stroke BCH-algebras. Then we show that the level-set of a neutrosophic $\mathcal{N}-$subalgebra (a neutrosophic $\mathcal{N}-$ideal) of a Sheffer stroke BCH-algebra is its subalgebra (an ideal) and vice versa. Also, we prove that the family of all neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke BCH-algebra forms a complete distributive modular lattice. We analyze the situations which $\mathcal{N}-$functions are constant. Moreover, we present new statements equivalent to the definition of a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCH-algebra and its properties. By defining a homomorphism on a Sheffer stroke BCH-algebra, we demonstrate relationships between neutrosophic $\mathcal{N}-$ideals of two Sheffer stroke BCH-algebras by means of a surjective homomorphism. We propound that every neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke BCH-algebra is its neutrosophic $\mathcal{N}-$subalgebra but the inverse is not true in general. Besides, the subsets A_{T_N}, A_{I_N} and A_{F_N} of a Sheffer stroke BCH-algebra are its ideals for the neutrosophic $\mathcal{N}-$ideal which is defined by means of the $\mathcal{N}-$functions T_N, I_N and F_N. After that we describe the subsets $A_N^{a_t}$, $A_N^{a_i}$ and $A_N^{a_f}$ of a Sheffer stroke BCH-algebra for its any elements a_t, a_i, a_f and state that these subsets are ideals of this algebraic structure if a neutrosophic $\mathcal{N}-$structure on this algebraic structure is the neutrosophic $\mathcal{N}-$ideal.

In future works, we wish to study on fuzzy and plithogenic structures on Sheffer stroke BCH-algebras.

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