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NeutroAlgebra of Idempotents in Group Rings

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Abstract: In this paper, the authors study the new concept of NeutroAlgebra of idempotents in group rings. It is assumed that *RG* is the group ring of a group G over the ring *R*. *R* should be a commutative ring with unit 1. *G* can be a finite or an infinite order group which can be commutative or non-commutative. We obtain conditions under which the idempotents of the group rings *ZG*, Z_nG, and *QG* form a NeutroAlgebra under the operations + or ×. Some collection of idempotents in these group rings form an AntiAlgebra. We propose some open problems which has resulted from this study.

Keywords: Symmetric group; NeutroAlgebra; AntiAlgebra; group ring, NeutrosubAlgebra, Partial Algebra.

1. Introduction

In this paper, we study the NeutroAlgebra of idempotent elements of the group ring RG, where R is a commutative ring with unit 1 (R can be Z or R or Q or Z_n; n a composite or a prime number) and G is a commutative or a non-commutative group of finite order. We only study the NeutroAlgebra of idempotent elements in the group ring under '+' and '×' operations inherited from the group ring *RG*.

The study of neutrosophy was first carried out by [1]. This concept can analyze real-world data's uncertainty, inconsistency, and indeterminacy. The new notion of NeutroAlgebraic structures and AntiAlgebraic structures was first introduced in [2] in 2019. There are several interesting results in this direction, like NeutroAlgebra as a generalization of partial algebra [6-7], Neutro-BE-Algebra and Anti-BE-Algebra, Neutro-BCK Algebra introduced in [8]. [9] has analyzed NeutroAlgebras in the context of number systems Neutrosophic triplets as NeutroAlgebra was carried out in [11-19]. [20] introduces Neutrosophic quadruple vector spaces. Extended Neutrosophic triplets are introduced and analyzed in [21-24]. Various researchers studied other unique properties of Neutrosophic triplets in [25-30]. Application of Neutrosophic theory is carried out in [31- 36], has been extended to the study of neutrosophic vector spaces, and algebraic codes.

This paper is organized into five sections. The first section is introductory. The second section presents the basic concepts needed to make this paper a self-contained one. Section three discusses and describes the NeutroAlgebra of idempotents in the group rings *ZG* and *QG* and the NeutroAlgebra of idempotents in the group ring Z_nG. The final section gives the conclusions based on the study and suggests a few open conjectures which will be taken for future research.

2. Basic Concepts

This section gives a few essential concepts for this paper to be self-contained. First, we recall the concept of the group ring, then recall the definitions and describe a few properties of the NeutroAlgebra and AntiAlgebra by some illustrative examples.

Definition 2.1. *Let R be a commutative ring with unit 1 and G be a multiplicative group. The group RG of the group G over the ring R consists of all finite formal sums of the form* $\sum_i \alpha_i g_i$ *(i – runs over a finite number) where* $\alpha_i \in G$ *and* $g_i \in G$ *satisfy the following conditions.*

i) $\sum_{i=1}^{n} \alpha_i g_i = \sum_{i=1}^{n} \beta_i g_i \Leftrightarrow \alpha_i, \beta_i \in R;\ \alpha_i = \beta_i$ *for i = 1, 2, …, n;* $g_i \in G$.

ii) $\left(\sum_{i=1}^{n} \alpha_i g_i\right) + \left(\sum_{i=1}^{n} \beta_i g_i\right) = \sum_{i=1}^{n} (\alpha_i + \beta_i)\ g_i,\ g_i \in G;\ \alpha_i, \beta_i \in R$

iii) $\left(\sum_{i=1}^{n} \alpha_i g_i\right)\left(\sum_{i=1}^{n} \beta_i g_i\right) = \sum_k \gamma_k m_k$ *where* $\gamma_k = \sum \alpha_i \beta_j$ *and* $g_i = m_k$

iv) $rg = gr$ *for all* $r \in R$ *and* $g \in G$

v) $r\sum_{i=1}^{n} r_i g_i = \sum_{i=1}^{n} (rr_i) g_i$ *for* $r_i, r_i \in R,\ g_i \in G$ *and* $\sum r_i g_i \in RG$.

RG is a ring with $0 \in R$*, which acts as the identity for addition. Since* $1 \in R$ *and we have 1.* $G = G \subseteq G$ *and* $\mathrm{Re} = R \subseteq G$*, where e is the identity of G.*

For more about grouprings and their properties refer [3].

Example 2.1. Let Z_4 = *{0, 1, 2, 3}* be the ring of modulo integers. $G = \langle g \mid g^2 = 1 \rangle$ be the cyclic group of order 2. Then the group ring Z_4G = *{1, 0, 2, 3, g, 2g, 3g, 1 + g, 2 + g, 3 + g, 1 + 2g, 1 + 3g, 2 + 2g, 2 + 3g, 3 + 2g, 3 + 3g}.*

We now proceed to recall the definition of support of α in a group ring *RG* where $\alpha \in RG$. We denote support of α by *supp* α = {all group elements in α with non-zero coefficients from R } and $|$*supp* $\alpha|$ = {number of group elements in α which has non-zero coefficient}.

Suppose $\alpha = 1 + 3g + 0g + 5g^3 + 0g^4 + 6g^5 \in RG$ where $R = G$ and $G = \langle g \mid g^6 = 1 \rangle$ then *supp* α = *{1, g,* g^3*,* g^5*}* of the group ring *RG* of the group *G* over the ring *R*; which is subset of the group *G* and $|$*supp* $\alpha| = 4$.

Now we recall the definition of NeutroAlgebra and describe this concept as in [2].

A NeutroAlgebra is an algebra with at least one Neutro-operation or one Neutroaxiom (axiom that is true for some elements, indeterminate or false for other elements) [2]. A partial algebra has at the minimum one partial operation, and all axioms are classical. [6] has described NeutroAlgebra that are partial algebras.

Similarly, an AntiAlgebra is a non-empty set endowed with at least one anti operation (or anti operations) or at least one anti axiom.

We proceed to give examples of NeutroAlgebra and AntiAlgebra.

Example 2.2. Let Z_{12} be the ring of modulo integers 12. The idempotents of Z_{12} are *{4, 9}* = W ; 0 and 1 in Z_{12} are defined as trivial idempotents of Z_{12}.

The Cayley table of *W* is as follows under +.

Table 1 Cayley table of *{W, +}*

+	4	9
4	od	od
9	od	od

So $\{W,+\}$ is an AntiAlgebra. The Cayley table of W under × is as follows.

Table 2 Cayley table of *{W, ×}*

×	4	9
4	4	od
9	od	9

Clearly if *V* = *{0, 1, 9, 4}* then the Cayley table of *V* under + is as follows.

Table 3 Cayley table of *{V, +}*

+	0	1	4	9
0	0	1	4	9
1	1	od	od	od
4	4	od	od	1
9	9	od	1	od

$\{V,+\}$ is a NeutroAlgebra of idempotents under +. Clearly $\{V,\times\}$ is a commutative semigroup of order 4.

3. NeutroAlgebra of idempotents in the group ring *ZG(QG)*

This section deals with NeutroAlgebra of idempotents in the group ring RG, where R is the ring of integers Z or the field of rationals Q of characteristic zero. This section finds the NeutroAlgebra of idempotents in the group ring ZG and QG, where G is taken as a commutative or a non-commutative group of finite order.

Example 3.1. Let QG be the group ring of G over Q where $G=\langle g \mid g^2=1\rangle$ is a cyclic group of order 2. A few of the idempotents of G are $\alpha=\frac{1}{2}(1-g)$ that is

$$\alpha^2=\frac{1}{4}(1-2g+g^2) = \frac{1}{4}\times\{2(1-g)\} = \frac{1}{2}(1-g) \text{ (using the fact } g^2=1).$$

If $\beta=\frac{1}{2}(1+g)\in QG$ then

$$\beta^2=\left\{\frac{1}{2}(1+g)\right\}^2 = \frac{1}{4}(1+2g+g^2)=\frac{1}{4}(2+2g)=\frac{1}{2}(1+g) \text{ as } g^2=1.$$

Now $QG=\{\alpha+\beta g \mid \alpha,\beta\in Q,\ g^2=1\}$.

Thus, the only two non-trivial idempotents of QG are $\alpha = \frac{1-g}{2}$ and $\beta = \frac{1+g}{2}$. QG has no other non-trivial idempotents. For if $x+yg$ is a nontrivial idempotent in QG with $x, y \in Q \setminus \{0\}$.

If $x+yg=t$ is an idempotent in QG then $t^2=(x+yg)^2=x+yg=t$. This implies

$$t^2=(x^2+2xyg+y^2)=x+yg=t \text{ as } g^2=1.$$

$$(x^2+y^2)+2xyg=x+yg$$

By equating the like terms.

$$x^2+y^2=x \qquad (1)$$

and

$$2xy=y \qquad (2)$$

Since $x, y \in Q \setminus \{0\};\ y \neq 0$ so $y^{-1} \in Q$.

Hence $2xy=y$ implies $(2x-1)y=0$ as $y \neq 0$. $2x=1$ or $x=\frac{1}{2}$. Using $x=\frac{1}{2}$ in equation (1) we get

$\left(\frac{1}{2}\right)^2+y^2=\frac{1}{2}$ so that $y^2+\frac{1}{2}-\frac{1}{4}$ or $y=\frac{1}{\pm 2}$.

Thus, the element $x=yg$ is an idempotent if and only if

$$x=y=\frac{1}{2} \text{ or } x=\frac{1}{2} \text{ and } y=-\frac{1}{2}.$$

That is $\alpha=\frac{1}{2}(1+g)$ or $\beta=\frac{1}{2}(1-g)$.

Other possibilities are $x=y=-\frac{1}{2}$ in this case $a=\frac{-1}{2}(1+g)$ but

$$a^2=\frac{1}{4}(1+2g+g^2)=\frac{2(1+g)}{4}=\frac{(1+g)}{2} \neq a.$$

Hence $a=\frac{-1}{2}(1+g)$ is not an idempotent of QG. So, if $x=y=\frac{-1}{2}$ does not yield an idempotent.

Suppose $x=\frac{-1}{2}$ and $y=\frac{1}{2}$ then $b=\frac{-1+g}{2}$. Now

$$b^2=\frac{1}{4}\left[1+g^2-2g\right]=\frac{1}{4}\left[2-2g\right]=\frac{1}{2}\left[1-g\right] \neq b.$$

So $b=\frac{-1+g}{2}$ too is not an idempotent of QG. Thus $\alpha=\frac{1}{2}(1+g)$ and $\beta=\frac{1}{2}(1-g)$ are the only nontrivial idempotents of QG.

Let $V=\left\{\frac{1}{2}(1+g), \frac{1}{2}(1-g)\right\}$ be the collection of all non-trivial idempotents of QG.

We give the Cayley table of V under +.

Table 4 Cayley table of *{V, +}*.

$+$	$\frac{1}{2}(1+g)$	$\frac{1}{2}(1-g)$
$\frac{1}{2}(1+g)$	*od*	*od*
$\frac{1}{2}(1-g)$	*od*	*od*

So, V under + is an AntiAlgebra of idempotents in QG. (od denotes the term outerdefined).

Now consider V under ×. The Cayley table of V is as follows:

Table 5 Cayley table of *{V, ×}*

$\times$	$\frac{1}{2}(1+g)$	$\frac{1}{2}(1-g)$
$\frac{1}{2}(1+g)$	$\frac{1}{2}(1+g)$	*od*
$\frac{1}{2}(1-g)$	*od*	$\frac{1}{2}(1-g)$

V under × is a NeutroAlgebra of idempotents of G.

Suppose $W=\left\{\frac{1+g}{2},\frac{1-g}{2},0,1\right\}$; now we find the Cayley table under +.

Table 6 Cayley table of *{W, +}*.

$+$	*1*	*0*	$\frac{1+g}{2}$	$\frac{1-g}{2}$
1	*od*	*1*	*od*	*od*
0	*1*	*0*	$\frac{1+g}{2}$	$\frac{1-g}{2}$
$\frac{1+g}{2}$	*od*	$\frac{1+g}{2}$	*od*	*od*
$\frac{1-g}{2}$	*od*	$\frac{1-g}{2}$	*od*	*od*

Clearly, W under + is a NeutroAlgebra of idempotents in QG under the + operation.

Consider the Cayley table under × of W given in the following:

Table 7 Cayley Table of W under ×

$\times$	*0*	*1*	$\frac{1+g}{2}$	$\frac{1-g}{2}$
0	*0*	*0*	*0*	*0*

1	0	1	$\frac{1+g}{2}$	$\frac{1-g}{2}$
$\frac{1+g}{2}$	0	$\frac{1+g}{2}$	$\frac{1+g}{2}$	0
$\frac{1-g}{2}$	0	$\frac{1-g}{2}$	0	$\frac{1-g}{2}$

Thus $\{W,\times\}$ is a semigroup of idempotents in QG.

Example 3.2. Let QS_3 be the group ring of the symmetric group S_3 over the field of rationals. Here the Cayley table for S_3 is as follows.

$$S_3 = \left\{ e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \; p_1 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \; p_2 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \; p_3 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \right.$$

$$\left. p_4 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \; p_5 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$$ is the permutation group of degree 3.

The Cayley table of the group S_3 under composition 'o' of maps is as follows:

Table 8. Cayley table of S_3 under 'o'.

o	e	p_1	p_2	p_3	p_4	p_5
e	e	p_1	p_2	p_3	p_4	p_5
p_1	p_1	e	p_5	p_4	p_3	p_2
p_2	p_2	p_4	e	p_5	p_1	p_3
p_3	p_3	p_5	p_4	e	p_2	p_1
p_4	p_4	p_2	p_3	p_1	p_5	e
p_5	p_5	p_3	p_1	p_2	e	p_4

The nontrivial idempotents of QS_2 are $\alpha_1 = \frac{1}{2}(1+p_1)$, $\alpha_2 = \frac{1}{2}(1 \pm p_2)$, $\alpha_3 = \frac{1}{2}(1 \pm p_3)$, $\alpha_4 = \frac{1}{3}(1+p_4+p_5)$ and $\alpha_5 = \frac{1}{6}(1+p_1+p_2+p_3+p_4+p_5)$.

Let $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ be the set of some nontrivial idempotents in QG.

Now we find the Cayley table of B under + in the following.

Thus, B under

Table 9. Cayley table of B under +

+	α_1	α_2	α_3	α_4	α_5
α_1	od	od	od	od	od
α_2	od	od	od	od	od
α_3	od	od	od	od	od
α_4	od	od	od	od	od

α_5	od	od	od	od	od

Thus, *B* under + is an AntiAlgebra of idempotents in QS_3.

Now we consider the Cayley table of B under ×.

Table 10. Cayley table of B under ×.

×	α_1	α_2	α_3	α_4	α_5
α_1	α_1	od	od	od	α_5
α_2	od	α_2	od	od	α_5
α_3	od	od	α_3	od	α_5
α_4	od	od	od	α_4	α_5
α_5	α_5	α_5	α_5	α_5	α_5

Clearly, *B* under × is NeutroAlgebra of idempotents of QS_3.

We give yet another example of a cyclic group of composite order. Based on these examples, we will proceed onto prove the following results.

Example 3.3. Let $G = \langle g \mid g^{24} = 1 \rangle$ 1 be the cyclic group of order 24. *Q* be the field of rationals. *QG* be the group ring of *G* order *Q*.

The idempotents of *QG* are

$$x_1 = \frac{1}{2}(1+g^{12})$$

$$x_2 = \frac{1}{3}(1+g^{8}+g^{16})$$

$$x_3 = \frac{1}{4}(1+g^{6}+g^{12}+g^{18})$$

$$x_4 = \frac{1}{6}(1+g^{4}+g^{6}+g^{12}+g^{16}+g^{20})$$

$$x_5 = \frac{1}{8}(1+g^{3}+g^{6}+g^{9}+g^{12}+g^{15}+g^{18}+g^{21})$$

$$x_6 = \frac{1}{12}(1+g^{2}+g^{4}+g^{6}+g^{8}+g^{14}+g^{10}+g^{12}+g^{16}+g^{18}+g^{20}+g^{22})$$

and $$x_7 = \frac{1}{24}(1+g+g^{2}+\ldots g^{23}).$$

Now let $W = (x_1, x_2, x_3, \ldots, x_6, x_7)$ be the collection of some set of idempotents in QG.

We see $y_1 = \frac{1}{2}(1-g^{12})$

$$y_2 = \frac{1}{4}(1-g^{6}+g^{12}-g^{18})$$

$$y_3 = \frac{1}{6}(1 - g^4 + g^3 - g^{12} + g^{16} - g^{20})$$

$$y_4 = \frac{1}{8}(1 - g^3 + g^6 - g^9 + g^{12} - g^{15} + g^{18} - g^{21})$$

$$y_5 = \frac{1}{12}(1 - g^2 + g^4 - g^6 + g^8 - g^{10} + g^{12} - g^{14} + g^{16} - g^{18} + g^{20} - g^{22})$$

are also idempotents of QG.

Now we find the Cayley tables of W under + and ×.

Let $M = \{y_1, y_3, y_4, y_5\}$ be the set of some idempotents of QG we find the Cayley table of M also under + and × is given in Tables 14 and 15 respectively.

First, the Cayley table of W under + is as follows.

Table 11. Cayley table of W under +.

+	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	od	od	od	od	od	od	od
x_2	od	od	od	od	od	od	od
x_3	od	od	od	od	od	od	od
x_4	od	od	od	od	od	od	od
x_5	od	od	od	od	od	od	od
x_6	od	od	od	od	od	od	od
x_7	od	od	od	od	od	od	od

Clearly, the set W of idempotents of QG is an AntiAlgebra under + as every term is outer defined in W. Now we give the table of W under product.

Table 12. Cayley table of W under ×.

×	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	x_1	x_4	x_3	x_4	x_5	x_6	x_7
x_2	x_4	x_2	x_6	x_4	x_7	x_6	x_7
x_3	x_3	x_6	x_3	x_6	x_5	x_6	x_7
x_4	x_4	x_4	x_6	x_4	x_7	x_6	x_7
x_5	x_5	x_7	x_5	x_7	x_5	x_7	x_7
x_6	x_6	x_6	x_6	x_6	x_7	x_6	x_7
x_7	x_7	x_7	x_7	x_7	x_7	x_7	x_7

Clearly, W under × is a semigroup and is not a NeutroAlgebra or AntiAlgebra.

If, on the other hand, x_7, the whole group sum is deleted as the support of x_7 is G, we will get for the corresponding set $\{W \setminus x_7\}$ the Cayley table under × which is as follows.

Table 13. Cayley table $W \setminus \{x_7\}$ under ×.

×	x_1	x_2	x_3	x_4	x_5	x_6
x_1	x_1	x_4	x_3	x_4	x_5	x_6
x_2	x_4	x_2	x_6	x_4	*od*	x_6
x_3	x_3	x_6	x_3	x_6	x_5	x_6
x_4	x_4	x_4	x_6	x_4	*od*	x_6
x_5	x_5	*od*	x_5	*od*	x_5	*od*
x_6	x_6	x_6	x_6	x_6	*od*	x_6

Thus $\{W \setminus x_7\}$ is a NeutroAlgebra of idempotents in QG.

The Cayley table of *M* under + is as follows.

Table 14: Cayley table of *M* under +

+	y_1	y_2	y_3	y_4	y_5
y_1	*od*	*od*	*od*	*od*	*od*
y_2	*od*	*od*	*od*	*od*	*od*
y_3	*od*	*od*	*od*	*od*	*od*
y_4	*od*	*od*	*od*	*od*	*od*
y_5	*od*	*od*	*od*	*od*	*od*

Thus, *M* under + is an AntiAlgebra of idempotents of QG.

Now we find the Cayley table of *M* under × which is as follows.

Table 15. Cayley table of M under ×.

×	y_1	y_2	y_3	y_4	y_5
y_1	y_1	*od*	y_3	*od*	*od*
y_2	*od*	y_2	*od*	*od*	y_5
y_3	y_3	*od*	y_3	*od*	*od*
y_4	*od*	*od*	*od*	y_4	*od*
y_6	*od*	y_5	*od*	*od*	y_5

Thus, the set M under × is a NeutroAlgebra of idempotents.

Now we proceed on to prove the following results.

Let G be a cyclic group of order *n, n* a composite number. Q be the field of rationals QG be the group ring of G over Q.

i) All proper idempotents in QG are obtained from the proper subgroups of G.

ii) If p1 is the order of the subgroup H of G, then $|\text{supp } p_1| < O(G)$ and $p_1 / O(G)$.

iii) The idempotents α formed by the subgroups of G will have a $|\text{supp } \alpha| < O(G)$

iv) If $|\text{supp } \alpha| = n$; α ∈ QG then this idempotent for all practical situations will be taken as a trivial idempotent. Similarly, $1 \in G$ is an idempotent, which is trivial. Also, $0,1 \in Q$ are trivial idempotents of QG.

These four conditions are strictly adhered to while finding the NeutroAlgebra of idempotents of the group ring QG under the operations + and ×.

v) When n, order of the cyclic group G is a product of odd prime then $n = p_1^{\alpha_1} \ldots p_t^{\alpha_t}$ where p_i's are distinct primes; $1 \le i \le t$ and $\alpha_i \ge 1$; $1 \le i \le t$.

vi) We see all subgroups of G are again an odd prime or a power of a prime or the product of some primes less than n.

vii) Furtherer if $\alpha = \frac{1}{p_i}\left(1 + g^{p_i} + \ldots + g^{p_i^{\alpha_i} - p_i}\right)$ is an idempotent then $\frac{1}{p_i}\left(1 - g^{p_i} + \ldots \mp \frac{1}{p_i^{\alpha_i}} g^{p_i^{\alpha_i - p_i}}\right)$ in general is not an idempotent.

To this effect, we propose an open problem in the section on the conclusion of this paper.

Suppose G is a cyclic group of order n; then G can have subgroups of both even and odd order unless $|G| = 2^n$.

If $|G| = 2^n$ and if $x = 1 + h + \ldots + h^t$ is an idempotent of QG then so is $y = 1 - h + h^2 - \ldots - h^t$ where h is a suitable power of g is the cyclic subgroup of G. In this case x an idempotent of QG with support of $x = \{1, h, \ldots, h^t\}$.

However, product of these two idempotents $x \times y = 0$ is not a proper idempotent of QG, only the trivial idempotent zero.

Theorem 3.1. *Let G be a cyclic group of odd order; QG be the group ring of G over Q.*

i) QG has only idempotents of the form $\frac{1}{t}(1 + h + h^2 + \ldots + h^{t-1})$ where $h \in G$ and $t < n$, and $\{1, h, h^2, \ldots, h^{t-1}\}$ is subgroup of G of order t.

ii) If W = {collection of nontrivial idempotents of G }, then

a) $\{W, +\}$ is an AntiAlgebra of idempotents of QG and

b) $\{W, \times\}$ is a NeutroAlgebra of idempotents of QG

$\left(0, 1 \text{ and } \frac{1}{n}(1 + g + \ldots + g^{n-1}) \text{ are the trivial idempotents of } QG\right)$.

Proof of (i). Given G is a cyclic group of odd order with $|G| = n$ (n a non-prime). So, G has only subgroups H_t^t of odd order, say t where t / n (t can be prime or non-prime).

Clearly $\alpha = \frac{1}{t}(1 + h + \ldots + h^{t-1})$ is an idempotent of QG, where $h \in G$.

Now 0, 1 and $x = \frac{1}{n}(1 + g + ... + g^{n-1})$ are assumed to be trivial idempotents of QG as $|\text{supp } x| = |G| = n$.

The other type of idempotents can be $\beta = \frac{1}{t}(1 - h + h^2 - ... + h^{t-1})$ but β^2 is not an idempotent easily verified using number theoretic or group theoretic properties.

$$\Big(\beta^2 = \frac{1}{t^2}\Big[1 - h + h^2 - ... - h^{t-2} + h^{t-1} - h + h^2 - h^3 - ... + h^{t-1} - 1$$

$$+h^2 - h^3 + h^4 - ... - 1 + h - h^3 + h^4 - ... - 1 + h - h^2 + h^{t-1} - 1 + ... + h^{t-2}\Big]$$

$$= \frac{1}{t^2}\Big[-(t-2) + (t-2)h ... + th^{t+1}\Big] \neq \beta .$$

Hence the claim.

Proof of (ii). Given W is the collection of all non-trivial idempotents of QG, so $\frac{1}{n}(1 + g + ... + g^{n-1}) \notin W$. $(W, +)$ is an AntiAlgebra.

For if $\alpha = \frac{1}{t}(1 + h + ... + h^{t-1})$, then $2\alpha = \frac{2}{t}(1 + h + ... + h^{t-1}) \notin W$.

Similarly, if $\beta \in W (\alpha \neq \beta)$ we see $\alpha + \beta \notin W$.

So, under +, every pair is outer defined.

Hence $(W, +)$ is an AntiAlgebra. Thus (a) of (ii) is proved. $(W, \times)$ is a NeutroAlgebra of idempotents of QG.

For if α and $\beta \in W$ such that $|\text{supp}\alpha| = m$ and $|\text{supp}\beta| = p$ such that $pm = n$ then $\alpha\beta = \frac{1}{n}(1 + g + ... + g^{n-1})$ which is a trivial idempotent of RG. As $n = pm$ can be written in a different way we have in the Cayley table of W under $\times$ has several od(outer defined) terms. Hence (b) of (ii) is proved.

Corollary 3.1. Let QG be as in the above theorem. If D, the trivial idempotent is taken in W, $(W, +)$ is a NeutroAlgebra of idempotents of QG.

Proof. If $0 \in W$ for every $\alpha \in W$, $\alpha + 0 = \alpha \in W$, so W under + is a NeutroAlgebra as we have some elements to be defined in W. Hence the claim.

Corollary 3.2. Let QG be as in the above theorem.

If the trivial idempotent $\alpha = \frac{1}{n}(1 + g + ... + g^{n-1}) \in W$ that is $|\text{supp}\alpha| = n$ then W under product $\times$ is not a NeutroAlgebra is a semigroup under $\times$.

Proof. Let $x=\frac{1}{t}(1+h+...+h^{t-1})$ and $y=\frac{1}{m}(1+K+...+K^{m-1})$ where h and K are powers of g and is in G. $x,y\in W$ with $|\operatorname{supp} x|=t$ and $|\operatorname{supp} y|=m$ with $mt=n$..

Thus $xy=\frac{1}{n}(1+g+...+g^{n-1})$ as $\operatorname{supp} x$ and $\operatorname{supp} y$ are subgroups of G.

Hence $(W,\times)$ is a semigroup, so W under $\times$ is not a NeutroAlgebra of idempotents.

Now we consider a cyclic group G of even order and obtain analogous results as in theorem for this QG when G is an odd composite number.

Theorem 3.2. *Let G be a cyclic group of even order say m; QG be the group ring of G over Q.*

i) The nontrivial idempotents of QG are of the form $\alpha=\frac{1}{t}(1+h+...+h^{t-1})$ or $\alpha'=\frac{1}{t}(1-h+h^2...-h^{t-1}+h^{t-1})$ where $h\in G$ with $\{1,h,...,h^{t-1}\}$ forming a proper subgroup of G of order t,t an even value (t can be only of even order if α' is to exist if t is of odd order; α' does not exist).

ii) If W = {collection of all idempotents of the form α and α' } then whenever α' is given as in (i) for the α given.

a) $\{W,\times\}$ is a NeutroAlgebra of idempotents of QG.

b) $\{W,+\}$ is an AntiAlgebra of idempotents of QG.

Proof. Given proper subgroups of G say of order t; t even, we have for $\alpha=\frac{1}{t}(1+h+...+h^{t-1})$ and $\alpha'=\frac{1}{t}(1-h+h^2-...-h^{t-2}+h^{t-1})$ are non-trivial idempotents of G.

Taking all even ordered subgroups of G, we have a collection of idempotents of the form α and α'. If the proper subgroup of G is odd-order say m then $\alpha=\frac{1}{m}(1+K+...+K^{m-1})$ are the only idempotents of QG.

If W is the collection of all idempotents of the form α, α' and so on then $(W,+)$ is an AntiAlgebra as no sum is defined.

If on the other hand, we include the trivial idempotent $0=0+0g+0g^2+...+0g^n$ then we see W under + is a NeutroAlgebra of idempotents of QG as $0+\alpha=\alpha$ for all $\alpha\in W$.

Now W under $\times$ is a NeutroAlgebra of idempotents for if α and β are two idempotents in W such that $|\operatorname{supp}\alpha|=K$ and $|\operatorname{supp}\beta|=m$ with $Km=n$ then $\alpha\times\beta=\frac{1}{n}(1+g+...+g^{n-1})$ the trivial idempotent of QG but by definition $|\operatorname{supp}\alpha\beta|=n$, the order of the whole group.

Thus, W under $\times$ is only a NeutroAlgebra, but if we allow the whole group idempotent $\frac{1}{n}(1+g+g^2+...+g^{n-1})$ in W, then W under $\times$ is not a NeutroAlgebra, in fact a semigroup. Hence the theorem.

Next, we proceed to prove the group ring QS_n has some idempotents sets W which forms AntiAlgebra under + and W under $\times$ happens to be a NeutroAlgebra.

We work mainly for this group S_n as every group G has a subgroup H of S_n, which is isomorphic with G [4, 5].

***Theorem* 3.3**. *Let S_n be the symmetric group of degree n (S_n, in particular, be a permutation on (1, 2, 3, …, n)) Q be the field of rationals. QS_n the group ring of the group S_n over Q. QS_n has subsets of nontrivial idempotents, which under ×, is a NeutroAlgebra and under addition + is an AntiAlgebra of idempotents of QS_n.*

Proof. Every subgroup *H* in S_n for an appropriate *n* there exists a group *G* isomorphic with *H.* Thus, if *H* be a cyclic group say of some order *m,* then $G \cong H \subseteq S_n$ for some appropriate cyclic subgroup of order *m*.

Now, apart from this, S_n has ${}_nC_2$ number of subgroups of order two.

All elements of the form $W = \left\{\frac{1}{2}(1-p_1), \frac{1}{2}(1+p_1), \frac{1}{2}(1-p_2), \frac{1}{2}(1+p_2), \ldots, \frac{1}{2}(1-p_n), \frac{1}{2}(1+p_n)\right\} \subseteq QS_n$ are idempotents where p_i 's are permutations in S_n such that $p_i.p_i$ = (1, 2, 3, …, *n*) the identity permutation of S_n.

$\{W,+\}$ can easily be realized as an AntiAlgebra as no element under + in W is in W.
Now similarly $\{W,\times\}$ is a NeutroAlgebra as

$$(1-p_i)\times 1+p_i = 0 \notin W \text{ and } (1-p_i)(1-p_j) = 1-p_i-p_j+p_jp_i \text{ and so on.}$$

Thus $\{W,\times\}$ is only a NeutroAlgebra of idempotents from QS_n.

Hence the theorem.
Based on this study, we propose a few open problems in the last section of this paper.

4. NeutroAlgebra of idempotents in the group ring Z_nG

Next, we study idempotents in the group ring Z_nG where Z_n is the ring of modulo integers and n a prime or a composite number and G a group of finite order. Thus, the group rings in this section are of finite order.

We will first illustrate this situation with some examples.

Example 4.1. Let Z_2 be the field of order two $G = \langle g \mid g^3 = 1 \rangle$ be the cyclic group of order 3.

Z_2G be the group ring of G order Z_2

$\alpha = 1 + g + g^2$ is the only non-trivial idempotent of Z_2G; for $(1+g+g^2)^2 = 1+g+g^2$.

Remark 4.1. *Let Z_p be the field of primes. G be the cyclic group of order $p+1$ (or any other group which has subgroups of order $p+1$), then Z_pG has an idempotent of the form $\alpha = 1 + g + ... + g^p$.*

Proof. For the group ring Z_pG; $\alpha = \left(1 + g + ... + g^p\right)$ is the non-trivial idempotent of Z_pG.

Example 4.2. $Z_{11}G$ be the group ring of G over Z_{11}. $G = \langle g \mid g^{12} = 1 \rangle$ be a cyclic group of order 12.

$\alpha = 1 + g + ... + g^{11} \in Z_{11}G$ is an idempotent of $Z_{11}G$.

$\beta = (6 + 5g^6) \in Z_{11}G$ is also an idempotent of $Z_{11}G$.

$\gamma = (6 + 6g^6)$ is an idempotent of $Z_{11}G$.

Let $W = \{\alpha, \beta, \gamma\}$ be the 3 nontrivial idempotents of $Z_{11}G$.

We give the Cayley table for W under + given by Table 17 in the following.

Table 17: Table of $\{W, +\}$

+	α	β	γ
α	*od*	*od*	*od*
β	*od*	*od*	*od*
γ	*od*	*od*	*od*

$(W, +)$ is an AntiAlgebra of idempotents of the group ring $Z_{11}G$.

The Cayley table of W under × is as follows.

Table 18: Table of $\{W, \times\}$

×	α	β	γ
α	α	*od*	γ
β	*od*	β	*od*

γ	γ	od	γ

Thus $(W,\times)$ is a NeutroAlgebra of idempotents of the group ring $Z_{11}G$.

Example 4.3. Let Z_7 be the field of prime order 7. S_8 be the permutation group of degree 8. Z_7S_8 be the group ring of S_8 over Z_7.

Let $H=\{1,p_1,p_2,...,p_7\}$ be the cyclic group generated by

$p=\begin{pmatrix}1&2&3&4&5&6&7&8\\8&1&2&3&4&5&6&7\end{pmatrix}$ and H is of order 8.

Clearly $\alpha=(1+p+p^2+...+p^7)\in Z_7S_8$ is a nontrivial idempotent of Z_7S_8

Consider $\beta=4+3p^4\in Z_7S_8$, we have $\beta^2=(4+3p^4)^2$

$= (16+9p^8+24p^4)$ (using $p^8=1$)

$= (25+24p^4)$

$= 4+3p^4=\beta$.

Thus, β is an idempotent of Z_7S_8.

Take $g=\begin{pmatrix}1&2&3&4&\dots&8\\2&1&3&4&\dots&8\end{pmatrix}\in S_8$

$g^2=\begin{pmatrix}1&2&3&4&\dots&8\\1&2&3&4&\dots&8\end{pmatrix}$;

the identity of S_8. Thus, we have ${}_8C_2$ number of such elements of order two in S_8.

Consider m = $4+3g\in Z_7S_8$, we see m² = m is an idempotent of Z_7S_8. In fact, we have ${}_8C_2$ number of such type of idempotents in the group ring Z_7S_8, where $g\in S_8$ is such that $g^2=\begin{pmatrix}1&2&3&4&\dots&8\\1&2&3&4&\dots&8\end{pmatrix}$.

Consider

$$t=4(1+g)\in Z_7S_8$$

$$t^2 = 16(1+g^2+2g)(g^2=1) = 16(2+2g) = 32(1+g) = 4(1+g) \in Z_7S_8$$

is an idempotent of the group ring Z_7S_8.

Thus, by this method also we have at least ${}_8C_2$ number of idempotents in Z_7S_8.

Now, if W is the collection of all idempotents of form $4(1+g)$ and $4+3g$ for varying $g \in S_8$ such that, $g^2 = id$ of S_8.

We see sum of $4+4g+4+3g=1$ is only a trivial idempotent.

$4+4g+4+4g=1+g$ is not an idempotent of this group ring Z_7S_8.

$3g+4+3g+4=6g+1$ is not an idempotent of Z_7S_8.

Thus, if $V = \{$collection of all non-trivial idempotents of the group ring Z_7S_8 of form $4(1+g)$ and $4+3g$ with all $g \in S_8$ such that g^2 is the identity element of $S_8\}$; then $(V,+)$ an AntiAlgebra of idempotents in Z_7S_8.

Also, we consider $4+3g \times 4+4g = 16+12g+16g+12 = 28+28g = 0$ is only a trivial idempotent of V and $0 \notin V$.

Consider $4+3g \times 4+4h\ (h^2=1)$.

We see $16+12g+12h+12gh \notin V$.

Thus, V under $\times$ is a NeutroAlgebra of idempotents of Z_7S_8 as $(4+3g)^2 = (4+3g)$ and $(4+4h)^2 = (4+4h)$.

Based on all these we have the following results.

Example 4.4. Let Z_{11} be the finite prime field of order 11. S_{12} be the symmetric group of order 12!. The group ring $Z_{11}S_{12}$ has a collection W of nontrivial idempotents from $Z_{11}S_{12}$ such that W under + is an AntiAlgebra of idempotents of the group ring $Z_{11}S_{12}$ and W under $\times$ is a NeutroAlgebra of idempotents of the group ring.

S_{12} has ${}_{12}C_2$ number of elements of order two. That is if

$$g_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 12 \\ 2 & 1 & 3 & 4 & \dots & 12 \end{pmatrix} \text{ is such that } g_1^2 = \begin{pmatrix} 1 & 2 & 3 & \dots & 12 \\ 1 & 2 & 3 & \dots & 12 \end{pmatrix}$$

= identity permutation.

$$g_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 12 \\ 3 & 2 & 1 & 4 & \dots & 12 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 12 \\ 4 & 2 & 3 & 1 & 5 & \dots & 12 \end{pmatrix}$$

and so $g_{11} = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 12 \\ 12 & 2 & 3 & 4 & \dots & 1 \end{pmatrix}$

Now $g_{12} = g_0 = \begin{pmatrix} 1 & 2 & 3 & \dots & 12 \\ 1 & 3 & 2 & \dots & 12 \end{pmatrix}$ identity element of S_{12}. Likewise, in W have ${}_{12}C_2$ number of such elements which are of order two.

Of course, there are other types of elements of order two also.

Our primary purpose is to prove the existence of some set of idempotents W of the group ring $Z_{11}S_{12}$ such that $(W,+)$ is an AntiAlgebra of idempotents and $(W,\times)$ is a NeutroAlgebra of idempotents.

So if we consider $W = \{6+6g_i, 6+5g_i \mid g_i$ is an element of order two in S_{12} described above} then first we show W is a collection of idempotents; then prove $\{W,+\}$ is an AntiAlgebra and {W, ×} is a NeutroAlgebra of idempotents under ×. Consider

$$x = 6+6g_i \text{ in } W,\ x^2 = (6+6g_i)^2 = 36+72g_i+36g_i^2 = = 72+72g_i (g_i^2 = 1) = 6+6g_i = x.$$

On similar lines it can be easily proved

$$y = (6+5g_i), y^2 = (6+5g_i)^2 = 36+60g_i+25g_i^2 = 61+60g_i = 6+5g_i = y.$$

So, W is the collection of idempotents.

Now W under + is not even closed for any pair. So $(W,+)$ is an AntiAlgebra of idempotents.

Further W under × is closed only for $(x \in W, x^2 = x)$ and not for any other pair.

So $(W,\times\}$ is a NeutroAlgebra of idempotents of the group ring $Z_{11}S_{12}$. Hence the claim.

However, for general group ring Z_pS_{p+1} (p a prime) we suggest it as an open problem in section 5.

Example 4.3. Let $G = \langle g \mid g^{10} = 1 \rangle$ be a cyclic group of order 10 and Z_{10} be the ring of integers modulo 10. $Z_{10}G$ be the group ring of G over Z_{10}.

Consider $\alpha = 3 + 2g^5 \in Z_{10}G$. We see

$$\alpha^2 = (3 + 2g^5) = 9 + 4 + 12g^5 = 3 + 2g^5 = \alpha$$

is an idempotent. of $Z_{10}G$.

Also $\beta = 3 + 8g$ in $Z_{10}G$ is such that

$$\beta^2 = (3 + 8g^5) = (9 + 64 + 48g^5) = 3 + 8g = \beta.$$

Hence β is an idempotent.

Let $a = 5(1 + g^2 + g^4 + g^6 + g^8) \in Z_{10}G$ where $a^2 = a$ so is an idempotent of $Z_{10}G$.

Take $b = 8 + 2g^5 \in Z_{10}G$; clearly

$$b^2 = (8 + 2g^5) = 64 + 4 + 32g^5 = 8 + 2g^5 = b.$$

Suppose we take the collection of some idempotents W in this group ring $Z_{10}G$; where

$$W = \{8 + 2g^5, 3 + 8g^5, 3 + 2g^5, 5(1 + g^2 + g^4 + g^6 + g^8)\}.$$

The Cayley table of W under + is given below.

Table 19. Cayley table of W under +.

+	α	β	a	b
α	*od*	*od*	*od*	*od*
β	*od*	*od*	*od*	*od*
a	*od*	*od*	*od*	*od*
b	*od*	*od*	*od*	*od*

The Cayley table of W are under × is given below.

Table 20. Cayley table with ×.

×	α	β	a	b
α	*od*	*od*	*od*	*od*
β	*od*	β	*od*	*od*
a	*od*	*od*	a	*od*

b	*od*	*od*	*od*	*b*

Thus, W under × is a NeutroAlgebra of idempotents of the group ring $Z_{10}G$ under product (×) operation.

We propose some open problems in the following section on conclusions.

5. Conclusions

In this section, we prove in general the a set of all non-trivial idempotents W in a group ring RG of a group G over a ring R have $\{W,+\}$ to be an AntiAlgebra of idempotents under + and $\{W,\times\}$ to be a NeutroAlgebra of idempotents under × for depending on R to be a ring of rationals or modulo integers Z_n (n a prime or a composite number) and G an appropriate finite group in the case of Z_n. Several examples are provided in the earlier for easy understanding.

We suggest some open problems for researchers in this direction, which will be taken by the authors for the future research.

Problem 5.1: Let Z_m be the ring of modulo integers n. S_n be the permutation group of degree n.

Given n and m fixed integers (we can find the solution for both small m and n; but finding for big m and n or a general m and n is challenging). We leave it as an open problem to find a collection of idempotents of the form.

$$W = \left\{(p+qg_i)/p^2+q^2 = p(\text{mod } n) \text{ and } 2pq = q(\text{mod } n) \text{ and } g_i \in S_m \text{ with } g_1^2 = \begin{pmatrix} 1 & 2 & 3 & ... & m \\ 1 & 2 & 3 & ... & , \end{pmatrix}\right\}$$

i) Further prove or disprove $(W,+)$ is an AntiAlgebra of idempotents of the group ring Z_mS_n.

ii) Prove or disprove $\{W,\times\}$ is a NeutroAlgebra of idempotents of the group ring Z_mS_n.

Can $\left\{\left(\frac{p+1}{2}+\frac{p+1}{2}g_i\right) \text{ and } \left(\frac{p+1}{2}+\frac{p-1}{2}g_i\right) \middle| g_i \in S_n \text{ set of group elements of order two in } S_{p+1}\right\}$, where the group ring is taken as Z_pS_{p+1} ; p is a prime? In the problem 5.1 we are replacing *m* = *p*(*p* is a prime) and *n* = *p* + *1*.

Problem 5.2. Can QG and RG have idempotents (nontrivial) other than those mentioned in this paper to form a NeutroAlgebra or AntiAlgebra of idempotents of QG and RG under × or + respectively?

Problem 5.3. Can we have idempotents of the form $a_1 + a_2 g + a_3 g^2 + \ldots + a_n g^{n-1}$ with $g^n = 1;\ a_i \in Q \setminus \{0,1\};\ 1 \le i \le n$ in the group ring QG where $G = \langle g \mid g^n = 1 \rangle$ is a cyclic group of order n ?

Problem 5.4. Let QG be the group ring. Can $\alpha g + \beta h \in QG$ where g and h are some two elements of $G(\alpha, \beta \in Q \setminus \{0,1\})$ be an idempotent for suitable α and β ?

Problem 5.5. Let Z_n be the ring of integers modulo n (n a composite number). Prove there exists two integers p and q (p and q need not be prime in Z_n) such that $p^2 + q^2 = p \pmod{n}$ and $2pq = q (\mathrm{mod}\, n)$.

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