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## Fixed Point Theorem of Weak Compatible Maps of Type $(\gamma)$ in Neutrosophic Metric Space

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**Abstract:** In this paper, we give definitions of compatible mappings of type  $(\gamma)$  in neutrosophic metric space, and obtain a common fixed point theorem under the conditions of weakly compatible mappings of type  $(\gamma)$  in complete neutrosophic metric spaces. Our research generalizes, extends and improves the results given by Sedghi et al.[19].

**Keywords:** Fixed point, Neutrosophic metric Space, Compatible Mappings, Weak Compatible Mappings of Type  $(\gamma)$ .

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### 1. Introduction :

Fuzzy set was presented by Zadeh [30] as a class of elements with a grade of membership. Kramosil and Michalek [9] defined new notion called Fuzzy Metric Space (FMS). Later on, many authors have examined the concept of fuzzy metric in various aspects. Since then, many authors have obtained fixed point results in fuzzy metric space using these compatible notions. Also, Kutukcu et al.[11] obtained the common fixed points of compatible maps of type  $(\beta)$  on fuzzy metric spaces, and Sedghi et al.[19] studied the common fixed point of compatible maps of type  $(\gamma)$  in complete fuzzy metric spaces.

Atanassov [1] introduced and studied the notion of intuitionistic fuzzy set by generalizing the notion of fuzzy set. Recently, Park[14] and Park et al. [17] defined the intuitionistic fuzzy metric space. Many authors [15, 16, 17] obtained a fixed point theorems in this space. Also, Park et al. [17] introduced the concept of compatible mappings of type  $(\alpha)$  and type  $(\beta)$ , and obtained common fixed point theorems in intuitionistic fuzzy metric space.

In 1998, Smarandache [20, 21, 22] characterized the new concept called neutrosophic logic and neutrosophic set and explored many results in it. In the idea of neutrosophic sets, there is T degree of membership, I degree of indeterminacy and F degree of non-membership. Basset et al. [3] Explored the neutrosophic applications in dif and only iferent fields such as model for sustainable supply chain risk management, resource levelling problem in construction projects, Decision Making. In 2020, Kirisci et al [10] defined NMS as a generalization of IFMS and brings about fixed point theorems in complete NMS. In 2020, Sowndrarajan et al. [23] proved some fixed point results for contraction theorems in neutrosophic metric spaces.

In this paper, we give definitions of compatible mappings of type  $(\gamma)$  in neutrosophic metric space and obtain common fixed point theorem under the conditions of weak compatible mappings of type  $(\gamma)$  in complete neutrosophic metric space.

## 2. Some Relevent Results:

### Definition: 2.1.[18]

A binary operation  $*$  :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is a continuous t-norm [CTN] if it satisfies the following conditions :

1.  $*$  is commutative and associative,
2.  $*$  is continuous,
3.  $\varepsilon_1 * 1 = \varepsilon_1$  for all  $\varepsilon_1 \in [0, 1]$ ,
4.  $\varepsilon_1 * \varepsilon_2 \leq \varepsilon_3 * \varepsilon_4$  whenever  $\varepsilon_1 \leq \varepsilon_3$  and  $\varepsilon_2 \leq \varepsilon_4$  , for each  $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0, 1]$ .

### Definition: 2.2.[18]

A binary operation  $\diamond$  :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is a continuous t-conorm [CTC] if it satisfies the following conditions:

1.  $\diamond$  is commutative and associative,
2.  $\diamond$  is continuous,
3.  $\varepsilon_1 \diamond 0 = \varepsilon_1$  for all  $\varepsilon_1 \in [0, 1]$ ,
4.  $\varepsilon_1 \diamond \varepsilon_2 \leq \varepsilon_3 \diamond \varepsilon_4$  whenever  $\varepsilon_1 \leq \varepsilon_3$  and  $\varepsilon_2 \leq \varepsilon_4$  , for each  $\varepsilon_1, \varepsilon_2, \varepsilon_3$  and  $\varepsilon_4 \in [0, 1]$ .

### Definition: 2.3.[23]

A 6-tuple  $(\Sigma, \Xi, \Theta, Y, *, \diamond)$  is said to be an Neutrosophic Metric Space (shortly NMS), if  $\Sigma$  is an arbitrary non empty set,  $*$  is a neutrosophic CTN,  $\diamond$  is a neutrosophic CTC and  $\Xi, \Theta$  and  $Y$  are neutrosophic on  $\Sigma^2 \times \mathbb{R}^+$  satisfying the following conditions:

For all  $\zeta, \eta, \delta, \omega \in \Sigma, \lambda \in \mathbb{R}^+$ .

1.  $0 \leq \Xi(\zeta, \eta, \lambda) \leq 1; 0 \leq \Theta(\zeta, \eta, \lambda) \leq 1; 0 \leq Y(\zeta, \eta, \lambda) \leq 1;$
2.  $\Xi(\zeta, \eta, \lambda) + \Theta(\zeta, \eta, \lambda) + Y(\zeta, \eta, \lambda) \leq 3;$
3.  $\Xi(\zeta, \eta, \lambda) = 1$  if and only if  $\zeta = \eta;$
4.  $\Xi(\zeta, \eta, \lambda) = \Xi(\eta, \zeta, \lambda),$
5.  $\Xi(\zeta, \eta, \lambda) * \Xi(\eta, \delta, \mu) \leq \Xi(\zeta, \delta, \lambda + \mu),$  for all  $\lambda, \mu > 0;$
6.  $\Xi(\zeta, \eta, .) : (0, \infty) \rightarrow (0, 1]$  is neutrosophic continuous ;
7.  $\lim_{\lambda \rightarrow \infty} \Xi(\zeta, \eta, \lambda) = 1$  for all  $\lambda > 0;$
8.  $\Theta(\zeta, \eta, \lambda) = 0$  if and only if  $\zeta = \eta;$
9.  $\Theta(\zeta, \eta, \lambda) = \Theta(\eta, \zeta, \lambda);$
10.  $\Theta(\zeta, \eta, \lambda) \diamond \Theta(\eta, \delta, \mu) \geq \Theta(\zeta, \delta, \lambda + \mu),$  for all  $\lambda, \mu > 0;$

11.  $\Theta (\zeta, \eta, \cdot) : (0, \infty) \rightarrow (0, 1]$  is neutrosophic continuous;
12.  $\lim_{\lambda \rightarrow \infty} \Theta (\zeta, \eta, \lambda) = 0$  for all  $\lambda > 0$ ;
13.  $Y (\zeta, \eta, \lambda) = 0$  if and only if  $\zeta = \eta$ ;
14.  $Y (\zeta, \eta, \lambda) = Y (\eta, \zeta, \lambda)$ ;
15.  $Y (\zeta, \eta, \lambda) \diamond Y (\eta, \delta, \mu) \geq Y (\zeta, \delta, \lambda + \mu)$ , for all  $\lambda, \mu > 0$ ;
16.  $Y (\zeta, \eta, \cdot) : (0, \infty) \rightarrow (0, 1]$  is neutrosophic continuous;
17.  $\lim_{\lambda \rightarrow \infty} Y (\zeta, \eta, \lambda) = 0$  for all  $\lambda > 0$ ;
18. If  $\lambda \leq 0$  then  $\Xi (\zeta, \eta, \delta, \lambda) = 0$ ;  $\Theta (\zeta, \eta, \delta, \lambda) = 1$ ;  $Y (\zeta, \eta, \delta, \lambda) = 1$ .

Then,  $(\Xi, \Theta, Y)$  is called an NMS on  $\Sigma$ . The functions  $\Xi, \Theta$  and  $Y$  denote degree of closedness, naturalness and non-closedness between  $\zeta$  and  $\eta$  with respect to  $\lambda$  respectively.

**Example: 2.4.[23]**

Let  $(\Sigma, d)$  be a metric space. Define  $\omega * \tau = \min \{ \omega, \tau \}$  and  $\omega \diamond \tau = \max \{ \omega, \tau \}$  and  $\Xi, \Theta, Y : \Sigma^2 \times \mathbb{R}^+ \rightarrow [0, 1]$  defined by, we define  $\Xi (\zeta, \eta, \lambda) = \frac{\lambda}{\lambda + d(\zeta, \eta)}$ ;  $\Theta (\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda + d(\zeta, \eta)}$ ;  $Y (\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda}$ , for all  $\zeta, \eta \in \Sigma$  and  $\lambda > 0$ . Then  $(\Sigma, \Xi, \Theta, Y, *, \diamond)$  is called NMS induced by a metric  $d$  the standard neutrosophic metric.

**Definition 2.5.[23]**

Let  $\Sigma$  be an NMS. Then  $\Xi, \Theta$  are said to be continuous on  $\Sigma^2 \times \mathbb{R}^+$  if  $\lim_{n \rightarrow \infty} \Xi(\zeta_n, \eta_n, \lambda_n) = \Xi(\zeta, \eta, \lambda)$ ;  $\lim_{n \rightarrow \infty} \Theta(\zeta_n, \eta_n, \lambda_n) = \Theta(\zeta, \eta, \lambda)$ ;  $\lim_{n \rightarrow \infty} Y(\zeta_n, \eta_n, \lambda_n) = Y(\zeta, \eta, \lambda)$ ,  
Whenever a sequence  $\{(\zeta_n, \eta_n, \lambda_n)\} \subset \Sigma^2 \times \mathbb{R}^+$  converges to a point  $(\zeta, \eta, \lambda) \in \Sigma^2 \times \mathbb{R}^+$ .

**Definition 2.6.**

Let  $\Gamma$  and  $\Omega$  be mappings from an NMS  $\Sigma$  into itself. Then the mappings are said to be compatible if  $\lim_{n \rightarrow \infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1$ ,  $\lim_{n \rightarrow \infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0$ ,  $\lim_{n \rightarrow \infty} Y(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0$ ,  $\forall \lambda > 0$ , whenever  $\{\zeta_n\}$  is a sequence in  $\Sigma$  such that  $\lim_{n \rightarrow \infty} \Gamma\zeta_n = \lim_{n \rightarrow \infty} \Omega\zeta_n = \zeta \in \Sigma$ .

**Example 2.7.**

Let  $\Sigma$  be an NMS, where  $\Sigma = [0, 2], *, \diamond$  defined  $a * b = \min\{a, b\}$ ,  $a \diamond b = \max\{a, b\}$  for all  $a, b \in [0, 1]$  and  $\Xi (\zeta, \eta, \lambda) = \frac{\lambda}{\lambda + d(\zeta, \eta)}$ ;  $\Theta (\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda + d(\zeta, \eta)}$ ;  $Y (\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda}$ , for all  $\lambda > 0$  and  $\zeta, \eta \in \Sigma$ . Define self maps  $\Gamma$  and  $\Omega$  on  $\Sigma$  as follows:

$$\Gamma \zeta = \begin{cases} 2 & \text{if } 0 \leq \zeta \leq 1 \\ \frac{\zeta}{2} & \text{if } 1 < \zeta \leq 2 \end{cases} \quad \text{and} \quad \Omega \zeta = \begin{cases} 2 & \text{if } \zeta = 1 \\ \frac{\zeta + 3}{3} & \text{otherwise} \end{cases}$$

and  $\zeta_n = 2 - \frac{1}{2n}$ . Then we have  $\Omega 1 = 2 = \Gamma 1$  and  $\Omega 2 = 1 = \Gamma 2$ .

Also,  $\Omega \Gamma 2 = \Omega 1 = 2$ ,  $\Gamma \Omega 2 = \Gamma 1 = 2$  ( $\Omega \Gamma 2 = \Gamma \Omega 2 = 2$ ), thus  $\Gamma$  and  $\Omega$  are weak compatible.

Also, since  $\Gamma \zeta_n = \frac{1}{2} (2 - \frac{1}{2n}) = 1 - \frac{1}{2n}$ ,  $\Omega \zeta_n = \frac{1}{2} (2 - \frac{1}{2n} + 3) = 1 - \frac{1}{10n}$ . Thus  $\lim_{n \rightarrow \infty} \Gamma \zeta_n = 1 = \lim_{n \rightarrow \infty} \Omega \zeta_n$ .

Furthermore,  $\Omega \Gamma \zeta_n = \Omega (1 - \frac{1}{4n}) = \frac{1}{5} (1 - \frac{3}{4n} + 3) = \frac{4}{5} - \frac{1}{20n}$ ,  $\Gamma \Omega \zeta_n = \Gamma (1 - \frac{1}{10n}) = 2$ .

Now,

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \lambda) &= \lim_{n \rightarrow \infty} \Xi (2, \frac{4}{5} - \frac{1}{20n}, \lambda) = \frac{5t}{5t+6}, \\ \lim_{n \rightarrow \infty} \Theta (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \lambda) &= \lim_{n \rightarrow \infty} \Theta (2, \frac{4}{5} - \frac{1}{20n}, \lambda) = \frac{6}{5t+6}, \\ \lim_{n \rightarrow \infty} Y (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \lambda) &= \lim_{n \rightarrow \infty} Y (2, \frac{4}{5} - \frac{1}{20n}, \lambda) = \frac{6}{5t}, \end{aligned}$$

Hence  $\Gamma$  and  $\Omega$  is not compatible.

### 3. Weak Compatible mappings of type $(\gamma)$ :

#### Definition 3.1.

Let  $\Gamma$  and  $\Omega$  be mappings from an NMS  $\Sigma$  into itself. Then the mappings  $\Gamma$  and  $\Omega$  are said to be compatible maps of type  $(\gamma)$  if satisfying:

1.  $\Gamma$  and  $\Omega$  are compatible, that is,  $\lim_{n \rightarrow \infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1$ ,  $\lim_{n \rightarrow \infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0$ ,  
 $\lim_{n \rightarrow \infty} Y(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0, \forall \lambda > 0$ .  
 Whenever  $\{\zeta_n\} \subset \Sigma$  such that  $\lim_{n \rightarrow \infty} \Gamma\zeta_n = \lim_{n \rightarrow \infty} \Omega\zeta_n = \zeta \in \Sigma$ .
2. They are continuous at  $\zeta$ . On the other hand, we have  
 $\Gamma\zeta = \Gamma(\lim_{n \rightarrow \infty} \Gamma\zeta_n) = \Gamma(\lim_{n \rightarrow \infty} \Omega\zeta_n) = (\lim_{n \rightarrow \infty} \Omega\Gamma\zeta_n) = \Omega(\lim_{n \rightarrow \infty} \Gamma\zeta_n) = \Omega\zeta$ .

#### Definition 3.2.

Let  $\Gamma$  and  $\Omega$  be mappings from an NMS  $\Sigma$  into itself. The mappings  $\Gamma$  and  $\Omega$  are said to be weak - compatible of type  $(\gamma)$  if  $\lim_{n \rightarrow \infty} \Gamma\zeta_n = \lim_{n \rightarrow \infty} \Omega\zeta_n = \zeta$  for some  $\zeta \in \Sigma$  implies that  $\Gamma\zeta = \Omega\zeta$ .

#### Remark 3.3.

If self maps  $\Gamma$  and  $\Omega$  of an NMS  $\Sigma$  are compatible of type  $(\gamma)$ , then they are weak compatible type  $(\gamma)$ . But the converse is not true.

#### Lemma 3.5.

Let  $\Sigma$  be an NMS,

1. If we define  $E_\alpha : \Sigma^2 \times \mathbb{R}^+$  by  
 $E_\alpha(\zeta, \eta) = \inf \{ \lambda > 0 ; \Xi(\zeta, \eta, \lambda) > 1 - \lambda, \Theta(\zeta, \eta, \lambda) < \lambda \text{ and } Y(\zeta, \eta, \lambda) < \lambda \}$  for each  $\mu \in (0,1)$  there exists  $\alpha \in (0,1)$  such that  $E_\alpha(\zeta_1, \zeta_n) \leq E_\alpha(\zeta_1, \zeta_2) + E_\alpha(\zeta_2, \zeta_3) + \dots + E_\alpha(\zeta_{n-1}, \zeta_n)$  for any  $\zeta_1, \zeta_2 \dots \zeta_n \in \Sigma$ .
2. The sequence  $\{\zeta_n\}_{n \in \mathbb{N}}$  is convergent in NMS  $\Sigma$  if and only if  $E_\alpha(\zeta_n, \zeta) \rightarrow 0$ .  
 Also, the sequence  $\{\zeta_n\}_{n \in \mathbb{N}}$  is Cauchy sequence if and only if it is Cauchy sequence with  $E_\alpha$ .

#### Lemma 3.6.

Let  $\Sigma$  be an NMS.  $\Xi(\zeta_n, \zeta_{n+1}, \lambda) \geq \Xi(\zeta_0, \zeta_1, k^n\lambda)$ ,  $\Theta(\zeta_n, \zeta_{n+1}, \lambda) \leq \Theta(\zeta_0, \zeta_1, k^n\lambda)$  and  $Y(\zeta_n, \zeta_{n+1}, \lambda) \leq Y(\zeta_0, \zeta_1, k^n\lambda)$  for some  $k > 1$  and for every  $n \in \mathbb{N}$ . Then sequence  $\{\zeta_n\}$  is a Cauchy sequence.

#### Lemma 3.7.

Let  $\Sigma$  be an NMS. If there exists a number  $k \in (0, 1)$  such that for all  $\zeta, \eta \in \Sigma$  and  $\lambda > 0$ .  $\Xi(\zeta, \eta, k\lambda) \geq \Xi(\zeta, \eta, \lambda)$ ,  $\Theta(\zeta, \eta, k\lambda) \leq \Theta(\zeta, \eta, \lambda)$  and  $Y(\zeta, \eta, k\lambda) \leq Y(\zeta, \eta, \lambda)$  then  $\zeta = \eta$ .

## 4. Main Results

#### Lemma 4.1.

Let  $\Gamma$  and  $\Omega$  are self – mappings of a complete NMS  $\Sigma$  satisfying:  
 There exists a constant  $k \in (0, 1)$  such that

$$\begin{aligned} & \Xi^2(\Gamma\zeta, \Omega\eta, k\lambda) * [\Xi(\zeta, \Gamma\zeta, k\lambda) \Xi(\eta, \Omega\eta, k\lambda)] * \Xi^2(\eta, \Omega\eta, k\lambda) + a \Xi(\eta, \Omega\eta, k\lambda) \\ & \Xi(\zeta, \Omega\eta, 2k\lambda) \geq [p \Xi(\zeta, \Gamma\zeta, \lambda) + q \Xi(\zeta, \eta, \lambda)] \Xi(\zeta, \Omega\eta, 2k\lambda) \end{aligned} \tag{4.1.1}$$

$$\begin{aligned} & \Theta^2(\Gamma\zeta, \Omega\eta, k\lambda) \diamond [\Theta(\zeta, \Gamma\zeta, k\lambda) \Theta(\eta, \Omega\eta, k\lambda)] \diamond \Theta^2(\eta, \Omega\eta, k\lambda) + a \Theta(\eta, \Omega\eta, k\lambda) \\ & \Theta(\zeta, \Omega\eta, 2k\lambda) \leq [p \Theta(\zeta, \Gamma\zeta, \lambda) + q \Theta(\zeta, \eta, \lambda)] \Theta(\zeta, \Omega\eta, 2k\lambda) \end{aligned} \tag{4.1.2}$$

$$\begin{aligned} & Y^2(\Gamma\zeta, \Omega\eta, k\lambda) \diamond [Y(\zeta, \Gamma\zeta, k\lambda) Y(\eta, \Omega\eta, k\lambda)] \diamond Y^2(\eta, \Omega\eta, k\lambda) + a Y(\eta, \Omega\eta, k\lambda) \\ & Y(\zeta, \Omega\eta, 2k\lambda) \leq [p Y(\zeta, \Gamma\zeta, \lambda) + q Y(\zeta, \eta, \lambda)] Y(\zeta, \Omega\eta, 2k\lambda) \end{aligned} \tag{4.1.3}$$

for every  $\zeta, \eta \in \Sigma$  and  $\lambda > 0$ , where  $0 < p, q < 1, 0 \leq a < 1$  such that  $p + q - a = 1$ . Then  $\Gamma$  and  $\Omega$  have a unique common fixed point in  $\Sigma$ .

**Proof:** Let  $\zeta_0 \in \Sigma$  be an arbitrary point, there exist  $\zeta_1 \in \Sigma$  such that  $\Gamma\zeta_0 = \zeta_1, \Omega\zeta_0 = \zeta_2$ . Inductively, construct the sequences  $\{\zeta_n\} \subset \Sigma$  such that  $\zeta_{2n+1} = \Gamma\zeta_{2n}, \zeta_{2n+2} = \Omega\zeta_{2n+1}$  for  $n = 0, 1, 2, \dots$ . Then we prove that  $\{\zeta_n\}$  is a Cauchy sequence.

For  $\zeta = \zeta_{2n}, \eta = \zeta_{2n+1}$  by we have

$$\begin{aligned} & \Xi^2(\Gamma\zeta_{2n}, \Omega\zeta_{2n+1}, k\lambda) * [\Xi(\zeta_{2n}, \Gamma\zeta_{2n}, k\lambda) \Xi(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)] * \Xi^2(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) \\ & + a \Xi(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) \Xi(\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda) \geq [p \Xi(\zeta_{2n}, \Gamma\zeta_{2n}, \lambda) + q \Xi(\zeta_{2n}, \zeta_{2n+1}, \lambda)] \times \\ & \Xi(\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda) \text{ and} \end{aligned}$$

$$\begin{aligned} & \Xi^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) * [\Xi(\zeta_{2n}, \zeta_{2n+1}, k\lambda) \Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)] * \Xi^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \\ & + a \Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \Xi(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda) \geq [p \Xi(\zeta_{2n}, \zeta_{2n+1}, \lambda) + q \Xi(\zeta_{2n}, \zeta_{2n+1}, \lambda)] \times \\ & \Xi(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda), \end{aligned}$$

$$\begin{aligned} & \Theta^2(\Gamma\zeta_{2n}, \Omega\zeta_{2n+1}, k\lambda) \diamond [\Theta(\zeta_{2n}, \Gamma\zeta_{2n}, k\lambda) \Theta(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)] \diamond \Theta^2(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) \\ & + a \Theta(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) \Theta(\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda) \leq [p \Theta(\zeta_{2n}, \Gamma\zeta_{2n}, \lambda) + q \Theta(\zeta_{2n}, \zeta_{2n+1}, \lambda)] \times \\ & \Theta(\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda) \text{ and} \end{aligned}$$

$$\begin{aligned} & \Theta^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \diamond [\Theta(\zeta_{2n}, \zeta_{2n+1}, k\lambda) \Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)] \diamond \Theta^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \\ & + a \Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \Theta(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda) \leq [p \Theta(\zeta_{2n}, \zeta_{2n+1}, \lambda) + q \Theta(\zeta_{2n}, \zeta_{2n+1}, \lambda)] \times \\ & \Theta(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda). \end{aligned}$$

$$\begin{aligned} & Y^2(\Gamma\zeta_{2n}, \Omega\zeta_{2n+1}, k\lambda) \diamond [Y(\zeta_{2n}, \Gamma\zeta_{2n}, k\lambda) Y(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)] \diamond Y^2(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) \\ & + a Y(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) Y(\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda) \leq [p Y(\zeta_{2n}, \Gamma\zeta_{2n}, \lambda) + q Y(\zeta_{2n}, \zeta_{2n+1}, \lambda)] \times \\ & Y(\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda) \text{ and} \end{aligned}$$

$$\begin{aligned} & Y^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \diamond [Y(\zeta_{2n}, \zeta_{2n+1}, k\lambda) Y(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)] \diamond Y^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \\ & + a Y(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) Y(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda) \leq [p Y(\zeta_{2n}, \zeta_{2n+1}, \lambda) + q Y(\zeta_{2n}, \zeta_{2n+1}, \lambda)] \times \\ & Y(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda). \end{aligned}$$

Then

$$\begin{aligned} & \Xi^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) * [\Xi(\zeta_{2n}, \zeta_{2n+1}, k\lambda) \Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)] + a \Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \\ & \Xi(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda) \geq (p + q) \Xi(\zeta_{2n}, \zeta_{2n+1}, \lambda) \Xi(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda), \end{aligned}$$

$$\begin{aligned} & \Theta^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \diamond [\Theta(\zeta_{2n}, \zeta_{2n+1}, k\lambda) \Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)] + a \Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \\ & \Theta(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda) \leq (p + q) \Theta(\zeta_{2n}, \zeta_{2n+1}, \lambda) \Theta(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda), \end{aligned}$$

$$\begin{aligned} & Y^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \diamond [Y(\zeta_{2n}, \zeta_{2n+1}, k\lambda) Y(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)] + a Y(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \\ & Y(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda) \leq (p + q) Y(\zeta_{2n}, \zeta_{2n+1}, \lambda) Y(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda). \end{aligned}$$

So,

$$\Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) + a \Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \geq (p + q) \Xi(\zeta_{2n}, \zeta_{2n+1}, \lambda)$$

$$\Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) + a \Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \leq (p + q) \Theta(\zeta_{2n}, \zeta_{2n+1}, \lambda)$$

$$Y(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) + a Y(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \leq (p + q) Y(\zeta_{2n}, \zeta_{2n+1}, \lambda).$$

Therefore

$$\Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \geq \Xi(\zeta_{2n}, \zeta_{2n+1}, \lambda), \Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \leq \Theta(\zeta_{2n}, \zeta_{2n+1}, \lambda) \text{ and}$$

$$Y(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \leq Y(\zeta_{2n}, \zeta_{2n+1}, \lambda).$$

Similarly, we also have

$$\begin{aligned} \Xi(\zeta_{2n+2}, \zeta_{2n+3}, k\lambda) &\geq \Xi(\zeta_{2n+1}, \zeta_{2n+2}, \lambda), \Theta(\zeta_{2n+2}, \zeta_{2n+3}, k\lambda) \leq \Theta(\zeta_{2n+1}, \zeta_{2n+2}, \lambda), \\ Y(\zeta_{2n+2}, \zeta_{2n+3}, k\lambda) &\leq Y(\zeta_{2n+1}, \zeta_{2n+2}, \lambda). \end{aligned}$$

For  $k \in (0, 1)$  if  $k_1 = \frac{1}{k} > 1$  and  $\lambda = k_1 \lambda_1$ , then we have

$$\begin{aligned} \Xi(\zeta_n, \zeta_{n+1}, \lambda) &\geq \Xi(\zeta_0, \zeta_1, k_1^n \lambda_1), \Theta(\zeta_n, \zeta_{n+1}, \lambda) \leq \Theta(\zeta_0, \zeta_1, k_1^n \lambda_1) \text{ and} \\ Y(\zeta_n, \zeta_{n+1}, \lambda) &\leq Y(\zeta_0, \zeta_1, k_1^n \lambda_1). \end{aligned}$$

By Lemma 3.6, since  $\{\zeta_n\}$  is a Cauchy sequence in  $\Sigma$  which is complete,  $\{\zeta_n\}$  converges to  $\omega$  in  $\Sigma$ . Hence

$$\lim_{n \rightarrow \infty} \Gamma \zeta_{2n} = \lim_{n \rightarrow \infty} \zeta_{2n+1} = \lim_{n \rightarrow \infty} \zeta_{2n+2} = \lim_{n \rightarrow \infty} \Omega \zeta_{2n+1} = \omega.$$

Now, taking  $\zeta = \omega$  and  $\eta = \zeta_{2n+1}$  in (i), we have as  $n \rightarrow \infty$ ,

$$\begin{aligned} \Xi^2(\Gamma\omega, \omega, k\lambda) * [\Xi(\omega, \Gamma\omega, k\lambda) \Xi(\omega, \omega, k\lambda)] * \Xi^2(\omega, \omega, k\lambda) + a \Xi(\omega, \omega, k\lambda) \Xi(\omega, \omega, 2k\lambda) \\ \geq [p\Xi(\omega, \Gamma\omega, \lambda) + q\Xi(\omega, \omega, \lambda)] \Xi(\omega, \omega, 2k\lambda), \\ \Theta^2(\Gamma\omega, \omega, k\lambda) * [\Theta(\omega, \Gamma\omega, k\lambda) \Theta(\omega, \omega, k\lambda)] \diamond \Theta^2(\omega, \omega, k\lambda) + a \Theta(\omega, \omega, k\lambda) \Theta(\omega, \omega, 2k\lambda) \\ \leq [p\Theta(\omega, \Gamma\omega, \lambda) + q\Theta(\omega, \omega, \lambda)] \Theta(\omega, \omega, 2k\lambda), \\ Y^2(\Gamma\omega, \omega, k\lambda) * [Y(\omega, \Gamma\omega, k\lambda) Y(\omega, \omega, k\lambda)] \diamond Y^2(\omega, \omega, k\lambda) + a Y(\omega, \omega, k\lambda) \Theta(\omega, \omega, 2k\lambda) \\ \leq [pY(\omega, \Gamma\omega, \lambda) + qY(\omega, \omega, \lambda)] Y(\omega, \omega, 2k\lambda). \end{aligned}$$

Therefore

$$\Xi(\Gamma\omega, \omega, k\lambda) + a \geq p\Xi(\omega, \Gamma\omega, \lambda) + q, \Theta(\Gamma\omega, \omega, k\lambda) \leq 0, Y(\Gamma\omega, \omega, k\lambda) \leq 0,$$

for all  $\lambda > 0$ , so  $\Gamma\omega = \omega$ . Taking  $\zeta = \zeta_{2n}$  and  $\eta = \omega$  in (i), we have as  $n \rightarrow \infty$ ,

$$\Xi(\omega, \Omega\omega, \lambda) + a \geq p + q, \Theta(\omega, \Omega\omega, \lambda) + a \Theta(\omega, \Omega\omega, \lambda) \leq 0 \text{ and } Y(\omega, \Omega\omega, \lambda) + a Y(\omega, \Omega\omega, \lambda) \leq 0,$$

for all  $\lambda > 0$ , so  $\Omega\omega = \omega$ . Thus  $\omega$  is a common fixed point of  $\Gamma$  and  $\Omega$ ,

Let  $\beta$  be another common fixed point of  $\Gamma$  and  $\Omega$ . Then using (i), we have

$$\begin{aligned} \Xi^2(\omega, \beta, k\lambda) + a \Xi(\omega, \beta, 2k\lambda) &\geq [p + q \Xi(\omega, \beta, \lambda)] \Xi(\omega, \beta, 2k\lambda), \\ \Theta^2(\omega, \beta, k\lambda) &\leq q \Theta(\omega, \beta, \lambda) \Theta(\omega, \beta, 2k\lambda) \text{ and } Y^2(\omega, \beta, k\lambda) \leq q Y(\omega, \beta, \lambda) Y(\omega, \beta, 2k\lambda) \end{aligned}$$

and

$$\begin{aligned} \Xi(\omega, \beta, \lambda) \Xi(\omega, \beta, 2k\lambda) + a \Xi(\omega, \beta, 2k\lambda) &\geq [p + q \Xi(\omega, \beta, \lambda)] \Xi(\omega, \beta, 2k\lambda), \\ \Theta(\omega, \beta, \lambda) \Theta(\omega, \beta, 2k\lambda) &\leq q \Theta(\omega, \beta, \lambda) \Theta(\omega, \beta, 2k\lambda), \\ Y(\omega, \beta, \lambda) Y(\omega, \beta, 2k\lambda) &\leq q Y(\omega, \beta, \lambda) Y(\omega, \beta, 2k\lambda). \end{aligned}$$

Thus, it follows that

$$\Xi(\omega, \beta, \lambda) \geq \frac{p-a}{1-q} = 1, \Theta(\omega, \beta, \lambda) \leq 0, Y(\omega, \beta, \lambda) \leq 0,$$

for all  $\lambda > 0$ , so  $\omega = \beta$ . Hence  $\Gamma$  and  $\Omega$  have a unique common fixed point in  $\Sigma$ .

**Theorem 4.2.**

Let  $\Gamma, \Omega, \Lambda$  and  $V$  be self mappings of a complete NMS  $\Sigma$  satisfying

- $\Gamma(\Sigma) \subseteq V(\Sigma), \Omega(\Sigma) \subseteq \Lambda(\Sigma),$

- There exists a constant  $k \in (0, 1)$  such that

$$\begin{aligned} \Xi^2(\Gamma\zeta, \Omega\eta, k\lambda) * [\Xi(\Lambda\zeta, \Gamma\zeta, k\lambda) \Xi(V\eta, \Omega\eta, k\lambda)] * \Xi^2(V\eta, \Omega\eta, k\lambda) + a \Xi(V\eta, \Omega\eta, k\lambda) \\ \Xi(\Lambda\zeta, \Omega\eta, 2k\lambda) \geq [p\Xi(\Lambda\zeta, \Gamma\zeta, \lambda) + q\Xi(\Lambda\zeta, V\eta, \lambda)] \Xi(\Lambda\zeta, \Omega\eta, 2k\lambda) \end{aligned} \tag{4.2.1}$$

$$\begin{aligned} \Theta^2(\Gamma\zeta, \Omega\eta, k\lambda) \diamond [\Theta(\Lambda\zeta, \Gamma\zeta, k\lambda) \Theta(V\eta, \Omega\eta, k\lambda)] \diamond \Theta^2(V\eta, \Omega\eta, k\lambda) + a \Theta(V\eta, \Omega\eta, k\lambda) \\ \Theta(\Lambda\zeta, \Omega\eta, 2k\lambda) \leq [p\Theta(\Lambda\zeta, \Gamma\zeta, \lambda) + q\Theta(\Lambda\zeta, V\eta, \lambda)] \Theta(\Lambda\zeta, \Omega\eta, 2k\lambda) \end{aligned} \tag{4.2.2}$$

$$\begin{aligned} Y^2(\Gamma\zeta, \Omega\eta, k\lambda) \diamond [Y(\Lambda\zeta, \Gamma\zeta, k\lambda) Y(V\eta, \Omega\eta, k\lambda)] \diamond Y^2(V\eta, \Omega\eta, k\lambda) + a Y(V\eta, \Omega\eta, k\lambda) \\ Y(\Lambda\zeta, \Omega\eta, 2k\lambda) \leq [pY(\Lambda\zeta, \Gamma\zeta, \lambda) + qY(\Lambda\zeta, V\eta, \lambda)] Y(\Lambda\zeta, \Omega\eta, 2k\lambda) \end{aligned} \tag{4.2.3}$$

for every  $\zeta, \eta \in \Sigma$  and  $\lambda > 0$ , where  $0 < p, q < 1, 0 \leq a < 1$  such that  $p + q - a = 1$ ,

- The pairs  $(\Gamma, \Lambda)$  and  $(\Omega, V)$  are weak compatible of type  $(\gamma)$ .

Then  $\Gamma, \Omega, \Lambda$  and  $V$  have a unique common fixed point in  $\Sigma$ .

**Proof:** Let  $\zeta_0 \in \Sigma$  be an arbitrary point. Since  $\Gamma(\Sigma) \subseteq V(\Sigma)$  and  $\Omega(\Sigma) \subseteq \Lambda(\Sigma)$ , there exists  $\zeta_1, \zeta_2 \in \Sigma$  such that  $\Gamma\zeta_0 = V\zeta_1 = \eta_1, \Omega\zeta_1 = \Lambda\zeta_2 = \eta_2$ . Because we can construct the sequences  $\{\zeta_n\}, \{\eta_n\} \subset \Sigma$  such that  $\eta_{2n+1} = \Gamma\zeta_{2n} = V\zeta_{2n+1}, \eta_{2n+2} = \Omega\zeta_{2n+1} = \Lambda\zeta_{2n+2}$ , for  $n = 0, 1, 2, \dots$ , we prove  $\{\eta_n\}$  is Cauchy sequence.

For  $\zeta = \zeta_{2n}, \eta = \zeta_{2n+1}$  by (ii), we have

$$\begin{aligned} & \Xi^2(\Gamma\zeta_{2n}, \Omega\zeta_{2n+1}, k\lambda) * [\Xi(\Lambda\zeta_{2n}, \Gamma\zeta_{2n}, k\lambda) \Xi(V\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) * \Xi^2(V\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) \\ & \quad + a \Xi(V\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) \Xi(\Lambda\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda)] \geq [p\Xi(\Lambda\zeta_{2n}, \Gamma\zeta_{2n}, \lambda) + q\Xi(\Lambda\zeta_{2n}, V\zeta_{2n+1}, \lambda)] \times \\ & \quad \Xi(\Lambda\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda), \\ & \Theta^2(\Gamma\zeta_{2n}, \Omega\zeta_{2n+1}, k\lambda) \diamond [\Theta(\Lambda\zeta_{2n}, \Gamma\zeta_{2n}, k\lambda) \Theta(V\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) \diamond \Theta^2(V\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) \\ & \quad + a \Theta(V\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) \Theta(\Lambda\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda)] \leq [p\Theta(\Lambda\zeta_{2n}, \Gamma\zeta_{2n}, \lambda) + q\Theta(\Lambda\zeta_{2n}, V\zeta_{2n+1}, \lambda)] \times \\ & \quad \Theta(\Lambda\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda), \\ & Y^2(\Gamma\zeta_{2n}, \Omega\zeta_{2n+1}, k\lambda) \diamond [Y(\Lambda\zeta_{2n}, \Gamma\zeta_{2n}, k\lambda) Y(V\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) \diamond Y^2(V\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) \\ & \quad + a Y(V\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) Y(\Lambda\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda)] \leq [pY(\Lambda\zeta_{2n}, \Gamma\zeta_{2n}, \lambda) + qY(\Lambda\zeta_{2n}, V\zeta_{2n+1}, \lambda)] \times \\ & \quad Y(\Lambda\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda). \end{aligned}$$

Hence

$$\begin{aligned} & \Xi(\eta_{2n+1}, \eta_{2n+2}, k\lambda) \Xi(\eta_{2n}, \eta_{2n+2}, 2k\lambda) + a\Xi(\eta_{2n+1}, \eta_{2n+2}, k\lambda) \Xi(\eta_{2n}, \eta_{2n+2}, 2k\lambda) \\ & \quad \geq (p + q) \Xi(\eta_{2n}, \eta_{2n+1}, \lambda) \Xi(\eta_{2n}, \eta_{2n+2}, 2k\lambda), \\ & \Theta(\eta_{2n+1}, \eta_{2n+2}, k\lambda) \Theta(\eta_{2n}, \eta_{2n+2}, 2k\lambda) + a \Theta(\eta_{2n+1}, \eta_{2n+2}, k\lambda) \Theta(\eta_{2n}, \eta_{2n+2}, 2k\lambda) \\ & \quad \leq (p + q) \Theta(\eta_{2n}, \eta_{2n+1}, \lambda) \Theta(\eta_{2n}, \eta_{2n+2}, 2k\lambda), \\ & Y(\eta_{2n+1}, \eta_{2n+2}, k\lambda) Y(\eta_{2n}, \eta_{2n+2}, 2k\lambda) + a Y(\eta_{2n+1}, \eta_{2n+2}, k\lambda) Y(\eta_{2n}, \eta_{2n+2}, 2k\lambda) \\ & \quad \leq (p + q) Y(\eta_{2n}, \eta_{2n+1}, \lambda) Y(\eta_{2n}, \eta_{2n+2}, 2k\lambda). \end{aligned}$$

So, we have

$$\begin{aligned} & \Xi(\eta_{2n+1}, \eta_{2n+2}, k\lambda) \geq \Xi(\eta_{2n}, \eta_{2n+1}, \lambda), \Theta(\eta_{2n+1}, \eta_{2n+2}, k\lambda) \leq \Theta(\eta_{2n}, \eta_{2n+1}, \lambda) \text{ and} \\ & Y(\eta_{2n+1}, \eta_{2n+2}, k\lambda) \leq Y(\eta_{2n}, \eta_{2n+1}, \lambda). \end{aligned}$$

Similarly, also we have

$$\begin{aligned} & \Xi(\eta_{2n+2}, \eta_{2n+3}, k\lambda) \geq \Xi(\eta_{2n+1}, \eta_{2n+2}, \lambda), \Theta(\eta_{2n+2}, \eta_{2n+3}, k\lambda) \leq \Theta(\eta_{2n+1}, \eta_{2n+2}, \lambda), \\ & Y(\eta_{2n+2}, \eta_{2n+3}, k\lambda) \leq Y(\eta_{2n+1}, \eta_{2n+2}, \lambda), \text{ for } k \in (0, 1), \text{ if } k_1 = \frac{1}{k} > 1 \text{ and } \lambda = k_1 \lambda_1, \text{ then} \\ & \Xi(\eta_n, \eta_{n+1}, \lambda) \geq \Xi(\eta_{n-1}, \eta_n, k_1 \lambda_1) \geq \dots \geq \Xi(\eta_0, \eta_1, k_1^n \lambda_1), \\ & \Theta(\eta_n, \eta_{n+1}, \lambda) \leq \Theta(\eta_{n-1}, \eta_n, k_1 \lambda_1) \leq \dots \leq \Theta(\eta_0, \eta_1, k_1^n \lambda_1), \\ & Y(\eta_n, \eta_{n+1}, \lambda) \leq Y(\eta_{n-1}, \eta_n, k_1 \lambda_1) \leq \dots \leq Y(\eta_0, \eta_1, k_1^n \lambda_1). \end{aligned}$$

Thus  $\{\eta_n\}$  is a Cauchy sequence and completeness of  $\Sigma, \{\eta_n\}$  converges to  $\omega \in \Sigma$ .

Hence

$$\lim_{n \rightarrow \infty} \Gamma\zeta_{2n} = \lim_{n \rightarrow \infty} \eta_{2n+1} = \lim_{n \rightarrow \infty} V\zeta_{2n+1} = \lim_{n \rightarrow \infty} \eta_{2n+2} = \lim_{n \rightarrow \infty} \Omega\zeta_{2n+1} = \lim_{n \rightarrow \infty} \Lambda\zeta_{2n+2} = \lim_{n \rightarrow \infty} \Lambda\zeta_{2n} = \omega.$$

Since  $\Gamma, \Lambda$  are weak compatible of type  $(\gamma), A\omega = \Lambda\omega$ .

Now, taking  $\zeta = \omega$  and  $\eta = \zeta_{2n+1}$  in (ii), we have as  $n \rightarrow \infty$ .

$$\begin{aligned} & \Xi^2(\Gamma\omega, \omega, k\lambda) * [\Xi(\Lambda\omega, \Gamma\omega, k\lambda) \Xi(\omega, \omega, k\lambda)] * \Xi^2(\omega, \omega, k\lambda) + a\Xi(\omega, \omega, k\lambda) \Xi(\Lambda\omega, \omega, 2k\lambda) \\ & \quad \geq [p\Xi(\Lambda\omega, \Gamma\omega, \lambda) + q\Xi(\Lambda\omega, \omega, \lambda)] \Xi(\Lambda\omega, \omega, 2k\lambda), \\ & \Theta^2(\Gamma\omega, \omega, k\lambda) \diamond [\Theta(\Lambda\omega, \Gamma\omega, k\lambda) \Theta(\omega, \omega, k\lambda)] \diamond \Theta^2(\omega, \omega, k\lambda) + a \Theta(\omega, \omega, k\lambda) \Theta(\Lambda\omega, \omega, 2k\lambda) \\ & \quad \leq [p\Theta(\Lambda\omega, \Gamma\omega, \lambda) + q\Theta(\Lambda\omega, \omega, \lambda)] \Theta(\Lambda\omega, \omega, 2k\lambda), \\ & Y^2(\Gamma\omega, \omega, k\lambda) \diamond [Y(\Lambda\omega, \Gamma\omega, k\lambda) Y(\omega, \omega, k\lambda)] \diamond Y^2(\omega, \omega, k\lambda) + a Y(\omega, \omega, k\lambda) Y(\Lambda\omega, \omega, 2k\lambda) \\ & \quad \leq [pY(\Lambda\omega, \Gamma\omega, \lambda) + qY(\Lambda\omega, \omega, \lambda)] Y(\Lambda\omega, \omega, 2k\lambda). \end{aligned}$$

It follows that

$$\begin{aligned} & \Xi^2(\Gamma\omega, \omega, k\lambda) + a\Xi(\Gamma\omega, \omega, 2k\lambda) \geq [p + q\Xi(\Gamma\omega, \omega, \lambda)] \Xi(\Gamma\omega, \omega, 2k\lambda), \\ & \Theta^2(\Gamma\omega, \omega, k\lambda) \leq q \Theta(\Gamma\omega, \omega, \lambda) \Theta(\Gamma\omega, \omega, 2k\lambda), Y^2(\Gamma\omega, \omega, k\lambda) \leq q Y(\Gamma\omega, \omega, \lambda) Y(\Gamma\omega, \omega, 2k\lambda). \end{aligned}$$



Since  $\Xi(\zeta, \eta, \cdot)$  is nondecreasing,  $\Theta(\zeta, \eta, \cdot)$  is nonincreasing and  $Y(\zeta, \eta, \cdot)$  is nonincreasing for all  $\zeta, \eta \in \Sigma$ , we have

$$\Xi(\Gamma\omega, \omega, \lambda) \geq \frac{p-a}{1-q} = 1, \Theta(\Gamma\omega, \omega, \lambda) \leq \frac{0}{1-q} = 0, Y(\Gamma\omega, \omega, \lambda) \leq \frac{0}{1-q} = 0.$$

for all  $\lambda > 0$ , So  $\Gamma\omega = \omega$ . Hence  $\Gamma\omega = \Lambda\omega = \omega$ .

Similarly, since  $\Omega, V$  are weak compatible of type  $(\gamma)$ , we get  $\Omega\omega = V\omega$ .

For taking  $\zeta = \zeta_{2n}$  and  $\eta = \omega$  in (ii), we have as  $n \rightarrow \infty$ ,

$$\begin{aligned} \Xi^2(\omega, \Omega\omega, k\lambda) * [\Xi(\omega, \omega, k\lambda) \Xi(V\omega, \Omega\omega, k\lambda)] * \Xi^2(V\omega, \Omega\omega, k\lambda) + a\Xi(V\omega, \Omega\omega, k\lambda) \\ \Xi(\omega, \Omega\omega, 2k\lambda) \geq [p\Xi(\omega, \omega, \lambda) + q\Xi(\omega, V\omega, \lambda)] \Xi(\omega, \Omega\omega, 2k\lambda), \\ \Theta^2(\omega, \Omega\omega, k\lambda) \diamond [\Theta(\omega, \omega, k\lambda) \Theta(V\omega, \Omega\omega, k\lambda)] \diamond \Theta^2(V\omega, \Omega\omega, k\lambda) + a\Theta(V\omega, \Omega\omega, k\lambda) \\ \Theta(\omega, \Omega\omega, 2k\lambda) \leq [p\Theta(\omega, \omega, \lambda) + q\Theta(\omega, V\omega, \lambda)] \Theta(\omega, \Omega\omega, 2k\lambda), \\ Y^2(\omega, \Omega\omega, k\lambda) \diamond [Y(\omega, \omega, k\lambda) Y(V\omega, \Omega\omega, k\lambda)] \diamond Y^2(V\omega, \Omega\omega, k\lambda) + aY(V\omega, \Omega\omega, k\lambda) \\ Y(\omega, \Omega\omega, 2k\lambda) \leq [pY(\omega, \omega, \lambda) + qY(\omega, V\omega, \lambda)] Y(\omega, \Omega\omega, 2k\lambda). \end{aligned}$$

Then,

$$\begin{aligned} \Xi^2(\omega, \Omega\omega, k\lambda) + a\Xi(\omega, \Omega\omega, 2k\lambda) \geq [p+q\Xi(\omega, V\omega, \lambda)] \Xi(\omega, \Omega\omega, 2k\lambda), \\ \Theta^2(\omega, \Omega\omega, k\lambda) \leq q\Theta(\omega, V\omega, \lambda) \Theta(\omega, \Omega\omega, 2k\lambda), Y^2(\omega, \Omega\omega, k\lambda) \leq qY(\omega, V\omega, \lambda) Y(\omega, \Omega\omega, 2k\lambda). \end{aligned}$$

Thus it follows that

$$\Xi(\omega, \Omega\omega, \lambda) \geq \frac{p-a}{1-q} = 1, \Theta(\omega, \Omega\omega, \lambda) \leq \frac{0}{1-q} = 0 \text{ and } Y(\omega, \Omega\omega, \lambda) \leq 0, \text{ for all } \lambda > 0, \text{ so } \Omega\omega = \omega.$$

Hence  $\Omega\omega = V\omega = \omega$ .

Therefore  $\omega$  is a common fixed point of  $\Gamma, \Omega, \Lambda$  and  $V$ .

Let  $\beta$  be another common fixed point of  $\Gamma, \Omega, \Lambda$  and  $V$ . Then we have

$$\begin{aligned} \Xi^2(\Gamma\omega, \Omega\beta, k\lambda) * [\Xi(\Lambda\omega, \Gamma\omega, k\lambda) \Xi(V\beta, \Omega\beta, k\lambda)] * \Xi^2(V\beta, \Omega\beta, k\lambda) \\ + a\Xi(V\beta, \Omega\beta, k\lambda) \Xi(\Lambda\beta, \Omega\beta, 2k\lambda) \geq [p\Xi(\Lambda\omega, \Gamma\omega, \lambda) + q\Xi(\Lambda\omega, V\beta, \lambda)] \Xi(\Lambda\omega, \Omega\beta, 2k\lambda), \\ \Theta^2(\Gamma\omega, \Omega\beta, k\lambda) \diamond [\Theta(\Lambda\omega, \Gamma\omega, k\lambda) \Theta(V\beta, \Omega\beta, k\lambda)] \diamond \Theta^2(V\beta, \Omega\beta, k\lambda) \\ + a\Theta(V\beta, \Omega\beta, k\lambda) \Theta(\Lambda\beta, \Omega\beta, 2k\lambda) \leq [p\Theta(\Lambda\omega, \Gamma\omega, \lambda) + q\Theta(\Lambda\omega, V\beta, \lambda)] \Theta(\Lambda\omega, \Omega\beta, 2k\lambda), \\ Y^2(\Gamma\omega, \Omega\beta, k\lambda) \diamond [Y(\Lambda\omega, \Gamma\omega, k\lambda) Y(V\beta, \Omega\beta, k\lambda)] \diamond Y^2(V\beta, \Omega\beta, k\lambda) \\ + aY(V\beta, \Omega\beta, k\lambda) Y(\Lambda\beta, \Omega\beta, 2k\lambda) \leq [pY(\Lambda\omega, \Gamma\omega, \lambda) + qY(\Lambda\omega, V\beta, \lambda)] Y(\Lambda\omega, \Omega\beta, 2k\lambda), \end{aligned}$$

So,

$$\begin{aligned} \Xi^2(\omega, \beta, k\lambda) + a\Xi(\omega, \beta, 2k\lambda) \geq [p + q\Xi(\omega, \beta, \lambda)] \Xi(\omega, \beta, 2k\lambda), \\ \Theta^2(\omega, \beta, k\lambda) \leq q\Theta(\omega, \beta, \lambda) \Theta(\omega, \beta, 2k\lambda) \text{ and } Y^2(\omega, \beta, k\lambda) \leq qY(\omega, \beta, \lambda) Y(\omega, \beta, 2k\lambda). \end{aligned}$$

Therefore

$$\Xi(\omega, \beta, \lambda) \geq \frac{p-a}{1-q} = 1, \Theta(\omega, \beta, \lambda) \leq \frac{0}{1-q} = 0, Y(\omega, \beta, \lambda) \leq 0,$$

for all  $\lambda > 0$ , so  $\omega = \beta$ , hence  $\Gamma, \Omega, \Lambda$  and  $V$  have unique common fixed point on  $\Sigma$ .

**Example 4.3.**

Let  $(\Sigma, d)$  be a metric space with  $\Sigma = [0,1]$ . Denote  $\omega * \tau = \min \{ \omega, \tau \}$  and  $\omega \diamond \tau = \max \{ \omega, \tau \}$  for  $\omega, \tau \in [0,1]$  and let  $\Xi_d, \Theta_d, Y_d$  be neutrosophic sets on  $\Sigma^2 \times [0, \infty]$  defined as follows;

$$\Xi_d(\zeta, \eta, \lambda) = \frac{\lambda}{\lambda + d(\zeta, \eta)}; \Theta_d(\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda + d(\zeta, \eta)}; Y_d(\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda}.$$

Then  $(\Xi_d, \Theta_d, Y_d)$  is an NMS on  $\Sigma$  and  $(\Sigma, \Xi_d, \Theta_d, Y_d, *, \diamond)$  is an NMS.

Define self mappings  $\Gamma, \Omega, \Lambda$  and  $V$  by

$$\Gamma(\Sigma) = 1; \Omega(\Sigma) = 1; \Lambda(\Sigma) = \begin{cases} 1 & \text{if } \zeta \text{ is rational} \\ 0 & \text{if } \zeta \text{ is irrational} \end{cases}; V(\Sigma) = \frac{\zeta+1}{2}$$

If we define  $\{\zeta_n\} \subset \Sigma$  by  $\zeta_n = 1 - \frac{1}{n}$ , then we have for  $\lim_{n \rightarrow \infty} \Gamma\zeta_n = \lim_{n \rightarrow \infty} \Omega\zeta_n = 1$  and  $\Gamma 1 = 1 = \Lambda 1$ .

$$\lim_{n \rightarrow \infty} \Xi (\wedge \Gamma \zeta_n, 1, \lambda) \leq \Xi (\Gamma 1, 1, \lambda) = 1 ; \quad \lim_{n \rightarrow \infty} \Theta (\wedge \Gamma \zeta_n, 1, \lambda) \geq \Theta (\Gamma 1, 1, \lambda) = 0 ;$$

$$\lim_{n \rightarrow \infty} \Upsilon (\wedge \Gamma \zeta_n, 1, \lambda) \geq \Upsilon (\Gamma 1, 1, \lambda) = 0 .$$

Also, for  $\lim_{n \rightarrow \infty} \Omega \zeta_n = \lim_{n \rightarrow \infty} \vee \zeta_n = 1$  and  $\Omega 1 = 1 = \vee 1$ .

$$\lim_{n \rightarrow \infty} \Xi (\vee \Omega \zeta_n, 1, \lambda) \leq \Xi (\Omega 1, 1, \lambda) = 1 ; \quad \lim_{n \rightarrow \infty} \Theta (\vee \Omega \zeta_n, 1, \lambda) \geq \Theta (\Omega 1, 1, \lambda) = 0 ;$$

$$\lim_{n \rightarrow \infty} \Upsilon (\vee \Omega \zeta_n, 1, \lambda) \geq \Upsilon (\Omega 1, 1, \lambda) = 0 .$$

Therefore,  $(\Gamma, \wedge)$  and  $(\Omega, \vee)$  are weak compatible of type  $(\gamma)$ . Then all the conditions of Theorem 4.2. are satisfied and 1 is a unique common fixed point of  $\Gamma, \Omega, \wedge$  and  $\vee$  on  $\Sigma$ .

**Conclusion:** In this study, we have made common fixed point results for weak compatible maps of type in neutrosophic metric Space. There is a degree to set up many fixed point brings about the spaces like fuzzy metric, generalized fuzzy metric, bipolar and partial fuzzy metric spaces by utilizing the idea of Neutrosophic Set.

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