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Neutrosophic Kumaraswamy Distribution with Engineering Application

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Abstract: In this study, Neutrosophic Kumaraswamy (NKw) distribution was proposed to analyze bounded data sets under an indeterminacy environment. Mathematical properties of NKw distribution are derived including, moments, mean, variance, Shannon entropy, reliability measures. We also present the graphical representation of density curves, cumulative distribution function, and hazard rate function. The parameters of NKw distribution are estimated using the maximum likelihood technique. We perform a simulation study to see the performance of maximum likelihood estimates. Eventually, the proposed model is applied to real data set. It has been concluded that NKw distribution provides better results than Neutrosophic beta distribution.

Keywords: Neutrosophic; indeterminacy; Kumaraswamy distribution; MLE; Simulation

1. Introduction

In 1995, Neutrosophic statistics was originally introduced by [1]. It is a new branch of philosophy, presented as a generalization for fuzzy logic and as a generalization for intuitionistic fuzzy logic. Neutrosophic statistics can be applied in an uncertain environment. Neutrosophic statistics acquire significance because of their ability to manage sets of values more explicitly an interval. To be more exact, when the values or parameters have disarray attached with them, then that particular value or parameter is replaced with a set of values [2, 3]. Further, [4-9] presented some more interesting fundamental concepts of the neutrosophic set.

Nowadays, authors contributed to the field of neutrosophic statistics methodologically as well as applied it in various fields. Alhabib and Salama [10] introduced time-series theory under indeterminacy. Aslam [11—13] extend the neutrosophic statistics in the field of total quality control. He proposed control charts under an indeterminacy environment. He presented several neutrosophic sampling plans.

The classical probability distributions applicable when the sample is selected from the population having uncertain observations. So there is an essential need to introduced probability models under an indeterminacy environment. Several authors introduced neutrosophic probability distributions, for example, Neutrosophic Weibull by [14], Neutrosophic Uniform, Neutrosophic exponential, and Neutrosophic Poisson [15], Normal distribution and binomial distribution by [16], Neutrosophic Raleigh distribution by [17] and Neutrosophic Beta distribution by [18].

1.1. Neutrosophic Approach

Neutrosophic statistics is the extended form of classical statistics. In classical statistics, we are dealing with specific values or crisp values but in neutrosophic statistics, the sample observations are taken from a population having uncertainty in observations. In the field of neutrosophic statistics, the data information might be vague, imprecise, ambiguous, uncertain, incomplete, even unknown. The shape of the neutrosophic number has a standard form in terms of the extension of the classical statistics and is shown below

$$X_N = E + i$$

where E is the exact or determined part of data information and i is the uncertain, inexact, or indeterminacy part of data. To differentiate the neutrosophic random variable the subscript N is used such as XN.

1.2. Kumaraswamy distribution

The Kumaraswamy (Kw) distribution is one of the most important and flexible distribution to analyze unit interval (0, 1) data sets. The Kw distribution was originally introduced by Kumaraswamy in 1980 [19]. The Kw distribution contained two positive shape parameters. The Kw distribution is applicable in the fields of reliability analysis, atmosphere temperatures, scores acquired in the test, hydrological, and economic data, etc. Jones [20] derived mathematical properties of Kumaraswamy distribution.

2. Neutrosophic Kumaraswamy distribution

A neutrosophic Kumaraswamy distribution (NKw) of a continuous variable X is a classical Kumaraswamy distribution of x, but such parameters are imprecise. The probability density function (pdf) of NKw distribution is

$$f_N(X) = \alpha_N \beta_N X^{\alpha_N - 1}(1 - X^{\alpha_N})^{\beta_N - 1}, \;\; X \in (0,1) \tag{1}$$

where α_N and β_N are the shape parameters. Figure 1 shows the pdf plots for various values of parameters.

The NKw distribution is flexible due to its variable shapes of the density function. The PDF curves showing exponentially decreasing behavior and start from the infinite point for $\alpha_N < 1$. For $\alpha_N = 1$, its behavior is exponentially decreasing but starts from a specific point on the y-axis. For $\alpha_N > 1$, the density curves showing unimodal behavior.

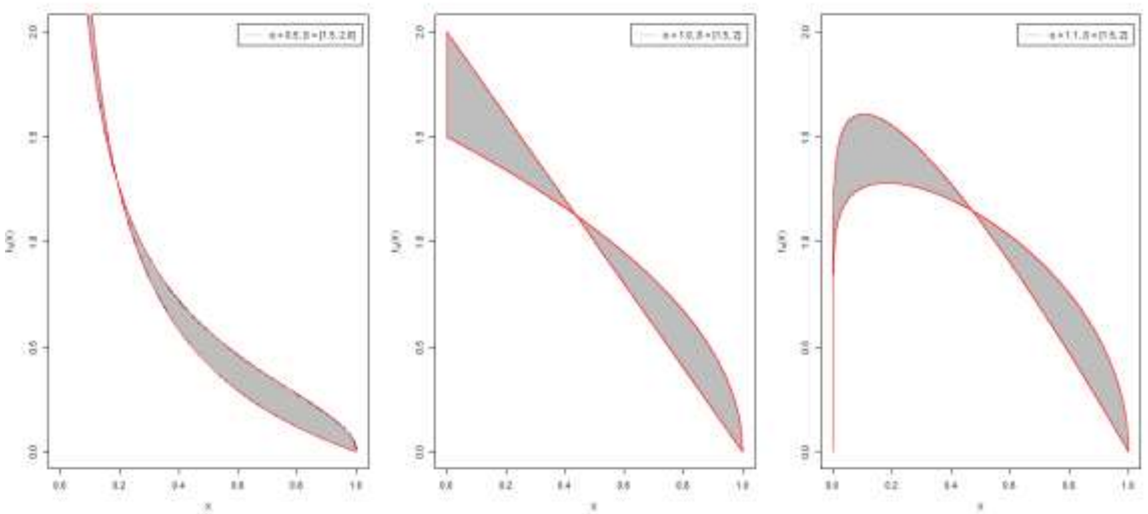

Figure 1(a). Density function plots of NKw distribution

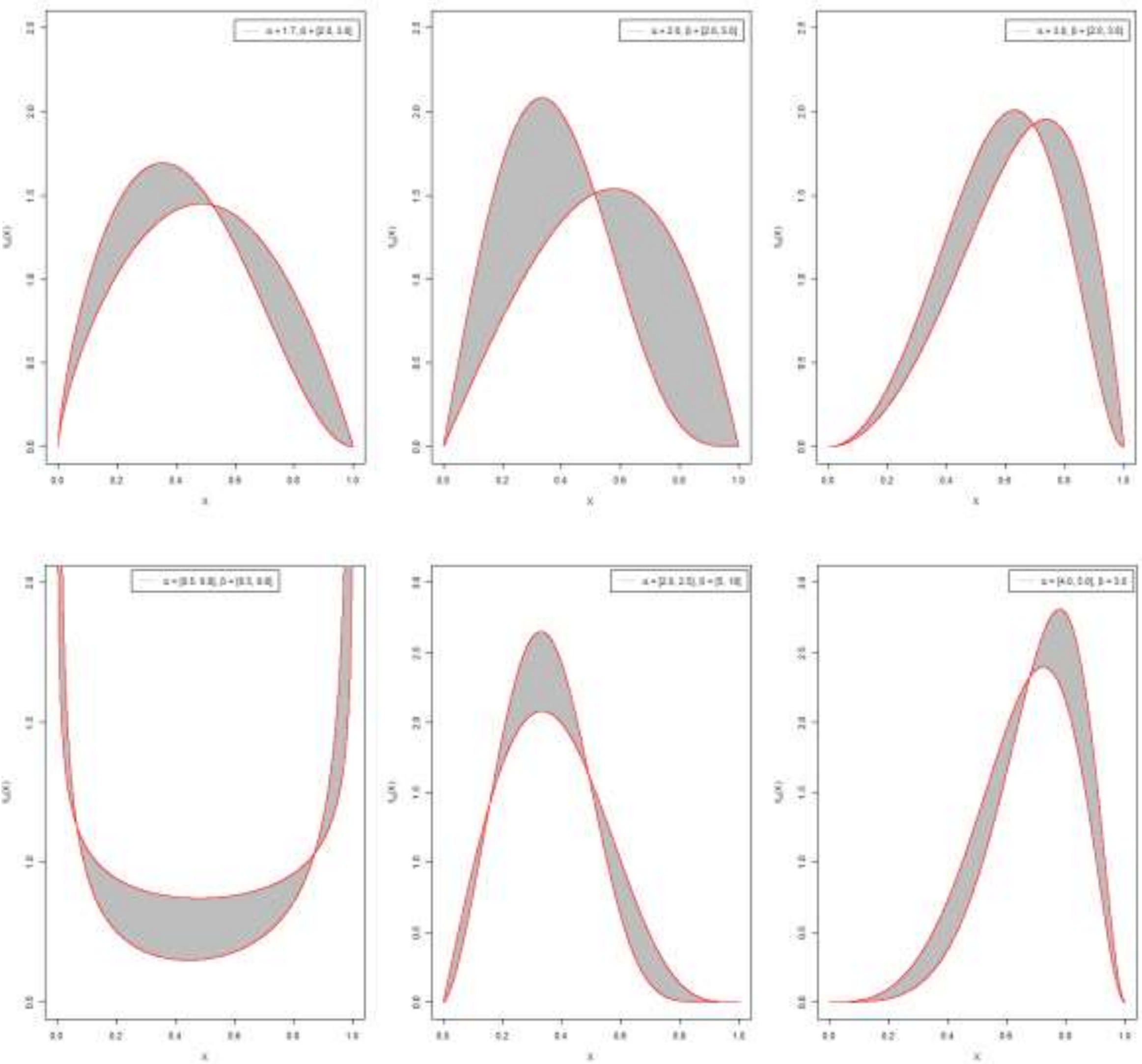

Figure 1(b). Density function plots of NKw distribution

The cumulative distribution function (CDF) of NKw distribution is

$$F_N(X) = 1 - (1 - X^{\alpha_N})^{\beta_N} \tag{2}$$

We plot CDF curves for some selected values of parameters, see Figure 2.

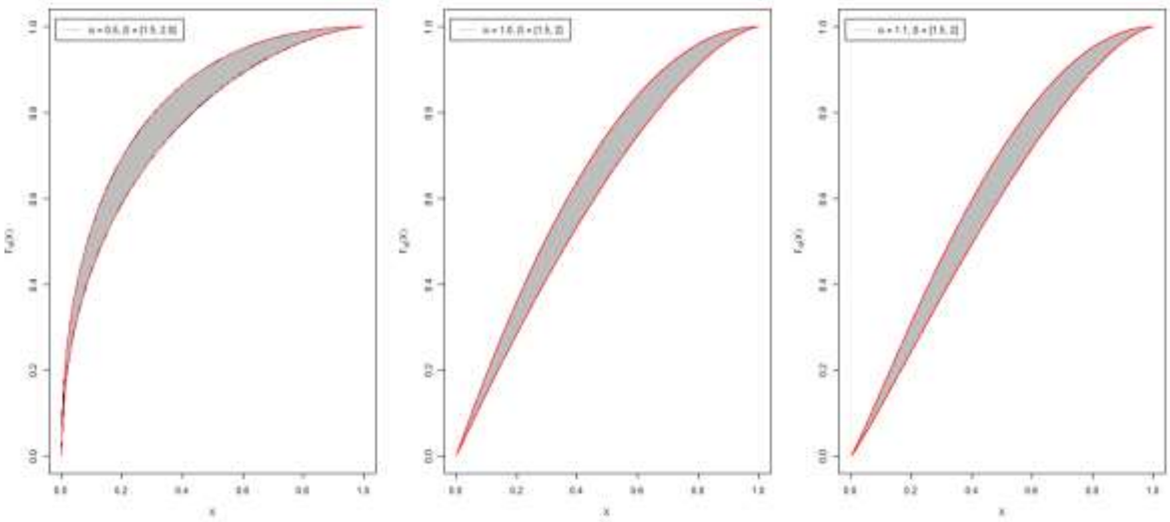

Figure 2. The CDF curves of NKw distribution

The survival function and hazard function of NKw distribution expressed as

$$S_N(X) = (1 - X^{\alpha_N})^{\beta_N} \tag{3}$$

and

$$F_N(X) = \frac{\alpha_N \beta_N X^{\alpha_N - 1}}{(1 - X^{\alpha_N})} \tag{4}$$

The HRF curves of the NKw distribution are presented in Figure 3.

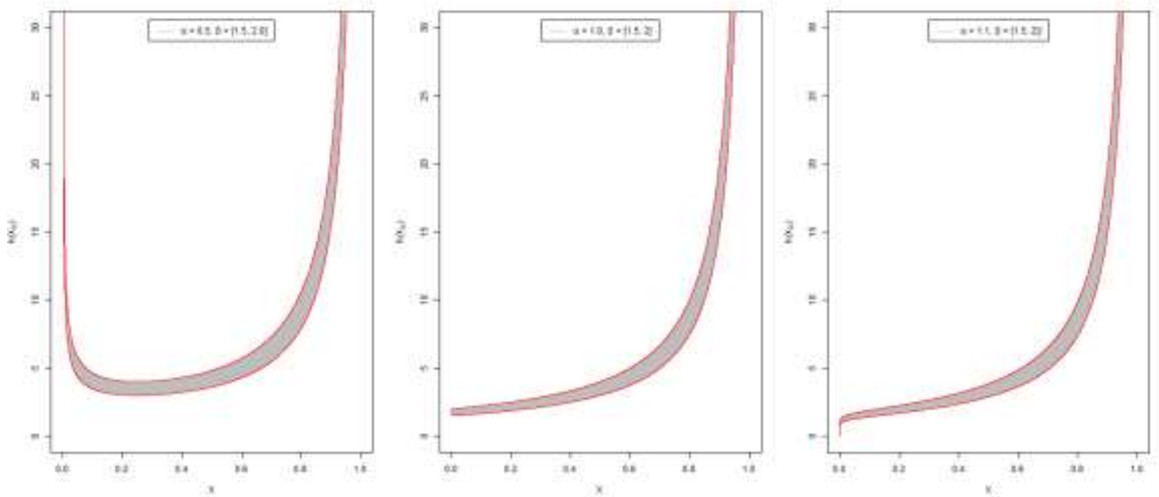

Figure 3. HRF curves for NKw distribution

From Figure 3, it is interesting to note the NKw distribution has variable shapes. The failure rate of NKw distribution is a bathtub and increasing behavior, which is very important to analyze data sets in various fields.

3. Mathematical Properties

In this section, we discussed some mathematical properties of the NKw distribution.

3.1. The rth moments:

$$E_N(X^r) = \beta_N \mathrm{B}\left[1 + \frac{r}{\alpha_N}, \beta_N\right]$$

3.2. The mean and variance:

$$E_N(X) = \beta_N \mathrm{B}\left[1 + \frac{1}{\alpha_N}, \beta_N\right]$$

and

$$Var_N(X) = \beta_N \mathrm{B}\left[1 + \frac{2}{\alpha_N}, \beta_N\right] - \left\{\beta_N \mathrm{B}\left[1 + \frac{1}{\alpha_N}, \beta_N\right]\right\}^2$$

3.3. Median:

$$Q_{0.25} = \left(1 - \frac{1}{\sqrt[\beta_N]{2}}\right)^{\frac{1}{\alpha_N}}$$

3.4. Shannon entropy of NKw distribution is

$$H_N(X) = \int_0^1 f_N(X) \log\big(f_N(X)\big)\, dx$$

$$H_N(X) = \int_0^1 \left\{\alpha_N \beta_N X^{\alpha_N - 1}(1 - X^{\alpha_N})^{\beta_N - 1}\right\} \log\left\{\alpha_N \beta_N X^{\alpha_N - 1}(1 - X^{\alpha_N})^{\beta_N - 1}\right\} dx$$

$$H_N(X) = \left(1 - \frac{1}{\alpha_N}\right) + \left(1 + \frac{1}{\beta_N}\right) H_{\beta_N} - \log(\alpha_N \beta_N)$$

where H_i is the harmonic number.

4. Maximum Likelihood Estimation

The model parameters are estimated using the famous maximum likelihood approach. Let $X_{N1}, X_{N2}, \dots, X_{Nn}$ be a random sample of NKw distribution. The log-likelihood function can be written as

$$l(\alpha_N, \beta_N) = n_N \log(\alpha_N \beta_N) + (\alpha_N - 1) \sum_{i=1}^{n_N} \log(X_{Ni}) + (\beta_N - 1) \sum_{i=1}^{n_N} \log\big(1 - X_{Ni}^{\alpha_N}\big) \quad (5)$$

The MLEs, $\hat{\alpha}_N \in [\hat{\alpha}_L, \hat{\alpha}_U]$ and $\hat{\beta}_N \in [\hat{\beta}_L, \beta_U]$, can be obtained by maximizing the above log-likelihood function equation.

$$\frac{\partial l(\alpha_N, \beta_N)}{\partial l(\alpha_N)} = \frac{n_N}{\alpha_N} + \sum_{i=1}^{n_N} \log(X_{Ni}) - (\beta_N - 1) \sum_{i=1}^{n_N} \frac{X_{Ni}^{\alpha_N} \log(X_{Ni})}{\big(1 - X_{Ni}^{\alpha_N}\big)} \quad (6)$$

$$\frac{\partial l(\alpha_N,\beta_N)}{\partial l(\beta_N)}=\frac{n_N}{\beta_N}+\sum_{i=1}^{n_N}\log\left(1-X_{Ni}^{\alpha_N}\right) \quad (7)$$

The maximum likelihood estimates can be obtained using the above equations.

5. Simulation Study

In this section, we carry out a simulation study to check the behavior of proposed estimators for NKw distribution. We generate 10,000 samples of sizes, $n = 30, 50, 100, 200,$ and 250 from NKw distribution with different combinations of parameters. The sample we generated from a random number generator. The average bias and Mean Square Error (MSEs) are used to check the properties of the best estimator. The results of the simulation study are listed in Tables 1-2.

Table 1. Parameter Estimates for $\alpha_N = 0.5,\ \beta_N = [1.5,2.0]$.

n	AEs		Avg. Biases		MSEs	
	$\hat{\alpha}_N$	$\hat{\beta}_N$	$\hat{\alpha}_N$	$\hat{\beta}_N$	$\hat{\alpha}_N$	$\hat{\beta}_N$
30	0.5334	[1.6469, 2.2859]	0.0334	[0.1469, 0.2859]	0.0141	[0.2436, 0.6027]
50	0.5180	[1.6045, 2.1481]	0.0180	[0.1045, 0.1481]	0.0071	[0.1293, 0.2584]
100	0.5102	[1.5492, 2.0636]	0.0102	[0.0492, 0.0636]	0.0036	[0.0524, 0.1129]
200	0.5059	[1.5173, 2.0368]	0.0059	[0.0173, 0.0368]	0.0017	[0.0236, 0.0483]
250	0.5033	[1.5256, 2.0378]	0.0033	[0.0256, 0.0378]	0.0013	[0.0214, 0.0403]

Table 2. Parameter Estimates for$\alpha_N = [1.5,2.0],\ \beta_N = 0.5$

n	AEs		Avg. Biases		MSEs	
	$\hat{\alpha}_N$	$\hat{\beta}_N$	$\hat{\alpha}_N$	$\hat{\beta}_N$	$\hat{\alpha}_N$	$\hat{\beta}_N$
30	[1.6603, 2.2442]	0.5346	[0.1603, 0.2442]	0.0346	[0.2798, 0.5358]	0.0171
50	[1.6107, 2.1833]	0.5249	[0.1107, 0.1833]	0.0249	[0.1624, 0.3348]	0.0091
100	[1.5449, 2.0835]	0.5091	[0.0449, 0.0835]	0.0091	[0.0647, 0.1206]	0.0039
200	[1.5314, 2.0356]	0.5062	[0.0314, 0.0356]	0.0062	[0.0332, 0.0550]	0.0019
250	[1.5209, 2.0233]	0.5058	[0.0209, 0.0233]	0.0058	[0.0251, 0.0400]	0.0015

From the above tables, it is seen that the ML estimators are consistent. The average bias and MSE decrease with an increase in sample size.

6. Application

In this section, a data set is analyzed to demonstrate the applicability and flexibility of the newly neutrosophic probability distribution over well-known existing probability distribution. The considered data set is about the ball-bearing data and from [21]. The selection of the best fit model shall be considered using the following model selection standards, log-likelihood value (Log-Lik.), and Akaike Information Criteria (AIC), and Kolmogorov Smirnov test. The maximum values of log-Likelihood and minimum values of AIC and KS statistic indicate that the model provides the best fit. The ML estimates along with goodness of fit measures are presented in Table 3.

Table 3. MLEs and model adequacy measures for ball bearing data.

Model		Estimates	LogLik.	AIC	KS
NKw	$\hat{\alpha}$	[2.0758, 2.2871]	[43.828, 41.623]	[-83.656, -79.246]	[0.600, 0.270]
	$\hat{\beta}$	[207.91, 164.94]			
NBD	$\hat{\alpha}$	[3.5523, 11.570]	[42.533, 39.262]	[-81.066, -74.525]	[0.510, 0.192]
	$\hat{\beta}$	[48.377, 103.14]			

From the above Table 3, it is tracked down that the new proposed neutrosophic distribution gives more efficient results than the neutrosophic beta distribution.

7. Conclusion

In this work, a new generalization of classical Kumaraswamy distribution is proposed for interval form of data sets. The proposed distribution is known as Neutrosophic Kumaraswamy distribution. Some mathematical properties of the NKw distribution are derived. The parameters are estimated using the maximum likelihood method. In the end, a real data set have been utilized to demonstrate the usefulness of the proposed distribution over Neutrosophic beta distribution. Numerical findings show that the NKw distribution provides better results than the Neutrosophic beta distribution.

Our future research will use the neutrosophic probability distributions NP of an event E defined as follows: NP(E)=(chance that the event E occurs (T), indeterminate-chance that the event E occurs (I), chance that the event E does not occur (F))

where T,I,F in [0, 1] and 0 <= T+I+F <= 3.

Therefore, we'll need to graph three curves for each neutrosophic probability distribution.

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