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Abstract. The notion of neutrosophic triplet, in the form of (p, n_p, a_p) is a recent subject of neutrosophy, where n_p is the neutral of the element p and a_p is the opposite of p. In this paper, neutrosophic triplet b-metric spaces are investigated. Then some new definitions and examples are given for neutrosophic triplet b-metric space. Based on these definitions, new theorems are given and proven. A neutrosophic triplet topology induced by neutrosophic triplet b-metric is obtained. Furthermore, a contraction map is defined for neutrosophic triplet b-metric space, and finally, a fixed point theorem is given for it.

Keywords: Neutrosophic triplet set; neutrosophic triplet topology; neutrosophic triplet metric; neutrosophic triplet b-metric.

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1. Introduction

The classical mathematical methods may not be sufficient to answer some of the complex problems encountered in such disciplines as usual set theory, fuzzy set theory, and probability theory. Many fields, such as economics, engineering, and environmental science, need to model linguistic values and uncertainties mathematically to continue their studies.

Zadeh defined the fuzzy set concept for handling uncertainty problems in 1965 [1]. A fuzzy set is substantially a function that appoints a membership degree (truth value) from the range [0,1] to each element in the universal set. In 1986 [2], Atanasov, as an expansion of the fuzzy set, described the notion of intuitionistic fuzzy set, via defining a non-membership degree in addition to the membership degree.

These theories can deal with real-world problems, but not with the indeterminate data. Based on this, Smarandache published his work in 2005 [3], containing the concept of neutrosophic sets and some of its applications. In this theory, truth, falsity, and indeterminacy are defined as independent of each other. A neutrosophic set is a triplet of the functions

(T, I, F), each of them is from the universal set to [0,1]; T assigns truth-value, I assigns indeterminate-value and F assigns false-value.

Lately, some researchers have been preoccupied with neutrosophic set theory [4–9].

By using the basic rule of neutrosophy, Smarandache established neutrosophic triplet set theory as a generalization of the classical group in 2017 [10]. An neutrosophic triplet space is dissimilar from the usual group because for all element p in a neutrosophic triplet space A with an operation $*$, there is a neutral of p, indicated by n_p and opposite of p, indicated by a_p such that $p * n_p = n_p * p = p$ and $p * a_p = a_p * p = n_p$. The neutral of any element p is distinct from the standard algebraic unit element, and it is not unique. A neutrosophic triplet is of the form $< p, n_p, a_p >$.

For a neutrosophic triplet space $(A, *)$ and a subset $B \subset A$, if B also forms a neutrosophic triplet space under the operation $*$, then $(B, *)$ is called a neutrosophic triplet subset of A.

Smarandache and Ali established a group structure on neutrosophic triplet space in 2018 [11]. Therefore, a new group form is assigned to some non-classical group structures.

Şahin, Kargın and Smarandache introduced neutrosophic triplet topology in [12]. Neutrosophic triplet base was defined for a neutrosophic triplet topology in [13]. Şahin and Kargin defined neutrosophic triplet metric space in 2017 [14] and neutrosophic triplet normed ring space in 2018 [15]. Also, in 2019 [16], they described neutrosophic triplet metric topology and investigated some features of metric topology in neutrosophic triplet metric space. In [17–20], researchers studied neutrosophic triplet metric spaces.

In recent years, many generalizations of classical metric spaces appeared. b-metric space is one of these generalizations, which was introduced by Czerwik in 1993 [21]. Then some fixed point theorems were extended to b-metric spaces. Şahin and Kargin extended b-metric to neutrosophic triplet metric space and defined convergence sequence and Cauchy sequence in neutrosophic triplet b-metric spaces [22]. In 2019 [23], Şahin and Kargın defined neutrosophic triplet b-metric topology by using the open balls.

The main idea of this paper is to describe a contraction for neutrosophic triplet b-metric space and prove a fixed point theorem for neutrosophic triplet b-metric space. Section 2 gives some basic definitions and examples for b-metric and neutrosophic triplet space. Section 3 presents the neutrosophic triplet b-metric space. Many examples are given in this section to illustrate the difference between neutrosophic triplet metric and neutrosophic triplet b-metric. Also, a sufficient condition is given for a convergence sequence in a neutrosophic triplet b-metric space to be a Cauchy sequence. Then a neutrosophic triplet topological structure is established by using neutrosophic triplet b-metric. Section 4 defines a contraction for neutrosophic triplet b-metric space. It is proven that the contraction is continuous respect to the induced topology by neutrosophic triplet b-metric. Also, a fixed point theorem is given and proven for neutrosophic triplet b-metric space. Finally, Section 5 and 6 present a brief of the paper and some future works.

2. Preliminaries

A review of essential concepts of neutrosophic triplet set are presented in this section.

Definition 2.1. [21] Let A be set and $d : A \times A \to \mathbb{R}^+ \cup \{0\}$. If the following statements are satisfied, then d is called a b-metric;

bm$_1$) $u = v$ if and only if $d(u, v) = 0$

bm$_2$) $d(u, v) = d(v, u)$

bm$_3$) for all $u, v, w \in A$, there is a real number $b \geq 1$ such that $d(u, v) \leq b\,[d(u, w) + d(w, v)]$.

The pair (A, d) is called a b-metric space with coefficient b.

It is clear that every metric space is a b-metric space with coefficient 1, but the converse is not always true. For example, $d : A \times A \to \mathbb{R}, d(u, v) = (u - v)^2$ is a b-metric with a coefficient $b \geq 2$, but it is not a metric. Also, the reader can realize that if d is a b-metric with the coefficient b, then d is a b-metric with any coefficient $k \geq b$.

Definition 2.2. [11] Let A be a set and $*$ be a binary operation on A. If for each $p \in A$ there exists n_p and a_p in A such that

$$p * n_p = n_p * p = p$$

and

$$p * a_p = a_p * p = n_p$$

then $(A, *)$ is called neutrosophic triplet set. Also, n_p and a_p are called neutral and opposite of p, respectively. A neutrosophic triplet p is indicated by $p = < p, n_p, a_p >$. Besides, $p_1 = p_2$ if and only if $n_{p_1} = n_{p_2}$ and $a_{p_1} = a_{p_2}$.

Definition 2.3. [13] Let $(A, *)$ be a neutrosophic triplet set, $P(A)$ be set of each subset of A and $\tau \subset P(A)$. If

1) $\varnothing, A \in \tau$
2) The intersection of a finite number of sets in τ is also in τ
3) The union of an arbitrary number of sets in τ is also in τ

then τ is called as a neutrosophic triplet topology on A and, $((A, *), \tau)$ is called as a neutrosophic triplet topological space.

Definition 2.4. [15] A neutrosophic triplet metric on a neutrosophic triplet space $(A, *)$ is a function $d_T : A \times A \to \mathbb{R}$ such that, for all $u, v \in A$,

m$_1$) $u * v \in A$

m$_2$) $d_T(u, v) \geq 0$

m$_3$) If $u = v$ then $d_T(u, v) = 0$

m$_4$) $d_T(u, v) = d_T(v, u)$

m$_5$) If there exists at least an element $w \in A$ for each $u, v \in A$ such that $d_T(u, v) \leq d_T(u, v * n_w)$, then

$$d_T(u, v * n_w) \leq d_T(u, w) + d_T(w, v).$$

$((A, *), d_T)$ is called a neutrosophic triplet metric space.

Example 2.5. [14] Let A be a set. Then $(P(A), \cup)$ is a neutrosophic triplet space with $n_U = a_U = U$ for all $U \in P(A)$, where $P(A)$ is the power set of A.

Define a map $d : P(A) \times P(A) \to \mathbb{R}$ such that $d(U, V) = |m(U) - m(V)|$, where $m(U)$ and $m(V)$ denote the numbers of elements of U and V, respectively. Then $((P(A), \cup), d)$ is a neutrosophic triplet metric space.

3. Neutrosophic Triplet b-metric

This section gives some new examples and definitions for neutrosophic triplet b-metric space. Also it defines neutrosophic triplet b-metric topology.

Definition 3.1. [22] A neutrosophic triplet b-metric is a function $d_b : A \times A \to \mathbb{R}$ such that, for all $u, v \in A$,

b$_1$) $u * v \in A$

b$_2$) $d_b(u, v) \geq 0$

b$_3$) if $u = v$ then $d_b(u, v) = 0$

b$_4$) $d_b(u, v) = d_b(v, u)$

b$_5$) If there exists at least an element $w \in A$ for each $u, v \in A$ such that $d_b(u, v) \leq d_b(u, v * n_w)$, then for a real number $b \geq 1$,

$$d_b(u, v * n_w) \leq b[d_b(u, w) + d_b(w, v)].$$

$((A, *), d_b)$ is called a neutrosophic triplet b-metric space.

neutrosophic triplet b-metric is distinct from the usual b-metric because of the binary operation and neutral element.

It is clear that all neutrosophic triplet metric space is a neutrosophic triplet b-metric space with coefficient 1, but the opposite is not true always like the following example indicates.

Example 3.2. Let $A = \{0, 2, 3, 4\}$. Then A is a neutrosophic triplet space with the multiplication module 6 in $\mathbb{Z}$. Neutrosophic triplets are $< 0, 0, 0 >, < 2, 4, 2 >, < 3, 3, 3 >$ and $< 4, 4, 4 >$. Let $d : A \times A \to \mathbb{R}$ be defined such that $d(u, v) = (u - v)^2$, for every $u, v \in A$, . Then it is clear that d_b satisfies conditions b_1, b_2, b_3 and b_4.

b_5) For $u = 0, v = 2$,

$d'(0, 2) \leq d'(0, 2.n_2)$. Also $d'(0, 2.n_2) \leq d'(0, 2) + d'(2, 2)$.

$d'(0, 2) \leq d'(0, 2.n_4)$. Also $d'(0, 2.n_4) \leq d'(0, 4) + d'(4, 2)$.

For $u = 0, v = 3$,

$d'(0, 3) \leq d'(0, 3.n_3)$. Also $d'(0, 3.n_3) \leq d'(0, 3) + d'(3, 3)$.

For $u = 0, v = 4$

$d'(0, 4) \leq d'(0, 4.n_2)$. Also $d'(0, 4.n_2) \leq 2\,[d'(0, 2) + d'(2, 4)]$.

$d'(0, 4) \leq d'(0, 4.n_4)$. Also $d'(0, 4.n_4) \leq d'(0, 4) + d'(4, 4)$.

For $u = 2, v = 3$,

$d'(2, 3) \leq d'(2, 3.n_0)$. Also $d'(2, 3.n_0) \leq d'(2, 0) + d'(0, 3)$.

$d'(2, 3) \leq d'(2, 3.n_2)$. Also $d'(2, 3.n_2) \leq 4\,[d'(2, 2) + d'(2, 3)]$.

$d'(2, 3) \leq d'(2, 3.n_4)$. Also $d'(2, 3.n_4) \leq 2\,[d'(2, 4) + d'(4, 3)]$.

For $u = 2, v = 4$,

$d'(2, 4) \leq d'(2, 4.n_0)$. Also $d'(2, 4.n_0) \leq d'(2, 0) + d'(0, 4)$.

$d'(2, 4) \leq d'(2, 4.n_2)$. Also $d'(2, 4.n_2) \leq d'(2, 2) + d'(2, 4)$.

$d'(2, 4) \leq d'(2, 4.n_3)$. Also $d'(2, 4.n_3) \leq 2\,[d'(2, 3) + d'(3, 4)]$.

$d'(2, 4) \leq d'(2, 4.n_4)$. Also $d'(2, 4.n_4) \leq d'(2, 4) + d'(4, 4)$.

For $u = 3, v = 4$,

$d'(3, 4) \leq d'(3, 4.n_0)$. Also $d'(3, 4.n_0) \leq d'(3, 0) + d'(0, 4)$.

$d'(3, 4) \leq d'(3, 4.n_3)$. Also $d'(3, 4.n_3) \leq d'(3, 3) + d'(3, 4)$.

Thus d is not a neutrosophic triplet metric, but a neutrosophic triplet b-metric with a coefficient $b \geq 4$.

Example 3.3. Let $A \neq \varnothing$ be a finite subset of the set of natural numbers $\mathbb{N}$ and $P(A)$ be the power set of A. Then $(P(A), \cup)$ is a neutrosophic triplet space. Let $d : P(A) \times P(A) \to \mathbb{R}$ be defined as, if $U = V, d(U,V) = 0$ and if $U \neq V$,

$$d(U,V) = \begin{cases} \left|2^{m(U)} - 2^{m(V)}\right|, & \text{if } m(U) \text{ and } m(V) \text{ are even} \\ 3, & \text{if } m(U) \text{ and } m(V) \text{ are odd} \\ 1, & \text{otherwise} \end{cases},$$

where $m(U)$ denotes the number of elements of U. Then d is not a neutrosophic triplet metric but a neutrosophic triplet b-metric. For example let $m(A) = 10$ and let $m(U) = 8, m(V) = 6, m(W) = 3$ and $W \subset V$. Then

$$d(U,V) \leq d(U, V \cup n_W) = d(U, V \cup W) = d(U,V).$$

But $d(U,V) \geq d(U,W) + d(W,V)$ since $d(U,V) = 192, d(U,W) = 1$ and $d(W,V) = 1$. So d is not a neutrosophic triplet metric. But for $b \geq 1023$, d is a neutrosophic triplet b-metric. (If we take $U = A$ and $V = W = \varnothing$, then b must be at least $\left|2^{10} - 2^0\right|$).

For generally d is a neutrosophic triplet b-metric with a coefficient $b \geq |2^{m(A)} - 2|$.

Example 3.4. Let $A = \{1,2,3,4,5,6,7,8,9,10\}$. Then $(P(A), \cap)$ is a neutrosophic triplet space with neutrosophic triplets in the form $< U, U, U >$.

Let define a map $d : P(A) \times P(A) \to \mathbb{R}$ such that

$$d(U,V) = \begin{cases} 1, & \text{if } U \cap V = \varnothing, \\ \frac{1}{m(U \cap V)}, & \text{if } U \cap V \neq \varnothing, U \neq V, \\ 0, & \text{if } U = V. \end{cases}$$

If we take $U = \{1,2,3\}$, $V = \{4,5,6,7\}$, and $W = \{1,2,3,4,5,6,7\}$. Then

$$d(U,V) = 1 \leq d(U, V \cap n_W) = d(U,V).$$

But

$$d(U, V \cap n_W) \geq d(U,W) + d(W,V) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

Then d is not a neutrosophic triplet metric. But $((P(A), \cap), d)$ is a neutrosophic triplet b-metric space for any coefficient $b \geq \frac{9}{2}$. For generally, d_b is a neutrosophic triplet b-metric with a coefficient $b \geq \frac{m(A)-1}{2}$.

Now, let give the concept of convergent sequence in neutrosophic triplet b-metric space and some properties.

Definition 3.5. [22] Let $((A,*), d_b)$ be a neutrosophic triplet b-metric space, $\{p_n\}$ be a sequence in A. If for all $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, $d_b(p, p_n) < \varepsilon$, then $\{p_n\}$ converges to p in $((A,*), d_b)$, denoted by $\lim_{n\to\infty} p_n = p$ or $p_n \to p$.

Definition 3.6. [22] Let $((A,*),d_b)$ be a neutrosophic triplet b-metric space, $\{p_n\}$ be a sequence in A. If for all $\varepsilon>0$, there exists an $M\in\mathbb{N}$ such that, for all $n,m\geq M$, $d_b(p_n,p_m)\leq\varepsilon$, then $\{p_n\}$ is a Cauchy sequence.

Theorem 3.7. *Let $((A,*),d_b)$ be a neutrosophic triplet b-metric space, $\{p_n\}$ be a sequence in A. If for all $\varepsilon>0$, there exists an $M\in\mathbb{N}$ such that, for all $n,m\geq M$, $d_b(p_n,p)<\varepsilon$ and $d_b(p_n,p_m)\leq d_b(p_n,p_m*n_p)$, then $\{p_n\}$ is a Cauchy sequence.*

Proof. By the hypothesis, for all $\varepsilon>0$, and for all $n,m\geq M$, $d_b(p,p_n)<\frac{\varepsilon}{2b}$ and $d_b(p,p_m)<\frac{\varepsilon}{2b}$. Since $d_b(p_n,p_m)\leq d_b(p_n,p_m*n_p)$, it is clear that

$$\begin{aligned} d_b(p_n,p_m*n_p) &\leq b\left[d_b(p_n,p)+d_b(p,p_m)\right] \\ &< b\left[\frac{\varepsilon}{2b}+\frac{\varepsilon}{2b}\right]=\varepsilon. \end{aligned}$$

Thus $\{p_n\}$ is a Cauchy sequence. □

Definition 3.8. Let $((A,*),d_b)$ be a neutrosophic triplet b-metric space. If every Cauchy sequent $\{x_n\}$ in A is convergent in A, then $((A,*),d_b)$ is called a complete neutrosophic triplet b-metric space.

Now, let define neutrosophic triplet b-metric topology and give some properties. First, we recall the definition of neutrosophic triplet topology.

Definition 3.9. Let $(A,*)$ be a neutrosophic triplet space, $P(A)$ be the power set of A and $\tau\subseteq P(A)$. If the following conditions are satisfied, then τ is called a neutrosophic triplet topology on A,

T$_1$) $U*V\in A$, for all $U,V\in A$

T$_2$) $\varnothing, A\in\tau$

T$_3$) If $U_i\in\tau$, for all $i\in I$, then $\bigcup\limits_{i\in I}U_i\in A$

T$_4$) If $U_i\in\tau$, for all $i\in J$ ($J\subset I$, J is finite), then $\bigcap\limits_{i\in J}U_i\in A$.

$((A,*),\tau)$ is called a neutrosophic triplet topological space.

Now let recall the definition of neutrosophic triplet b-metric topology in [23].

Definition 3.10. [23] Let $((A,*),\tau)$ be a neutrosophic triplet topological space and $\mathcal{B}\subset P(A)$. Then the family

$$\mathcal{B}^{\#}=\{Y\subset A\mid Y=\cup Z, Z\subset Y\}$$

is called as the base of the neutrosophic topology τ.

Definition 3.11. [23] Let $((A,*),d_b)$ be a neutrosophic triplet b-metric space, $a\in A$ and $r>0$. Then

i) $B(a,r) = \{x \in A \mid d_b(a,x) < r\}$ is called as an open ball with center a and radius r.

ii) $B[a,r] = \{x \in A \mid d_b(a,x) \leq r\}$ is called as a closed ball with center a and radius r.

iii) $S(a,r) = \{x \in A \mid d_b(a,x) = r\}$ is called as a disk with center a and radius r.

Definition 3.12. [23] Let $((A,*),d_b)$ be a neutrosophic triplet b-metric space and $\mathcal{B} = \{B(a,\epsilon) \mid a \in A, \epsilon > 0\}$. Then the family $\mathcal{B}^{\#} = \{Y \subset A \mid Y = \cup Z, Z \subset Y\}$ is a neutrosophic triplet topology on A, called as neutrosophic triplet b-metric topology.

Now, let define a neutrosophic triplet topology that arises from neutrosophic triplet b-metric.

For a neutrosophic triplet b-metric $d_b : A \times A \to \mathbb{R}$, let $B_\varepsilon^n(p) = \{q \in A \mid nd_b(p,q) < \varepsilon,\ \text{for any } n \in \mathbb{N}\}$.

Theorem 3.13. *Let $((A,*),d_b)$ be a neutrosophic triplet b-metric space. Then*

$$\tau_{d_b} = \{U \subset A \mid \exists\varepsilon > 0, \exists n \in \mathbb{N},\ B_\varepsilon^n(p) \subset U \text{ for all } p \in U\}$$

is a neutrosophic triplet topology.

Proof. T_1 and T_2 are clear. So let prove the other conditions.

T_3) Let $U_i \in \tau_{d_b}$ and $p \in \bigcup_{i\in I} U_i$. Then there is $j \in I$ such that $p \in U_j$. Since $U_j \in \tau_{d_b}$, $\exists\varepsilon > 0$ such that

$$B_\varepsilon^n(p) \subseteq U_j \Rightarrow B_\varepsilon^n(p) \subseteq \bigcup_{i\in I} U_i.$$

Therefore, $\bigcup_{i\in I} U_i \in \tau_{d_b}$.

T_4) Let $U_i \in \tau_{d_b}$, for all $i \in J (J \subset I, J$ is finite) and $p \in \bigcap_{i\in J} U_i$. Then $p \in U_i$ for all $i \in J$. Also $B_{\varepsilon_i}^{n_i}(p) \subseteq U_i$ for $\exists\varepsilon_i > 0, \exists n_i \in \mathbb{N}$, since $U_i \in \tau_{d_b}$.

Let $\inf\{\varepsilon_i \mid i \in J\} = \varepsilon$ and $q \in B_\varepsilon^{\sum_{i\in J} n_i}(p)$. Then for all $i \in J$,

$$n_i d_b(p,q) < \left(\sum_{i\in J} n_i\right).d_b(p,q) < \varepsilon.$$

So $n_i d_b(p,q) < \varepsilon_i$ for all $i \in J$. Thus $q \in B_{\varepsilon_i}^{n_i}(p) \Rightarrow q \in U_i$, for all $i \in J$, since $B_{\varepsilon_i}^{n_i}(p) \subseteq U_i$. That means $q \in \bigcap_{i\in J} U_i$ and thus $B_{\varepsilon_i}^{n_i}(p) \subseteq \bigcap_{i\in J} U_i$, for all $i \in J$. So $\bigcap_{i\in J} U_i \in \tau_{d_b}$.

Consequently, τ_{d_b} is an neutrosophic triplet topology.

This topology called as neutrosophic triplet topology induced by neutrosophic triplet b-metric. □

4. **Fixed Point Theorem For Neutrosophic Triplet b-Metric Space**

In this section, first, a contraction is defined for neutrosophic triplet b-metric space and a fixed point theorem for neutrosophic triplet b-metric space is given and proven.

Definition 4.1. Let $((A, *), d_b)$ be a neutrosophic triplet b-metric space and $S : A \to A$ be a map. If for all $p, q \in A$,

i) there exists any element $z \in A$ such that $d_b(p, q) \leq d_b(p, q * n_z)$
ii) there is $t \in [0, 1)$ such that $d_b(S(p), S(q)) \leq t.d_b(p, q)$, where $t < \frac{1}{b^2}$

then S is called a contraction with the bound t for $((A, *), d_b)$ if for all $p, q \in A$.

Theorem 4.2. *Let $((A, *), d_b)$ be a neutrosophic triplet b-metric space and S be a contraction with the bound t. Then S is continuous respect to τ_{d_b}.*

Proof. Let for each $r > 0$ and $p \in A$, there be $k > 0$ such that $d_b(p, q) \leq k$. Let $k = \frac{r}{t}$, then $d_b(S(p), S(q)) < t.d_b(p, q) < t.k = t.\frac{r}{t} = r$. Therefore, S is continuous. □

Theorem 4.3. *Let $((A, *), d_b)$ be a neutrosophic triplet b-metric space with the coefficient $b > 1$ and $S : A \to A$ be a contraction with a bound t for $((A, *), d_b)$.*

If $d_b(p, S^m(p)) \leq d_b(p, S^m(p) * n_{S(p)})$ for all $p \in A$ and for all $m \in \mathbb{N} \cup \{0\}$, then,

$$d_b(p, S^m(p)) \leq \sum_{i=1}^{m} b^i t^{i-1} d_b(p, S(p)).$$

Proof. Let prove the theorem by induction on m.

For $m = 0$ and for all $p \in A$, $d_b(p, S^0(p)) = d_b(p, p) = 0 \leq \sum_{i \in \varnothing} d(p, S(p))$.

By the hypothesis and the condition b_5 of neutrosophic triplet b-metric,

$$\begin{aligned} d_b\left(p, S^{m+1}(p)\right) &\leq d_b\left(p, S^{m+1}(p) * n_{S(p)}\right) \leq b\left[d_b(p, S(p)) + d_b\left(S(p), S^{m+1}(p)\right)\right] \\ &= b\left[d_b(p, S(p)) + d_b\left(S(p), S^m(S(p))\right)\right] \\ &= b.d_b(p, S(p)) + b.d_b\left(S(p), S^m(S(p))\right). \end{aligned}$$

By the inductive hypothesis, $d_b(S(p), S^m(S(p))) \leq \left(\sum_{i=1}^{m} b^i t^{i-1}\right) d_b(S(p), S(S(p)))$.

Since S is a contraction with the bound t, $d_b(S(p), S(S(p))) \leq t.d_b(p, S(p))$. Therefore

$$\begin{aligned} d_b\left(p, S^{m+1}(p)\right) &\leq b.d_b(p, S(p)) + b\left(\sum_{i=1}^{m} b^i t^{i-1}\right) t.d_b(p, S(p)) \\ &= b.d_b(p, S(p)) + \left(\sum_{i=1}^{m} b^{i+1} t^i\right) d_b(p, S(p)) \\ &= b.d_b(p, S(p)) + \sum_{i=2}^{m+1} b^i t^{i-1} (d_b(p, S(p))) \\ &= \left(b + \sum_{i=2}^{m+1} b^i t^{i-1}\right) d_b(p, S(p)) \\ &= \sum_{i=1}^{m+1} b^i t^{i-1} d_b(p, S(p)). \end{aligned}$$

This completes the proof. □

Next, we give a fixed point theorem for neutrosophic triplet b-metric space.

Theorem 4.4. *Let $((A, *), d_b)$ be a complete neutrosophic triplet b-metric space. If $S : A \to A$ is a contraction with a bound t, then S has a fixed point.*

Proof. Let $p_0 \in A$ be an unique element and define a sequence $\{p_n\}$ such that $p_n = S(p_{n-1})$, for all $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, $p_n = S^n(p_0)$.

Now, let show p_n is a Cauchy sequence. For $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that

$$\frac{1}{b^{2M-2}} \cdot \frac{1}{b-1} . d_b(p_0, p_1) < \varepsilon.$$

Let $m \geq n \geq M$. Since S is a contraction,

$$\begin{aligned} d_b(p_n, p_m) &= d_b(S^n(p_0), S^m(p_0)) = d_b\left(S\left(S^{n-1}(p_0)\right), S\left(S^{m-1}(p_0)\right)\right) \\ &\leq t.d_b\left(S^{n-1}(p_0), S^{m-1}(p_0)\right) \\ &\leq t^2.d_b\left(S^{n-2}(p_0), S^{m-2}(p_0)\right) \\ &. \\ &. \\ &\leq t^n.d_b\left(p_0, S^{m-n}(p_0)\right). \end{aligned}$$

By Theorem 4.3,

$$d_b\left(p_0, S^{m-n}(p_0)\right) \leq \left(\sum_{i=1}^{m-n} b^i t^{i-1} d_b(p_0, S(p_0))\right).$$

Therefore, since $t < \frac{1}{b^2}$,

$$\begin{aligned} d_b\left(p_n, p_m\right) &\leq t^n.\left(\sum_{i=1}^{m-n} b^i t^{i-1} d_b\left(p_0, S\left(p_0\right)\right)\right) \leq \frac{1}{b^{2n}}\left(\sum_{i=1}^{m-n} \frac{1}{b^{i-2}}\right) d_b\left(p_0, p_1\right) \\ &\leq \frac{1}{b^{2n}}\left(\sum_{i=1}^{\infty} \frac{1}{b^{i-2}}\right) d_b\left(p_0, p_1\right) = \frac{1}{b^{2n-2}}\left(\sum_{i=1}^{\infty} \frac{1}{b^{i}}\right) d_b\left(p_0, p_1\right) \\ &= \frac{1}{b^{2n-2}}.\frac{1}{b-1} d_b\left(p_0, p_1\right) \\ &\leq \frac{1}{b^{2M-2}}.\frac{1}{b-1}.d_b\left(p_0, p_1\right) \\ &< \varepsilon. \end{aligned}$$

So $\{p_n\}$ is a Cauchy sequence. Then $\{p_n\}$ is converges a point $p \in A$, i.e $p_n \to p$ because $((A, *), d_b)$ is a complete neutrosophic triplet b-metric space. Since S is continuous, $S(p_n) \to S(p)$. □

5. Conclusion

In this paper, some new concepts, properties, and examples were given for neutrosophic triplet b-metric spaces as follows:

1) The neutrosophic triplet space $A = \{0, 2, 3, 4\}$, with the multiplication module 6 in $\mathbb{Z}$, is a neutrosophic triplet b-metric space with $d : A \times A \to \mathbb{R}$, $d(u, v) = (u - v)^2$ where $b \geq 4$.
2) The neutrosophic triplet space $(P(A), \cup)$ is a neutrosophic triplet b-metric space with $d : P(A) \times P(A) \to \mathbb{R}$ defined as, if $U = V$, $d(U, V) = 0$ and if $U \neq V$,

$$d(U,V) = \begin{cases} \left|2^{m(U)} - 2^{m(V)}\right|, & \text{if } m(U) \text{ and } m(V) \text{ are even} \\ 3, & \text{if } m(U) \text{ and } m(V) \text{ are odd} \\ 1, & \text{otherwise} \end{cases},$$

where $b \geq (2^{|(A)|} - 2)$.
3) The neutrosophic triplet space $(P(A), \cap)$ is a neutrosophic triplet b-metric space where $A \subset \mathbb{N}$ is finite, with

$$d(U,V) = \begin{cases} 1, & \text{if } U \cap V = \varnothing, \\ \frac{1}{m(U\cap V)}, & \text{if } U \cap V \neq \varnothing, U \neq V, \\ 0, & \text{if } U = V. \end{cases}$$

where $b \geq \frac{m(A)-1}{2}$.

4) A sequence $\{p_n\}$ in a neutrosophic triplet b-metric space $((A,*),d_b)$ is a Cauchy sequence if for all $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that, for all $n, m \geq M$, $d_b(p_n, p) < \varepsilon$ and

$$d_b(p_n, p_m) \leq d_b(p_n, p_m * n_p).$$

Since S is continuous. Since $S(p_{n-1}) = p_n$ then $S(p_n) = p_{n+1} \to p$. Thus $S(p) = p$.

5) The family $\tau_{d_b} = \{U \subset A \mid \exists \varepsilon > 0, \exists n \in \mathbb{N},\ B^n_\varepsilon(p) \subset U \text{ for all } p \in U\}$ is a neutrosophic triplet topology on a neutrosophic triplet b-metric space $((A,*),d_b)$, called as neutrosophic triplet topology induced by neutrosophic triplet b-metric d_b.
6) A map $S : A \to A$ is called a contraction with the bound $t > 0$ if,
   i) there exists any element $z \in A$ such that $d_b(p,q) \leq d_b(p, q * n_z)$
   ii) there is $t \in [0,1)$ such that $d_b(S(p), S(q)) \leq t.d_b(p,q)$, where $t < \frac{1}{b^2}$
7) A contraction S on a neutrosophic triplet b-metric space $((A,*),d_b)$ is continuous respect to τ_{d_b}.
8) For contraction S on neutrosophic triplet b-metric space $((A,*),d_b)$, if $d_b(p, S^m(p)) \leq d_b(p, S^m(p) * n_{S(p)})$ for all $p \in A$ and for all $m \in \mathbb{N} \cup \{0\}$, then, $d_b(p, S^m(p)) \leq \sum_{i=1}^m b^i t^{i-1} d_b(p, S(p))$.

$$d_b(p, S^m(p)) \leq \sum_{i=1}^{m} b^i t^{i-1} d_b(p, S(p)).$$

9) A contraction S on a neutrosophic triplet b-metric space $((A,*),d_b)$ has a fixed point.

6. Future Work

By utilizing these results, the researcher can define neutrosophic triplet partial b-metric, neutrosophic triplet quasi-partial b-metric, neutrosophic triplet rectangular b-metric and can investigate fixed point theorems in these spaces.

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