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On Neutrosophic Vague Binary $\mathbf{BZMV^{dM}}$ Sub - algebra of $\mathbf{BZMV^{dM}}$- algebra In Neutrosophic Vague Binary Sets

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Abstract: In Model theory, common algebraic structures found are Lattices and Boolean Algebras. In the broad field of research, various algebraic structures can be introduced for a set. BCK, BCI, BCH, BH etc. are some of them. In this paper, a comparatively novel mixed structure namely, de Morgan BZMV – algebra, is presented for neutrosophic vague binary sets. Obviously, this is a mixed output pattern and more effective than the already existing single output approaches. Instead of our usual Boolean approach, this model is a kind of de Morgan lattice extension. Additionally, it takes the effects of Lukasiewicz many- valued logic, combined with BZ –lattices. The logical connective operators, MV conjunction operator $\odot$ and MV disjunction operator $\oplus$ have shown the behavior of idempotency, as same as, their underlying logical patterns, framed of 'usual conjunction $\wedge$ and disjunction $\vee$ '. Both kind of orthocomplementations or negations, one as a fuzzy type and other one as an intuitionistic type are implemented by BZ lattices. That is, Kleene (fuzzy or Zadeh) orthocomplementation $\neg$ and Brouwer orthocomplementation $\sim$ are got implemented in BZ- structure. Here $\neg$ is a fuzzy type negation and $\sim$ is an intuitionistic type negation. In our new type, de Morgan effects are given to these orthocomplementations and hence in this paper, instead of usual negations, de Morgan negations or orthocomplementations are used. Some ideals for this new concept have also got constructed. Behaviour of its direct sum, properties and some of its related theorems are also mentioned in this paper.

Keywords: vague $\mathrm{BZMV^{dM}}$ – subalgebra, neutrosophic $\mathrm{BZMV^{dM}}$ – subalgebra , neutrosophic vague $\mathrm{BZMV^{dM}}$ – subalgebra, neutrosophic vague binary $\mathrm{BZMV^{dM}}$ – subalgebra, prime ideal of $\mathrm{BZMV^{dM}}$ – subalgebra, normal ideal of $\mathrm{BZMV^{dM}}$ – subalgebra, $\sim$ ideal of $\mathrm{BZMV^{dM}}$ – subalgebra, direct sum of neutrosophic vague binary $\mathrm{BZMV^{dM}}$ – subalgebra

Notations: NVBS - neutrosophic vague binary set, NVBSS - neutrosophic vague binary subset, NVBI - neutrosophic vague binary ideal. Throughout this paper NVB indicates neutrosophic vague binary, NV denotes neutrosophic vague, N denotes neutrosophic and V denote vague

1. Introduction

For a set, different kinds of structures can be defined. Poset or partially ordered set is one such! In a poset, elements may or may not be comparable. $(P, \leq)$ is a poset indicated by a relation $\leq$ defined on P and does not give any sense to the actual meaning of 'less than or equal to'. Chang [4] invented MV

(Many − Valued) – algebra in 1958. As algebraic semantics of Lukasiewicz many- valued (or multi-valued) logic, MV-algebra is considered to be an algebraic system with one binary operation, 2 unary operations and with one constant which satisfy certain axioms. In 1994, Vijay K. Khanna [24] explored lattices and Boolean algebra in his book. In 2010, N.O. Alshehri [1] introduced a new concept called an additive derivation of MV – algebra and investigated its several properties. It is possible to introduce a lattice structure from any MV – algebra. In 2017, Jean B. Nganou [13] introduced Stone MV – algebras and strongly complete MV – algebras. In 2008, Shokoofeh Ghorbani, Hasankhani. A and Esfandiar Eslami [23] introduced hyper MV – algebras. In 2012, Musa Hasankhani. M and Borumand Saeid. A [21] studied, hyper MV – algebras defined by bipolar-valued fuzzy sets. In 2014, Yongwei Yang, Xiaolong Xin and Pengfei He [28] gave some characterizations of MV – Algebras based on the theory of falling shadows. In 2003, Gianpiero Cattaneo and David Ciucci [11] gave an algebraic approach to Shadowed sets. In 2003, Gianpiero Cattaneo and David Ciucci [12] also discussed on shadowed sets and related algebraic structures. In 2004, Cattaneo. G, Ciucci.D, Giuntini. R and Konig. M [3] discussed on algebraic structures related to many valued logical systems. In 1984, Komori. Y [15] introduced the concept of a BCC algebra. In 2018, Mozahir Anwar and Jay Nandan Prasad Singh [20] discussed on the direct sum of BCC algebra. In 1991, Ye. R. F [27] discussed on BZ-algebras. Several authors like Zhang. X. H in Korea and China termed weak BCC algebra as BZ algebra. In 2003, Xiaohong Zhang, Yongquan Wang and Wieslaw. A. Dudek [26] studied T-ideals in BZ (Brower − Zadeh) -algebras and T-type BZ-algebras. In 2009, Wieslaw A. Dudek, Xiaohong Zhang, Yongquan Wang [25] gave a study on ideals and atoms of BZ Algebras. In 2017, Yousefi. A and Borumand.Saeid. A [29] discussed on various types of ideals in $BZMV^{dM}$-algebra. In 2020, Mohamed Abdel-Basset, Abduallah Gamal, Le Hoang Son and Florentin Smarandache [16] provided a new approach for professional selection using bipolar neutrosophic multi criteria decision making method. In 2020, Mohamed Abdel – Basset, Weiping Ding, Rehab Mohamed and Noura Metawa [17] explained a new plithogenic Multicriteria approach, which is beneficial to the manufacturing industries, for evaluation of financial performance. For smart product service systems, in 2020, Mohamed Abdel-Basst, Rehab Mohamed, Mohamed Elhoseny [18] discussed on a novel framework to evaluate innovative value proposition. In 2020, Mohamed Abdel-Basset, Rehab Mohamed, Abd El-Nasser, H. Zaied, Abduallah Gamal, Florentin Smarandache [19] developed a plithogenic model based on best – worst method in supply chain problem.

Heyting Algebra is introduced by a dutch mathematician and logician, Arend Heyting [29] in 1930, to formulize intuitionistic logic. Wajesberg algebra is introduced in 1984 by Joseph. Maria Font, Antonio. J. Rodriguez, Antonio Torrens [14] and it is an extension of BCK algebras. Ortho-algebra is introduced by Foulis, D. P, Greechie. R. J and Rüttimann. G. T [5] in the year 1992. An effect algebra is considered as a generalization of ortho-algebra. MV – algebra too, is a generalization of ortho- algebra. Effect algebra, which is a tool for the studies of unsharp elements in quantum mechanics, is introduced by Foulis. D. J and Bennett. M.K [6] in 1994. A special example for MV – algebra is effect algebra. An orthogonal algebra is an effect algebra. Heyting Wajesberg algebra is introduced by Gianpiero Cattaneo and Davide Ciucci [11] in the year 2002. Sometimes, HW algebra is called as pseudo Boolean algebra or even Brower Lattice. Boolean algebra, which is a generalization of power set algebra, is considered to be a complemented distributed lattice in an abstract algebra.

In 1999, G. Cattaneo, R. Giuntini, R. Pilla [2] introduced $BZMV^{dM}$-algebra. It is an algebraic structure with a binary operation $\oplus$ (which is both commutative and associative), with 2 unary operations $\neg$ (Kleene orthocomplementation) and $\sim$ (Brouwer orthocomplementation) and with two constants 0 and 1, satisfying certain axioms. $\sim$ is also known as intutionistic complement, since neither excluded middle law, $(a \vee \neg a = 0)$ nor double negation law, $(\neg \neg a = a)$ got satisfied by it, but it behaves well with non – contradiction law, $(a \wedge \neg a = 0)$. Any brouwer distributive lattice (resp. de Morgan Brouwer distributive lattice) becomes pre –brouwer distributive lattice (resp. de Morgan pre - Brouwer distributive lattice), if it satisfies additionally the above non – contradiction law. So any Brouwer lattice is pre –Brouwer but the converse does not hold, generally. An open problem is given in paper [2] which rightly points towards the behavioral difference between MV – algebras and $BZMV^{dM}$-algebras. [0, 1] is an example for a $BZMV^{dM}$-algebra but it is not a $MVBZ^3$ – algebra. Every equation which holds in every MV – algebra based on [0, 1] holds in every MV – algebra. But the open problem is that it is not known still, whether it is true for $BZMV^{dM}$-algebras. But it is true for every $MVBZ^3$- algebra and the concrete $MVBZ^3$- algebra based on $\left\{0, \left(\frac{1}{2}\right), 1\right\}$, that is, shadowed sets. Any stonian MV-algebra will act as a $BZMV^{dM}$-algebra and vice versa.

In 1993, Gau. W. L and Buehrer. D. J [9] introduced vague set theory. In 2005, Florentin Smarandache [8] introduced neutrosophic sets. In 2019, Emimanancy. M and Francina Shalini. A [7] introduced Bipolar-valued fuzzy $BZMV^{dM}$ sub- algebra. In 2019, Remya. P.B and Francina Shalini. A [22] developed neutrosophic vague binary sets. These sets are more useful than fuzzy, since neutrosophic criteria is implemented and it will provide three values for each data set values collected. Fuzzy sets limitation is got reduced in neutrosophic zone. In this paper, $BZMV^{dM}$-algebraic concepts are developed to neutrosophic vague binary sets. Ideals, direct sum, theorems are also got discussed to this novel concept.

In this paper the following points are newly presented

- V $BZMV^{dM}$ – subalgebra for Different sets [Section 3]
  - ✓ V $BZMV^{dM}$ subalgebra [Definition 3.1]
  - ✓ N $BZMV^{dM}$ subalgebra [Definition 3.2]
  - ✓ NV $BZMV^{dM}$ subalgebra [Definition 3.3]
- NVB $BZMV^{dM}$ – subalgebra [Section 4]
  - ✓ NVB $BZMV^{dM}$ – subalgebra [Definition 4.1]
- Ideal in Neutrosophic Vague Binary $BZMV^{dM}$ Sub - algebra [Section 5]
  - ✓ NVB $BZMV^{dM}$- Ideal [Definition 5.1]
- Various Ideals in Neutrosophic Vague Binary $BZMV^{dM}$- subalgebra [Section 5]
  - ✓ Prime ideal, normal ideal, $\sim$ ideal of NVB $BZMV^{dM}$–subalgebra [Definition 5.2]
- Direct sum of two NVB $BZMV^{dM}$ – subalgebra [Section 6]

2. Preliminaries

In this section some preliminaries are given.

Definition 2.1 [22] (Neutrosophic vague binary set)
A neutrosophic vague binary set M_{NVB} (NVBS in short) over a common universe $\left\{U_1 = \{x_j / 1 \le j \le n\}; U_2 = \{y_k/1 \le k \le p\}\right\}$ is an object of the form

$$M_{NVB} = \left\{\langle\frac{\hat{T}_{M_{NVB}}(x_j),\ \hat{I}_{M_{NVB}}(x_j),\ \hat{F}_{M_{NVB}}(x_j)}{x_j};\ \forall\, x_j \in U_1\rangle \langle\frac{\hat{T}_{M_{NVB}}(y_k),\ \hat{I}_{M_{NVB}}(y_k),\ \hat{F}_{M_{NVB}}(y_k)}{y_k};\ \forall\, y_k \in U_2\rangle\right\}$$

is defined as

$\hat{T}_{M_{NVB}}(x_j) = [T^-(x_j), T^+(x_j)]$, $\hat{I}_{M_{NVB}}(x_j) = [I^-(x_j), I^+(x_j)]$ and $\hat{F}_{M_{NVB}}(x_j) = [F^-(x_j), F^+(x_j)]$; $x_j \in U_1$ and
$\hat{T}_{M_{NVB}}(y_k) = [T^-(y_k), T^+(y_k)]$, $\hat{I}_{M_{NVB}}(y_k) = [I^-(y_k), I^+(y_k)]$ and $\hat{F}_{M_{NVB}}(y_k) = [F^-(y_k), F^+(y_k)]$; $y_k \in U_2$
where (1) $T^+(x_j) = 1 - F^-(x_j)$; $F^+(x_j) = 1 - T^-(x_j)$; $\forall\, x_j \in U_1$ and
$T^+(y_k) = 1 - F^-(y_k)$; $F^+(y_k) = 1 - T^-(y_k)$; $\forall\, y_k \in U_2$

(2) $^-0 \le T^-(x_j) + I^-(x_j) + F^-(x_j) \le 2^+$; $^-0 \le T^-(y_k) + I^-(y_k) + F^-(y_k) \le 2^+$
or
$^-0 \le T^-(x_j) + I^-(x_j) + F^-(x_j) + T^-(y_k) + I^-(y_k) + F^-(y_k) \le 4^+$

and

$$^{-}0 \leq T^{+}(x_j) + I^{+}(x_j) + F^{+}(x_j) \leq 2^{+} ; \quad ^{-}0 \leq T^{+}(y_k) + I^{+}(y_k) + F^{+}(y_k) \leq 2^{+}$$

or

$$^{-}0 \leq T^{+}(x_j) + I^{+}(x_j) + F^{+}(x_j) + T^{+}(y_k) + I^{+}(y_k) + F^{+}(y_k) \leq 4^{+}$$

(3) $T^{-}(x_j),\ I^{-}(x_j),\ F^{-}(x_j) : V(U_1) \longrightarrow [0,1]$ and $T^{-}(y_k),\ I^{-}(y_k),\ F^{-}(y_k) : V(U_2) \longrightarrow [0,1]$

$T^{+}(x_j),\ I^{+}(x_j),\ F^{+}(x_j) : V(U_1) \longrightarrow [0,1]$ and $T^{+}(y_k),\ I^{+}(y_k),\ F^{+}(y_k) : V(U_2) \longrightarrow [0,1]$

Here $V(U_1),\ V(U_2)$ denotes power set of vague sets on $U_1,\ U_2$ respectively.

Definition 2.2 [24]

A poset $(L, \leq)$ is said to form a lattice if for every a, b $\in$ L, Sup $\{a, b\}$ and Inf $\{a, b\}$ exist in L

In that case, we write

Sup $\{a, b\} = (a \vee b)$ [read a join b]

Inf $\{a, b\} = (a \wedge b)$ [read a meet b]

Other notations like $(a + b)$ and $(a\,.\,b)$ or $(a \cup b)$ and $(a \cap b)$ are also used for Sup$\{a, b\}$& Inf $\{a, b\}$

Definition 2.3 [1] (MV- algebra)

An MV-algebra is a structure (M, $\oplus$, $\star$, 0) where $\oplus$ is a binary operation, $\star$ is a unary operation, and 0 is a constant such that the following axioms are satisfied for any a, b $\in$ M:

(MV1) (M, $\oplus$, 0) is a commutative monoid

(MV2) $(a^{\star})^{\star} = a$

(MV3) $(0^{\star} \oplus a) = 0^{\star}$

(MV4) $(a^{\star} \oplus b)^{\star} \oplus b = (b^{\star} \oplus a)^{\star} \oplus a$

If we define the constant 1 = $0^{\star}$ and the auxiliary operations $\odot$, $\vee$ and $\wedge$ by

$a \odot b = (a^{\star} \oplus b^{\star})^{\star}$, $a \vee b = a \oplus (b \odot a^{\star})$, $a \wedge b = a \odot (b \oplus a^{\star})$,

then (M, $\odot$, 1) is a commutative monoid and the structure (M, $\vee$, $\wedge$, 0, 1) is a bounded distributive lattice. Also, we define the binary operation $\ominus$ by $x \ominus y = x \odot y^{\star}$. A subset X of an MV- algebra M is called sub algebra of M if and only if X is closed under the MV-operations defined in M. In any MV-algebras, one can define a partial order $\leq$ by putting $x \leq y$ if and only if $x \wedge y = x$ for each x, y $\in$ M. If the order relation $\leq$, defined over M, is total, then we say that M is linearly ordered. For an MV-algebra M, if we define $B(M) = \{x \in M : x \oplus x = x\} = \{x \in M : x \odot x = x\}$. Then, (B(M), $\oplus$, $\star$, 0) is both a largest subalgebra of M and a Boolean algebra

An MV-algebra M has the following properties for all x, y, z $\in$ M,

(1) $(x \oplus 1) = 1$

(2) $(x \oplus x*) = 1$

(3) $(x \odot x^{*}) = 0$

(4) If $(x \odot y) = 0$, then $x = y = 0$

(5) If $(x \odot y) = 1$, then $x = y = 1$

(6) If $x \leq y$, then $(x \vee z) \leq (y \vee z)$ and $(x \wedge z) \leq (y \wedge z)$

(7) If $x \leq y$, then $(x \oplus z) \leq (y \oplus z)$ and $(x \odot z) \leq (y \odot z)$

(8) $x \leq y$ if and only $y^{*} \leq x^{*}$

(9) $(x \oplus y) = y$ if and only if $(x \odot y) = x$

Definition 2.4 [20] (BZ-algebra)

An algebra $\langle X, *, 0\rangle$ of type $(2, 0)$ is called a BZ- algebra if it satisfies the following axioms :

For any $x, y, z \in X$,

(1) $[(x * z) * (y * z)] * (x * y) = 0$

(2) $(x * 0) = x$

(3) $(x * y) = (y * x) = 0$ implies $x = y$

A partial ordering $\leq$ can be defined by $x \leq y$ if and only if $(x * y) = 0$

Remark 2.5 [20]

A weak BCC algebra was also termed as BZ algebra in Korea and China by several authors e.g., X. H. Zhang.

Definition 2.6 [25, 26] $\left(\textbf{distributive Brouwer} - \textbf{Zadeh} - \textbf{lattice}\ (\mathrm{BZ} - \mathrm{lattice})\right)$

A distributive Brouwer - Zadeh (BZ)-lattice is a structure $\langle \sum, \vee, \wedge, \neg, \sim, 0\rangle$, where

(a) $\langle \sum, \vee, \wedge, 0\rangle$ is a (nonempty) distributive lattice with minimum element 0

(b) The mapping $\neg : \sum \rightarrow \sum$ is a Kleene orthocomplementation, that is

(doc-1) $\neg(\neg a) = a$

(doc-2) $\neg(a \vee b) = (\neg a \wedge \neg b)$

(re) $(a \wedge \neg a \leq b \vee \neg b)$

(c) The mapping $\sim : \sum \rightarrow \sum$ is a Brouwer orthocomplementation, that is

(woc-1) $(a \wedge \sim\sim a) = a$

(woc-2) $\sim (a \vee b) = (\sim a \wedge \sim b)$

(woc-3) $(a \wedge \sim a) = 0$

(d) The two orthocomplementations are linked by the following interconnection rule:

(in) $\neg \sim a = \sim\sim a$

The mapping $\neg$ is also called the Lukasiewicz [or fuzzy (Zadeh)] orthocomplementation while the mapping $\sim$ is an intuitionistic – like orthocomplementation. The element $1 := \sim 0 = \neg 0$ is the greatest element of $\sum$

Definition 2.7 [2] $\left(\textbf{distributive de Morgan BZ} - \textbf{lattice}(\mathbf{BZ^{dM}} - \textbf{lattice})\right)$

A distributive de Morgan BZ -lattice $\left(\mathrm{BZ^{dM}} - \mathrm{lattice}\right)$ is a distributive BZ- lattice for which the following hold: $\sim (a \wedge b) = (\sim a \vee \sim b)$

Definition 2.8 [11, 12] (Brouwer – Zadeh many – valued algebra (BZMV - algebra))

By pasting of BZ -lattices and MV – algebras one obtains so – called BZMV – algebra

A Brouwer Zadeh Many Valued (BZMV) algebra is a system $\mathcal{A} = \langle A, \oplus, \neg, \sim, 0\rangle$ where A is a non-empty set, 0 is a constant, $\neg$ and $\sim$ are unary operations, $\oplus$ a binary operator, obeying the following axioms:

(BZMV1) $(a \oplus b) \oplus c = (b \oplus c) \oplus a$ (BZMV2) $(a \oplus 0) = a$

(BZMV3) $\neg(\neg a) = a$ (BZMV4) $\neg(\neg a \oplus b) \oplus b = \neg(a \oplus \neg b) \oplus a$

(BZMV5) $\sim a \oplus \sim\sim a = \neg 0$ (BZMV6) $a \oplus \sim\sim a = \sim\sim a$

(BZMV7) $\sim \neg[(\neg(a \oplus \neg b) \oplus b] = \neg(\sim\sim a \oplus \neg \sim\sim b) \oplus \neg \sim\sim b$

Definition 2.9 [2] (de Morgan BZMV ($\mathbf{BZMV^{dM}}$) algebra

A de Morgan BZMV ($\mathrm{BZMV^{dM}}$) algebra, is a BZMV algebra $\mathcal{A} = \langle A, \oplus, \neg, \sim, 0\rangle$ where axiom (BZMV7) is replaced by the following:

(BZMV7′) $\sim \neg[(\neg(a \oplus \neg b) \oplus \neg b)] = \neg(\sim\sim a \oplus \neg \sim\sim b) \oplus \neg \sim\sim b$

Connectives $\vee$ and $\wedge$ are the algebraic realization of logical disjunction and conjunction of a distributive lattice. In particular, they are idempotent operators. Connectives $\oplus$ and $\odot$ are the well known MV disjunction and MV conjunction operators, which are idempotent.

$$(a \odot b) \coloneqq \neg\ (\neg a\ \oplus \neg b)$$
$$(a \vee b) := \neg\ (\neg a \oplus b) \oplus b \quad ; \quad (a \wedge b) := \neg\ (a \oplus \neg b) \oplus \neg b$$

A partial order can be naturally induced by the lattice operators as:

$a \leq b$ iff $(a\ \wedge\ b) = a$ (equivalently, $a \vee b = b$)

Let us notice that, since it is possible to prove that $\sim 0 = \neg 0$, in the sequel we set $1 := \sim 0 = \neg 0$
With respect to the just defined partial order we have that the lattice is bounded: $\forall\ a \in A,\ 0\ \leq\ a\ \leq\ 1$

The unary operation $\neg : A \rightarrow A$ is a Kleene (or Zadeh) orthocomplementation (negation). In other words, it satisfies the properties:

(K1) $\neg\ (\neg a) = a$ (K2) $\neg\ (a \vee b) = \neg a\ \wedge\ \neg\ b$ (K3) $a\ \wedge\ \neg a\ \leq\ b\ \vee \neg\ b$

Let us recall that under (K1), condition (K1), condition (K2) is equivalent to the dual de Morgan law. In general, neither the non-contradiction law,

$\forall\ a : a\ \wedge\ \neg a\ = 0$, nor the excluded middle law, $\forall\ a : a\ \vee\ \neg a\ = 1$, are satisfied by this negation

The unary operation $\sim : A \rightarrow A$ is a Brouwer orthocomplementation (negation). In other words, it satisfies the properties:

(B1) $a\ \wedge \sim\ \sim\ a = a$ (equivalently, $a\ \leq\ \sim\ \sim\ a$) (B2) $\sim\ (a \vee b) = \sim\ a\ \wedge \sim\ b$ (B3) $a\ \wedge \sim\ a = 0$

Remark 2. 10 [29]

BZW and BZMV – algebras are equivalent structures.

Definition 2.11 [29] (Linear $\mathbf{BZMV^{dM}}$-algebra)

In linear $BZMV^{dM}$-algebra A, Brouwer orthocomplementation $\sim$ is uniquely defined in the following way for all $a \in A$.

$$\sim a = \begin{cases} 1 & ;\ \text{if}\ a = 0 \\ 0 & ;\ \text{otherwise} \end{cases}$$

Theorem 2.12 [9]

If A is a $BZMV^{dM}$--algebra then the following results are true:

(1) $(x \oplus y) = (y \oplus x)$ (2) $(x \oplus y) \oplus z\ =\ x \oplus (y \oplus z)$ (3) $(x \oplus 1) =\ 1$

(4) $(x \oplus \neg\ x) =\ 1$ (5) $\neg\ (x \oplus \sim\ \sim x)\ \oplus \sim\ \sim x\ =\ 1$ (6) $\neg\ x \oplus \sim\ \sim x\ = 1$

(7) $x\ \wedge\ \sim\ \sim x\ =\ x$ (8) $\neg \sim x\ =\ \sim\ \sim x$ (9) $\sim (x\ \wedge\ y)\ =\ \sim x\ \vee \sim y$

(10) $\sim (x\ \vee\ y)\ =\ \sim x\ \wedge \sim y$ (Equivalently, $x \leq\ y$ implies $\sim y \leq \sim x$)

(11) $x\ \wedge\ \sim x =\ 0$ (12) $\sim x\ =\ \sim \sim \sim x$ (13) $\sim x \oplus \sim x\ =\ \sim x$ (14) $\neg\ 0\ =\ \sim 0$

Definition 2.13 [29] (Ideal in $\mathbf{BZMV^{dM}}$- algebra)

Let A be an $BZMV^{dM}$--algebra. An ideal of A is any subset I of A such that the following conditions are satisfied:

(i) $0 \in I$ (ii) $x, y \in I$ imply $(x \oplus y) \in I$ (iii) $x \in I$ and $y \in A$ then $(x \odot y) \in I$

Definition 2.14 [29] (prime ideal of $\mathbf{BZMV^{dM}}$ algebra)

An ideal I is called a prime ideal if and only if $\forall\ x, y \in A$, $(x \odot \neg\ y) \in I$ or $(\neg\ x\ \odot\ y) \in I$

Definition 2.15 [29] (normal ideal of $\mathbf{BZMV^{dM}}$ algebra)

Let I be an ideal of A. I is called normal ideal whenever, $\forall\ x, y \in A$, $(\neg\ x \odot y) \in I$ if and only if $(\sim x \odot y\ \in I)$

Definition 2.16 [29] ($\sim$ ideal of $\mathbf{BZMV^{dM}}$ algebra)

Let A be a $BZMV^{dM}$ – algebra. Let I be an ideal of A. Then it is called $\sim$ ideal of A if it satisfies the following condition: $\forall\ x,\ y \in A$, if $(x \odot y) \in I$, then $(\sim \sim x \odot\ \sim \neg\ y \in I)$

Lemma 2.17 [21] (Fuzzy sub – hyper MV algebra)

A fuzzy subset μ of M is called a fuzzy sub-hyper-MV -algebra of M if it satisfies:

(1) $(\forall\ x \in\ M),\ \mu(x)\ \leq\ \mu(x^*)$ (2) $(\forall\ x, y \in\ M),\ \inf_{v\ \in\ x \oplus y}\ \mu(v)\ \geq\ \min\ \{\ \mu(x),\ \mu(y)\}$

Definition 2.18 [7] (Bipolar valued fuzzy BZMV algebra)

Let $M = \langle M, \oplus, \neg, \sim, 0\rangle$ be a BZMV – algebra with a non – empty set M, a binary operation $\oplus$, two unary operations $\neg$ and $\sim$ and with a constant 0. A bipolar valued fuzzy set $B = (\mu^{-1}, \mu^{+})$ is called bipolar valued fuzzy BZMV algebra of M, if for every x, y in M it satisfies:

(1) $\mu^{+}(x) \leq \mu^{+}(\neg x)$ (2) $\mu^{-}(x) \geq \mu^{+}(\neg x)$ (3) $\mu^{+}(x) \leq \mu^{+}(\sim x)$ (4) $\mu^{-}(x) \geq \mu^{+}(\sim x)$

(5) $\inf_{v \in x \oplus y} \mu^{+}(v) \geq \min\{\mu^{+}(x), \mu^{+}(y)\}$ (6) $\sup_{v \in x \oplus y} \mu^{-}(v) \leq \max\{\mu^{-}(x), \mu^{-}(y)\}$

3. $\mathbf{BZMV^{dM}}$ – algebra for different uncertain sets

In this section $\mathbf{BZMV^{dM}}$ algebraic structure is developed for vague and neutrosophic sets. Then it is extended to its hybrid set 'neutrosophic vague' with single universe.

Definition 3. 1 (Vague $\mathbf{BZMV^{dM}}$ subalgebra)

A vague $BZMV^{dM}$ – subalgebra (V $BZMV^{dM}$ - subalgebra, in short) is a structure $\mathfrak{M}_A = \left(U^{\mathfrak{M}_A}, \oplus, \neg, \sim, 0\right)$ which satisfies, the following two $\mathfrak{M}_A$ inequalities:

$\boldsymbol{\mathfrak{M}_A}$ ***inequality* (1) :** $\left(\inf_{\mathbf{u_v} \in \mathbf{u_x} \oplus \mathbf{u_y}} \mathbf{V_A}(\mathbf{u_v})\right) \succcurlyeq \text{r}\min\{\mathbf{V_A}(\mathbf{u_x}), \mathbf{V_A}(\mathbf{u_y})\}$; $\forall$ $\mathbf{u_v}, \mathbf{u_x}, \mathbf{u_y} \in \mathbf{U}$

$$\text{i.e., } \left(\inf_{u_v \in (u_x \oplus u_y)} t_A(u_v)\right) \geq \min\{t_A(u_x), t_A(u_y)\} \;;$$

$$\left(\inf_{u_v \in (u_x \oplus u_y)} f_A(u_v)\right) \leq \max\{f_A(u_x), f_A(u_y)\}$$

$\boldsymbol{\mathfrak{M}_A}$ ***inequality*** **(2)** :

(i) $\mathbf{V_A}(\mathbf{u_x}) \succcurlyeq \mathbf{V_A}(\neg\, \mathbf{u_x})$

i.e., $t_A(u_x) \geq t_A(\neg u_x)$ & $f_A(u_x) \leq f_A(\neg u_x)$

(ii) $\mathbf{V_A}(\mathbf{u_x}) \succcurlyeq \mathbf{V_A}(\sim \mathbf{u_x})$

i.e., $t_A(u_x) \geq t_A(\sim u_x)$ & $f_A(u_x) \leq f_A(\sim u_x)$

Here,

- A is a non- empty vague set with universe **U**
- $U^{\mathfrak{M}_A} = (U, \oplus, \neg, \sim, 0)$ is a $BZMV^{dM}$– algebraic structure of the underlying universal set U of a vague set A, with a binary operation $\oplus$, two unary operations $\neg, \sim$ and a constant 0 which satisfies the following $BZMV^{dM}$ axioms : $\forall\ u_x, u_y, u_z \in U$

  (1) $(u_x \oplus u_y) \oplus u_z = (u_y \oplus u_z) \oplus u_x$ (2) $(u_x \oplus 0) = u_x$ (3) $\neg(\neg u_x) = u_x$

  (4) $\neg(\neg u_x \oplus u_y) \oplus u_y = \neg(u_x \oplus \neg u_y) \oplus u_x$ (5) $\sim u_x \oplus \sim\sim u_x = \neg 0$ (6) $u_x \oplus \sim\sim u_x = \sim\sim u_x$

  (7) $\sim\neg[(\neg(u_x \oplus \neg u_y) \oplus \neg u_y)] = \neg(\sim\sim u_x \oplus \neg\sim\sim u_y) \oplus \neg\sim\sim u_y$

Definition 3.2 (Neutrosophic $\mathbf{BZMV^{dM}}$ subalgebra)

A neutrosophic $BZMV^{dM}$– subalgebra (N $BZMV^{dM}$- sub algebra, in short) is a structure $\mathfrak{M}_{M_N} = \left(U^{\mathfrak{M}_{M_N}}, \oplus, \neg, \sim, 0\right)$ which satisfies, the following two $\mathfrak{M}_{M_N}$ inequalities:

$\mathfrak{M}_{M_N}$ ***inequality* (1):**

$$\left(\inf_{u_v \in (u_x \oplus u_y)} N_{M_N}(v)\right) \succcurlyeq r \min\{N_{M_N}(u_x), N_{M_N}(u_y)\} \; ; \; \forall \; u_v, \; u_x, \; u_y \in U$$

$$\left(\inf_{u_v \in (u_x \oplus u_y)} T_{M_N}(u_v)\right) \geq \min\{T_{M_N}(u_x), \; T_{M_N}(u_y)\};$$
$$\left(\inf_{u_v \in (u_x \oplus u_y)} I_{M_N}(u_v)\right) \leq \max\{I_{M_N}(u_x), \; I_{M_N}(u_y)\} \; ;$$
$$\left(\inf_{u_v \in (u_x \oplus u_y)} F_{M_N}(u_v)\right) \leq \max\{F_{M_N}(u_x), \; F_{M_N}(u_y)\}$$

$\mathfrak{M}_{M_N}$***inequality* (2) :**

(i) $N_{M_N}(u_x) \succcurlyeq N_{M_N}(\neg u_x)$; i.e., $T_{M_N}(u_x) \geq T_{M_N}(\neg u_x)$; $I_{M_N}(u_x) \leq I_{M_N}(\neg u_x)$; $F_{M_N}(u_x) \leq F_{M_N}(\neg u_x)$

(ii) $N_{M_N}(u_x) \succcurlyeq N_{M_N}(\sim u_x)$; i.e., $T_{M_N}(u_x) \geq T_{M_N}(\sim u_x)$; $I_{M_N}(u_x) \leq I_{M_N}(\sim u_x)$; $F_{M_N}(u_x) \leq F_{M_N}(\sim u_x)$

Here,

- M_N is a non- empty neutrosophic set with universe U
- $U^{\mathfrak{M}_{M_N}} = (U, \; \oplus, \; \neg, \; \sim, 0)$ is a $BZMV^{dM}$ – algebraic structure of the underlying universal set U of a neutrosophic set M_N with a binary operation $\oplus$, two unary operations $\neg$, $\sim$ and a constant 0 which satisfies the following $BZMV^{dM}$ - axioms : $\forall \; u_x, \; u_y, \; u_z \in U$

  (1) $(u_x \oplus u_y) \oplus u_z = (u_y \oplus u_z) \oplus u_x$ (2) $(u_x \oplus 0) = u_x$ (3) $\neg(\neg u_x) = u_x$

  (4) $\neg(\neg u_x \oplus u_y) \oplus u_y = \neg(u_x \oplus \neg u_y) \oplus u_x$ (5) $\sim u_x \oplus \sim\sim u_x = \neg 0$ (6) $u_x \oplus \sim\sim u_x = \sim\sim u_x$

  (7) $\sim\neg[(\neg(u_x \oplus \neg u_y) \oplus \neg u_y)] = \neg(\sim\sim u_x \oplus \neg\sim\sim u_y) \oplus \neg\sim\sim u_y$

Definition 3.3 $\left(\textbf{Neutrosophic Vague } \mathbf{BZMV^{dM}} - \textbf{subalgebra}\right)$

A neutrosophic vague $BZMV^{dM}$ – subalgebra (NV $BZMV^{dM}$ sub algebra, in short) is a structure $\mathfrak{M}_{M_{NV}} = \left(U^{\mathfrak{M}_{M_{NV}}}, \; \oplus, \; \neg, \; \sim, 0\right)$ with, $\hat{T} = [T^{-1}, T^{+}]$; $\hat{I} = [I^{-1}, I^{+}]$; $\hat{F} = [F^{-1}, F^{+}]$, which satisfies, the following two $\mathfrak{M}_{M_{NV}}$ inequalities:

$\mathfrak{M}_{M_{NV}}$ ***inequality* (1)** :

$$\left(\inf_{u_v \in (u_x \oplus u_y)} NV_{M_{NV}}(u_v)\right) \succcurlyeq r \min\{NV_{M_{NV}}(u_x), NV_{M_{NV}}(u_y)\}; \; \forall \; u_v, \; u_x, \; u_y \in U$$

i.e.,

$$\left(\inf_{u_v \in (u_x \oplus u_y)} \hat{T}_{M_{NV}}(u_v)\right) \geq \min\{\hat{T}_{M_{NV}}(u_x), \; \hat{T}_{M_{NV}}(u_y)\};$$
$$\left(\inf_{u_v \in (u_x \oplus u_y)} \hat{I}_{M_{NV}}(u_v)\right) \leq \max\{\hat{I}_{M_{NV}}(u_x), \; \hat{I}_{M_{NV}}(u_y)\} \; ;$$
$$\left(\inf_{u_v \in (u_x \oplus u_y)} \hat{F}_{M_{NV}}(u_v)\right) \leq \max\{\hat{F}_{M_{NV}}(u_x), \; \hat{F}_{M_{NV}}(u_y)\}$$

$\mathfrak{M}_{M_{NV}}$ ***inequality* (2)** :

(i) $NV_{M_{NV}}(u_x) \succcurlyeq NV_{M_{NV}}(\neg u_x)$ i.e., $\hat{T}_{M_{NV}}(u_x) \geq \hat{T}_{M_{NV}}(\neg u_x)$; $\hat{I}_{M_{NV}}(u_x) \leq \hat{I}_{M_{NV}}(\neg u_x)$; $\hat{F}_{M_{NV}}(u_x) \leq \hat{F}_{M_{NV}}(\neg u_x)$

(ii) $NV_{M_{NV}}(u_x) \succcurlyeq NV_{M_{NV}}(\sim u_x)$ i.e., $\hat{T}_{M_{NV}}(u_x) \geq \hat{T}_{M_{NV}}(\sim u_x)$; $\hat{I}_{M_{NV}}(u_x) \leq \hat{I}_{M_{NV}}(\sim u_x)$; $\hat{F}_{M_{NV}}(u_x) \leq \hat{F}_{M_{NV}}(\sim u_x)$

Here,

- M_{NV} is a non- empty neutrosophic vague set with a single universe U

- $U^{\mathfrak{M}_{M_{NV}}} = (U, \oplus, \neg, \sim, 0)$ is a $BZMV^{dM}$ – algebraic structure of the underlying single universal set U of a neutrosophic vague set M_{NV}, with a binary operation $\oplus$, two unary operations $\neg, \sim$ and a constant 0 which satisfies the following $BZMV^{dM}$ - axioms: $\forall\ u_x, u_y, u_z \in U$

  (1) $(u_x \oplus u_y) \oplus u_z = (u_y \oplus u_z) \oplus u_x$ (2) $(u_x \oplus 0) = u_x$ (3) $\neg(\neg u_x) = u_x$

  (4) $\neg(\neg u_x \oplus u_y) \oplus u_y = \neg(u_x \oplus \neg u_y) \oplus u_x$ (5) $\sim u_x \oplus \sim\sim u_x = \neg 0$ (6) $u_x \oplus \sim\sim u_x = \sim\sim u_x$

  (7) $\sim\neg[(\neg(u_x \oplus \neg u_y) \oplus \neg u_y)] = \neg(\sim\sim u_x \oplus \neg\sim\sim u_y) \oplus \neg\sim\sim u_y$

4. Neutrosophic Vague Binary $BZMV^{dM}$-subalgebra

In this section neutrosophic vague $\mathbf{BZMV^{dM}}$-subalgebra is extended to its binary concept.

Definition 4.1 (Neutrosophic Vague Binary $BZMV^{dM}$ subalgebra)

A neutrosophic vague binary $BZMV^{dM}$- subalgebra (NVB $BZMV^{dM}$ Sub-algebra, in short) is a structure $\mathfrak{M}_{M_{NVB}} = \left(U^{\mathfrak{M}_{M_{NVB}}}, \oplus, \neg, \sim, 0\right)$ with, $\hat{T} = [T^{-1}, T^{+}]$; $\hat{I} = [I^{-1}, I^{+}]$; $\hat{F} = [F^{-1}, F^{+}]$, which satisfies, the following two $\mathfrak{M}_{M_{NVB}}$ inequalities:

$\boldsymbol{\mathfrak{M}_{M_{NVB}}}$ ***inequality* (1)** :

$$\left(\inf_{u_v \in (u_x \oplus u_y)} NVB_{M_{NVB}}(u_v)\right) \succcurlyeq r\min\left\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\right\};\ \forall\ u_v, u_x, u_y \in U$$

i.e.,

$$\left(\inf_{u_v \in (u_x \oplus u_y)} \hat{T}_{M_{NVB}}(u_v)\right) \geq \min\left\{\hat{T}_{M_{NVB}}(u_x), \hat{T}_{M_{NVB}}(u_y)\right\}$$
$$\left(\inf_{u_v \in (u_x \oplus u_y)} \hat{I}_{M_{NVB}}(u_v)\right) \leq \max\left\{\hat{I}_{M_{NVB}}(u_x), \hat{I}_{M_{NVB}}(u_y)\right\}$$
$$\left(\inf_{u_v \in (u_x \oplus u_y)} \hat{F}_{M_{NVB}}(u_v)\right) \leq \max\left\{\hat{F}_{M_{NVB}}(u_x), \hat{F}_{M_{NVB}}(u_y)\right\}$$

$\mathfrak{M}_{M_{NVB}}$ *inequality* (2) :

(i) $NVB_{M_{NVB}}(u_x) \succcurlyeq NVB_{M_{NVB}}(\neg u_x)$

i.e., $\hat{T}_{M_{NVB}}(u_x) \geq \hat{T}_{M_{NVB}}(\neg u_x)$; $\hat{I}_{M_{NVB}}(u_x) \leq \hat{I}_{M_{NVB}}(\neg u_x)$; $\hat{F}_{M_{NVB}}(u_x) \leq \hat{F}_{M_{NVB}}(\neg u_x)$

(ii) $NVB_{M_{NVB}}(u_x) \succcurlyeq NVB_{M_{NVB}}(\sim u_x)$

i.e., $\hat{T}_{M_{NVB}}(u_x) \geq \hat{T}_{M_{NVB}}(\sim u_x)$; $\hat{I}_{M_{NVB}}(u_x) \leq \hat{I}_{M_{NVB}}(\sim u_x)$; $\hat{F}_{M_{NVB}}(u_x) \leq \hat{F}_{M_{NVB}}(\sim u_x)$

Here,

- M_{NVB} is a non- empty neutrosophic vague binary set with two universes U_1, U_2

- $U^{\mathfrak{M}_{M_{NVB}}} = (U = \{U_1 \cup U_2\}, \oplus, \neg, \sim, 0)$ is a $BZMV^{dM}$ – algebraic structure of the underlying universal set U which is got by combining the two universes of the given neutrosophic vague binary set M_{NVB}, with a binary operation $\oplus$, two unary operations $\neg, \sim$ and a constant 0 which satisfies the following $BZMV^{dM}$ - axioms: $\forall\ u_x, u_y, u_z \in U$

  (1) $(u_x \oplus u_y) \oplus u_z = (u_y \oplus u_z) \oplus u_x$ (2) $(u_x \oplus 0) = u_x$ (3) $\neg(\neg u_x) = u_x$

  (4) $\neg(\neg u_x \oplus u_y) \oplus u_y = \neg(u_x \oplus \neg u_y) \oplus u_x$ (5) $\sim u_x \oplus \sim\sim u_x = \neg 0$ (6) $u_x \oplus \sim\sim u_x = \sim\sim u_x$

  (7) $\sim\neg[(\neg(u_x \oplus \neg u_y) \oplus \neg u_y)] = \neg(\sim\sim u_x \oplus \neg\sim\sim u_y) \oplus \neg\sim\sim u_y$

Remark 4.2

(i) In a NVB $BZMV^{dM}$ – subalgebra, possible operations can also be further derived

(1) $(u_x \odot u_y) = \neg(\neg u_x \odot \neg u_y)$

(2) $(u_x \vee u_y) = \neg(\neg u_x \oplus u_y) \oplus u_y$

(3) $\left(u_x \wedge u_y\right) = \neg\left(\neg\left(u_x \oplus \neg u_y\right) \oplus \neg u_y\right)$

(ii) In neutrosophic vague binary concept $U=\{U_1 \cup U_2\}$. But in neutrosophic vague it is single universe U

Example 4.3

Let $U_1 = \{0,\ u_p,\ u_q, 1\}$ and $U_2 = \{0,\ u_r,\ u_s,\ 1\}$ be two universes with neutrosophic vague binary membership grades as given below:

$$\forall\ u_p \in U_1,\ NVB_{M_{NVB}}(u_p)=\begin{cases}[0.8, 0.9][0.1, 0.6][0.1, 0.2] & ;\ u_p = 0 \text{ and } 1\\ [0.9, 0.9][0.1, 0.2][0.1, 0.1] & ;\ 0 < u_p < 1\end{cases}$$

$$\forall\ u_q \in U_2,\ NVB_{M_{NVB}}(u_q)=\begin{cases}[0.7, 0.9][0.2, 0.5][0.1, 0.3] & ;\ u_q = 0 \text{ and } 1\\ [0.8, 0.9][0.1, 0.4][0.1, 0.2] & ;\ 0 < u_q < 1\end{cases}$$

Corresponding neutrosophic vague binary set (in short, NVBS) is given as below:

$$\therefore\ M_{NVB} = \left\{\begin{matrix}\left\langle \frac{[0.8,0.9][0.1,0.6][0.1,0.2]}{0}, \frac{[0.9,0.9][0.1,0.2][0.1,0.1]}{u_p}, \frac{[0.9,0.9][0.1,0.2][0.1,0.1]}{u_q}, \frac{[0.8,0.9][0.1,0.6][0.1,0.2]}{1}\right\rangle \\ \left\langle \frac{[0.7,0.9][0.2,0.5][0.1,0.3]}{0}, \frac{[0.8,0.9][0.1,0.4][0.1,0.2]}{u_r}, \frac{[0.8,0.9][0.1,0.4][0.1,0.2]}{u_s}, \frac{[0.7,0.9][0.2,0.5][0.1,0.3]}{1}\right\rangle\end{matrix}\right\}$$

Combined universal set is $U = \{U_1 \cup U_2\} = \{0, u_p, u_q, u_r, u_s, 1\}$ & $\{U_1 \cap U_2\} = \{0, 1\}$

Neutrosophic vague binary union of common elements are given by,

$NVB_{M_{NVB}}(0) = [0.8, 0.9][0.1, 0.6][0.1, 0.2] \cup [0.7, 0.9][0.2, 0.5][0.1, 0.3]$

$= [0.8, 0.9][0.1, 0.5][0.1, 0.2] = NVB_{M_{NVB}}(1)$

Combined neutrosophic vague binary membership grades are given as follows:

$$NVB_{M_{NVB}}(u_t) = \begin{cases}[0.8, 0.9][0.1, 0.5][0.1, 0.2]\ ;\ u_t = 0\\ [0.9, 0.9][0.1, 0.2][0.1, 0.1]\ ;\ u_t = u_p\\ [0.9, 0.9][0.1, 0.2][0.1, 0.1]\ ;\ u_t = u_q\\ [0.8, 0.9][0.1, 0.4][0.1, 0.2]\ ;\ u_t = u_r\\ [0.8, 0.9][0.1, 0.4][0.1, 0.2]\ ;\ u_t = u_s\\ [0.8, 0.9][0.1, 0.5][0.1, 0.2]\ ;\ u_t = 1\end{cases}$$

Algebraic structure $U^{\mathfrak{M}_{M_{NVB}}} = (U = \{U_1 \cap U_2\},\ *,\ \neg,\ \sim, 0)$ with binary and unary operations defined as in Cayley table given below clearly indicates a $BZMV^{dM}$ – subalgebra.

Cayley table for unary operations $\neg$ and $\sim$ are given below:

	0	$\mathbf{u_p}$	$\mathbf{u_q}$	$\mathbf{u_r}$	$\mathbf{u_s}$	1
$\neg$	1	$\mathbf{u_r}$	$\mathbf{u_s}$	$\mathbf{u_s}$	$\mathbf{u_r}$	0
$\sim$	1	**0**	**0**	0	**0**	**0**

Cayley table for binary operation '$*$' is given below

$*$	0	$\mathbf{u_p}$	$\mathbf{u_q}$	$\mathbf{u_r}$	$\mathbf{u_s}$	1
0	**0**	$\mathbf{u_p}$	$\mathbf{u_q}$	$\mathbf{u_r}$	$\mathbf{u_s}$	1
$\mathbf{u_p}$	$\mathbf{u_p}$	$\mathbf{u_p}$	$\mathbf{u_r}$	$\mathbf{u_r}$	1	1

$\mathbf{u_q}$	$\mathbf{u_q}$	$\mathbf{u_r}$	$\mathbf{u_s}$	1	$\mathbf{u_s}$	1
$\mathbf{u_r}$	$\mathbf{u_r}$	$\mathbf{u_r}$	1	1	1	1
$\mathbf{u_s}$	$\mathbf{u_s}$	**1**	$\mathbf{u_s}$	1	$\mathbf{u_s}$	1
1	1	1	1	1	1	1

Now have to verify, neutrosophic vague binary concept!
For that check the inequalities given in definition 4.1.

$\mathfrak{M}_{M_{NVB}}$ inequality (1): (Binary Operation)

$$\forall\, u_x,\ u_y \in U,\ \left(\inf_{u_v \in (u_x * u_y)} NVB_{M_{NVB}}(u_v)\right) \geqslant r\min\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\}$$

$$\Rightarrow \left(\inf_{u_v \in \{0,\ u_p,\ u_q,\ u_r,\ u_s, 1\}} NVB_{M_{NVB}}(u_v)\right) \geqslant r\min\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\}$$

$$\Rightarrow \left(\operatorname{glb}_{u_v \in \{0,\ u_p,\ u_q,\ u_r,\ u_s, 1\}} NVB_{M_{NVB}}(u_v)\right) \geqslant r\min\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\}$$

$$\Rightarrow \left(\min_{u_v \in \{0,\ u_p,\ u_q,\ u_r,\ u_s, 1\}} NVB_{M_{NVB}}(u_v)\right) \geqslant r\min\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\}$$

$$= \min\begin{Bmatrix} [0.8, 0.9][0.1, 0.5][0.1, 0.2]\ ;\ u_v = 0 \\ [0.9, 0.9][0.1, 0.2][0.1, 0.1]\ ;\ u_v = u_p \\ [0.9, 0.9][0.1, 0.2][0.1, 0.1]\ ;\ u_v = u_q \\ [0.8, 0.9][0.1, 0.4][0.1, 0.2]\ ;\ u_v = u_r \\ [0.8, 0.9][0.1, 0.4][0.1, 0.2]\ ;\ u_v = u_s \\ [0.8, 0.9][0.1, 0.5][0.1, 0.2]\ ;\ u_v = 1 \end{Bmatrix} = \cap \begin{Bmatrix} [0.8, 0.9][0.1, 0.5][0.1, 0.2]\ ;\ u_v = 0 \\ [0.9, 0.9][0.1, 0.2][0.1, 0.1]\ ;\ u_v = u_p \\ [0.9, 0.9][0.1, 0.2][0.1, 0.1]\ ;\ u_v = u_q \\ [0.8, 0.9][0.1, 0.4][0.1, 0.2]\ ;\ u_v = u_r \\ [0.8, 0.9][0.1, 0.4][0.1, 0.2]\ ;\ u_v = u_s \\ [0.8, 0.9][0.1, 0.5][0.1, 0.2]\ ;\ u_v = 1 \end{Bmatrix} = [0.8, 0.9][0.1, 0.5][0.1, 0.2]$$

[In neutrosophic concept, minimum concept has been taken as intersection. ie., in this case, (Min, Max, Max)]

$$\therefore \left(\inf_{u_v \in (u_x * u_y)} NVB_{M_{NVB}}(u_v)\right) = [0.8, 0.9][0.1, 0.5][0.1, 0.2]$$

In this case, for any pair of elements from U,

$\left(\inf_{u_v \in (u_x \oplus u_y)} NVB_{M_{NVB}}(u_v)\right) \geqslant r\min\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\}$, got satisfied.

$\mathfrak{M}_{M_{NVB}}$ inequality (2): (Unary operations)

Next to check, the 2 inequalities of $\mathfrak{M}_{M_{NVB}}(2)$ for all elements of U.
(i) From Cayley table for unary operation $\neg$ (Kleene or Zadeh or fuzzy orthocomplementation)

$NVB_{M_{NVB}}(u_x) \geqslant NVB_{M_{NVB}}(\neg\, u_x)\ ;\ \forall\ \ u_x \in U$ as showed in table.

u_x	$NVB_{M_{NVB}}(u_x)$	$NVB_{M_{NVB}}(\neg\, u_x)$
0	[0.8, 0.9][0.1, 0.5][0.1, 0.2]	[0.8, 0.9][0.1, 0.5][0.1, 0.2]
u_p	[0.9, 0.9][0.1, 0.2][0.1, 0.1]	[0.8, 0.9][0.1, 0.4][0.1, 0.2]
u_q	[0.9, 0.9][0.1, 0.2][0.1, 0.1]	[0.8, 0.9][0.1, 0.4][0.1, 0.2]
u_r	[0.8, 0.9][0.1, 0.4][0.1, 0.2]	[0.8, 0.9][0.1, 0.4][0.1, 0.2]

u_s	[0.8, 0.9][0.1, 0.4][0.1, 0.2]	[0.8, 0.9][0.1, 0.4][0.1, 0.2]
1	[0.8, 0.9][0.1, 0.5][0.1, 0.2]	[0.8, 0.9][0.1, 0.5][0.1, 0.2]

(ii) From Cayley table for unary operation $\sim$ (Brower orthocomplementation)

$NVB_{M_{NVB}}(u_x) \succcurlyeq NVB_{M_{NVB}}(\sim u_x)$; $\forall$ $u_x \in U$ as showed in table

u_x	$NVB_{M_{NVB}}(u_x)$	$NVB_{M_{NVB}}(\sim u_x)$
0	[0.8, 0.9][0.1, 0.5][0.1, 0.2]	[0.8, 0.9][0.1, 0.5][0.1, 0.2]
u_p	[0.9, 0.9][0.1, 0.2][0.1, 0.1]	[0.8, 0.9][0.1, 0.5][0.1, 0.2]
u_q	[0.9, 0.9][0.1, 0.2][0.1, 0.1]	[0.8, 0.9][0.1, 0.5][0.1, 0.2]
u_r	[0.8, 0.9][0.1, 0.4][0.1, 0.2]	[0.8, 0.9][0.1, 0.5][0.1, 0.2]
u_s	[0.8, 0.9][0.1, 0.4][0.1, 0.2]	[0.8, 0.9][0.1, 0.5][0.1, 0.2]
1	[0.8, 0.9][0.1, 0.5][0.1, 0.2]	[0.8, 0.9][0.1, 0.5][0.1, 0.2]

So given example is a $\mathfrak{M}_{M_{NVB}}$ with structure $\left(U^{\mathfrak{M}_{M_{NVB}}}, *, \neg, \sim, 0\right)$

Remark 4.4

It is to be noted that,

(i) first column of the Cayley table for binary operation will be a copy of column of operands, using definition 2.8 (BZMV2)

(ii) last row and column of the Cayley table for binary operation will be always 1 for a $BZMV^{dM}-$ algebra, by using (3) and (4) of theorem 2.12

Theorem 4.5

If M_{NVB} is a $\mathfrak{M}_{M_{NVB}}$ then the following results are true:

(1) $NVB_{M_{NVB}}\left(u_x \oplus u_y\right) = NVB_{M_{NVB}}\left(u_y \oplus u_x\right)$ [i.e., commutative law holds for Binary Operation]

(2) $NVB_{M_{NVB}}\left(\left(u_x \oplus u_y\right) \oplus u_z\right) = NVB_{M_{NVB}}\left(u_x \oplus \left(u_y \oplus u_z\right)\right)$

[i.e., associative law holds for Binary Operation]

(3) $NVB_{M_{NVB}}(u_x \oplus 1) = NVB_{M_{NVB}}(1)$

(4) $NVB_{M_{NVB}}(u_x \oplus \neg u_x) = NVB_{M_{NVB}}(1)$

[neutrosophic vague binary membership grade of an element binary operated with it's kleene complement always produce the neutrosophic vague binary membership grade of the maximum element 1]

(5) $NVB_{M_{NVB}}(\neg (u_x \oplus \sim\sim u_x) \oplus \sim\sim u_x) = NVB_{M_{NVB}}(1)$

(6) $NVB_{M_{NVB}}(\neg u_x \oplus \sim\sim u_x) = NVB_{M_{NVB}}(1)$

(7) $NVB_{M_{NVB}}(u_x \wedge \sim\sim u_x) = NVB_{M_{NVB}}(u_x)$

(8) $NVB_{M_{NVB}}(\neg \sim u_x) = NVB_{M_{NVB}}(\sim \sim u_x)$

(9) $NVB_{M_{NVB}}\left(\sim (u_x \wedge u_y)\right) = NVB_{M_{NVB}}(\sim u_x \vee \sim u_y)$

(10) $NVB_{M_{NVB}}\left(\sim (u_x \vee u_y)\right) = NVB_{M_{NVB}}(\sim u_x \wedge \sim u_y)$

(Equivalently, $u_x \leq u_y$ implies $\sim u_y \leq \sim u_x$)

(11) $NVB_{M_{NVB}}(u_x \wedge \sim u_x) = NVB_{M_{NVB}}(0)$

(12) $NVB_{M_{NVB}}(\sim u_x) = NVB_{M_{NVB}}(\sim \sim \sim u_x)$

(13) $NVB_{M_{NVB}}(\sim u_x \oplus \sim u_x) = NVB_{M_{NVB}}(\sim u_x)$

(14) $NVB_{M_{NVB}}(\neg 0) = NVB_{M_{NVB}}(\sim 0)$

Proof

(1) $NVB_{M_{NVB}}(u_x \oplus u_y) = NVB_{M_{NVB}}\left((u_x \oplus u_y) \oplus 0\right)$,

by putting $u_x = (u_x \oplus u_y)$ in definition 4.1 (2), $(u_x \oplus 0) = u_x$

$= NVB_{M_{NVB}}\left((u_y \oplus 0) \oplus u_x\right)$, by using definition 4.1 (1)

$= NVB_{M_{NVB}}(u_y \oplus u_x)$, by using definition 4.1 (2)

(2) $NVB_{M_{NVB}}\left((u_x \oplus u_y) \oplus u_z\right) = NVB_{M_{NVB}}\left((u_y \oplus u_z) \oplus u_x\right)$, by using definition 4.1 (1)

$= NVB_{M_{NVB}}\left(u_x \oplus (u_y \oplus u_z)\right)$, by using theorem 4.5 (1)

(3) $NVB_{M_{NVB}}(u_x \oplus 1) = NVB_{M_{NVB}}(u_x \oplus \neg 0)$, since $1 = \neg 0$

$= NVB_{M_{NVB}}\left(u_x \oplus (\sim u_x \oplus \sim \sim u_x)\right)$, by using definition 4.1 (5)

$= NVB_{M_{NVB}}\left((\sim u_x \oplus \sim \sim u_x) \oplus u_x\right)$, by using theorem 4.5 (1)

$= NVB_{M_{NVB}}\left((\sim \sim u_x \oplus u_x) \oplus \sim u_x\right)$, by using definition 4.1 (1)

$= NVB_{M_{NVB}}\left((u_x \oplus \sim \sim u_x) \oplus \sim u_x\right)$, by using theorem 4.5 (1)

$= NVB_{M_{NVB}}(\sim \sim u_x \oplus \sim u_x)$, by using theorem 4.1 (6)

$= NVB_{M_{NVB}}(\sim u_x \oplus \sim \sim u_x)$, by theorem 4.5 (1)

$= NVB_{M_{NVB}}(\neg 0)$, by using definition 4.1 (5)

$= NVB_{M_{NVB}}(1)$, since $\neg 0 = 1$

(4) $NVB_{M_{NVB}}(u_x \oplus \neg u_x)$

$= NVB_{M_{NVB}}(\neg \neg u_x \oplus \neg u_x)$, by using definition 4.1 (3)

$= NVB_{M_{NVB}}(\neg (\neg u_x \oplus 0) \oplus \neg u_x)$, by using definition 4.1 (2)

$= NVB_{M_{NVB}}(\neg (0 \oplus \neg u_x) \oplus \neg u_x)$, by theorem 4.5 (1)

$= NVB_{M_{NVB}}(\neg (u_x \oplus \neg 0) \oplus \neg 0)$, by theorem 4.1 (4)

$= NVB_{M_{NVB}}(\neg (u_x \oplus 1) \oplus 1)$, since $\neg 0 = 1 = NVB_{M_{NVB}}(\neg 0)$

$= NVB_{M_{NVB}}(1)$, since $\neg 0 = 1$

(5) $NVB_{M_{NVB}}(\neg (u_x \oplus \sim \sim u_x) \oplus \sim \sim u_x)$

$= NVB_{M_{NVB}}(\neg (\sim \sim u_x) \oplus \sim \sim u_x)$, by using definition 4.1 (6)

$= NVB_{M_{NVB}}\left(\sim \sim u_x \oplus \neg (\sim \sim u_x)\right)$, by theorem 4.5 (1)

$= NVB_{M_{NVB}}(1)$, by theorem 4.5 (4)

(6) $NVB_{M_{NVB}}(\neg u_x \oplus \sim \sim u_x)$

$= NVB_{M_{NVB}}\left(\neg u_x \oplus (u_x \oplus \sim \sim u_x)\right)$, by using definition 4.1(6)

$= NVB_{M_{NVB}}\left((\neg u_x \oplus u_x) \oplus \sim \sim u_x\right)$, by theorem 4.5 (2)

$= NVB_{M_{NVB}}\left((u_x \oplus \neg u_x) \oplus \sim \sim u_x\right)$, by theorem 4.5 (1)

$= \mathrm{NVB}_{M_{NVB}}(1 \oplus \sim\sim u_x)$, by theorem 4.5 (4)
$= \mathrm{NVB}_{M_{NVB}}(\sim\sim u_x \oplus 1)$, by theorem 4.5 (1)
$= \mathrm{NVB}_{M_{NVB}}(1)$, by putting $u_x = \sim\sim u_x$ in theorem 4.5 (3)
(7) $\mathrm{NVB}_{M_{NVB}}(u_x \wedge \sim\sim u_x)$
$= \mathrm{NVB}_{M_{NVB}}\big(\neg(\neg(u_x \oplus \neg\sim\sim u_x) \oplus \neg\sim\sim u_x)\big)$

$\big[\text{From remark 4.2 (i) 3, we have, } (u_x \wedge u_y) = \neg\big(\neg(u_x \oplus \neg u_y) \oplus \neg u_y\big), \text{ by putting } u_y = \sim\sim u_x\big]$

$= \mathrm{NVB}_{M_{NVB}}\big(\neg(\neg(\neg\sim\sim u_x \oplus u_x) \oplus \neg\sim\sim u_x)\big)$

$= \mathrm{NVB}_{M_{NVB}}\big(\neg(\neg(u_x \oplus \neg\sim\sim u_x) \oplus \neg\sim\sim u_x)\big) = \mathrm{NVB}_{M_{NVB}}\big(\neg(\neg(u_x \oplus \neg\sim\sim u_x) \oplus \sim\sim\sim u_x)\big)$

$= \mathrm{NVB}_{M_{NVB}}\big(\neg(\neg(u_x \oplus \neg\sim\sim u_x) \oplus u_x)\big) = \mathrm{NVB}_{M_{NVB}}\big(\neg(\neg(u_x \oplus \neg\sim\sim u_x) \oplus u_x)\big)$

$= \mathrm{NVB}_{M_{NVB}}\big(\neg(\neg(\neg u_x \oplus \sim\sim u_x) \oplus \sim\sim u_x)\big)$

$\big[\text{From definition 4.1(4)}, \neg\big(\neg u_x \oplus u_y\big) \oplus u_y = \neg\big(u_x \oplus \neg u_y\big) \oplus u_x\big]$

$= \mathrm{NVB}_{M_{NVB}}\big(\neg(\neg 1 \oplus \sim\sim u_x)\big)$, by definition 4.5 (6) $= \mathrm{NVB}_{M_{NVB}}\big(\neg(0 \oplus \sim\sim u_x)\big)$, since $\neg 1 = 0$

$= \mathrm{NVB}_{M_{NVB}}\big(\neg(\sim\sim u_x \oplus 0)\big)$, from theorem 4.5 (1) $= \mathrm{NVB}_{M_{NVB}}\big(\neg(\sim\sim u_x)\big)$, from theorem 4.1 (1)

$= \mathrm{NVB}_{M_{NVB}}(\sim\sim\sim u_x)$, since $\neg(\sim\sim u_x) = \sim\sim\sim u_x$ $= \mathrm{NVB}_{M_{NVB}}(u_x)$, since $\sim\sim\sim u_x = u_x$

(8) Consider definition 4.1 (7),
$\mathrm{NVB}_{M_{NVB}}\big(\sim\neg\big[\neg\big(u_x \oplus \neg u_y\big) \oplus \neg u_y\big]\big) = \mathrm{NVB}_{M_{NVB}}\big(\neg\big(\sim\sim u_x \oplus \neg\sim\sim u_y\big) \oplus \neg\sim\sim u_y\big)$
By putting $u_y = u_x$ in the above,
$\Rightarrow \mathrm{NVB}_{M_{NVB}}(\sim\neg[\neg(u_x \oplus \neg u_x) \oplus \neg u_x]) = \mathrm{NVB}_{M_{NVB}}(\neg(\sim\sim u_x \oplus \neg\sim\sim u_x) \oplus \neg\sim\sim u_x)$,
$\Rightarrow \mathrm{NVB}_{M_{NVB}}(\sim\neg[\neg 1 \oplus \neg u_x]) = \mathrm{NVB}_{M_{NVB}}(\neg 1 \oplus \neg\sim\sim u_x)$, by using definition 4.5 (4)
$\Rightarrow \mathrm{NVB}_{M_{NVB}}(\sim\neg[0 \oplus \neg u_x]) = \mathrm{NVB}_{M_{NVB}}(0 \oplus \neg\sim\sim u_x)$
$\Rightarrow \mathrm{NVB}_{M_{NVB}}(\sim\neg[\neg u_x \oplus 0]) = \mathrm{NVB}_{M_{NVB}}(\neg\sim\sim u_x \oplus 0)$, by theorem 4.5 (1)
$\Rightarrow \mathrm{NVB}_{M_{NVB}}(\sim\neg[\neg u_x]) = \mathrm{NVB}_{M_{NVB}}(\neg\sim\sim u_x)$, by definition 4.1 (2)
$\Rightarrow \mathrm{NVB}_{M_{NVB}}(\sim u_x) = \mathrm{NVB}_{M_{NVB}}(\neg\sim\sim u_x)$, by definition 4.1 (3)
$\Rightarrow \mathrm{NVB}_{M_{NVB}}(\neg\sim u_x) = \mathrm{NVB}_{M_{NVB}}(\neg\neg\sim\sim u_x)$, by applying $\neg$ on both sides.
$\Rightarrow \mathrm{NVB}_{M_{NVB}}(\neg\sim u_x) = \mathrm{NVB}_{M_{NVB}}(\sim\sim u_x)$, by using definition 4.1(3)
(9) Consider definition 4.1 (7)
$\Rightarrow \mathrm{NVB}_{M_{NVB}}\big(\sim\neg\big[\neg\big(u_x \oplus \neg u_y\big) \oplus \neg u_y\big]\big) = \mathrm{NVB}_{M_{NVB}}\big(\neg\big(\sim\sim u_x \oplus \neg\sim\sim u_y\big) \oplus \neg\sim\sim u_y\big)$

$\Rightarrow \mathrm{NVB}_{M_{NVB}}\Big(\sim\big(u_x \wedge u_y\big)\Big) = \mathrm{NVB}_{M_{NVB}}\big(\neg\big(\neg\sim u_x \oplus \neg\neg\sim u_y\big) \oplus \neg\neg\sim u_y\big)$

$\Big[\text{By using auxiliary operation, } \big(u_x \wedge u_y\big) = \neg\big[\neg\big(u_x \oplus \neg u_y\big) \oplus \neg u_y\big]\Big]$ &

[By using theorem 4.1 (8), $\neg\sim u_x = \sim\sim u_x$]

$\Rightarrow \mathrm{NVB}_{M_{NVB}}\Big(\sim\big(u_x \wedge u_y\big)\Big) = \mathrm{NVB}_{M_{NVB}}\big(\neg\big(\sim\sim u_x \oplus \sim u_y\big) \oplus \sim u_y\big)$, by using definition 4.1 (3)

$\Rightarrow NVB_{M_{NVB}}\left(\sim\left(u_x \wedge u_y\right)\right) = NVB_{M_{NVB}}\left(\neg\left(\neg\sim u_x \oplus \sim u_y\right) \oplus \sim u_y\right)$, by using $\sim\sim u_x = \neg\sim u_x$

$\Rightarrow NVB_{M_{NVB}}\left(\sim\left(u_x \wedge u_y\right)\right) = NVB_{M_{NVB}}\left(\sim u_x \vee \sim u_y\right)$

[By using auxiliary operation, $NVB_{M_{NVB}}\left(u_x \vee u_y\right) = NVB_{M_{NVB}}\left(\neg\left[\neg\left(u_x \oplus u_y\right) \oplus u_y\right]\right)$]

(10) Let $NVB_{M_{NVB}}(u_x) \leq NVB_{M_{NVB}}(u_y) \Rightarrow NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_x \wedge u_y)$

$\Rightarrow NVB_{M_{NVB}}(\sim u_x) = NVB_{M_{NVB}}\left(\sim\left(u_x \wedge u_y\right)\right)$, by applying $\sim$ on both sides and by using (9)

$\Rightarrow NVB_{M_{NVB}}(\sim u_x) = NVB_{M_{NVB}}(\sim u_x \vee \sim u_y)$, by theorem 4.5 (9)

$\Rightarrow NVB_{M_{NVB}}(\sim u_y) \leq NVB_{M_{NVB}}(\sim u_x)$.

Using theorem 4.5 (7), $\forall\ u_x$, $NVB_{M_{NVB}}(u_x) \leq NVB_{M_{NVB}}(\sim\sim u_x)$.

So now the contraposition law is equivalent to the de Morgan law:

$NVB_{M_{NVB}}\left(\sim\left(u_x \vee u_y\right)\right) = NVB_{M_{NVB}}(\sim u_x \wedge \sim u_y)$

(11) $NVB_{M_{NVB}}(u_x \wedge \sim u_x) = NVB_{M_{NVB}}\left(\neg(\neg(u_x \oplus \neg\sim u_x) \oplus \neg\sim u_x)\right)$

$= NVB_{M_{NVB}}\left(\neg(\neg(u_x \oplus \sim\sim u_x) \oplus \sim\sim u_x)\right)$, by theorem 2.12 (8) & by definition 4.5 (8)

$= NVB_{M_{NVB}}(\neg 1)$, by definition 4.5 (5) $= NVB_{M_{NVB}}(0)$

(12) From definition 4.5 (8), $NVB_{M_{NVB}}(\neg\sim u_x) = NVB_{M_{NVB}}(\sim\sim u_x)$

Put $u_x = \sim u_x$, in the above then, $NVB_{M_{NVB}}(\neg\sim\sim u_x) = NVB_{M_{NVB}}(\sim\sim\sim u_x)$

$\Rightarrow NVB_{M_{NVB}}(\neg\neg\sim u_x) = NVB_{M_{NVB}}(\sim\sim\sim u_x)$, since $\sim\sim u_x = \neg\sim u_x$

$\Rightarrow NVB_{M_{NVB}}(\sim u_x) = NVB_{M_{NVB}}(\sim\sim\sim u_x)$, since $\neg\neg\sim u_x = \sim u_x$

(13) $NVB_{M_{NVB}}(u_x \wedge \sim\sim u_x) = NVB_{M_{NVB}}(\sim\sim u_x)$, by definition 4.1 (6)

Taking Brouwerian orthocomplementation to both sides,

$NVB_{M_{NVB}}(\sim u_x \wedge \sim\sim\sim u_x) = NVB_{M_{NVB}}(\sim\sim\sim u_x)$

$\Rightarrow NVB_{M_{NVB}}(\sim u_x \wedge \sim u_x) = NVB_{M_{NVB}}(\sim u_x)$ [By theorem 4.5(12), $\sim\sim\sim u_x = \sim u_x$]

(14) To prove that, $NVB_{M_{NVB}}(\neg 0) = NVB_{M_{NVB}}(\sim 0)$.

It is enough to prove that,

$NVB_{M_{NVB}}(\neg 0) \preccurlyeq NVB_{M_{NVB}}(\sim 0)$ and $NVB_{M_{NVB}}(\sim 0) \preccurlyeq NVB_{M_{NVB}}(\neg 0)$

We know that, $NVB_{M_{NVB}}(1) = NVB_{M_{NVB}}(\neg 0)$

$\forall\ u_x \in M_{NVB}$, where M_{NVB} is a $\mathfrak{M}_{M_{NVB}}$,

$NVB_{M_{NVB}}(u_x) \preccurlyeq NVB_{M_{NVB}}(1)$, since 1 is the maximum element;

In particular, $NVB_{M_{NVB}}(\sim 0) \preccurlyeq NVB_{M_{NVB}}(1) \Rightarrow NVB_{M_{NVB}}(\sim 0) \preccurlyeq NVB_{M_{NVB}}(\neg 0)$

Similarly, being the least element, $\forall\ u_x \in M_{NVB}$, where M_{NVB} is a $\mathfrak{M}_{M_{NVB}}$,

$NVB_{M_{NVB}}(u_x) \preccurlyeq NVB_{M_{NVB}}(\sim\sim u_x) \preccurlyeq NVB_{M_{NVB}}(\sim 0)$.

In particular, $NVB_{M_{NVB}}(\neg 0) \preccurlyeq NVB_{M_{NVB}}(\sim 0)$

Theorem 4.6

Let M_{NVB} is a $\mathfrak{M}_{M_{NVB}}$. Then

(i) $\forall\ u_x, u_y \in M_{NVB}$, $NVB_{M_{NVB}}(u_x \wedge u_y) = NVB_{M_{NVB}}(0) \Leftrightarrow NVB_{M_{NVB}}(y) \preccurlyeq NVB_{M_{NVB}}(\sim u_x)$

Equivalently, $NVB_{M_{NVB}}(u_x \wedge u_y) = NVB_{M_{NVB}}(0) \Leftrightarrow NVB_{M_{NVB}}(u_x) \preccurlyeq NVB_{M_{NVB}}(\sim u_y)$

(ii) Let $u_x \in M_{NVB}$ be such that $NVB_{M_{NVB}}(u_x \oplus u_x) = NVB_{M_{NVB}}(u_x)$,

Then $\forall\ u_y \in M_{NVB}$, $NVB_{M_{NVB}}(u_x \wedge u_y) = NVB_{M_{NVB}}(0) \Leftrightarrow NVB_{M_{NVB}}(u_x) \preccurlyeq NVB_{M_{NVB}}(\neg u_y)$

Proof

(i) Assume $\left(u_x \wedge u_y\right) = 0$.

Now, $NVB_{M_{NVB}}(u_y \wedge \sim u_x) = NVB_{M_{NVB}}\left((u_y \wedge \sim u_x) \vee 0\right)$, since in any lattice $(u_x \vee 0) = u_x$

$\Rightarrow NVB_{M_{NVB}}(u_y \wedge \sim u_x) = NVB_{M_{NVB}}\left((u_y \wedge \sim u_x) \vee (u_y \wedge \sim u_y)\right)$,

by a result $\left(u_y \wedge \sim u_y\right) = 0$, of BZMV^{dM} algebra

$= NVB_{M_{NVB}}\left(u_y \wedge (\sim u_x \vee \sim u_y)\right)$, by theorem 4.5 (9) $= NVB_{M_{NVB}}\left(u_y \wedge \sim (u_x \wedge u_y)\right)$, by theorem 4.5 (9)

$= NVB_{M_{NVB}}(u_y \wedge \sim 0)$ [by assumption]

$= NVB_{M_{NVB}}(u_y) = NVB_{M_{NVB}}(u_y \wedge 1)$, since $\sim 0 = \neg\ 0 = 1$

$= NVB_{M_{NVB}}(u_y)$, [since in any lattice $(y \wedge 1) = y \Rightarrow NVB_{M_{NVB}}(y \wedge 1) = NVB_{M_{NVB}}(y)$]

$\Rightarrow NVB_{M_{NVB}}(u_y) \preccurlyeq NVB_{M_{NVB}}(\sim u_x)$

Conversely, suppose $NVB_{M_{NVB}}(y) \preccurlyeq NVB_{M_{NVB}}(\sim u_x)$ then $NVB_{M_{NVB}}(u_x \wedge y)$

$= NVB_{M_{NVB}}\left(u_x \wedge (u_y \wedge \sim u_x)\right) = NVB_{M_{NVB}}\left(u_y \wedge (u_x \wedge \sim u_x)\right)$, by associativity

$= NVB_{M_{NVB}}(u_y \wedge 0)$ [by theorem 4.5 (11)]

$= NVB_{M_{NVB}}(0)$, since in any lattice $\left(u_y \wedge 0\right) = u_y$

Equivalently, $NVB_{M_{NVB}}(u_x \wedge u_y) = NVB_{M_{NVB}}(0) \Leftrightarrow NVB_{M_{NVB}}(u_x) \preccurlyeq NVB_{M_{NVB}}(\sim u_y)$, can be proved

(ii) Suppose $NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_x \oplus u_x)$.

Then, $NVB_{M_{NVB}}(u_x \odot u_x) = NVB_{M_{NVB}}(u_x \wedge u_y)$.

Thus we got, $NVB_{M_{NVB}}(u_x) \preccurlyeq NVB_{M_{NVB}}(\neg\ u_y)$ iff $NVB_{M_{NVB}}(u_x \wedge u_y) = NVB_{M_{NVB}}(0)$

Theorem 4. 7

In a $\mathfrak{M}_{M_{NVB}}$ the following holds :

$NVB_{M_{NVB}}(\sim \sim u_x) = NVB_{M_{NVB}}(u_x) \Leftrightarrow NVB_{M_{NVB}}(\sim u_x \oplus u_x) = NVB_{M_{NVB}}(1)$

$\Leftrightarrow NVB_{M_{NVB}}(u_x \oplus u_x) = NVB_{M_{NVB}}(u_x)$

Proof

Assume $\sim \sim u_x = u_x$. Then, $NVB_{M_{NVB}}(\sim \sim u_x) = NVB_{M_{NVB}}(u_x)$,

Definition 4.1 (6) $\Rightarrow NVB_{M_{NVB}}(u_x \oplus \sim \sim u_x) = NVB_{M_{NVB}}(\sim \sim u_x)$

$\Rightarrow NVB_{M_{NVB}}(u_x \oplus u_x) = NVB_{M_{NVB}}(u_x)$

Again, from definition 4.1 (5) $\Rightarrow NVB_{M_{NVB}}(\sim u_x \oplus \sim \sim u_x) = NVB_{M_{NVB}}(\neg\ 0)$

$\Rightarrow NVB_{M_{NVB}}(\sim u_x \oplus u_x) = NVB_{M_{NVB}}(1)$

Assume, $NVB_{M_{NVB}}(u_x \oplus u_x) = NVB_{M_{NVB}}(u_x)$. Then by theorem 4.6 (ii), since $\neg\ u_x \in \mathfrak{M}_{M_{NVB}}$ and $(u_x \wedge \neg\ u_x) = 0 \Rightarrow NVB_{M_{NVB}}(u_x \wedge \neg\ u_x) = NVB_{M_{NVB}}(0) \Rightarrow NVB_{M_{NVB}}(\neg\ u_x) \preccurlyeq NVB_{M_{NVB}}(\sim u_x)$

$\Rightarrow NVB_{M_{NVB}}(\neg \sim u_x) \preccurlyeq NVB_{M_{NVB}}(\sim \sim u_x)$, by putting $u_x = \sim u_x$

$\Rightarrow NVB_{M_{NVB}}(\sim \sim u_x) = NVB_{M_{NVB}}(u_x)$

Under condition $NVB_{M_{NVB}}(\sim u_x \oplus u_x) = NVB_{M_{NVB}}(1)$, it is clear that,

$NVB_{M_{NVB}}(u_x \wedge \sim \sim u_x) = NVB_{M_{NVB}}(\sim \sim u_x) \Rightarrow NVB_{M_{NVB}}(\sim \sim u_x) \preccurlyeq NVB_{M_{NVB}}(u_x)$.

In fact, $NVB_{M_{NVB}}(u_x \wedge \sim \sim u_x) = NVB_{M_{NVB}}(\neg\ [\neg\ (u_x \oplus \neg \sim \sim u_x) \oplus \neg \sim \sim u_x])$

[Using definition of $\wedge$]

$= NVB_{M_{NVB}}(\neg\ [\neg\ (u_x \oplus \neg\ \neg \sim u_x) \oplus \neg\ \neg \sim u_x])$, [since $\neg \sim u_x = \sim \sim u_x$]

$= NVB_{M_{NVB}}(\neg\ [\neg\ (u_x \oplus \sim u_x) \oplus \sim u_x])$, [since $\neg\ \neg\ u_x = u_x$]

$= NVB_{M_{NVB}}\left(\neg\ (0 \oplus \sim u_x)\right)$

$= NVB_{M_{NVB}}(\neg(\sim u_x \oplus 0))$, [by theorem 4.5 (1)]
$= NVB_{M_{NVB}}(\neg \sim u_x)$, [by definition 4.1 (2)]
$= NVB_{M_{NVB}}(\sim \sim u_x)$. [since $\neg \sim u_x = \sim \sim u_x$]

Similarly, we get, $NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(\sim \sim u_x)$. Hence proved.

Theorem 4.8

Let M_{NVB} be a $\mathfrak{M}_{M_{NVB}}$; then for any $u_x, u_y, u_z \in M_{NVB}$,

$NVB_{M_{NVB}}\left((u_x \oplus u_y) \wedge u_z\right) = NVB_{M_{NVB}}(0)$ iff $NVB_{M_{NVB}}(u_x \wedge u_z) = NVB_{M_{NVB}}(0)$ and $NVB_{M_{NVB}}(u_y \wedge u_z) = NVB_{M_{NVB}}(0)$

Proof

Suppose, $NVB_{M_{NVB}}\left((u_x \oplus u_y) \wedge u_z\right) = NVB_{M_{NVB}}(0)$

$\Rightarrow NVB_{M_{NVB}}(u_x \wedge u_z) \preccurlyeq NVB_{M_{NVB}}\left((u_x \oplus u_y) \wedge u_z\right) = NVB_{M_{NVB}}(0)$ and

$NVB_{M_{NVB}}(u_y \wedge u_z) \preccurlyeq NVB_{M_{NVB}}\left((u_x \oplus u_y) \wedge u_z\right) = NVB_{M_{NVB}}(0)$, by theorem 4.6

$\Rightarrow NVB_{M_{NVB}}(u_x \wedge u_z) = NVB_{M_{NVB}}(0) = NVB_{M_{NVB}}(u_y \wedge u_z)$, trivial

Conversely, let $NVB_{M_{NVB}}(u_x \wedge u_z) = NVB_{M_{NVB}}(0)$ and $NVB_{M_{NVB}}(u_y \wedge u_z) = NVB_{M_{NVB}}(0)$.

$NVB_{M_{NVB}}(u_x \wedge \neg u_z) = NVB_{M_{NVB}}(\neg u_z) = NVB_{M_{NVB}}(u_y \wedge \neg u_z)$.

So, $NVB_{M_{NVB}}\left((u_x \oplus u_y) \oplus \neg u_z\right) = NVB_{M_{NVB}}\left(u_x \oplus (u_y \oplus \neg u_z)\right)$, by definition 4.5 (2)

$= NVB_{M_{NVB}}(u_x \oplus \neg u_z) = NVB_{M_{NVB}}(\neg u_z)$, from assumption

$\therefore NVB_{M_{NVB}}\left((u_x \oplus u_y) \wedge u_z\right) = NVB_{M_{NVB}}(\neg [\neg (u_x \oplus \neg \sim \sim u_x) \oplus \neg \neg \sim u_x])$

$= NVB_{M_{NVB}}\left(\left((u_x \oplus u_y) \oplus \neg u_z\right) \odot \neg u_z\right) = NVB_{M_{NVB}}(0 \odot \neg u_z) = NVB_{M_{NVB}}(u_x \odot 0) = NVB_{M_{NVB}}(0)$

5. Ideals in Neutrosophic Vague Binary $BZMV^{dM}$ subalgebra

Concept of ideal with three different kinds are developed in this section

Definition 5.1 (NVB $BZMV^{dM}$- ideal)

Let M_{NVB} be a $\mathfrak{M}_{M_{NVB}}$ and I_{NVB} be a nonempty subset of M_{NVB}. Then I_{NVB} is a neutrosophic vague binary $BZMV^{dM}$- ideal $(NVB - BZMV^{dM}$ ideal$)$ if the following inequalities got satisfied:

(i) $NVB_{I_{NVB}}(0) \succcurlyeq NVB_{I_{NVB}}(u_x)$; $\forall\ u_x \in I_{NVB}$

(ii) $NVB_{I_{NVB}}(u_x \oplus u_y) \succcurlyeq r\min\left\{NVB_{I_{NVB}}(u_x), NVB_{I_{NVB}}(u_y)\right\}$; $\forall\ u_x, u_y \in I_{NVB}$

(iii) $NVB_{I_{NVB}}(u_y) \succcurlyeq r\min\left\{NVB_{I_{NVB}}(u_x), NVB_{I_{NVB}}(u_y \le u_x)\right\}$; $\forall\ u_x, u_y \in I_{NVB}$

Definition 5.2 (prime ideal, ~ ideal, normal ideal of a NVB $BZMV^{dM}$ − subalgebra)

Let M_{NVB} be a $\mathfrak{M}_{M_{NVB}}$ and I_{NVB} be a NVB $BZMV^{dM}$- ideal of M_{NVB}. I_{NVB} is called,

(i) a neutrosophic vague binary $BZMV^{dM}$- prime ideal (NVB $BZMV^{dM}$ − prime ideal) of M_{NVB}

$\Leftrightarrow \{NVB_{I_{NVB}}(u_x \odot \neg u_y) \in M_{NVB}$ or $NVB_{I_{NVB}}(\neg u_x \odot u_y) \in M_{NVB}$; $\forall u_x, u_y \in M_{NVB}$

(ii) a neutrosophic vague binary $BZMV^{dM}$ ~ ideal (NVB $BZMV^{dM}$ ~ ideal) of M_{NVB} if it satisfies:

$NVB_{I_{NVB}}(\sim \sim u_x \odot \sim \neg u_y \in I_{NVB}) \succcurlyeq NVB_{I_{NVB}}(u_x \odot u_y)$; $\forall\ u_x, u_y \in M_{NVB}$

(iii) a neutrosophic vague binary $BZMV^{dM}$ normal ideal (NVB $BZMV^{dM}$ normal ideal) of M_{NVB} whenever

$NVB_{I_{NVB}}(\neg u_x \odot u_y) \succcurlyeq NVB_{I_{NVB}}(\sim u_x \odot u_y)$ and $NVB_{I_{NVB}}(\neg u_x \odot u_y) \preccurlyeq NVB_{I_{NVB}}(\sim u_x \odot u_y)$

i.e., $NVB_{I_{NVB}}(\neg u_x \odot u_y) \Leftrightarrow NVB_{I_{NVB}}(\sim u_x \odot u_y)$, $\forall u_x, u_y \in M_{NVB}$

Theorem 5.3

I_{NVB} is a NVB $BZMV^{dM}$– p ideal of $\mathfrak{M}_{M_{NVB}} \Leftrightarrow \begin{Bmatrix} NVB_{I_{NVB}}(u_x) \succcurlyeq NVB_{I_{NVB}}(u_x \wedge u_y) \\ \text{or} \\ NVB_{I_{NVB}}(u_y) \succcurlyeq NVB_{I_{NVB}}(u_x \wedge u_y) \end{Bmatrix}$

Proof

Assume I_{NVB} is a NVB $BZMV^{dM}$ – p ideal of $\mathfrak{M}_{M_{NVB}}$

$$\Rightarrow \begin{cases} NVB_{I_{NVB}}(u_x \odot \neg u_y) \succcurlyeq r\min\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\};\ \forall u_x,\ u_y \in M_{NVB} \\ or \\ NVB_{I_{NVB}}(\neg u_x \odot u_y) \succcurlyeq r\min\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\};\ \forall u_x,\ u_y \in M_{NVB} \end{cases}$$

Without loss of generality, consider $NVB_{I_{NVB}}(\neg u_x \odot u_y) \succcurlyeq r\min\{NVB_{M_{NVB}}(u_x),\ NVB_{M_{NVB}}(u_y)\}$

$NVB_{I_{NVB}}(u_x \wedge u_y) = NVB_{I_{NVB}}(u_y \wedge u_x)$ [using commutativity of $\wedge$]

$= NVB_{I_{NVB}}(u_y \odot (u_x \oplus \neg u_y))$ [by using $(u_a \wedge u_b) = u_a \odot (u_b \oplus \neg u_a)$]

$= NVB_{I_{NVB}}(\neg(\neg u_y \oplus \neg(u_x \oplus \neg u_y)))$ [by using $(u_a \odot u_b) = \neg(\neg u_a \oplus \neg u_b)$]

$= NVB_{I_{NVB}}(\neg(\neg u_y \oplus \neg(\neg u_y \oplus u_x)))$ [by commutativity of $\oplus$]

$= NVB_{I_{NVB}}(\neg(\neg u_y \oplus \neg(\neg u_y \oplus \neg\neg u_x)))$ [since $\neg\neg u_x = u_x$]

$= NVB_{I_{NVB}}(\neg(\neg u_y \oplus (u_y \odot \neg u_x)))$ [since $\neg(\neg u_y \oplus \neg\neg u_x) = (u_y \odot \neg u_x)$] $\in I_{NVB}$

$NVB_{I_{NVB}}(\neg(\neg u_y \oplus (u_y \odot \neg u_x))) \succcurlyeq NVB_{I_{NVB}}(u_x \wedge u_y)$ and

$NVB_{I_{NVB}}(\neg u_x \odot u_y) \succcurlyeq r\min\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\}$

$\Rightarrow NVB_{I_{NVB}}(\neg(\neg u_y \oplus (u_y \odot \neg u_x)) \oplus (\neg u_x \odot u_y)) \in I_{NVB}$

$\Rightarrow NVB_{I_{NVB}}(\neg(\neg u_y \oplus (u_y \odot \neg u_x)) \oplus (u_y \odot \neg u_x)) \in I_{NVB}$

Hence, $NVB_{I_{NVB}}(u_y \vee (u_y \odot \neg u_x)) \in I_{NVB} \Rightarrow NVB_{I_{NVB}}(u_y) \in I_{NVB}$, since $(u_y \odot \neg u_x) \leq u_y$

$\therefore NVB_{I_{NVB}}(u_y) \succcurlyeq NVB_{I_{NVB}}(\neg u_x \odot u_y)$

Similarly, if $NVB_{I_{NVB}}(\neg u_y \odot u_x) \in I_{NVB} \Rightarrow NVB_{I_{NVB}}(u_x) \in I_{NVB}$

$\Rightarrow NVB_{I_{NVB}}(u_x) \succcurlyeq NVB_{I_{NVB}}(\neg u_y \odot u_x)$

Conversely, $NVB_{I_{NVB}}((\neg u_x \odot u_y) \wedge (u_x \odot \neg u_y)) = NVB_{I_{NVB}}(0) \in I_{NVB}$

[from definition of NVB $BZMV^{dM}$ ideal]

$\Rightarrow NVB_{I_{NVB}}(\neg u_x \odot u_y) \in I_{NVB}$ or $NVB_{I_{NVB}}(u_x \odot \neg u_y) \in I_{NVB}$

$\Rightarrow I_{NVB}$ is a NVB $BZMV^{dM}$ prime ideal of $\mathfrak{M}_{M_{NVB}}$

Theorem 5.4

Let I_{NVB} be an NVB ideal of a neutrosophic vague binary $BZMV^{dM}$ – subalgebra $\mathfrak{M}_{M_{NVB}}$ and $\sim\sim u_x = u_x$ for all $u_x \in M_{NVB}$. Then the following conditions are equivalent :

(1) I_{NVB} is a NVB $BZMV^{dM}$ normal ideal (2) I_{NVB} is a NVB $BZMV^{dM}$ $\sim$ ideal

(3) $NVB_{I_{NVB}}(\sim u_x) \in I_{NVB} \Leftrightarrow NVB_{I_{NVB}}(\neg u_x) \in I_{NVB}$

Proof

(1) $\Rightarrow$ (2)

Let I_{NVB} is a NVB $BZMV^{dM}$- normal ideal of $\mathfrak{M}_{M_{NVB}}$

Then, $\forall\ u_x, u_y \in M_{NVB}$, $NVB_{M_{NVB}}(u_x \odot u_y) \in I_{NVB}$

$\Rightarrow NVB_{M_{NVB}}(\neg\neg u_x \odot u_y) \in I_{NVB}$ [by using $u_x = \neg\neg u_x$]

$\Rightarrow NVB_{M_{NVB}}(\sim\neg u_x \odot u_y) \in I_{NVB}$ [by property of $BZMV^{dM}$ – subalgebra $\neg\neg u_x = \sim\neg u_x$]

$\Rightarrow NVB_{M_{NVB}}(\sim\neg u_x \odot \sim\sim u_y) \in I_{NVB}$ [since given $\sim\sim u_x = u_x$]

$\Rightarrow NVB_{M_{NVB}}(\sim\sim u_y \odot \sim\neg u_x) \in I_{NVB}$ [by commutativity]

$\therefore NVB_{M_{NVB}}(\sim\sim u_y \odot \sim\neg u_x) \geqslant NVB_{M_{NVB}}(u_x \odot u_y)$, $\forall\ u_x, u_y \in M_{NVB}$

$\Rightarrow I_{NVB}$ is a neutrosophic vague binary $BZMV^{dM} \sim$ ideal of M_{NVB}[by definition 5.2 (ii)]

(2) $\Rightarrow$ (1)

Let I_{NVB} be a neutrosophic vague binary $BZMV^{dM} \sim$ ideal.

Then, $\forall\ u_x, u_y \in M_{NVB}$, $NVB_{M_{NVB}}(\neg u_x \odot u_y) \in I_{NVB}$

$\Rightarrow NVB_{M_{NVB}}(\sim\neg(\neg u_x) \odot \sim\sim u_y) \in I_{NVB} \Rightarrow NVB_{M_{NVB}}(\sim(\neg\neg u_x) \odot \sim\sim u_y) \in I_{NVB}$

$\Rightarrow NVB_{M_{NVB}}(\sim u_x \odot u_y) \in I_{NVB} \Rightarrow NVB_{M_{NVB}}(\sim\neg(\sim u_x) \odot \sim\sim u_y) \in I_{NVB}$

$\Rightarrow NVB_{M_{NVB}}(\sim(\neg\sim u_x) \odot \sim\sim u_y) \in I_{NVB} \Rightarrow NVB_{M_{NVB}}(\sim\sim\sim u_x \odot \sim\sim u_y) \in I_{NVB}$, since $[\neg\sim u_x = \sim\sim u_x]$

$\Rightarrow NVB_{M_{NVB}}(\neg(\sim\sim u_x) \odot \sim\sim u_y) \in I_{NVB}$

$\Rightarrow NVB_{M_{NVB}}(\neg u_x \odot u_y) \in I_{NVB}$, so I_{NVB} is a NVB $BZMV^{dM}$ normal ideal of M_{NVB}.

(1) $\Rightarrow$ (3)

Let I_{NVB} be a NVB $BZMV^{dM}$ normal ideal of M_{NVB}

$\Rightarrow NVB_{I_{NVB}}(\neg u_x \odot u_y) \Leftrightarrow NVB_{I_{NVB}}(\sim u_x \odot u_y)$, $\forall u_x, u_y \in M_{NVB}$

$\Rightarrow NVB_{I_{NVB}}(\neg u_x \odot 1) \Leftrightarrow NVB_{I_{NVB}}(\sim u_x \odot 1)$, $\forall u_x, u_y \in M_{NVB}$ [by putting $u_y = 1$]

$\Rightarrow NVB_{I_{NVB}}(\neg(\neg\neg u_x \oplus \neg 1)) \Leftrightarrow NVB_{I_{NVB}}(\neg(\neg\sim u_x \oplus \neg 1))$, $\forall u_x, u_y \in M_{NVB}$ [by definition of $\odot$]

$\Rightarrow NVB_{I_{NVB}}(\neg(\neg\neg u_x \oplus 0)) \Leftrightarrow NVB_{I_{NVB}}(\neg(\neg\sim u_x \oplus 0))$, $\forall u_x, u_y \in M_{NVB}$

$\Rightarrow NVB_{I_{NVB}}(\neg(\neg\neg u_x)) \Leftrightarrow NVB_{I_{NVB}}(\neg(\neg\sim u_x))$, $\forall u_x, u_y \in M_{NVB}$ [since $(u_x \oplus 0) = u_x$]

$\Rightarrow NVB_{I_{NVB}}(\neg\neg(\neg u_x)) \Leftrightarrow NVB_{I_{NVB}}(\neg\neg(\sim u_x))$, $\forall u_x, u_y \in M_{NVB}$

$\Rightarrow NVB_{I_{NVB}}(\neg u_x) \Leftrightarrow NVB_{I_{NVB}}(\sim u_x)$, $\forall u_x, u_y \in M_{NVB}$ [since $\neg\neg(u_x) = u_x$]

(3) $\Rightarrow$ (1)

Suppose, $NVB_{I_{NVB}}(\sim u_x) \in I_{NVB} \Leftrightarrow NVB_{I_{NVB}}(\neg u_x) \in I_{NVB}$

$NVB_{I_{NVB}}(\sim u_x \odot u_y) \in I_{NVB} \Leftrightarrow NVB_{I_{NVB}}(\sim u_x \odot \sim\sim u_y) \in I_{NVB}$ [since $u_y = \sim\sim u_y$, by definition]

$\Rightarrow NVB_{I_{NVB}}(\neg(\neg\sim u_x \oplus \neg u_y)) \in I_{NVB}$ [by definition of $\odot$]

$\Rightarrow NVB_{I_{NVB}}(\neg(\sim\sim u_x \oplus \neg u_y)) \in I_{NVB}$ $[\neg\sim u_x = \sim\sim u_x]$

$\Rightarrow NVB_{I_{NVB}}(\neg(u_x \oplus \sim u_y)) \in I_{NVB}$ [given $\sim\sim u_x = u_x$] $\Rightarrow NVB_{I_{NVB}}(\neg(u_x \odot \neg\sim u_y)) \in I_{NVB}$

$\Rightarrow NVB_{I_{NVB}}(\neg u_x \odot \neg\sim u_y) \in I_{NVB}$ [by property of $\odot$]$\Rightarrow NVB_{I_{NVB}}(\neg u_x \odot \sim\sim u_y) \in I_{NVB}$ $[\neg\sim u_y = \sim\sim u_y]$

$\Rightarrow NVB_{I_{NVB}}(\neg u_x \odot u_y) \in I_{NVB}$ [by using the given property $\sim\sim u_x = u_x$]. $\therefore$ I_{NVB} is a NVB $BZMV^{dM}$- ideal

6. Direct sum of neutrosophic vague binary $BZMV^{dM}$ – subalgebra

In this section, a method is provided as a theorem to obtain a NVB $BZMV^{dM}$ - subalgebra by joining two NVB $BZMV^{dM}$ - subalgebras having $\{0, 1\}$ as common elements.

Theorem 6.1

Let $\mathfrak{M}_{M_{NVB}} = \langle U^{\mathfrak{M}_{M_{NVB}}}, \oplus_1, \neg_1, \sim_1, 0, 1\rangle$ and $\mathfrak{M}_{P_{NVB}} = \langle U^{\mathfrak{M}_{P_{NVB}}}, \oplus_2, \neg_2, \sim_2, 0, 1\rangle$ be two NVB $BZMV^{dM}$ – subalgebras such that $\left(U^{\mathfrak{M}_{M_{NVB}}} \cap U^{\mathfrak{M}_{P_{NVB}}}\right) = \{0, 1\}$. Let $U^{\mathfrak{M}_{W_{NVB}}} = \left(U^{\mathfrak{M}_{M_{NVB}}} \cup U^{\mathfrak{M}_{P_{NVB}}}\right)$ and let a binary operation $\ominus$ be defined on Z as follows:

$$(u_a \ominus u_b) = \begin{cases} u_a \oplus_1 u_b & \text{if } u_a,\ u_b \in U^{\mathfrak{M}_{M_{NVB}}} \\ u_a \oplus_2 u_b & \text{if } u_a,\ u_b \in U^{\mathfrak{M}_{P_{NVB}}} \\ u_a & \text{otherwise} \end{cases}$$

$$\neg^{\circledast} u_a = \begin{cases} \neg_1 u_a & \text{if } u_a \in U^{\mathfrak{M}_{M_{NVB}}} \\ \neg_2 u_a & \text{if } u_a \in U^{\mathfrak{M}_{P_{NVB}}} \end{cases}$$

$$\sim^{\circledast} u_a = \begin{cases} \sim_1 u_a & \text{if } u_a \in U^{\mathfrak{M}_{M_{NVB}}} \\ \sim_2 u_a & \text{if } u_a \in U^{\mathfrak{M}_{P_{NVB}}} \end{cases}$$

Then, $\langle U^{\mathfrak{M}_{W_{NVB}}}, \ominus, \neg^{\circledast}, \sim^{\circledast}, 0, 1\rangle$ is a neutrosophic vague binary $BZMV^{dM}$ – subalgebra $\mathfrak{M}_{Z_{NVB}}$.
Here, $\ominus$ denotes direct sum.

Proof

(1) Let $u_a,\ u_b \in U^{\mathfrak{M}_{M_{NVB}}}$ and $u_c \in U^{\mathfrak{M}_{P_{NVB}}}$

$NVB_{M_{NVB}}\left((u_a \oplus u_b) \oplus u_c\right) = NVB_{M_{NVB}}\left((u_a \oplus_1 u_b) \oplus u_c\right) = NVB_{M_{NVB}}(u_a \oplus_1 u_b)$

$NVB_{M_{NVB}}\left((u_b \oplus u_c) \oplus u_a\right) = NVB_{M_{NVB}}(u_b \oplus u_a) = NVB_{M_{NVB}}(u_a \oplus u_b) = NVB_{M_{NVB}}(u_a \oplus_1 u_b)$

(2) Let $u_a \in U^{\mathfrak{M}_{M_{NVB}}}$ and $0 \in U^{\mathfrak{M}_{P_{NVB}}}$. $NVB_{M_{NVB}}(u_a \oplus 0) = NVB_{M_{NVB}}(u_a)$

Similarly, all the axioms for a $BZMV^{dM}$– subalgebra can be verified.

Case (i) : $u_a,\ u_b \in U^{\mathfrak{M}_{M_{NVB}}}$, $\left(\forall\, u_a,\ u_b \in U^{\mathfrak{M}_{M_{NVB}}}\right)$

(1) $\left(\inf_{u_v \in (u_a \ominus u_b)} NVB_{M_{NVB}}(u_v)\right)$

$= \left(\inf_{u_v \in (u_a \oplus_1 u_b)} NVB_{M_{NVB}}(u_v)\right) \succcurlyeq r\min\left\{NVB_{M_{NVB}}(u_a), NVB_{M_{NVB}}(u_b)\right\}$

(2) $\left(\forall\, u_a \in U^{\mathfrak{M}_{M_{NVB}}}\right)$

(i) $NVB_{M_{NVB}}(u_a) \succcurlyeq NVB_{M_{NVB}}\left(\neg^{\circledast} u_a\right) \Rightarrow NVB_{M_{NVB}}(u_a) \succcurlyeq NVB_{M_{NVB}}(\neg_1 u_a)$

(ii) $NVB_{M_{NVB}}(u_a) \succcurlyeq NVB_{M_{NVB}}\left(\sim^{\circledast} u_a\right) \Rightarrow NVB_{M_{NVB}}(u_a) \succcurlyeq NVB_{M_{NVB}}(\sim_1 u_a)$

[Since, $\langle U^{\mathfrak{M}_{M_{NVB}}}, \oplus_1, \neg_1, \sim_1, 0, 1\rangle$ is a NVB $BZMV^{dM}$ – subalgebra]

Case (ii) : $u_a,\ u_b \in U^{\mathfrak{M}_{P_{NVB}}}$, $\left(\forall\, u_a,\ u_b \in U^{\mathfrak{M}_{P_{NVB}}}\right)$

(1) $\left(\inf_{u_v \in (u_a \ominus u_b)} NVB_{M_{NVB}}(u_v)\right)$

$= \left(\inf_{u_v \in (u_a \oplus_2 u_b)} NVB_{M_{NVB}}(u_v)\right) \succcurlyeq r\min\left\{NVB_{M_{NVB}}(u_a), NVB_{M_{NVB}}(u_b)\right\}$

(2) $\left(\forall\, u_a \in U^{\mathfrak{M}_{P_{NVB}}}\right)$

(i) $NVB_{M_{NVB}}(u_a) \succcurlyeq NVB_{M_{NVB}}\left(\neg^{\circledast} u_a\right) \Rightarrow NVB_{M_{NVB}}(u_a) \succcurlyeq NVB_{M_{NVB}}(\neg_2 u_a)$

(ii) $NVB_{M_{NVB}}(u_a) \succcurlyeq NVB_{M_{NVB}}\left(\sim^{\circledast} u_a\right) \Rightarrow NVB_{M_{NVB}}(u_a) \succcurlyeq NVB_{M_{NVB}}(\sim_2 u_a)$

[Since, $\langle U^{\mathfrak{M}_{P_{NVB}}}, \oplus_2, \neg_2, \sim_2, 0, 1\rangle$ is a NVB $BZMV^{dM}$– subalgebra]

Case (iii) : $\forall\, u_a \in U^{\mathfrak{M}_{M_{NVB}}}$, $u_b \in U^{\mathfrak{M}_{P_{NVB}}}$ or $u_a \in U^{\mathfrak{M}_{P_{NVB}}}$, $u_b \in U^{\mathfrak{M}_{M_{NVB}}}$

(1) $\left(\forall\, u_a \in U^{\mathfrak{M}_{M_{NVB}}},\ u_b \in U^{\ddagger} \text{ or } u_a \in U^{\mathfrak{M}_{P_{NVB}}},\ u_b \in U^{\mathfrak{M}_{M_{NVB}}}\right)$

$\left(\inf_{u_v \in (u_a \ominus u_b)} NVB_{M_{NVB}}(u_v)\right) = \left(\inf_{u_v \in (u_a)} NVB_{M_{NVB}}(v)\right) \succcurlyeq r\min\left\{NVB_{M_{NVB}}(u_a),\ NVB_{M_{NVB}}(b)\right\}$

Being a unary operation, 2nd axiom does not exists.

Clearly all the conditions for a NVB $\mathrm{BZMV^{dM}}$ – subalgebra is verified. It is clear that combining of two NVB $\mathrm{BZMV^{dM}}$– subalgebras, will produce the same.

7. Conclusions

Binary concept leads us to handle the situations with two universal sets which are found to be common in real-life. In this paper neutrosophic vague binary $\mathbf{BZMV^{dM}}$ Sub-algebra of $\mathbf{BZMV^{dM}}$- algebra is developed. This idea will provide a combined effect of the distributive Brouwer Zadeh lattice with Many –Valued or Multi - Valued algebra when stipulated into the de-Morgan's zone. Its basic-ideal with various sub kinds are also developed. Some theorems, properties, direct sum for this new concept are explored. This paper is an attempt to discuss with mixed patterns and an investigation towards its wide scope. It could be further extended to higher dimensions. Pre -$\mathbf{BZMV^{dM}}$ algebraic structure is a weaker structure than $\mathbf{BZMV^{dM}}$ – algebra. In some times, violation of theorems in stronger structure may come into a control mode in their relaxed forms. That also could be verified by comparing both these structures. $\mathbf{BZMV^{dM}}$ –algebra can be considered as a strong $\mathbf{MV}$- algebraic structure in de- Morgan's environment. Same way, one more strengthened $\mathbf{MV}$ – algebraic structure namely neutrosophic vague binary - $\mathbf{BZMV_{\Delta}}$ ($\mathbf{NVB - BZMV_{\Delta} - subalgebra}$) can also be developed. Here Δ is an additional unary operator in the context. It can be also extended towards the de – Morgan's atmosphere and can get a more tightened structure neutrosophic vague binary $\mathbf{BZMV_{\Delta}}$ $\mathbf{^{dM}}$ Sub-algebra of $\mathbf{BZMV_{\Delta}}$ $\mathbf{^{dM}}$ - algebra ($\mathbf{NVB\ BZMV_{\Delta}{}^{dM}\ Sub - algebra\ of\ BZMV_{\Delta}{}^{dM} - algebra}$). In future, its applications can be extended to a number of areas like geology, unmanned aerial vehicle, business analysis, chemistry, mechatronics, aerospace, biomedical etc and have to be discussed in detail. Numerous applications can be tried out in the field of working algorithms of vacuum cleaners, washing machines etc., and in stock trading, medical diagnosis and treatment plans, weather forecasting systems, 3D animations etc. Neutrosophic Vague Binary ideas with its logical $\mathbf{BZMV^{dM}}$ Sub - algebraic pattern can be hopefully developed towards this area and to its working algorithms to produce more accuracy in this digital world.

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