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Abstract. This paper presents the refinement of a type of neutrosophic hyperring in which $+'$ and $\cdot'$ are hyperopraetions and studied some of its properties. Several interesting results and examples are presented.

Keywords: .

Neutrosophic, neutrosophic hyperring, neutrosophic hypersubring, refined neutrosophic hyperring, refined neutrosophic hypersubhyperring , refined neutrosophic hyperring homomorphism.

1. Introduction

In a general sense the triple $(R, +, \cdot)$ is an hyperring if the hyperoperations $+$ and $\cdot$ are such that $(R, +)$ is a hypergroup, $(R, \cdot)$ is semihypergroup and $\cdot$ is distributive with respect to $+$. These structures are essentially rings with approximately modified axioms. Different notions of hyperrings have been investigated by researchers in the field of algebraic hyperstructures. For example, Krasner in [20] introduced a type of hyperring in which $''+''$ is an hyperoperation and $''\cdot''$ is a binary operation. This type of hyperring is referred to as a Krasner hyperring. In [24] a type of hyperring called multiplicatve hyperring was introduced by Rota. In this hyperring $''+''$ is considered as an ordinary addition and $''\cdot''$ as an hyperoperation. The type of hyperring in which $''+''$ and $''\cdot''$ were hyperoperations was studied by De Salvo in [14]. These classes of hyperrings were further studied by Barghi [12] , Asokkumar and Velrajan [9–11] .

In 1995, Smarandache generalized fuzzy logic/set and intuitionistic fuzzy logic/set by introducing a new branch of philosophy called Neutrosophy, which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. In neutrosophic logic, each proposition has a degree of truth (T), a degree of indeterminancy (I) and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]-0, 1+[$ as can be seen in [22,23] . Ever since the introduction of this theory, several neutrosophic structures have been introduced, some of which includes; neutrosophic group, neutrosophic rings, neutrosophic modules, neutrosophic hypergroups, neutrosophic hyperrings, neutrosophic loops and many more. Smarandache in [22] introduced the concept of refined neutrosophic logic and neutrosophic set which is basically the splitting of the components $< T, I, F >$ into subcomponents of the form $< T_1, T_2, \cdots, T_p; I_1, I_2, \cdots, I_r; F_1, F_2, \cdots, F_s >$. This concept inspired the work of Agboola in [5] where he introduced refined neutrosophic algebraic structures. A lot of results have been published on the refinement of some of the known neutrosophic algebraic structures/hyperstructures ever since the work of Agboola. A comprehensive review of refined neutrosophic structures/hyperstrutures, can be found in [1,2,8,15–19] .

In this paper, the refinement of neutrosophic hyperring is studied and several interesting results and examples are presented.

2. **Preliminaries**

In this section, we will give some definitions, examples and results that will be used in the sequel.

Definition 2.1. [13] Let H be a non-empty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation. The couple $(H, \circ)$ is called a hypergroupoid. For any two non-empty subsets A and B of H and $x \in H$, we define

$$A \circ B = \bigcup_{a\in A, b\in B} a \circ b, \quad A \circ x = A \circ \{x\} \quad \text{and} \quad x \circ B = \{x\} \circ B.$$

Definition 2.2. [13] Let H be a non-empty set and let $+$ be a hyperoperation on H. The couple $(H, +)$ is called a canonical hypergroup if the following conditions hold:

(1) $x + y = y + x$, for all $x, y \in H$,
(2) $x + (y + z) = (x + y) + z$, for all $x, y, z \in H$,
(3) there exists a neutral element $0 \in H$ such that $x + 0 = \{x\} = 0 + x$, for all $x \in H$,
(4) for every $x \in H$, there exists a unique element $-x \in H$ such that $0 \in x + (-x) \cap (-x) + x$,
(5) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$, for all $x, y, z \in H$. A nonempty subset A of H is called a subcanonical hypergroup if A is a canonical hypergroup under the same hyperaddition as that of H that is, for every $a, b \in A$, $a - b \in A$. If in addition $a + A - a \subseteq A$ for all $a \in H$, A is said to be normal.

Definition 2.3. A hyperring is a triple $(R, +, \cdot)$ satisfying the following axioms:

(1) $(R, +)$ is a canonical hypergroup.
(2) $(R, \cdot)$ is a semihypergroup such that $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$, that is, 0 is a bilaterally absorbing element,
(3) For all $x, y, z \in R$,
    (a) $x \cdot (y + z) = x \cdot y + x \cdot z$ and

(b) $(x+y)\cdot z = x\cdot z+y\cdot z$. That is, the hyperoperation . is distributive over the hyperoperation +.

Definition 2.4. Let $(R, +, \cdot)$ be a hyperring and let A be a nonempty subset of R. A is said to be a subhyperring of R if $(A, +, \cdot)$ is itself a hyperring.

Definition 2.5. Let A be a subhyperring of a hyperring R. Then,

(1) A is called a left hyperideal of R if $r \cdot a \subseteq A$ for all $r \in R, a \in A$.
(2) A is called a right hyperideal of R if $a \cdot r \subseteq A$ for all $r \in R, a \in A$. A is called a hyperideal of R if A is both left and right hyperideal of R.

Definition 2.6. Let A be a hyperideal of a hyperring R. A is said to be normal in R if $r + A - r \subseteq A$ for all $r \in R$.

It will be assumed that I splits into two sub-indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. With the properties that:

$$\begin{aligned} I_1I_1 &= I_1^2 &= I_1, \\ I_2I_2 &= I_2^2 &= I_2 \text{ and} \\ I_1I_2 &= I_2I_1 &= I_1. \end{aligned}$$

Definition 2.7. [4] If $* : X(I_1, I_2) \times X(I_1, I_2) \mapsto X(I_1, I_2)$ is a binary operation defined on $X(I_1, I_2)$, then the couple $(X(I_1, I_2), *)$ is called a refined neutrosophic algebraic structure and it is named according to the laws (axioms) satisfied by $*$.

Definition 2.8. [4] Let $(X(I_1, I_2), +, \cdot)$ be any refined neutrosophic algebraic structure where + and . are ordinary addition and multiplication respectively.
For any two elements $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$, we define

$$(a, bI_1, cI_2) + (d, eI_1, fI_2) = (a + d, (b + e)I_1, (c + f)I_2),$$

$$(a, bI_1, cI_2) \cdot (d, eI_1, fI_2) = (ad, (ae + bd + be + bf + ce)I_1, (af + cd + cf)I_2).$$

Definition 2.9. [4] If + and . are ordinary addition and multiplication, I_k with $k = 1, 2$ have the following properties:

(1) $I_k + I_k + \cdots + I_k = nI_k$.
(2) $I_k + (-I_k) = 0$.
(3) $I_k \cdot I_k \cdot \cdots \cdot I_k = I_k^n = I_k$ for all positive integers $n > 1$.
(4) $0 \cdot I_k = 0$.
(5) I_k^{-1} is undefined and therefore does not exist.

Definition 2.10. [4] Let $(G, *)$ be any group. The couple $(G(I_1, I_2), *)$ is called a refined neutrosophic group generated by G, I_1 and I_2. $(G(I_1, I_2), *)$ is said to be commutative if for all $x, y \in G(I_1, I_2)$, we have $x * y = y * x$. Otherwise, we call $(G(I_1, I_2), *)$ a non -commutative refined neutrosophic group.

Definition 2.11. [4] If $(X(I_1, I_2), *)$ and $(Y(I_1, I_2), *')$ are two refined neutrosophic algebraic structures, the mapping

$$\phi : (X(I_1, I_2), *) \longrightarrow (Y(I_1, I_2), *')$$

is called a neutrosophic homomorphism if the following conditions hold:

(1) $\phi((a, bI_1, cI_2) * (d, eI_1, fI_2)) = \phi((a, bI_1, cI_2)) *' \phi((d, eI_1, fI_2))$.
(2) $\phi(I_k) = I_k$ for all $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$ and $k = 1, 2$.

Example 2.12. [4] Let $\mathbb{Z}_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2)(0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$. Then $(\mathbb{Z}_2(I_1, I_2), +)$ is a commutative refined neutrosophic group of integers modulo 2. Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1, I_2), +)$ is a finite commutative refined neutrosophic group of integers modulo n.

Example 2.13. [4] Let $(G(I_1, I_2), *)$ and and $(H(I_1, I_2), *')$ be two refined neutrosophic groups. Let $\phi : G(I_1, I_2) \times H(I_1, I_2) \rightarrow G(I_1, I_2)$ be a mapping defined by $\phi(x, y) = x$ and let $\psi : G(I_1, I_2) \times H(I_1, I_2) \rightarrow H(I_1, I_2)$ be a mapping defined by $\psi(x, y) = y$. Then ϕ and ψ are refined neutrosophic group homomorphisms.

Definition 2.14. [6] Let $(H, +)$ be any canonical hypergroup and let I be an indeterminate.
Let $H(I) =< H \cup I >= \{(a, bI) : a, b \in H\}$ be a set generated by H and I. The hyperstructure $(H(I), +)$ is called a neutrosophic canonical hypergroup . For all $(a, bI), (c, dI) \in H(I)$ with $b \neq 0$ or $d \neq 0$, we define $(a, bI) + (c, dI) = \{(x, yI) : x \in a + c,\ y \in a + d \cup b + c \cup b + d\}$. An element $I \in H(I)$ is represented by $(0, I)$ in $H(I)$ and any element $x \in H$ is represented by $(x, 0)$ in $H(I)$. For any nonempty subset $A(I)$ of $H(I)$, we define $-A(I) = \{-(a, bI) = (-a, -bI) : a, b \in H\}$.

Definition 2.15. [6] Let $(H(I), +)$ be a neutrosophic canonical hypergroup .

(1) A nonempty subset $A(I)$ of $H(I)$ is called a neutrosophic subcanonical hypergroup of $H(I)$ if $(A(I), +)$ is itself a neutrosophic canonical hypergroup . It is essential that $A(I)$ must contain a proper subset which is a subcanonical hypergroup of H.
If $A(I)$ does not contain a proper subset which is a subcanonical hypergroup of H, then it is called a pseudo neutrosophic subcanonical hypergroup of $H(I)$.
(2) If $A(I)$ is a neutrosophic subcanonical hypergroup (pseudo neutrosophic subcanonical hypergroup), $A(I)$ is said to be normal in $H(I)$ if for all $(a, bI) \in H(I)$, $(a, bI) + A(I) - (a, bI) \subseteq A(I)$.

Definition 2.16. [6] Let $(R, +, \cdot)$ be any hyperring and let I be an indeterminate. The hyperstructure $(R(I), +, \cdot)$ generated by R and I, that is, $R(I) =< R \cup I >$, is called a neutrosophic hyperring. For all $(a, bI), (c, dI) \in R(I)$ with $b \neq 0$ or $d \neq 0$, we define

$$(a, bI) \cdot (c, dI) = \{(x, yI) : x \in a \cdot c, y \in a \cdot d \cup b \cdot c \cup b \cdot d\}.$$

Definition 2.17. [6] Let $(R(I), +, \cdot)$ be a neutrosophic hyperring and let $A(I)$ be a nonempty subset of $R(I)$. $A(I)$ is called a neutrosophic subhyperring of $R(I)$ if $(A(I), +, \cdot)$ is itself a neutrosophic hyperring . It is essential that $A(I)$ must contain a proper subset which is a hyperring. Otherwise, $A(I)$ is called a pseudo neutrosophic subhyperring of $R(I)$.

Definition 2.18. [6] Let $(R(I), +, \cdot)$ be a neutrosophic hyperring and let $A(I)$ be a neutrosophic subhyperring of $R(I)$.

(1) $A(I)$ is called a left neutrosophic hyperideal if $(r, sI) \cdot (a, bI) \subseteq A(I)$ for all $(r, sI) \in R(I)$ and $(a, bI) \in A(I)$.

(2) $A(I)$ is called a right neutrosophic hyperideal if $(a, bI) \cdot (r, sI) \subseteq A(I)$ for all $(r, sI) \in R(I)$ and $(a, bI) \in A(I)$.

(3) $A(I)$ is called a neutrosophic hyperideal if $A(I)$ is both a left and right neutrosophic hyperideal .

A neutrosophic hyperideal $A(I)$ of $R(I)$ is said to be normal in $R(I)$ if for all $(r, sI) \in R(I)$

$$(r, sI) + A(I) - (r, sI) \subseteq A(I).$$

Definition 2.19. [6] Let $(R_1(I), +, \cdot)$ and $(R_2(I), +, \cdot)$ be two neutrosophic hyperring and let $\phi : R_1(I) \longrightarrow R_2(I)$ be a mapping from $R_1(I)$ into $R_2(I)$.

(1) ϕ is called a homomorphism if :
   (a) ϕ is a hyperring homomorphism,
   (b) $\phi((0, I)) = (0, I)$.

(2) ϕ is called a good or strong homomorphism if:
   (a) ϕ is a good or strong hyperring homomorphism,
   (b) $\phi((0, I)) = (0, I)$.

(3) ϕ is called an isomorphism (strong isomorphism) if ϕ is a bijective homomorphism (strong homomorphism).

3. Formulation of a refined neutrosophic hyperrings

In this section, we study and present the development of refined neutrosophic hyperring $(R(I_1, I_2), +, \cdot)$ generated by R, I_1 and I_2 where the operations $''+''$ and $''\cdot''$ are hyperoperations. i.e.,

$$+, \cdot : R(I_1, I_2) \times R(I_1, I_2) \longrightarrow 2^{R(I_1, I_2)}.$$

For all $(a, bI_1, cI_2), (d, eI_1, fI_2) \in R(I_1, I_2)$ with $a, b, c, d, e, f \in R$, we define

$$(a, bI_1, cI_2) + (d, eI_1, fI_2) = \{(p, qI_1, rI_2) : p \in a + d, q \in (b + e), r \in (c + f)\},$$

$(a, bI_1, cI_2) \cdot (d, eI_1, fI_2) = \{(p, qI_1, rI_2) : p \in ad,\ q \in ae + bd + be + bf + ce, r \in af + cd + cf\}.$

Definition 3.1. A refined neutrosophic hyperring is a tripple $(R(I_1, I_2), +, \cdot)$ satisfying the following axioms:

(1) $(R(I_1, I_2), +)$ is a refined neutrosophic canonical hypergroup .
(2) $(R(I_1, I_2), \cdot)$ is a refined neutrosophic semihypergroup.
(3) For all $(a, bI_1, cI_2), (d, eI_1, fI_2), (g, hI_1, jI_2) \in R(I_1, I_2)$,
  (a) $(a, bI_1, cI_2) \cdot ((d, eI_1, fI_2) + (g, hI_1, jI_2)) = (a, bI_1, cI_2) \cdot (d, eI_1, fI_2) + (a, bI_1, cI_2) \cdot (g, hI_1, jI_2)$ and
  (b) $((d, eI_1, fI_2) + (g, hI_1, jI_2)) \cdot (a, bI_1, cI_2) = (d, eI_1, fI_2) \cdot (a, bI_1, cI_2) + (g, hI_1, jI_2) \cdot (a, bI_1, cI_2)$.

Definition 3.2. Let $(R(I_1, I_2), +, \cdot)$ be a refined neutrosophic hyperring. A non-empty subset $M(I_1, I_2)$ of $R(I_1, I_2)$ is called a refined neutrosophic subhyperring of $R(I_1, I_2)$ if $(M(I_1, I_2), +, \cdot)$ is itself a neutrosophic hyperring. It is essential that $M(I_1, I_2)$ must contain a proper subset which is a hyperring. Otherwise, $M(I_1, I_2)$ is called a refined pseudo neutrosophic subhyperring of $R(I_1, I_2)$.

Definition 3.3. Let $R(I_1, I_2)$ be a refined neutrosophic hyperring. The refined neutrosophic subhyperring $M(I_1, I_2)$ is said to be normal in $R(I_1, I_2)$ if and only if $(a, bI_1, cI_2) + M(I_1, I_2) - (a, bI_1, cI_2) \subseteq M(I_1, I_2)$ for all $(a, bI_1, cI_2) \in R(I_1, I_2)$.

Definition 3.4. Let $(R(I_1, I_2)), +, \cdot)$ be a refined neutrosophic hyperring and let $M(I_1, I_2)$ be a refined neutrosophic subhyperring of $R(I_1, I_2)$. $(M(I_1, I_2), +, \cdot)$ is a left(right) refined neutrosophic hyperideal of $R(I_1, I_2)$ if $x \cdot m \in M(I_1, I_2)[m \cdot x \in M(I_1, I_2)]$ for all $x = (a, bI_1, cI_2) \in R(I_1, I_2)$ and $m = (p, qI_1, sI_2) \in M(I_1, I_2)$. $M(I_1, I_2)$ is a refined neutrosophic hyperideal if $M(I_1, I_2)$ is both left and right refined neutrosophic hyperideal.

Remark 3.5. It should be noted that a refined neutrosophic hyperideal $H(I_1, I_2)$ of a refined neutrosophic hyperring $R(I_1, I_2)$ is normal in $R(I_1, I_2)$ only if hyperideal H is normal in hyperring R.

Proposition 3.6. *Let $(R(I_1, I_2), +, \cdot)$ be any refined neutrosophic hyperring. $(R(I_1, I_2), +, \cdot)$ is a hyperring.*

Proof. (1) That $(R(I_1, I_2), +)$ is a canonical hypergroup follows from Proposition 2.3 in [19].

(2) We show that $(R(I_1, I_2), \cdot)$ is a semihypergroup.

$$\begin{aligned} x \cdot (y \cdot z) &= (a, bI_1, cI_2) \cdot ((d, eI_1, fI_2) \cdot (g, hI_1, kI_2)) \\ &= (a, bI_1, cI_2) \cdot ((dg, (dh + eg + eh + ek + fh)I_1, (dk + fg + fk)I_2) \\ &= (a(dg), (a(dh) + a(eg) + a(eh) + a(ek) + a(fh) + b(dg) + b(dh) + b(eg) + b(eh) \\ &\quad +b(ek) + b(fh) + b(dk) + b(fg) + b(fk) + c(dh) + c(eg) + c(eh) + c(ek) + c(fh))I_1, \\ &\quad (a(dk) + a(fg) + a(fk) + c(dg) + c(dk) + c(fg) + c(fk))I_2) \\ &= (ad)g, ((aI_1, ((ad)k + (af)g + (af)k + (cd)g + (cd)k + (cf)g + (cf)k)I_2) \\ &= ((a, bI_1, cI_2) \cdot (d, eI_1, fI_2)) \cdot (g, hI_1, kI_2) \\ &= (x \cdot y) \cdot z. \end{aligned}$$

Accordingly, $(R(I_1, I_2), \cdot)$ is a semihypergroup. Also, for all $(a, bI_1, cI_2) \in R(I_1, I_2)$,

$$(a, bI_1, cI_2) \cdot (0, 0I_1, 0I_2) = \{(x, yI_1, zI_2) : x \in a \cdot 0,\ y \in a \cdot 0 + b \cdot 0 + c \cdot 0, z \in a \cdot 0 + c \cdot 0\} = \{(0, 0I_1, 0I_2)\}.$$

Similarly, it can be shown that $(0, 0I_1, 0I_2) \cdot (a, bI_1, cI_2) = \{(0, 0I_1, 0I_2)\}$. Hence, $(0, 0I_1, 0I_2)$ is a bilaterally absorbing element.

(3) For the distributivity of $\cdot$ over $+$.

Let $a = (x, yI_1, zI_2), b = (u, vI_1, sI_2), c = (k, mI_1, nI_2)$ be arbitrary elements in $R(I_1, I_2)$ with $x, y, z, u, v, s, k, m, n \in R$.

$$\begin{aligned} a \cdot (b + c) &= a \cdot \{(h_1, h_2I_1, h_3I_2) : h_1 \in u + k, h_2 \in v + m, h_3 \in s + n\} \\ &= \{(x, yI_1, zI_2) \cdot (h_1, h_2I_1, h_3I_2) : h_1 \in u + k, h_2 \in v + m, h_3 \in s + n\} \\ &= \{(p_1, p_2I_1, p_3I_2) : p_1 \in xh_1, p_2 \in xh_2 + yh_1 + yh_2 + yh_3 + zh_2, p_3 \in xh_3 + zh_1 + zh_3\} \\ &= \{(p_1, p_2I_1, p_3I_2) : p_1 \in xu + xk, p_2 \in xv + xm + yu + yk + yv + ym + ys + yn+ \\ &\quad zv + zm, p_3 \in xs + xn + zu + zk + zs + zn\}. \end{aligned}$$

Now if we take $p_1 = t_1 + t_1',\ p_2 = t_2 + t_2',\ p_3 = t_3 + t_3',$ then we have

$$\begin{aligned} a \cdot (b + c) &= \{(t_1 + t_1', (t_2 + t_2')I_1, (t_3 + t_3')I_2) : t_1 + t_1' \in xu + xk, \\ &\quad t_2 + t_2' \in xv + xm + yu + yk + yv + ym + ys + yn + zv + zm, \\ &\quad t_3 + t_3' \in xs + xn + zu + zk + zs + zn\} \\ &= \{(t_1, t_2I_1, t_3I_2) : t_1 \in xu,\ t_2 \in xv + yu + yv + ys + zv,\ t_3 \in xs + zu + zs\}+ \\ &\quad \{(t_1', t_2'I_1, t_3'I_2) : t_1' \in xk,\ t_2' \in xm + yk + ym + yn + zm,\ t_3' \in xn + zk + zn\} \\ &= (x, yI_1, zI2) \cdot (u, vI_1, sI_2) + (x, yI_1, zI_2) \cdot (k, mI_1, nI_2) \\ &= a \cdot b + a \cdot c. \end{aligned}$$

Similarly, we can show that $(b + c) \cdot a = b \cdot a + c \cdot a$. Therefore $\cdot$ is distributive over $+$.

Hence $R(I_1, I_2)$ is a hyperring. □

Example 3.7. Let $R(I_1, I_2) = \{a_1 = (s, sI_1, sI_2), a_2 = (s, sI_1, tI_2), a_3 = (s, tI_1, sI_2),$ $a_4 = (s, tI_1, tI_2),\ b_1 = (t, tI_1, tI_2), b_2 = (t, tI_1, sI_2), b_3 = (t, sI_1, tI_2), b_4 = (t, sI_1, sI_2)\}$ be a refined neutrosophic set and let $+$ be the hyperoperation on $R(I_1, I_2)$ defined as in the tables below. Let $a = \{a_1, a_2, a_3, a_4\}$ and $b = \{b_1, b_2, b_3, b_4\}$.

It is clear from Table 1 and 2 that $(R(I_1, I_2), +, \cdot)$ is a refined neutrosophic hyperring.

TABLE 1. Cayley table for the binary operation " + "

$+$	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
a_1	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
a_2	a_2	$\{a_1, a_2\}$	a_4	$\{a_3, a_4\}$	$\{b_1, b_2\}$	b_1	$\{b_3, b_4\}$	b_3
a_3	a_3	a_4	$\{a_1, a_3\}$	$\{a_2, a_4\}$	$\{b_1, b_3\}$	$\{b_2, b_4\}$	b_1	b_2
a_4	a_4	$\{a_3, a_4\}$	$\{a_2, a_4\}$	a	b	$\{b_1, b_3\}$	$\{b_1, b_2\}$	b_1
b_1	b_1	$\{b_1, b_2\}$	$\{b_1, b_3\}$	b	$R(I_1, I_2)$	$\{a_2, a_4, b_1, b_3\}$	$\{a_3, a_4, b_1, b_2\}$	$\{a_4, b_1\}$
b_2	b_2	b_1	$\{b_2, b_4\}$	$\{b_1, b_3\}$	$\{a_2, a_4, b_1, b_3\}$	$\{a_1, a_3, b_2, b_4\}$	$\{a_4, b_1\}$	$\{a_3, b_2\}$
b_3	b_3	$\{b_3, b_4\}$	b_1	$\{b_1, b_2\}$	$\{a_3, a_4, b_1, b_2\}$	$\{a_4, b_1\}$	$\{a_1, a_2, b_3, b_4\}$	$\{a_2, b_3\}$
b_4	b_4	b_3	b_2	b_1	$\{a_4, b_1\}$	$\{a_3, b_2\}$	$\{a_2, b_3\}$	$\{a_1, b_4\}$

Proposition 3.8. *Let $(R(I_1, I_2), +_1, \cdot_1)$ be a refined neutrosophic hyperring and let $(K, +_2, \cdot_2)$ be a hyperring. Define for all $(x_1, k_1), (x_2, k_2) \in R(I_1, I_2) \times K$ the hyperoperations $''+''$ and $''\cdot''$ by*

$$(x_1, k_1) + (x_2, k_2) = \{(x_3, k_3) : x_3 \in x_1 +_1 x_2,\ k_3 \in k_1 +_2 k_2\}$$

and

$$(x_1, k_1) \cdot (x_2, k_2) = \{(x_3, k_3) : x_3 \in x_1 \cdot_1 x_2,\ k_3 \in k_1 \cdot_2 k_2\}.$$

Then $(R(I_1, I_2) \times K, +, \cdot)$ is a refined neutrosophic hyperring.

Proof. (1) That $(R(I_1, I_2) \times K, +)$ is a canonical hypergroup follows from the proof of Proposition 2.6 in [19] .

(2) We shall show that $(R(I_1, I_2) \times K, \cdot)$ is a refined neutrosophic semihypergroup.

Let $(r_1, k_1), (r_2, k_2), (r_3, k_3) \in R(I_1, I_2) \times K$ where $r = (a, bI_1, cI_2)$.

$(r_1, k_1) \cdot ((r_2, k_2) \cdot (r_3, k_3)) =$

$((a_1, b_1I_1, c_1I_2), k_1) \cdot [((a_2, b_2I_1, c_2I_2), k_2) \cdot ((a_3, b_3I_1, c_3I_2), k_3)]$

$= ((a_1, b_1I_1, c_1I_2), k_1) \cdot \{((p, qI_1, sI_2), k_4) :\ p \in a_2 \cdot_1 a_3,$

$q \in a_2 \cdot_1 b_3 +_1 b_2 \cdot_1 a_3 +_1 b_2 \cdot_1 b_3 +_1 b_2 \cdot_1 c_3 +_1 c_2 \cdot_1 b_3, s \in a_2 \cdot_1 c_3 +_1 c_2 \cdot_1 a_3 +_1 c_2 \cdot_1 c_3,\ k_4 \in k_2 \cdot_2 k_3\}$

$= \{((x, yI_1, zI_2), k_5) : x \in a_1 \cdot_1 p,$

TABLE 2. Cayley table for the binary operation $''\cdot''$

$\cdot$	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1
a_2	a_1	$\{a_1, a_2\}$	$\{a_1, a_3\}$	a	a	a	$\{a_1, a_2\}$	$\{a_1, a_2\}$
a_3	a_1	$\{a_1, a_3\}$	$\{a_1, a_3\}$	$\{a_1, a_3\}$	$\{a_1, a_3\}$	$\{a_1, a_3\}$	$\{a_1, a_3\}$	$\{a_1, a_3\}$
a_4	a_1	a	$\{a_1, a_3\}$	a	a	a	a	a
b_1	a_1	a	$\{a_1, a_3\}$	a	$R(I_1, I_2)$	$R(I_1, I_2)$	$R(I_1, I_2)$	$R(I_1, I_2)$
b_2	a_1	a	$\{a_1, a_3\}$	a	$R(I_1, I_2)$	$\{a_1, a_3, b_2, b_4\}$	$R(I_1, I_2)$	$\{a_1, a_3, b_2, b_4\}$
b_3	a_1	$\{a_1, a_2\}$	$\{a_1, a_3\}$	a	$R(I_1, I_2)$	$R(I_1, I_2)$	$\{a_1, a_2, b_3, b_4\}$	$\{a_1, a_2, b_3, b_4\}$
b_4	a_1	$\{a_1, a_2\}$	$\{a_1, a_3\}$	a	$R(I_1, I_2)$	$\{a_1, a_3, b_2, b_4\}$	$\{a_1, a_2, b_3, b_4\}$	$\{a_1, b_4\}$

$y \in a_1 \cdot_1 q +_1 b_1 \cdot_1 p +_1 b_1 \cdot_1 q +_1 b_1 \cdot_1 s +_1 c_1 \cdot_1 q, z \in a_1 \cdot_1 s +_1 c_1 \cdot_1 p +_1 c_1 \cdot_1 s,\ k_5 \in k_1 \cdot_2 k_4\}$

$= \{((x, yI_1, zI_2), k_5) : x \in a_1 \cdot_1 (a_2 \cdot_1 a_3),$

$y \in a_1 \cdot_1 (a_2 \cdot_1 b_3 +_1 b_2 \cdot_1 a_3 +_1 b_2 \cdot_1 b_3 +_1 b_2 \cdot_1 c_3 +_1 c_2 \cdot_1 b_3) +_1 b_1 \cdot_1 (a_2 \cdot_1 a_3) +_1 b_1 \cdot_1 (a_2 \cdot_1 b_3 +_1 b_2 \cdot_1 a_3 +_1 b_2 \cdot_1 b_3 +_1 b_2 \cdot_1 c_3 +_1 c_2 \cdot_1 b_3) +_1 b_1 \cdot_1 (a_2 \cdot_1 c_3 +_1 c_2 \cdot_1 a_3 +_1 c_2 \cdot_1 c_3) +_1 c_1 \cdot_1 (a_2 \cdot_1 b_3 +_1 b_2 \cdot_1 a_3 +_1 b_2 \cdot_1 b_3 +_1 b_2 \cdot_1 c_3 +_1 c_2 \cdot_1 b_3),$

$z \in a_1 \cdot_1 (a_2 \cdot_1 c_3 +_1 c_2 \cdot_1 a_3 +_1 c_2 \cdot_1 c_3) +_1 c_1 \cdot_1 (a_2 \cdot_1 a_3) +_1 c_1 \cdot_1 (a_2 \cdot_1 c_3 +_1 c_2 \cdot_1 a_3 +_1 c_2 \cdot_1 c_3),$

$k_5 \in k_1 \cdot_2 (k_2 \cdot_2 k_3)\}$

$= \{((x, yI_1, zI_2), k_5) : x \in a_1 \cdot_1 a_2 \cdot_1 a_3,$

$y \in a_1 \cdot_1 a_2 \cdot_1 b_3 +_1 a_1 \cdot_1 b_2 \cdot_1 a_3 +_1 a_1 \cdot_1 b_2 \cdot_1 b_3 +_1 a_1 \cdot_1 b_2 \cdot_1 c_3 +_1 a_1 \cdot_1 c_2 \cdot_1 b_3 +_1 b_1 \cdot_1 a_2 \cdot_1 a_3 +_1 b_1 \cdot_1 a_2 \cdot_1 b_3 +_1 b_1 \cdot_1 b_2 \cdot_1 a_3 +_1 b_1 \cdot_1 b_2 \cdot_1 b_3 +_1 b_1 \cdot_1 b_2 \cdot_1 c_3 +_1 b_1 \cdot_1 c_2 \cdot_1 b_3 +_1 b_1 \cdot_1 a_2 \cdot_1 c_3 +_1 b_1 \cdot_1 c_2 \cdot_1 a_3 +_1 b_1 \cdot_1 c_2 \cdot_1 c_3 +_1 c_1 \cdot_1 a_2 \cdot_1 b_3 +_1 c_1 \cdot_1 b_2 \cdot_1 a_3 +_1 c_1 \cdot_1 b_2 \cdot_1 b_3 +_1 b_2 \cdot_1 c_3 +_1 c_2 \cdot_1 b_3,$

$z \in a_1 \cdot_1 a_2 \cdot_1 c_3 +_1 a_1 \cdot_1 c_2 \cdot_1 a_3 +_1 a_1 \cdot_1 c_2 \cdot_1 c_3 +_1 c_1 \cdot_1 a_2 \cdot a_3 +_1 c_1 \cdot_1 a_2 \cdot_1 c_3 +_1 c_1 \cdot_1 c_2 \cdot_1 a_3 +_1 c_1 \cdot_1 c_2 \cdot_1 c_3, k_5 \in k_1 \cdot_2 k_2 \cdot_2 k_3\}$

$= \{((x, yI_1, zI_2), k_5) : x \in (a_1 \cdot_1 a_2) \cdot_1 a_3,$

$y \in (a_1 \cdot_1 a_2) \cdot_1 b_3 +_1 (a_1 \cdot_1 b_2 +_1 b_1 \cdot_1 a_2 +_1 b_1 \cdot_1 b_2 +_1 b_1 \cdot_1 c_2 +_1 c_1 \cdot_1 b_2) \cdot_1 a_3 +_1 (a_1 \cdot_1 b_2 +_1 b_1 \cdot_1 a_2 +_1 b_1 \cdot_1 b_2 +_1 b_1 \cdot_1 c_2 +_1 c_1 \cdot_1 b_2) \cdot_1 b_3 +_1 (a_1 \cdot_1 b_2 +_1 b_1 \cdot_1 a_2 +_1 b_1 \cdot_1 b_2 +_1 b_1 \cdot_1 c_2 +_1 c_1 \cdot_1 b_2) \cdot_1$

$c_3 +_1 (a_1c_2 +_1 c_1a_2 +_1 c_1c_2) \cdot_1 b_3,$

$z \in (a_1 \cdot_1 a_2) \cdot_1 c_3 +_1 (a_1 \cdot_1 c_2 +_1 c_1 \cdot_1 a_2 +_1 c_1 \cdot_1 c_2) \cdot_1 a_3 +_1 (a_1 \cdot_1 c_2 +_1 c_1 \cdot_1 a_2 +_1 c_1 \cdot_1 c_2) \cdot_1 c_3, k_5 \in$

$(k_1 \cdot_2 k_2) \cdot_2 k_3\}$

$= \{((m, nI_1, hI_2), k) : m \in a_1 \cdot_1 a_2,\ n \in a_1 \cdot_1 b_2 +_1 b_1 \cdot_1 a_2 +_1 b_1 \cdot_1 b_2 +_1 b_1 \cdot_1 c_2 +_1 c_1 \cdot_1 b_2, h \in$

$a_1c_2 +_1 c_1a_2 +_1 c_1c_2, k \in k_1 \cdot_2 k_2\} \cdot ((a_3, b_3I_1, c_3I_2), k_3)$

$= [((a_1, b_1I_1, c_1I_2), k_1) \cdot ((a_2, b_2I_1, c_2I_2), k_2)] \cdot ((a_3, b_3I_1, c_3I_2))$

$= ((r_1, k_1) \cdot (r_2, k_2)) \cdot (r_3, k_3).$

Accordingly, $(R(I_1, I_2) \times K, \cdot)$ is a refined neutrosophic semihypergroup.

Also, for all $((a, bI_1, cI_2), k) \in R(I_1, I_2) \times K$,

$$\begin{aligned} ((a, bI_1, cI_2), k) \cdot ((0, 0I_1, 0I_2), 0) &= \{((x, yI_1, zI_2), k_1) : x \in a \cdot_1 0, y \in a \cdot_1 0 +_1 b \cdot_1 0 +_1 c \cdot_1 0, \\ &\quad z \in a \cdot_1 0 +_1 c \cdot 0,\ k_1 \in k \cdot_2 0\} \\ &= \{((0, 0I_1, 0I_2), 0)\}. \end{aligned}$$

Similarly, it can be shown that $((0, 0I_1, 0I_2), 0) \cdot ((a, bI_1, cI_2), k) = \{((0, 0I_1, 0I_2), 0)\}$.

Hence, $((0, 0I_1, 0I_2), 0)$ is a bilaterally absorbing element.

(3) For the distributivity of $\cdot$ over $+$.

Let $a = ((x, yI_1, zI_2), t_1),\ b = ((u, vI_1, sI_2), t_2),\ c = ((k, mI_1, nI_2), t_3)$ be arbitrary elements in $R(I_1, I_2) \times K$ with $x, y, z, u, v, s, k, m, n \in R$ and $t_1, t_2, t_3 \in K$.

$$\begin{aligned} a \cdot (b + c) &= a \cdot \{((h_1, h_2I_1, h_3I_2), t_4) : h_1 \in u +_1 k, h_2 \in v +_1 m, h_3 \in s +_1 n, t_4 \in t_2 +_2 t_3\} \\ &= \{((x, yI_1, zI_2), t_1) \cdot ((h_1, h_2I_1, h_3I_2), t_4) : h_1 \in u +_1 k, h_2 \in v +_1 m, h_3 \in s +_1 n, \\ &\quad t_4 \in t_2 +_2 t_3\} \\ &= \{((p_1, p_2I_1, p_3I_2), t_5) : p_1 \in x \cdot_1 h_1, p_2 \in x \cdot_1 h_2 +_1 y \cdot_1 h_1 +_1 y \cdot_1 h_2 +_1 y \cdot_1 h_3 \\ &\quad +_1 z \cdot_1 h_2, p_3 \in x \cdot_1 h_3 +_1 z \cdot_1 h_1 +_1 z \cdot_1 h_3, t_5 \in t_1 \cdot_2 t_4\} \\ &= \{((p_1, p_2I_1, p_3I_2), t_5) : p_1 \in x \cdot_1 u +_1 x \cdot_1 k, \\ &\quad p_2 \in x \cdot_1 v +_1 x \cdot_1 m +_1 y \cdot_1 u +_1 y \cdot_1 k +_1 y \cdot_1 v +_1 y \cdot_1 m +_1 y \cdot_1 s +_1 y \cdot_1 n +_1 \\ &\quad z \cdot_1 v +_1 z \cdot_1 m,\ p_3 \in x \cdot_1 s +_1 x \cdot_1 n +_1 z \cdot_1 u +_1 z \cdot_1 k +_1 z \cdot_1 s +_1 z \cdot_1 n, \\ &\quad t_5 \in t_1 \cdot_2 t_2 +_2 t_1 \cdot_2 t_3\}. \end{aligned}$$

Now if we take $p_1 = g_1 +_1 g_1',\ p_2 = g_2 +_1 g_2',\ p_3 = g_3 +_1 g_3',\ t_5 = h_1 +_2 h_1'$ then we have

$$\begin{aligned} a \cdot (b + c) &= \{((g_1 +_1 g_1', (g_2 +_1 g_2')I_1, (g_3 +_1 g_3')I_2), (h_1 +_2 h_1')) : g_1 +_1 g_1' \in x \cdot_1 u +_1 x \cdot_1 k, \\ &\quad g_2 +_1 g_2' \in x \cdot_1 v +_1 x \cdot_1 m +_1 y \cdot_1 u +_1 y \cdot_1 k +_1 y \cdot_1 v +_1 y \cdot_1 m +_1 y \cdot_1 s +_1 y \cdot_1 n +_1 \\ &\quad z \cdot_1 v +_1 z \cdot_1 m,\ g_3 +_1 g_3' \in x \cdot_1 s +_1 x \cdot_1 n +_1 z \cdot_1 u +_1 z \cdot_1 k +_1 z \cdot_1 s +_1 z \cdot_1 n, \\ &\quad h_1 +_2 h_1' \in t_1 \cdot_2 t_2 +_2 t_1 \cdot_2 t_3\} \end{aligned}$$

$$\begin{aligned} &= \{((g_1, g_2I_1, g_3I_2), h_1) : g_1 \in x \cdot_1 u, g_2 \in x \cdot_1 v +_1 y \cdot_1 u +_1 y \cdot_1 v +_1 y \cdot_1 s +_1 z \cdot_1 v, \\ &\quad g_3 \in x \cdot_1 s +_1 z \cdot_1 u +_1 z \cdot_1 s, h_1 \in t_1 \cdot_2 t_2\} + \\ &\quad \{((g_1', g_2'I_1, g_3'I_2), h_1') : g_1' \in x \cdot_1 k, g_2' \in x \cdot_1 m +_1 y \cdot_1 k +_1 y \cdot_1 m +_1 y \cdot_1 n +_1 z \cdot_1 m, \\ &\quad g_3' \in x \cdot_1 n +_1 z \cdot_1 k +_1 z \cdot_1 n, h_1' \in t_1 \cdot_2 t_3\} \\ &= a \cdot b + a \cdot c. \end{aligned}$$

Similarly, we can show that $(b + c) \cdot a = b \cdot a + c \cdot a$.

Therefore $\cdot$ is distributive over $+$. Hence $(R(I_1, I_2), \times K, +, \cdot)$ is a refined neutrosophic Hyperring. □

Proposition 3.9. *Let $(R(I_1, I_2), +_1, \cdot_1)$ and $(K(I_1, I_2), +_2, \cdot_2)$ be any two refined neutrosophic hyperring. Define for all $(x_1, k_1), (x_2, k_2) \in R(I_1, I_2) \times K(I_1, I_2)$ the hyperoperations $''+''$ and $''\cdot''$ by*

$$(x_1, k_1) + (x_2, k_2) = \{(x_3, k_3) : x_3 \in x_1 +_1 x_2,\ k_3 \in k_1 +_2 k_2\}$$

and

$$(x_1, k_1) \cdot (x_2, k_2) = \{(x_3, k_3) : x_3 \in x_1 \cdot_1 x_2,\ k_3 \in k_1 \cdot_2 k_2\}.$$

Then $(R(I_1, I_2) \times K(I_1, I_2), +, \cdot)$ is a refined neutrosophic hyperring.

Proof. The proof is similar to the proof of Proposition 3.8. □

Lemma 3.10. *Let $R(I_1, I_2)$ be a refined neutrosophic hyperring. A non-empty subset $M(I_1, I_2)$ of $R(I_1, I_2)$ is a left(right) refined neutrosophic hyperideal if and only if for $m_1 = (p_1, q_1I_1, s_1I_1), m_2 = (p_2, q_2I_1, s_2I_1) \in M(I_1, I_2)$ and $x = (a, bI_1, cI_2) \in R(I_1, I_2)$*

(1) $m_1 - m_2 \subseteq M(I_1, I_2)$,

(2) $x \cdot m_1 \in M(I_1, I_2)$ $[m_1 \cdot x \in M(I_1, I_2)]$.

Definition 3.11. Let $H(I_1, I_2)$ and $J(I_1, I_2)$ be any two nonempty subsets of a refined neutrosophic hyperring $R(I_1, I_2)$.

(1) The sum $H(I_1, I_2) + J(I_1, I_2) = \{(x, yI_1, zI_2) : x \in x_1 + x_2,\ y \in y_1 + y_2,\ z \in z_1 + z_2\}$.
For some $x_1, y_1, z_1 \in H,\ x_2, y_2, z_2 \in J$.

(2) The product

$$H(I_1, I_2)J(I_1, I_2) = \{(x, yI_1, zI_2) : (x, yI_1, zI_2) \in \sum_{i=1}^{n}(a_i, b_iI_1, c_iI_2) \cdot (d_i, e_iI_1, f_iI_1),\ n \in \mathbb{Z}^+\}.$$

Proposition 3.12. *Let $R(I_1, I_2)$ be a refined neutrosophic hyperring. Let $H(I_1, I_2)$ and $J(I_1, I_2)$ be refined neutrosophic hyperideals of $R(I_1, I_2)$ then :*

(1) *$H(I_1, I_2) + J(I_1, I_2)$ is a refined neutrosophic hyperideal.*

(2) *$H(I_1, I_2)J(I_1, I_2)$ is a refined neutrosophic hyperideal.*

Proof. (1) Let $x = (a, bI_1, cI_2), y = (d, eI_1, fI_2) \in H(I_1, I_2) + J(I_1, I_2)$ and let $r = (g, hI_1, kI_2) \in R(I_1, I_2)$.

$$\begin{aligned}
(i)\ x - y &= (a, bI_1, cI_2) - (d, eI_1, fI_2) = (a, bI_1, cI_2) + (-d, -eI_1, -fI_2) \\
&= \{(p, qI_1, rI_2) : p \in a + (-d), q \in b + (-e),\ r \in c + (-f)\} \\
&= \{(p_1 + p_2, (q_1 + q_2)I_1, (r_1 + r_2)I_2) : p_1 + p_2 \in (a_1 + a_2) + (-d_1 + (-d_2)), \\
&\quad q_1 + q_2 \in (b_1 + b_2) + (-e_1 + (-e_2)),\ r_1 + r_2 \in (c_1 + c_2) + (-f_1 + (-f_2))\} \\
&= \{(p_1, q_1I_1, r_1I_2) : p_1 \in a_1 + (-d_1),\ q_1 \in b_1 + (-e_1),\ r_1 \in c_1 + (-f_1)\} + \\
&\quad \{(p_2, q_2I_1, r_2I_2) : p_2 \in a_2 + (-d_2),\ q_2 \in b_2 + (-e_2),\ r_2 \in c_2 + (-f_2)\} \\
&= \{(p_1, q_1I_1, r_1I_2) : p_1 \in a_1 - d_1,\ q_1 \in b_1 - e_1, r_1 \in c_1 - f_1\} + \\
&\quad \{(p_2, q_2I_1, r_2I_2) : p_2 \in a_2 - d_2,\ q_2 \in b_2 - e_2, r_2 \in c_2 - f_2\} \\
&= (x_1 - y_1) + (x_2 - y_2) \\
&\subseteq H(I_1, I_2) + J(I_1, I_2).
\end{aligned}$$

$$
\begin{aligned}
(ii)\ r\cdot x &= (g,hI_1,kI_2)\cdot(a,bI_1,cI_2)\\
&= \{(u,vI_1,mI_2): u\in ga,\ v\in gb+ha+hb+hc+kb,\ m\in gc+ka+kc\}\\
&= \{(u_1+u_2,(v_1+v_2)I_1,(m_1+m_2)I_2): u_1+u_2\in g(a_1+a_2),\\
&\quad v_1+v_2\in g(b_1+b_2)+h(a_1+a_2)+h(b_1+b_2)+h(c_1+c_2)+k(b_1+b_2),\\
&\quad m_1+m_2\in g(c_1+c_2)+k(a_1+a_2)+k(c_1+c_2)\}\\
&= \{(u_1,v_1I_1,m_1I_2): u_1\in ga_1, v_1\in gb_1+ha_1+hb_1+hc_1+kb_1, m\in gc_1+ka_1+kc_1\}+\\
&\quad \{(u_2,v_2I_1,m_2I_2): u_2\in ga_2, v_2\in gb_2+ha_2+hb_2+hc_2+kb_2, m_2\in gc_2+ka_2+kc_2\}\\
&= r\cdot x_1+r\cdot x_2\\
&\subseteq H(I_1,I_2)+J(I_1,I_2).
\end{aligned}
$$

$$
\begin{aligned}
(iii)\ x\cdot r &= (a,bI_1,cI_2)\cdot(g,hI_1,kI_2)\\
&= \{(u,vI_1,mI_2): u\in ag,\ v\in ah+bg+bh+bk+ch,\ m\in ak+cg+ck\}\\
&= \{(u_1+u_2,(v_1+v_2)I_1,(m_1+m_2)I_2): u_1+u_2\in (a_1+a_2)g,\\
&\quad v_1+v_2\in (a_1+a_2)h+(b_1+b_2)g+(b_1+b_2)h+(b_1+b_2)k+(c_1+c_2)h,\\
&\quad m_1+m_2\in (a_1+a_2)k+(c_1+c_2)g+(c_1+c_2)k\}\\
&= \{(u_1,v_1I_1,m_1I_2): u_1\in a_1g, v_1\in a_1h+b_1g+b_1h+b_1k+c_1h, m_1\in a_1k+c_1g+c_1k\}+\\
&\quad \{(u_2,v_2I_1,m_2I_2): u_2\in a_2g, v_2\in a_2h+b_2g+b_2h+b_2k+c_2h, m_2\in a_2k+c_2g+c_2k\}\\
&= x_1\cdot r+x_2\cdot r\\
&\subseteq H(I_1,I_2)+J(I_1,I_2).
\end{aligned}
$$

(2) Let $x=(a,bI_1,cI_2), y=(d,eI_1,fI_2)\in H(I_1,I_2)J(I_1,I_2)$ and let $r=(g,hI_1,kI_2)\in R(I_1,I_2)$.

Here

$$(a,bI_1,cI_2)\in\sum_{i=1}^{n}(a_i,b_iI,c_iI)\cdot(a_i',b_i'I_1,c_i'I_2) \text{ and } (d,eI_1,fI_2)\in\sum_{i=1}^{n}(d_i,e_iI_1,f_iI_2)\cdot(d_i',e_i'I_1,f_i'I_2).$$

For $(a_i,b_iI_1,c_iI_2),(d_i,e_iI_1,f_iI_2)\in H(I_1,I_2)$, $(a_i',b_i'I_1,c_i'I_2),(d_i',e_i'I_1,f_i'I_2)\in J(I_1,I_2)$, $a_i,b_i,c_i,d_i,e_i,f_i\in H$ and $a_i',b_i',c_i',d_i',e_i',f_i'\in J$.

So we have

$$a\in\sum_{i=1}^{n}a_ia_i',\ b\in\sum_{i=1}^{n}(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i'),\ c\in\sum_{i=1}^{n}(a_ic_i'+c_ia_i'+c_ic_i')$$

and

$$d\in\sum_{i=1}^{n}d_id_i',\ e\in\sum_{i=1}^{n}(d_ie_i'+e_id_i'+e_ie_i'+e_if_i'+f_ie_i'),\ f\in\sum_{i=1}^{n}(d_if_i'+f_id_i'+f_if_i').$$

$$
\begin{aligned}
(i)\ x-y &= (a,bI_1,cI_2)-(d,eI_1,fI_2)=(a,bI_1,cI_2)+(-d,-eI_1,-fI_2)\\
&= \{(u,vI_1,mI_2): u\in a-d,\ v\in b-e,\ m\in c-f\}\\
&= \{(u,vI_1,mI_2): u\in \textstyle\sum_{i=1}^{n}a_ia_i'-\sum_{i=1}^{n}d_id_i',\\
&\quad v\in\textstyle\sum_{i=1}^{n}(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i')-\sum_{i=1}^{n}(d_ie_i'+e_id_i'+e_ie_i'+e_if_i'+f_ie_i'),\\
&\quad m\in\textstyle\sum_{i=1}^{n}(a_ic_i'+c_ia_i'+c_ic_i')-\sum_{i=1}^{n}(d_if_i'+f_id_i'+f_if_i')\}\\
&= \{(u,vI_1,mI_2): u\in\textstyle\sum_{i=1}^{n}(a_ia_i'+(-d_id_i')),\\
&\quad v\in\textstyle\sum_{i=1}^{n}(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i'+(-d_ie_i')+(-e_id_i')+(-e_ie_i')+(-e_if_i')\\
&\quad +(-f_ie_i')),\ m\in\textstyle\sum_{i=1}^{n}(a_ic_i'+c_ia_i'+c_ic_i'+(-d_if_i')+(-f_id_i')+(-f_if_i'))\}\\
&\subseteq H(I_1,I_2)J(I_1,I_2).
\end{aligned}
$$

$$\begin{aligned}
(ii)\ r\cdot x &= (g,hI_1,kI_2)\cdot(a,bI_1,cI_2)\\
&= \{(u,vI_1,mI_2): u\in ga,\ v\in gb+ha+hb+hc+kb,\ m\in gc+ka+kc\}\\
&= \{(u,vI_1,mI_2): u\in g\textstyle\sum_{i=1}^n a_ia_i',\\
&\quad v\in g\textstyle\sum_{i=1}^n(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i')+h\sum_{i=1}^n a_ia_i'+\\
&\quad h\textstyle\sum_{i=1}^n(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i')+h\sum_{i=1}^n(a_ic_i'+c_ia_i'+c_ic_i')+\\
&\quad k\textstyle\sum_{i=1}^n(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i'),\\
&\quad m\in g\textstyle\sum_{i=1}^n(a_ic_i'+c_ia_i'+c_ic_i')+k\sum_{i=1}^n a_ia_i'+k\sum_{i=1}^n(a_ic_i'+c_ia_i'+c_ic_i')\}\\
&= \{(u,vI_1,mI_2): u\in \textstyle\sum_{i=1}^n ga_ia_i',\\
&\quad v\in\textstyle\sum_{i=1}^n(ga_ib_i'+gb_ia_i'+gb_ib_i'+gb_ic_i'+gc_ib_i'+ha_ia_i'+ha_ib_i'+hb_ia_i'+hb_ib_i'+\\
&\quad hb_ic_i'+hc_ib_i'+ha_ic_i'+hc_ia_i'+hc_ic_i'+ka_ib_i'+kb_ia_i'+kb_ib_i'+kb_ic_i'+kc_ib_i'),\\
&\quad m\in\textstyle\sum_{i=1}^n(ga_ic_i'+gc_ia_i'+gc_ic_i'+ka_ia_i'+ka_ic_i'+kc_ia_i'+kc_ic_i')\}\\
&\subseteq H(I_1,I_2)J(I_1,I_2).
\end{aligned}$$

$$\begin{aligned}
(ii)\ x\cdot r &= (a,bI_1,cI_2)\cdot(g,hI_1,kI_2)\\
&= \{(u,vI_1,mI_2): u\in ag,\ v\in ah+bg+bh+bk+ch,\ m\in ak+cg+ck\}\\
&= \{(u,vI_1,mI_2): u\in \textstyle\sum_{i=1}^n a_ia_i'g,\\
&\quad v\in\textstyle\sum_{i=1}^n a_ia_i'h+\sum_{i=1}^n(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i')g+\\
&\quad \textstyle\sum_{i=1}^n(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i')h+\sum_{i=1}^n(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i')k+\\
&\quad \textstyle\sum_{i=1}^n(a_ic_i'+c_ia_i'+c_ic_i')h,\\
&\quad m\in\textstyle\sum_{i=1}^n a_ia_i'k+\sum_{i=1}^n(a_ic_i'+c_ia_i'+c_ic_i')g+\sum_{i=1}^n(a_ic_i'+c_ia_i'+c_ic_i')k\}\\
&= \{(u,vI_1,mI_2): u\in \textstyle\sum_{i=1}^n a_ia_i'g,\\
&\quad v\in\textstyle\sum_{i=1}^n(a_ia_i'h+a_ib_i'g+b_ia_i'g+b_ib_i'g+b_ic_i'g+c_ib_i'g+a_ib_i'h+b_ia_i'h+b_ib_i'h+\\
&\quad b_ic_i'h+c_ib_i'h+a_ib_i'k+b_ia_i'k+b_ib_i'k+b_ic_i'k+c_ib_i'k+a_ic_i'h+c_ia_i'h+c_ic_i'h),\\
&\quad m\in\textstyle\sum_{i=1}^n(a_ia_i'k+a_ic_i'g+c_ia_i'g+c_ic_i'g+a_ic_i'k+c_ia_i'k+c_ic_i'k)\}\\
&\subseteq H(I_1,I_2)J(I_1,I_2).
\end{aligned}$$

Hence $H(I_1,I_2)J(I_1,I_2)$ is a refined neutrosophic hyperideal of $R(I_1,I_2)$. □

Proposition 3.13. *Let $R(I_1,I_2)$ be a refined neutrosophic hyperrings and $J_i(I_1,I_2)_{i\in\Lambda}$ be a family of refined neutrosophic hyperideals of $R(I_1,I_2)$, then $\bigcap_{i\in\Lambda} J_i(I_1,I_2)$ is a refined neutrosophic hyperideal of $R(I_1,I_2)$.*

Proof. The proof is the same as the proof in classical case. □

Proposition 3.14. *Let $H(I_1,I_2)$ and $J(I_1,I_2)$ be a refined neutrosophic hyperideals of a refined neutrosophic hyperring $R(I_1,I_2)$ such that $J(I_1,I_2)$ is normal in $R(I_1,I_2)$. Then*

(1) *$H(I_1,I_2)\cap J(I_1,I_2)$ is a normal refined neutrosophic hyperideal of $H(I_1,I_2)$.*
(2) *$J(I_1,I_2)$ is a normal refined neutrosophic hyperideal of $H(I_1,I_2)+J(I_1,I_2)$.*

Proof. (1) That $H(I_1,I_2)\cap J(I_1,I_2)$ is a refined neutrosophic hyperideal of $H(I_1,I_2)$ can be easily established. So, it remains to show that $H(I_1,I_2)\cap J(I_1,I_2)$ is normal in $H(I_1,I_2)$.

Let $x = (a, bI_1, cI_2) \in H(I_1, I_2) \cap J(I_1, I_2)$, $h = (u, vI_1, tI_2) \in H(I_1, I_2)$ with $a, b, c \in H \cap J$ and $u, v, t \in H$. Then

$$\begin{aligned}
h + H(I_1,I_2) \cap J(I_1,I_2) - h &= h + x - h \; for \; x \in H(I_1,I_2) \cap J(I_1,I_2) \\
&= (u, vI_1, tI_2) + (a, bI_1, cI_2) - (u, vI_1, tI_2) \\
&= \{(p, qI_1, rI_2) : p \in u + a - u, \; q \in v + b - v, \; r \in t + c - t\} \\
&= \{(p, qI_1, rI_2) : p \in u + (H \cap J) - u, \; q \in v + (H \cap J) - v, \\
&\quad r \in t + (H \cap J) - t\} \\
&= \{(p, qI_1, rI_2) : p \in u + (H \cap J) - u \subseteq H \cap J, \\
&\quad q \in v + (H \cap J) - v \subseteq H \cap J, \; r \in t + (H \cap J) - t \subseteq H \cap J\} \\
&= \{(p, qI_1, rI_2) : p \in H \cap J, \; q \in H \cap J, \; r \in H \cap J\} \\
&\subseteq H(I_1,I_2) \cap J(I_1,I_2).
\end{aligned}$$

Accordingly, $H(I_1, I_2) \cap J(I_1, I_2)$ is a normal refined neutrosophic hyperideal of $H(I_1, I_2)$.

(2) That $J(I_1, I_2)$ is a refined neutrosophic hyperideal of $H(I_1, I_2) + J(I_1, I_2)$ can be easily established. So, it remains to show that $J(I_1, I_2)$ is normal in $H(I_1, I_2) + J(I_1, I_2)$. Let $x = (a, bI_1, cI_2) \in J(I_1, I_2)$, $h = (u, vI_1, tI_2) = (u_1+u_2, (v_1+v_2)I_1, (t_1+t_2)I_2) \in H(I_1, I_2)+J(I_1, I_2)$ with $a, b, c, u_2, v_2, t_2 \in J$ and $u_1, v_1, t_2 \in H$. Then

$$\begin{aligned}
h + J(I_1,I_2) - h &= h + x - h \; for \; x \in J(I_1,I_2) \\
&= (u, vI_1, tI_2) + (a, bI_1, cI_2) - (u, vI_1, tI_2) \\
&= ((u_1 + u_2), (v_1 + v_2)I_1, (t_1 + t_2)I_2) + (a, bI_1, cI_2) \\
&\quad -((u_1 + u_2), (v_1 + v_2)I_1, (t_1 + t_2)I_2) \\
&= \{(p, qI_1, rI_2) : p \in (u_1 + u_2) + a - (u_1 + u_2), \; q \in (v_1 + v_2) + b - (v_1 + v_2), \\
&\quad r \in (t_1 + t_2) + c - (t_1 + t_2) \\
&= \{(p, qI_1, rI_2) : p \in (u_1 + u_2) + J - (u_1 + u_2), \; q \in (v_1 + v_2) + J - (v_1 + v_2), \\
&\quad r \in (t_1 + t_2) + J - (t_1 + t_2)\} \\
&= \{(p, qI_1, rI_2) : p \in u_1 + (u_2 + J - u_2) - u_1, \; q \in v_1 + (v_2 + J - v_2) - v_1, \\
&\quad r \in t_1 + (t_2 + J - t_2) - t_1\} \\
&\subseteq \{(p, qI_1, rI_2) : p \in u_1 + J - u_1, \; q \in v_1 + J - v_1, r \in t_1 + J - t_1\} \\
&= \{(p, qI_1, rI_2) : p \in u_1 + J - u_1 \subseteq J, \; q \in v_1 + J - v_1 \subseteq J, \; r \in t_1 + J - t_1 \subseteq J\} \\
&= \{(p, qI_1, rI_2) : p \in J, \; q \in J, \; r \in J\} \\
&\subseteq J(I_1,I_2).
\end{aligned}$$

Accordingly, $J(I_1, I_2)$ is a normal refined neutrosophic hyperideal of $H(I_1, I_2) + J(I_1, I_2)$. □

Let $R(I_1, I_2)$ be a refined neutrosophic hyperring, and let $H(I_1, I_2)$ be a refined neutrosophic hyperideal of $R(I_1, I_2)$. Since $H(I_1, I_2)$ is a refined neutrosophic subcanonical hypergroup of $R(I_1, I_2)$, if $(R/H, +)$ is a canonical hypergroup then

$$R(I_1,I_2)/H(I_1,I_2) = \{\bar{(}x, yI_1, zI_2) : (x, yI_1, zI_2) \in R(I_1,I_2)\}$$

is a refined neutrosophic canonical hypergroup under the hyperaddition $+'$ defined for $r_1 + H(I_1, I_2), r_2 + H(I_1, I_2) \in R(I_1, I_2)/H(I_1, I_2)$ with $r_1 = (x_1, y_1I_1, z_iI_2), r_2 = (x_2, y_2I_1, z_2I_2)$, by

$$r_1 + H(I_1, I_2) +' r_2 + H(I_1, I_2) = (r_1 +' r_2) + H(I_1, I_2).$$

Define on $R(I_1, I_2)/H(I_1, I_2)$ a hypermultiplication $\cdot'$ by

$$r_1 + H(I_1, I_2) \cdot' r_2 + H(I_1, I_2) = (r_1r_2) + H(I_1, I_2).$$

It can be shown that $(R(I_1, I_2)/H(I_1, I_2), +', \cdot')$ is a refined neutrosophic hyperring if $(R/H, +, \cdot)$ is a hyperring.

Definition 3.15. Let $(R(I_1, I_2), +_1, \cdot_1)$ and $(P(I_1, I_2), +_2, \cdot_2)$ be any two refined neutrosophic hypergrings and let

$$\phi : R(I_1, I_2) \longrightarrow P(I_1, I_2)$$

be a mapping from $R(I_1, I_2)$ into $P(I_1, I_2)$.

(1) ϕ is called a refined neutrosophic hyperring homomorphism if:
   (a) ϕ is hyperring homomorphism,
   (b) $\phi(I_k) = I_k$ for $k = 1, 2$.
(2) ϕ is called a good refined neutrosophic hyperring homomorphism if:
   (a) ϕ is good hyperring homomorphism,
   (b) $\phi(I_k) = I_k$ for $k = 1, 2$.
(3) ϕ is called a refined neutrosophic hyperring isomorphism if ϕ is a refined neutrosophic hyperring homomorphism and ϕ^{-1} is also a refined neutrosophic hyperring homomorphism.

Definition 3.16. Let $\phi : R(I_1, I_2) \longrightarrow P(I_1, I_2)$ be a refined neutrosophic hyperring homomorphism from a refined neutrosophic hyperring $R(I_1, I_2)$ into a refined neutrosophic hyperring $P(I_1, I_2)$.

(1) The $Ker\phi = \{(u, vI_1, wI_2) \in R(I_1, I_2) : \phi((u, vI_1, wI_2)) = (0, 0I_1, 0I_2)\}$.
(2) The $Im\phi = \{\phi((u, vI_1, wI_2)) : (u, vI_1, wI_2) \in R(I_1, I_2)\}$.

Proposition 3.17. *Let $\phi : R(I_1, I_2) \longrightarrow P(I_1, I_2)$ be a refined neutrosophic homomorphism.*

(1) *The kernel of ϕ is not a neutrosophic subhyperring of $R(I_1, I_2)$.*
(2) *The kernel of ϕ is not a neutrosophic hyper ideal of $R(I_1, I_2)$.*
(3) *The image of ϕ is a neutrosophic subhyperring of $P(I_1, I_2)$.*

Proof. (1) It follows easily from 1 of definition 3.16.
(2) It follows from the Proof of 1.
(3) The proof is similar to the proof in classical case.

It can be shown that $ker\phi$ is just a subhyperrings of $R(I_1, I_2)$. □

4. Conclusions

This paper studied the refinement of a type of neutrosophic hyperrings in which $''+''$ and $''\cdot''$ are hyperoperations and presented their basic properties. It was established that every refined neutrosophic hyperring is a hyperring. It was also shown that the kernel of a refined neutrosophic hyperring homomorphism is not a refined neutrosophic hyperideal but the image is a refined neutrosophic subhyperring.

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