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NEUTROSOPHIC DISCRETE GEOMETRIC DISTRIBUTION

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Abstract

Uncertainty, vagueness, and ambiguity surround us in many real-life problems and, therefore, always remain under consideration for researchers to quantify them. This study proposed neutrosophic discrete probability distribution as a generalization of classical or existing probability distributions, named neutrosophic geometric distribution. Case studies presented in the paper will help understand the concept and application of the proposed distribution. Several properties are derived, like the proposed distribution's moment, characteristic, and probability-generating functions. Furthermore, the newly proposed distribution derives properties from the reliability analysis, such as survival function, hazard rate function, reversed hazard rate function, cumulative hazard rate function, mills ratio, and odds ratio. In addition, order statistics for NGD, including wth, the largest, and the smallest order statistics, are also derived from joint, median, minimum, and maximum order statistics. This examination opens the path for managing issues that follow traditional conveyances and simultaneously contain information that is not determined precisely.

1. Introduction

Smarandache first initiated neutrosophy in [1]. It is a new branch of the philosophy presented as a generalization of fuzzy logic and a generalization of intuitionistic fuzzy logic [2, 3]. The modern world is overfilled with uncertainty, ambiguity, fuzzy (problems, circumstances, and ideas) [4]. The classical probability ignores extreme, aberrant, unclear values, so a new suitable instrument had to be used [5]. The basic concept of neutrosophic sets was introduced by [6] in a well-defined book in 2014 purely in a statistics scenario, which presents a new base for dealing with several issues containing indeterminate data. The primary objective of neutrosophic logic is to characterize a logical statement in a 3-D neutrosophic space. Each dimension of the nutrosophic space represents, respectively, Truth (T), False (U), and Indeterminacy (I) of the statements under consideration. The terms T, I, and U are the standard, non-standard real subsets $(-0, 1+)$ without any specific connection. Many researchers extended the classical distributions neutrosophically, including neutrosophic binomial distribution and neutrosophic normal distribution, neutrosophic multinomial distribution, neutrosophic Poisson distribution, neutrosophic exponential distribution and neutrosophic uniform distribution, neutrosophic gamma distribution, neutrosophic Weibull distribution and its several families, etc [7-10]. The readers can see further information on neutrosophic statistics vs classical statistics in [5, 8, 11-15]. This study extended classical geometric distribution, which is neutrosophically named neutrosophic geometric distribution, using neutrosophic logic.

2. Neutrosophic Geometric Distribution (NGD)

Geometric distribution from the distribution theory is a classical discrete probability distribution [16]. The distribution is related to the binomial distribution in terms of the nature of the experimental trial, i.e., the trials are independent and have two possible outcomes (success or failure). The random variable used in the geometric distribution is the number of trials needed to attain a first success. The geometric distribution requires the exact number of failure attempts before obtaining a first success, but the precise number of failures is hard to find in some situations. In such situations, the experimenter is not sure about the classification of the outcome, i.e., either classify the outcome as a success or failure. This indecisive situation is called the *indeterminate state* of the experiment. It requires some specialized form of geometric distribution because the methodology of the classical geometric distribution does not support handling such situations. This section proposed a modified form of the classical geometric distribution called *neutrosophic geometric distribution*.

The neutrosophic geometric distribution of discrete r.v. X is well-defined as the extension of the classical geometric distribution of X with some indeterminacy or vagueness in the experiment. A neutrosophic geometric experiment may result in some failures as well as indeterminate outcomes till the occurrence of the first success; for example, tossing a coin on an unstable surface that may have cracks, a coin may fall on its edge inside the crack, and one may get neither head nor tale but some indeterminacy. The probability of success is labeled as $p_r(S)$, a failure $p_r(U)$ and indeterminate outcome as $p_r(I)$.

Neutrosophic geometric r.v. X represents the number of trials needed for first success when experimenting with a variable number of times - the neutrosophic probability distribution of r.v. X is called a *neutrosophic geometric probability distribution*.

For $x \in \{1, 2, 3, ..., \infty\}$, Np_r (occurrences of first success after x trials) $= (T_x, U_x, I_x)$.

For cases where *threshold* $> th^*$ will belong to indeterminate part and when *threshold* $< th^*$, the cases will belong to a determinate part. The probability mass function (T_x) and the cumulative distribution function $(F(x))$ of neutrosophic geometric distribution are given in equations (2.1) and (2.4):

$$T_x = p_r(S) \sum_{h=0}^{th^*} \binom{x-1}{h} (p_r(I))^h (p_r(U))^{x-h-1}. \tag{2.1}$$

The same will be true when trials result in failures as there is only one success, not a fixed number of successes, like in binomial distribution:

$$U_x = p_r(S)\sum_{h=0}^{th^*}\binom{x-1}{h}(p_r(I))^h(p_r(U))^{x-h-1} \tag{2.2}$$

and

$$I_x = p_r(S)\sum_{z^*=th^*+1}^{x-1}\binom{x-1}{z^*}(p_r(I))^{z^*}$$

$$\times\sum_{x=1}^{\infty}\sum_{h=0}^{x-z^*-1}\binom{x-z^*-1}{h}(p_r(U))^{x-z^*-h-1}. \tag{2.3}$$

The CDF corresponding to equation (2.1) is given by

$$F(x) = 1 - p_r(S)(p_r(U))^x(1-p_r(U))^{-1}\sum_{m'=0}^{\infty}\binom{x+m'}{m'}\left(\frac{p_r(I)}{p_r(U)}\right)^{m'}. \tag{2.4}$$

The CDF corresponding to equation (2.3) is given by

$$F(x) = 1 - p_r(S)\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\cdot\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}. \tag{2.5}$$

2.1. Physical conditions

(i) Each trial results in three mutually exclusive and exhaustive outcomes: success, failure, and indeterminacy.

(ii) All the trials must be independent.

(iii) The probability of success remains fixed or constant for every trial.

(iv) The experiment performs a variable number of times until the first success occurs.

2.2. Case studies

Case study 1. A physician is seeking an anti-depressant for five patients newly diagnosed. Assume that, of all available anti-depressant drugs, 60% were found effective, 20% were found ineffective, and 30% have no evidence about their effectiveness. What is the probability that the first drug is effective for the fifth patient?

Using the information in the problem mentioned above, we compute the probability for all three parts of PMF.

$X \rightarrow$ No of patients until we get the one for which ant-depressant drug found effective.

So $x = 1, 2, 3, 4, 5$ and we have to find $NGp_r\ (x = 5)$.

p_r (effective drug) $= p_r(S) = 0.6,$

p_r (ineffective drug) $= p_r(U) = 0.2,$

p_r (neither effective nor ineffective) $= p_r(I) = 0.3.$

Let indeterminacy threshold be 2. Then

$$T_5 = (0.6)\sum_{h=0}^{2}\binom{5-1}{h}(0.3)^h(0.2)^{5-h-1}$$

$$= (0.6)\left[\binom{5-1}{0}(0.3)^0(0.2)^{5-0-1} + \binom{5-1}{1}(0.3)^1(0.2)^{5-1-1} + \binom{5-1}{2}(0.3)^2(0.2)^{5-2-1}\right]$$

$$= (0.6)\left(\frac{1}{625} + \frac{6}{625} + \frac{27}{1250}\right)$$

$$= 0.01968,$$

$$I_5 = (0.6)\sum_{z^*=3}^{5-1=4}\frac{(5-1)!}{z^*!}(0.3)^{z^*}\sum_{h=0}^{5-z^*-1}\frac{(0.2)^{5-z^*-h-1}}{h!(5-z^*-h-1)!}$$

$$= (0.6)\left[\left\{\frac{(5-1)!}{3!}(0.3)^3\sum_{h=0}^{5-3-1=1}\frac{(0.2)^{5-3-h-1}}{h!(5-3-h-1)!}\right\}\right.$$

$$+\left\{\frac{(5-1)!}{4!}(0.3)^4 \sum_{h=0}^{5-4-1=0} \frac{(0.2)^{5-4-h-1}}{h!(5-4-h-1)!}\right\}\Bigg]$$

$$= (0.6)\left[4(0.3)^3\left\{\frac{(0.2)^{2-0-1}}{0!(2-0-1)!} + \frac{(0.2)^{2-1-1}}{1!(2-1-1)!}\right\}\right.$$

$$\left. + (0.3)^4\left\{\frac{(0.2)^{1-0-1}}{0!(1-0-1)!}\right\}\right]$$

$$= (0.6)(0.1296 + 8.1 \times 10^{-3})$$

$$= 0.08262,$$

$$U_5 = (p_r(S) + p_r(I) + p_r(U))^x - T_5 - I_5$$

$$= (0.6 + 0.3 + 0.2)^5 - 0.01968 - 0.08262$$

$$= 1.50821.$$

If normalized, the computed vector becomes

$$(T_5, I_5, U_5) = (0.01968, 0.08262, 1.50821).$$

By dividing each component of the vector by its total sum, we have

$$0.01968 + 0.08262 + 1.50821 = 1.61051.$$

Hence we get

$$(T_5, I_5, U_5) = (0.012219, 0.051301, 0.93648).$$

Case study 2. Suppose an investigator is searching for a student who lives five miles away from the investigator. The investigator knows that 52% of the 25000 live within five miles of him, 33% do not, and there is no information about 10% of students where they live. The investigator randomly calls college students until one says the student lives within 5 miles of him. What is the probability that six people need a contact?

Using the information in the abovementioned problem, we compute the probability for all three parts of PMF.

$X \rightarrow$ Number of students the investigator must contact till one says yes.

So, $x = 1, 2, 3, \ldots$ (total number of students), we need to find the $NGp_r(x = 6)$.

p_r (do live within five miles) $= p_r(S) = 0.52$,

p_r (do not live within five miles) $= p_r(U) = 0.33$,

p_r (no information about where they live) $= p_r(I) = 0.1$.

Let indeterminacy threshold be 2. Then

$$T_6 = (0.52)\sum_{h=0}^{2}\binom{6-1}{h}(P(I))^h(P(U))^{6-h-1}$$

$$= (0.52)\left\{\binom{6-1}{0}(0.1)^0(0.33)^{6-0-1} + \binom{6-1}{1}(0.1)^1(0.33)^{6-1-1} + \binom{6-1}{2}(0.1)^2(0.33)^{6-2-1}\right\}$$

$$= (0.52)\{(3.9135\times 10^{-3}) + (5.9296\times 10^{-3}) + (3.5937\times 10^{-3})\}$$

$$= 0.006987,$$

$$I_6 = (0.52)\sum_{z^*=3}^{6-1=5}\frac{(6-1)!}{z^*!}(0.1)^{z^*}\sum_{h=0}^{6-z^*-1}\frac{(0.33)^{6-z^*-h-1}}{h!(6-z^*-h-1)!}$$

$$= (0.52)\left[\left\{\frac{(6-1)!}{3!}(0.1)^3\sum_{k=0}^{6-3-1=2}\frac{(0.33)^{6-3-h-1}}{h!(6-3-h-1)!}\right\} + \left\{\frac{(6-1)!}{4!}(0.1)^4\sum_{h=0}^{6-4-1=1}\frac{(0.33)^{6-4-h-1}}{h!(6-4-h-1)!}\right\}\right.$$

$$+\left\{\frac{(6-1)!}{5!}(0.1)^5\sum_{h=0}^{6-5-1=0}\frac{(0.33)^{6-5-h-1}}{h!(6-5-h-1)!}\right\}\Bigg]\Bigg]$$

$$= (0.52)\Bigg[(0.02)\left\{\frac{(0.33)^{3-0-1}}{0!(3-0-1)!}+\frac{(0.33)^{3-1-1}}{1!(3-1-1)!}+\frac{(0.33)^{3-2-1}}{2!(3-2-1)!}\right\}$$

$$+(5\times 10^{-4})\cdot\left\{\frac{(0.33)^{2-0-1}}{0!(2-0-1)!}+\frac{(0.33)^{2-1-1}}{1!(2-1-1)!}\right\}$$

$$+(1\times 10^{-5})\left\{\frac{(0.33)^{1-0-1}}{1!(1-0-1)!}\right\}\Bigg]$$

$$= (0.52)(0.01769 + 6.65\times 10^{-4} + 0.1^5)$$

$$= 9.5498\times 10^{-3},$$

$$U_6 = (p_r(S) + p_r(I) + p_r(U))^x - T_6 - I_6$$

$$= (0.52 + 0.1 + 0.33)^6 - 0.006987 - 9.5498\times 10^{-3}$$

$$= 0.71856.$$

If normalized, the computed vector becomes

$$(T_6,\ I_6,\ U_6) = (0.006987,\ 0.71856,\ 9.5498\times 10^{-3}).$$

By dividing each component of the vector by its total sum, we have

$$0.006987 + 9.5498\times 10^{-3} + 0.71856 = 0.735097.$$

Hence we get

$$(T_6,\ I_6,\ U_6) = (9.50487\times 10^{-3},\ 0.012991,\ 0.97750).$$

Case study 3. Suppose that different computer components have a random selection, from which 30% were defective, 60% were non-defective, and 10% were unclear whether defective or not. How likely is the seventh component tested to cause the first defect?

Using the information in the abovementioned problem, we compute the probability for all three parts of PMF.

X takes on values 1, 2, 3, ..., and we have to find $NGp_r\ (x = 7)$.

p_r (defective components) $= p_r(S) = 0.3,$

p_r (non-defective components) $= p_r(U) = 0.6,$

p_r (neither defective nor non-defective) $= p_r(I) = 0.1.$

Let indeterminacy threshold be 3. Then

$$T_7 = (0.3)\sum_{h=0}^{3}\binom{7-1}{h}(0.1)^h(0.6)^{7-h-1}$$

$$= (0.3)\left[\binom{7-1}{0}(0.1)^0(0.6)^{7-0-1} + \binom{7-1}{1}(0.1)^1(0.6)^{7-1-1} + \binom{7-1}{2}(0.1)^2(0.6)^{7-2-1} + \binom{7-1}{3}(0.1)^3(0.6)^{7-3-1}\right]$$

$$= (0.3)\{0.046656 + 0.046656 + 0.01944 + (4.32\times 10^{-3})\}$$

$$= 0.0351216,$$

$$I_7 = (0.3)\sum_{z^*=4}^{7-1=5}\frac{(7-1)!}{z^*!}(0.1)^{z^*}\sum_{h=0}^{7-z^*-1}\frac{(0.6)^{7-z^*-h-1}}{h!(7-z^*-h-1)!}$$

$$= (0.3)\left[\left\{\frac{(7-1)!}{4!}(0.1)^4\sum_{h=0}^{7-4-1=2}\frac{(0.6)^{7-4-h-1}}{h!(7-4-h-1)!}\right\} + \left\{\frac{(7-1)!}{5!}(0.1)^5\sum_{h=0}^{7-5-1=1}\frac{(0.6)^{7-5-h-1}}{h!(7-5-h-1)!}\right\} + \left\{\frac{(7-1)!}{6!}(0.1)^6\sum_{h=0}^{7-6-1=0}\frac{(0.6)^{7-6-h-1}}{h!(7-6-h-1)!}\right\}\right]$$

$$= (0.3)\left[(30 \times 0.1^4)\left\{\frac{(0.6)^{3-0-1}}{0!(3-0-1)!} + \frac{(0.6)^{3-1-1}}{1!(3-1-1)!} + \frac{(0.6)^{3-2-1}}{2!(3-2-1)!}\right\}\right.$$

$$+ (6 \times 0.1^5)\left\{\frac{(0.6)^{2-0-1}}{0!(2-0-1)!} + \frac{(0.6)^{2-1-1}}{1!(2-1-1)!}\right\}$$

$$\left. + (0.1^6)\left\{\frac{(0.6)^{1-0-1}}{1!(1-0-1)!}\right\}\right]$$

$$= (0.3)(3.84 \times 10^{-3} + 9.6 \times 10^{-5} + 0.1^6)$$

$$= 1.1811 \times 10^{-3},$$

$$U_7 = (p_r(S) + p_r(I) + p_r(U))^x - T_7 - I_7$$

$$= (0.3 + 0.1 + 0.6)^7 - 0.0351216 - 1.1811 \times 10^{-3}$$

$$= 0.96530.$$

Hence we get

$$(T_7,\ I_7,\ U_7) = (0.0351216,\ 0.96530,\ 1.1811 \times 10^{-3}).$$

3. Main Properties of NGD

3.1. Moment generating function

Moment generating function (m.g.f) for the true part of *PMF* $X \sim NGD(x;\ p_r(S))$ is given by

$$M_0(t) = \left(1 - \frac{p_r(I)}{p_r(U)}\right)^{-1} \cdot \frac{p_r(S)}{(e^{-t} - p_r(U))}. \tag{3.1}$$

The indeterminate part of PMF is given as

$$M_0(t) = e^t \cdot p_r(S) \cdot B^* \cdot \left(\sum_{n=1}^{\infty} -\frac{(p_r(I))^n}{n}\right) \cdot \{1 - p_r(U) \cdot e^t\}^{-1}, \tag{3.2}$$

where

$$B^* = -\frac{1}{p_r(U)} + \frac{1}{2!(p_r(U))^3} - \frac{1}{2!3!(p_r(U))^5} + \cdots.$$

3.2. Characteristic function

The characteristic function of $X \sim NGD(x;\ p_r(S))$ the true part of PMF is given by

$$\phi_i(t) = \frac{p_r(S)}{p_r(U)} \cdot \left(1 - \frac{p_r(I)}{p_r(U)}\right)^{-1} \cdot (1 - p_r(U) \cdot e^{it})^{-1}. \tag{3.3}$$

The indeterminate part of PMF is given as

$$\phi_i(t) = p_r(S) \cdot B^* \cdot \left\{\sum_{n=1}^{\infty} -\frac{(p_r(I))^n}{n}\right\} \cdot \{1 - p_r(U) \cdot e^{it}\}^{-1}. \tag{3.4}$$

3.3. Probability generating function

The probability generating function for the true part of PMF $X \sim NGD(x;\ p_r(S))$ is given by

$$G(\theta) = \theta p_r(S)\left(1 - \frac{P(I)}{p_r(U)}\right)^{-1} \cdot (1 - \theta p_r(U))^{-1}. \tag{3.5}$$

The indeterminate part of PMF is given as

$$G(\theta) = \theta p_r(S)\left\{\sum_{n=1}^{\infty} -\frac{(p_r(I))^n}{n}\right\}\{1 - (p_r(U)\theta)\}^{-1}. \tag{3.6}$$

4. Reliability Analysis

This section finds various reliability properties like survival function, hazard rate function, reversed hazard rate function, and cumulative hazard rate function. In addition, the Mills ratio and the odd ratio for the new proposed distribution are derived.

4.1. Survival function

The survival function of r.v. $X \sim NGD(x;\ p_r(S))$ for the true part of PMF is given as:

$$S(x) = p_r(S)(p_r(U))^x (1 - p_r(U))^{-1} \sum_{i=0}^{\infty} \binom{x+i}{i} \left(\frac{p_r(I)}{p_r(U)} \right)^i. \tag{4.1}$$

The indeterminate part of PMF is given as:

$$S(x) = p_r(S) \sum_{m'=0}^{\infty} \binom{x+m'}{m'+1} (p_r(I))^{m'+1} \sum_{m'=0}^{\infty} \binom{x-1}{m'} (p_r(U))^{x-m'-1}. \tag{4.2}$$

4.2. Hazard rate or failure rate function

The hazard rate function of r.v. $X \sim NGD(x;\ p_r(S))$ for the true part of PMF is given as:

$$h(x) = \frac{\sum_{h=0}^{th^*} \binom{x-1}{h} p_r(I)^h p_r(U)^{x-h-1}}{(p_r(U))^x (1 - p_r(U))^{-1} \sum_{i=0}^{\infty} \binom{x+i}{i} \left(\frac{p_r(I)}{p_r(U)} \right)^i}. \tag{4.3}$$

The indeterminate part of PMF is given as:

$$h(x) = \frac{\sum_{z^*=th^*+1}^{x-1} \binom{x-1}{z^*} (p_r(I))^{z^*} \times \sum_{x=1}^{\infty} \sum_{h=0}^{x-z^*-1} \binom{x-z^*-1}{h} (p_r(U))^{x-z^*-h-1}}{\sum_{m'=0}^{\infty} \binom{x+m'}{m'+1} (p_r(I))^{m'+1} \sum_{m'=0}^{\infty} \binom{x-1}{m'} (p_r(U))^{x-m'-1}}. \tag{4.4}$$

4.3. Reversed hazard rate function

The reversed hazard rate function (RHRF) of r.v. $X \sim NGD(x;\ p_r(S))$ for the true part of PMF is given as:

$$\partial(x)=\frac{p_r(S)\sum_{h=0}^{th^*}\binom{x-1}{h}p_r(I)^h\,p_r(U)^{x-h-1}}{1-p_r(S)(p_r(U))^x(1-p_r(U))^{-1}\sum_{i=0}^{\infty}\binom{x+i}{i}\left(\frac{p_r(I)}{p_r(U)}\right)^i}. \quad (4.5)$$

For the indeterminate part of PMF, RHRF is given as:

$$\partial(x)=\frac{p_r(S)\sum_{z^*=th^*+1}^{x-1}\binom{x-1}{z^*}(p_r(I))^{z^*}\times\sum_{x=1}^{\infty}\sum_{h=0}^{x-z^*-1}\binom{x-z^*-1}{h}(p_r(U))^{x-z^*-h-1}}{1-p_r(S)\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}}.$$

(4.6)

4.4. Cumulative hazard rate function

The cumulative hazard rate function (CHRF) of r.v. $X \sim NGD(x;\ p_r(S))$ for true part of PMF is given as:

$$H(x)=-\ln\left\{p_r(S)(p_r(U))^x(1-p_r(U))^{-1}\sum_{i=0}^{\infty}\binom{x+i}{i}\left(\frac{p_r(I)}{p_r(F)}\right)^i\right\}. \quad (4.7)$$

For the indeterminate part of PMF, CHRF is given as:

$$H(x)=-\ln\left\{p_r(S)\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}\right\}.$$

(4.8)

4.5. Mills ratio

The Mills ratio of r.v. $X \sim NGD(x;\ p_r(S))$ for the true part of PMF is given as:

$$m(x) = \frac{\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}}{\sum_{h=0}^{th^*}\binom{x-1}{h}p_r(I)^h p_r(U)^{x-h-1}}. \quad (4.9)$$

For the indeterminate part of PMF, the Mills ratio is given as:

$$m(x) = \frac{\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}}{\sum_{z^*=th^*+1}^{x-1}\binom{x-1}{z^*}(p_r(I))^{z^*} \times\sum_{x=1}^{\infty}\sum_{h=0}^{x-z^*-1}\binom{x-z^*-1}{h}(p_r(U))^{x-z^*-h-1}}. \quad (4.10)$$

4.6. Odds ratio

An odds ratio of r.v. $X \sim NGD(x;\ p_r(S))$ for the true part of PMF is given as:

$$\varphi(x) = \frac{1-p_r(S)(p_r(U))^x(1-p_r(U))^{-1}\sum_{i=0}^{\infty}\binom{x+i}{i}\left(\frac{p_r(I)}{P(F)}\right)^i}{p_r(S)(p_r(U))^x(1-p_r(U))^{-1}\sum_{i=0}^{\infty}\binom{x+i}{i}\left(\frac{p_r(I)}{p_r(U)}\right)^i}. \quad (4.11)$$

For an indeterminate part of PMF, the odds ratio is given as:

$$\varphi(x) = \frac{1-p_r(S)\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}}{p_r(S)\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}}.$$

(4.12)

5. Order Statistics

In this section, we derived the order statistics for the new proposed distribution NGD, such as *w*th order statistics, joint, largest, and smallest order statistics, maximum and minimum, median order statistics, and smallest and largest order statistics.

5.1. *w*th **order statistics**

Let $X_1, X_2, ..., X_w$ be the random sample from NGD, and let $X_{(1)}, X_{(2)}, ..., X_{(w)}$ be the corresponding order statistics. The proposed *w*th order statistic for the true part of the NGD is:

$$f_{w:n}(x) = \frac{n!}{(w-1)!(n-w)!}(p_r(S))^{n-w+1}\sum_{h=0}^{th^*}\binom{x-1}{h}(p_r(I))^h(p_r(U))^{x-h-1}$$

$$\times\left[1-p_r(S)(p_r(U))^x(1-p_r(U))^{-1}\sum_{i=0}^{\infty}\binom{x+i}{i}\left(\frac{p_r(I)}{p_r(U)}\right)^i\right]^{w-1}$$

$$\times\left[(p_r(U))^x(1-p_r(U))^{-1}\sum_{i=0}^{\infty}\binom{x+i}{i}\left(\frac{p_r(I)}{p_r(U)}\right)^i\right]^{n-w}. \tag{5.1}$$

*w*th order statistics for the indeterminate part of NGD can be given as:

$$f_{w:n}(x) = \frac{n!}{(w-1)!(n-w)!}(p_r(S))^{n-w+1}$$

$$\times\sum_{z^*=th^*+1}^{x-1}\binom{x-1}{z^*}(p_r(I))^{z^*}\sum_{x=1}^{\infty}\sum_{h=0}^{x-z^*-1}\binom{x-z^*-1}{h}(p_r(U))^{x-z^*-h-1}$$

$$\times\left[-p_r(S)\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}\right]^{w-1}$$

$$\times\left[\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}\right]^{n-w}. \tag{5.2}$$

5.2. Joint order statistics

Joint order statistics of $y_{1:b}$ and $y_{u:b}$ for true part of NGD is derived as follows:

$$f_{a:u:b}(y)$$

$$= \frac{b!}{(a-1)!(u-a-1)!(b-u)!}(p_r(S))^{n-a+1}(1-p_r(U))^{m-n+1}$$

$$\times\left\{\sum_{h=0}^{th^*}\binom{y-1}{h}(p_r(I))^h(p_r(U))^{y-h-1}\right\}$$

$$\times\left\{\sum_{h=0}^{th^*}\binom{z-1}{h}(p_r(I))^h(p_r(U))^{z-h-1}\right\}$$

$$\times\left[1-p_r(S)(p_r(U))^x(1-p_r(U))^{-1}\sum_{i=0}^{\infty}\binom{x+i}{i}\left(\frac{p_r(I)}{p_r(U)}\right)^i\right]^{m-1}$$

$$\times\left[(p_r(U))^y\sum_{i=0}^{\infty}\binom{y+i}{i}\left(\frac{p_r(I)}{p_r(U)}\right)^i-(p_r(U))^z\sum_{i=0}^{\infty}\binom{z+i}{i}\left(\frac{p_r(I)}{p_r(U)}\right)^i\right]^{u-a-1}$$

$$\times\left[(p_r(U))^y\sum_{i=0}^{\infty}\binom{y+i}{i}\left(\frac{p_r(I)}{p_r(U)}\right)^i\right]^{b-u}. \tag{5.3}$$

Joint order statistic for the indeterminate part of NGD is given as follows:

$$f_{a:u:b}(x)$$

$$= \frac{b!}{(a-1)!(u-a-1)!(b-u)!}(p_r(S))^{s'(b-a+1)}$$

$$\times\left[\sum_{z^*=th+1}^{y-s'}\binom{y-s'}{z^*}(p_r(I))^{z^*}\sum_{x=s'}^{\infty}\sum_{t=0}^{y-s'-z^*}\binom{y-s'-z^*}{t}(p_r(U))^{y-s'-z^*-t}\right]$$

$$\times\left[\sum_{z^*=th^*+1}^{z-s'}\binom{z-s'}{z^*}(p_r(I))^{z^*}\sum_{z=s'}^{\infty}\sum_{t=0}^{z-s'-z^*}\binom{z-s'-z^*}{t}(p_r(U))^{z-s'-z^*-t}\right]$$

$$\times\left[\sum_{m'=1}^{\infty}\binom{y-s'+m'}{m'}(p_r(I))^{m'}\sum_{m'=0}^{\infty}\binom{y-s'}{m'}(p_r(U))^{y-s'-m'}\right.$$

$$\left.-\sum_{m'=1}^{\infty}\binom{z-s'+m'}{m'}(p_r(I))^{m'}\sum_{m'=0}^{\infty}\binom{z-s'}{m'}(p_r(U))^{x-s'-m'}\right]^{u-a-1}$$

$$\times\left[\sum_{m'=1}^{\infty}\binom{y-s'+m'}{m'}(p_r(I))^{m'}\sum_{m'=0}^{\infty}\binom{y-s'}{m'}(p_r(U))^{y-s'-m'}\right]^{b-u}. \quad (5.4)$$

5.3. Largest order statistics

For $u = \eta'$, the largest order statistic for the true part of NGD is given:

$$f_{\eta':\eta'}(x) = \eta' p_r(S)\sum_{h=0}^{th^*}\binom{x-1}{h}(p_r(I))^h(p_r(U))^{x-h-1}$$

$$\times\left[1-p_r(S)(p_r(U))^x(1-p_r(U))^{-1}\sum_{i=0}^{\infty}\binom{x+i}{i}\left(\frac{p_r(I)}{p_r(U)}\right)^i\right]^{\eta'-1}. \quad (5.5)$$

Largest order statistic for an indeterminate part of NGD is given as:

$$f_{\eta':\eta'}(x)$$

$$= \eta' p_r(S)\sum_{z^*=th^*+1}^{x-1}\binom{x-1}{z^*}(p_r(I))^{z^*}$$

$$\times \sum_{x=1}^{\infty} \sum_{h=0}^{x-z^*-1} \binom{x-z^*-1}{h} (p_r(U))^{x-z^*-h-1}$$

$$\times \left[1 - p_r(S) \sum_{m'=0}^{\infty} \binom{x+m'}{m'+1} (p_r(I))^{m'+1} \sum_{m'=0}^{\infty} \binom{x-1}{m'} (p_r(U))^{x-m'-1}\right]^{\eta'-1}. \tag{5.6}$$

5.4. Smallest order statistics

For $u = 1,$ smallest order statistics for the true part of NGD is given as:

$$f_{1:n}(x) = n(p_r(S))^n \sum_{h=0}^{th^*} \binom{x-1}{h} (p_r(I))^h (p_r(U))^{x-h-1}$$

$$\times \left[(p_r(U))^x (1 - p_r(U))^{-1} \sum_{i=0}^{\infty} \binom{x+i}{i} \left(\frac{p_r(I)}{p_r(U)}\right)^i\right]^{n-1}. \tag{5.7}$$

Smallest order statistics for the indeterminate part of NGD is given as:

$$f_{1:n}(x) = n(p_r(S))^n \sum_{z^*=th^*+1}^{x-1} \binom{x-1}{z^*} (p_r(I))^{z^*}$$

$$\times \sum_{x=1}^{\infty} \sum_{h=0}^{x-z^*-1} \binom{x-z^*-1}{h} (p_r(U))^{x-z^*-h-1}$$

$$\times \left[\sum_{m'=0}^{\infty} \binom{x+m'}{m'+1} (p_r(I))^{m'+1} \sum_{m'=0}^{\infty} \binom{x-1}{m'} (p_r(U))^{x-m'-1}\right]^{n-1}. \tag{5.8}$$

5.5. Median order statistics

For $u = m + 1,$ median order statistics for the true part of NGD is given as:

$$f_{m+1,n}(x) = \frac{(2m+1)!}{m!n!}(p_r(S))^{m+1}$$

$$\times\left[1 - p_r(S)(p_r(U))^x(1 - p_r(U))^{-1}\sum_{m'=0}^{\infty}\binom{x+m'}{m'}\left(\frac{p_r(I)}{p_r(U)}\right)^{m'}\right]^m$$

$$\times\left[(p_r(U))^x(1 - p_r(U))^{-1}\sum_{m'=0}^{\infty}\binom{x+m'}{m'}\left(\frac{p_r(I)}{p_r(U)}\right)^{m'}\right]^m$$

$$\times\sum_{h=0}^{th^*}\binom{x-1}{h}(p_r(I))^h(p_r(U))^{x-h-1}. \tag{5.9}$$

The median order statistic for the indeterminate part of NGD is given as:

$$f_{m+1,n}(x)$$

$$= \frac{(2m+1)!}{m!n!}(p_r(S))^{m+1}$$

$$\times\left[1 - p_r(S)\sum_{m'=0}^{\infty}\binom{x+m'}{m'+1}(P(I))^{j+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}\right]^m$$

$$\times\left[\sum_{j=0}^{\infty}\binom{x+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{x-1}{m'}(p_r(U))^{x-m'-1}\right]^m$$

$$\times\sum_{z^*=th^*+1}^{x-1}\binom{x-1}{z^*}(p_r(I))^{z^*}\sum_{x=1}^{\infty}\sum_{h=0}^{x-z^*-1}\binom{x-z^*-1}{h}(p_r(U))^{x-z^*-h-1}. \tag{5.10}$$

5.6. Minimum and maximum joint order statistics

Minimum and maximum joint order statistics for the true part of NGD are given as follows:

$$f_{1:a:a}(y)$$

$$= a(a-1)(p_r(S))^a(1-p_r(U))^{-1}$$

$$\times\left\{\sum_{h=0}^{th^*}\binom{y-1}{h}(p_r(I))^h(p_r(U))^{y-h-1}\right\}$$

$$\times\left\{\sum_{h=0}^{th^*}\binom{z-1}{h}(p_r(I))^h(p_r(U))^{z-h-1}\right\}$$

$$\times\left[(p_r(U))^y\sum_{i=0}^{\infty}\binom{y+i}{i}\left(\frac{p_r(I)}{p_r(U)}\right)^i-(p_r(U))^z\sum_{i=0}^{\infty}\binom{z+i}{i}\left(\frac{p_r(I)}{p_r(U)}\right)^i\right]^{a-2}. \tag{5.11}$$

Minimum and maximum joint order statistics for the indeterminate part of NGD are given as:

$$f_{1:a:a}(y)$$

$$= a(a-1)(p_r(S))^n$$

$$\times\left\{\sum_{z^*=th^*+1}^{y-1}\binom{y-1}{z^*}(p_r(I))^{z^*}\sum_{x=1}^{\infty}\sum_{h=0}^{y-z^*-1}\binom{y-z^*-1}{h}(p_r(U))^{y-z^*-h-1}\right\}$$

$$\times\left\{\sum_{z^*=th^*+1}^{z-1}\binom{z-1}{z^*}(p_r(I))^{z^*}\sum_{x=1}^{\infty}\sum_{h=0}^{z-z^*-1}\binom{z-z^*-1}{h}(p_r(U))^{z-z^*-h-1}\right\}$$

$$\times\left[\sum_{m'=0}^{\infty}\binom{y+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{y-1}{m'}(p_r(U))^{y-m'-1}\right.$$

$$\left.-\sum_{m'=0}^{\infty}\binom{z+m'}{m'+1}(p_r(I))^{m'+1}\sum_{m'=0}^{\infty}\binom{z-1}{m'}(p_r(U))^{z-m'-1}\right]^{a-2}. \tag{5.12}$$

6. Conclusion

This paper proposed one parameter discrete probability distribution named neutrosophic geometric distribution (NGD). One of the reasons for generalizing a classical geometric distribution to a neutrosophic geometric distribution is the property of dealing with the uncertain situation that classical geometric distribution fails to deal with, e.g., consider a coin toss problem where tossing a coin on an irregular surface in which each trial may result in success or failure. There may be an outcome we cannot examine whether its success or failure (almost pointing towards both); such outcome will be considered indeterminacy, which classical statistics exclude\ignored during the experiment. The proposed modified form of the geometric distribution is explained with the help of some case studies. Moreover, several distributional properties and characteristics were explored for the newly developed neutrosophic geometric distribution.

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