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A Generalized Neutrosophic Metric Space and Coupled Coincidence Point Results

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Abstract. This work introduces the notion of J-Neutrosophic Metric Space using the concept of Neutrosophic Sets. We analyze and extend some coupled coincidence point results on θ_{JN}-coupled and compatible mappings $\mathcal{G} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and $\mathfrak{h} : \mathcal{A} \to \mathcal{A}$ where $\mathcal{G}$ has the mixed $\mathfrak{h}$-monotone property. We support the proposed result with suitable example.

Keywords: J-Neutrosophic Metric Space; JN-Compatible; JN-Mixed $\mathfrak{h}$-Monotone Property; θ_{JN}-Coupled Mappings; JN-Coupled Coincidence Point

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1. Introduction

The classical set theory [3, 8] evolves through various extensions as it laid a foundation for modern mathematics. The especially notable extension is the concept of fuzzy sets [9, 22] which introduced graded identity in set theory. This significant feature of assigning graded membership polarized the researchers to come out with numerous analysis and applications over kinds of fuzzy metric spaces.

In 1983, Atanassov [1] made an excitement with the introduction of Intutionstic Fuzzy Sets by adding the idea of nonmembership grade to fuzzy set theory. Since then numerous work have been done to bring out new results and to extend existing concepts over intuitionistic fuzzy setting.

In the year 1995, Florentin Smarandache [17–19] introduced Neutrosophy, an extension of intuitionistic fuzzy set, which claims that between an idea and its opposite, there is a continuum-power spectrum of neutralities. As neutrosophy adds neutralities to intuitionistic fuzzy sets, it inspired the research community and the field is currently growing fruitfully with so many investigations, analysis, computing techniques and applications [4–7, 11, 13, 15, 20].

In the meanwhile, Mustafa and Sims [14] defined the following generalized metric space.

Definition 1.1. [14]Let $\mathcal{A}$ be a nonempty set. $G : \mathcal{A}^3 \to (-\infty, +\infty)$ is called a Generalized Metric (Shortly, G-metric) on $\mathcal{A}$ if for all $\mu, \rho, \upsilon, \gamma \in \mathcal{A}$,

(G1) $G(\mu, \mu, \rho) > 0$ if $\mu \neq \rho$,

(G2) $G(\mu, \rho, \upsilon) = 0$ if and only if $\mu = \rho = \upsilon$

(G3) $G(\mu, \mu, \rho) \leq G(\mu, \rho, \upsilon)$ if $\rho \neq \upsilon$

(G4) $G(\mu, \rho, \upsilon)$ is symmetry in all three variables.

(G5) $G(\mu, \rho, \upsilon) \leq G(\mu, \gamma, \gamma) + G(\gamma, \rho, \upsilon)$

The pair $(\mathcal{A}, G)$ is called generalized metric space.

This space was then used by Sun and Yang [21] to bring out the notion of generalized fuzzy metric space. Mohiuddine and Alotaibi [12] used it to introduce intuitionistic fuzzy metric space. As a consequence, numerous terms and definitions are introduced along with related results in these settings. Notable among them is the concept of common fixed point, coupled coincidence point and mixed $\mathfrak{h}$-monotone property that are given by Bhaskar and Lakshmikantham [2] and Lakshmikantham and Ciric [10].

Definition 1.2. [2]Let $\mathcal{A}$ be a set with partial order $\leq$. The mapping $\mathcal{G} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ is said to have the mixed monotone property if the following conditions hold.

(i) $\mu_1, \mu_2 \in \mathcal{A}$, $\mu_1 \leq \mu_2$ implies $\mathcal{G}(\mu_1, \rho) \leq \mathcal{G}(\mu_2, \rho)$ for all $\rho \in \mathcal{A}$;

(ii) $\rho_1, \rho_2 \in \mathcal{A}$, $\rho_1 \leq \rho_2$ implies $\mathcal{G}(\mu, \rho_1) \leq \mathcal{G}(\mu, \rho_2)$ for all $\mu \in \mathcal{A}$.

Definition 1.3. [2]$(\mu, \rho) \in \mathcal{A} \times \mathcal{A}$ is a coupled fixed point of $\mathcal{G} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ if $\mathcal{G}(\mu, \rho) = \mu$ and $\mathcal{G}(\rho, \mu) = \rho$.

Definition 1.4. [10]Let $\mathcal{A}$ be a set with partial order $\leq$. The mappings $\mathcal{G} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ and $\mathfrak{h} : \mathcal{A} \to \mathcal{A}$ have mixed $\mathfrak{h}$-monotone property if the following conditions hold.

(i) $\mu_1, \mu_2 \in \mathcal{A}$, $\mathfrak{h}(\mu_1) \leq \mathfrak{h}(\mu_2)$ implies $\mathcal{G}(\mu_1, \rho) \leq \mathcal{G}(\mu_2, \rho)$ for all $\rho \in \mathcal{A}$;

(ii) $\rho_1, \rho_2 \in \mathcal{A}$, $\mathfrak{h}(\rho_1) \leq \mathfrak{h}(\rho_2)$ implies $\mathcal{G}(\mu, \rho_1) \leq \mathcal{G}(\mu, \rho_2)$ for all $\mu \in \mathcal{A}$.

Definition 1.5. [10]$(\mu, \rho) \in \mathcal{A} \times \mathcal{A}$ is a coupled coincidence point of $\mathcal{G} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ and $\mathfrak{h} : \mathcal{A} \to \mathcal{A}$ if $\mathcal{G}(\mu, \rho) = \mathfrak{h}(\mu)$ and $\mathcal{G}(\rho, \mu) = \mathfrak{h}(\rho)$.

In this scenario, we present here the notion of J-Neutrosophy Metric Space. We propose coincidence point results for compatible, coupled mappings that are having a kind of mixed monotone property in the newly defined space with a partial order.

2. J-Neutrosophic Metric Space

Let us start with the definitions of following binary operations which will be the main frame in defining the J-Neutrosophic Metric Space.

Definition 2.1. [16] A binary operation $\odot : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm (Shortly, CTN) if

(i) $\odot$ is commutative, associative and continuous,

(ii) $t \odot 1 = t$ for all $t \in [0,1]$

(iii) $t \odot s \leq u \odot v$ whenever $t \leq u$ and $s \leq v$, and $s, t, u, v \in [0,1]$.

Definition 2.2. [16] A binary operation $\oslash : [0,1] \times [0,1]$ is a continuous t-conorm (Shortly, CTCN) if

(i) $\oslash$ is commutative, associative and continuous

(ii) $t \oslash 0 = t$ for all $t \in [0,1]$

(iii) $t \oslash s \leq u \oslash v$ whenever $t \leq u$ and $s \leq v$, and $s, t, u, v \in [0,1]$.

The following definition defines the new space, namely, J-Neutrosophic Metric Space.

Definition 2.3. Consider a nonempty set $\mathcal{A}$, a CTN $\odot$, two CTCNs $\oslash$, $\otimes$ and fuzzy sets J, S, F on $\mathcal{A}^3 \times [0,1]$. A 7-tuple $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ is called a J-Neutrosophic Metric Space (Shortly, JNMS) if for each $\mu, \rho, \upsilon, \gamma \in \mathcal{A}$ and $t > 0$,

(JN1) $J(\mu, \rho, \upsilon, t) + S(\mu, \rho, \upsilon, t) + F(\mu, \rho, \mu, t) \leq 3$,

(JN2) $J(\mu, \rho, \upsilon, t) > 0$,

(JN3) $J(\mu, \rho, \upsilon, t)$ is symmetry in μ, ρ and υ,

(JN4) $J(\mu, \mu, \rho, t) \geq J(\mu, \rho, \upsilon, t)$ if $\rho \neq \upsilon$,

(JN5) $J(\mu, \rho, \upsilon, t) = 1$ if and only if $\mu = \rho = \upsilon$,

(JN6) $J(\mu, \rho, \upsilon, t + s) \geq J(\mu, \gamma, \gamma, t) \odot J(\gamma, \rho, \upsilon, s)$,

(JN7) $J(\mu, \rho, \upsilon, t)$ is continuous with respect to t,

(JN8) J is nondecreasing on $[0, +\infty]$,
$\lim_{\mathrm{q}\to+\infty} J(\mu, \rho, \upsilon) = 1$, $\lim_{\mathrm{q}\to 0} J(\mu, \rho, \upsilon) = 0$,

(JN9) $S(\mu, \rho, \upsilon, t) < 1$,

(JN10) $S(\mu, \rho, \upsilon, t)$ is symmetry in μ, ρ and υ,

(JN11) $S(\mu, \mu, \rho, t) \leq S(\mu, \rho, \upsilon, t)$ if $\rho \neq \upsilon$,

(JN12) $S(\mu, \rho, \upsilon, t) = 0$ if and only if $\mu = \rho = \upsilon$,

(JN13) $S(\mu, \rho, \upsilon, t + s) \leq S(\mu, \gamma, \gamma, t) \oslash S(\gamma, \rho, \upsilon, s)$, (JN14) $S(\mu,\rho,\upsilon,t)$ is continuous with respect to t,

(JN15) S is nonincreasing on $[0,+\infty]$

$\lim_{q\to+\infty} S(\mu,\rho,\upsilon)=0$, $\lim_{q\to 0} S(\mu,\rho,\upsilon)=1$

(JN16) $F(\mu,\rho,\upsilon,t)<1$,

(JN17) $F(\mu,\rho,\upsilon,t)$ is symmetry in μ,ρ and υ,

(JN18) $F(\mu,\mu,\rho,t)\leq F(\mu,\rho,\upsilon,t)$ if $\rho\neq\upsilon$,

(JN19) $F(\mu,\rho,\upsilon,t)=0$ if and only if $\mu=\rho=\upsilon$,

(JN20) $F(\mu,\rho,\upsilon,t+s)\leq F(\mu,\gamma,\gamma,t)\otimes F(\gamma,\rho,\upsilon,s)$,

(JN21) $F(\mu,\rho,\upsilon,t)$ is continuous with respect to t.

(JN22) F is nonincreasing on $[0,+\infty]$,

$\lim_{q\to+\infty} F(\mu,\rho,\upsilon)=0$, $\lim_{q\to 0} F(\mu,\rho,\upsilon)=1$.

The triplet (J,S,F) is called J-Neutrosophic Metric on $\mathcal{A}$.

Remark 2.4. $J(\mu,\rho,\upsilon,t)$, $S(\mu,\rho,\upsilon,t)$ and $F(\mu,\rho,\upsilon,t)$ represent, respectively, the degree of nearness, the degree of non-nearness and the degree of indeterminacy between μ,ρ and υ with respect to t.

Example 2.5. Let $(\mathcal{A},G)$ be a generalized metric space. Define the fuzzy sets J,S,F by

$$J(\mu,\rho,\upsilon,t)=\frac{t}{t+G(\mu,\rho,\upsilon)},$$
$$S(\mu,\rho,\upsilon,t)=\frac{G(\mu,\rho,\upsilon)}{t+G(\mu,\rho,\upsilon)}\ and$$
$$F(\mu,\rho,\upsilon,t)=\frac{G(\mu,\rho,\upsilon)}{t}\text{ for all }\mu,\rho,\upsilon\in\mathcal{A}$$

Define $\odot$, $\oslash$ and $\otimes$ by $a\odot b=ab$, $a\oslash b=min\{a+b,1\}$ and $a\otimes b=max\{a,b\}$. Then $(\mathcal{A},J,S,F,\odot,\oslash,\otimes)$ is a JNMS.

Definition 2.6. Consider a JNMS $(\mathcal{A},J,S,F,\odot,\oslash,\otimes)$. Let $\mu\in\mathcal{A}$, $r\in(0,1)$ and $t>0$. The JN-open ball $B(\mu,r,t)$ with centre at μ and radius r is defined by

$$B(\mu,r,t)=\{\rho\in\mathcal{A}:J(\mu,\mu,\rho)>1-r,\ S(\mu,\mu,\rho)<r,\ F(\mu,\mu,\rho)<r\}.$$

Remark 2.7. The above definition leads to the following facts.

(1) Every JN-open ball is an open set.

(2) Every JNMS is Hausdorff.

It follows from the above remark that the collection $\{B(\mu,r,t):\mu\in\mathcal{A},\ r\in(0,1),\ t>0\}$ forms a base for the JN-metric topology on $\mathcal{A}$ and this topology coincides with the metric topology arising from the generalized metric.

Definition 2.8. A sequence $\{\mu_q\}$ in a JNMS $(\mathcal{A},J,S,F,\odot,\oslash,\otimes)$ is JN-Convergent to x if it converges to x in the JN-metric topology.

Remark 2.9. If $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ is a JNMS, $\{\mu_q\}$ is a sequence in $\mathcal{A}$ and $\mu \in \mathcal{A}$, then the following are equivalent:

(1) $\{\mu_q\}$ is JN-convergent to μ.
(2) $d_G(\mu_q, \mu) \longrightarrow 0$ as $n \longrightarrow +\infty$.
(3) $J(\mu_q, \mu_q, \mu, t) \longrightarrow 1$, $S(\mu_q, \mu_q, \mu, t) \longrightarrow 0$, $F(\mu_q, \mu_q, \mu, t) \longrightarrow 0$ as $n \longrightarrow +\infty$.
(4) $J(\mu_q, \mu, \mu, t) \longrightarrow 1$, $S(\mu_q, \mu, \mu, t) \longrightarrow 0$, $F(\mu_q, \mu, \mu, t) \longrightarrow 0$ as $n \longrightarrow +\infty$.
(5) $J(\mu_q, \mu_m, \mu, t) \longrightarrow 1$, $S(\mu_q, \mu_m, \mu, t) \longrightarrow 0$, $F(\mu_q, \mu_m, \mu, t) \longrightarrow 0$ as $n \longrightarrow +\infty$.

Definition 2.10. A sequence $\{\mu_q\}$ in a JNMS $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ is JN-Cauchy if for every $\epsilon > 0$ and $t > 0$ there exists $N \in \mathbb{N}$ such that $J(\mu_q, \mu_m, \mu_l, t) > 1 - \epsilon$, $S(\mu_q, \mu_m, \mu_l, t) < \epsilon$, $F(\mu_q, \mu_m, \mu_l, t) < \epsilon$ for all $n, m, l \in \mathbb{N}$. A JNMS is said to be JN-Complete if every JN-Cauchy sequence is JN-convergent.

Definition 2.11. A JNMS $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ is said to be regular if the following conditions hold:

(i) If a nondecreasing sequence μ_q in $\mathcal{A}$ JN-converges to μ, then $\mu_q \leq \mu$ for all n.
(ii) If a nonincreasing sequence μ_q in $\mathcal{A}$ JN-converges to μ, then $\mu_q \geq \mu$ for all n.

Definition 2.12. Let $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ be a JNMS. The mappings $\mathcal{G} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ and $\mathfrak{h} : \mathcal{A} \longrightarrow \mathcal{A}$ are said to be JN-Compatible if for all $t > 0$,

$$\lim_{q\to\infty} J(\mathfrak{h}\mathcal{G}(\mu_q, \rho_q), \mathfrak{h}\mathcal{G}(\mu_q, \rho_q), \mathcal{G}(\mathfrak{h}x_q, \mathfrak{h}\rho_q), t) = 1,$$
$$\lim_{q\to\infty} S(\mathfrak{h}\mathcal{G}(\mu_q, \rho_q), \mathfrak{h}\mathcal{G}(\mu_q, \rho_q), \mathcal{G}(\mathfrak{h}x_q, \mathfrak{h}\rho_q), t) = 0,$$
$$\lim_{q\to\infty} F(\mathfrak{h}\mathcal{G}(\mu_q, \rho_q), \mathfrak{h}\mathcal{G}(\mu_q, \rho_q), \mathcal{G}(\mathfrak{h}x_q, \mathfrak{h}\rho_q), t) = 0,$$
$$\lim_{q\to\infty} J(\mathfrak{h}\mathcal{G}(\rho_q, \mu_q), \mathfrak{h}\mathcal{G}(\rho_q, \mu_q), \mathcal{G}(\mathfrak{h}y_q, \mathfrak{h}\mu_q), t) = 1,$$
$$\lim_{q\to\infty} S(\mathfrak{h}\mathcal{G}(\rho_q, \mu_q), \mathfrak{h}\mathcal{G}(\rho_q, \mu_q), \mathcal{G}(\mathfrak{h}y_q, \mathfrak{h}\mu_q), t) = 0 \text{ and}$$
$$\lim_{q\to+\infty} F(\mathfrak{h}\mathcal{G}(\rho_q, \mu_q), \mathfrak{h}\mathcal{G}(\rho_q, \mu_q), \mathcal{G}(\mathfrak{h}y_q, \mathfrak{h}\mu_q), t) = 0,$$

where $\{\mu_q\}$ and $\{\rho_q\}$ are sequences in $\mathcal{A}$ such that $\lim_{q\to+\infty} \mathcal{G}(\mu_q, \rho_q) = \lim_{q\to+\infty} \mathfrak{h}\mu_q = \mu$ and $\lim_{q\to+\infty} \mathcal{G}(\rho_q, \mu_q) = \lim_{q\to+\infty} \mathfrak{h}\rho_q = \rho$ for some $\mu, \rho \in \mathcal{A}$.

To continue the work, we need to define the family Θ of strictly increasing, upper semi-continuous functions $\theta : [0, +\infty) \to [0, \infty)$ in which $\theta(0) = \{0\}, \theta(t) < t$ and $\sum_{q=1}^{+\infty} \theta^n(t) < +\infty$ for all $t > 0$.

Definition 2.13. Let $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ be a JNMS. The mapping $\mathcal{G} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ is said to be self $\theta_{JN}-$coupled if there exists $\theta \in \Theta$ such that

$$J(\mathcal{G}(\mu,\rho), \mathcal{G}(\mu,\rho), \mathcal{G}(\gamma,\sigma), \theta(t)) \geq \Big\{ J(\mu,\mu,\gamma,t) \odot J(\mu,\mu,\mathcal{G}(\mu,\rho),t) \odot J(\gamma,\gamma,\mathcal{G}(\gamma,\sigma),t) \Big\},$$

$$S(\mathcal{G}(\mu,\rho), \mathcal{G}(\mu,\rho), \mathcal{G}(\gamma,\sigma), \theta(t)) \leq \Big\{ S(\mu,\mu,\gamma,t) \oslash S(\mu,\mu,\mathcal{G}(\mu,\rho),t) \oslash S(\gamma,\gamma,\mathcal{G}(\gamma,\sigma),t),$$

$$F(\mathcal{G}(\mu,\rho), \mathcal{G}(\mu,\rho), \mathcal{G}(\gamma,\sigma), \theta(t)) \leq \Big\{ F(\mu,\mu,\gamma,t) \otimes F(\mu,\mu,\mathcal{G}(\mu,\rho),t) \otimes F(\gamma,\gamma,\mathcal{G}(\gamma,\sigma),t) \Big\},$$

for all $\mu, \rho, \gamma, \sigma \in \mathcal{A}$ and $t > 0$ with $\mu \leq \gamma,\ \rho \geq \sigma$ or $\mu \geq \gamma,\ \rho \leq \sigma$.

Definition 2.14. Let $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ be a JNMS. The mappings $\mathcal{G} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ and $\mathfrak{h} : \mathcal{A} \longrightarrow \mathcal{A}$ are said to be $\theta_{JN}-$coupled if there exists $\theta \in \Theta$ such that

$$J(\mathcal{G}(\mu,\rho), \mathcal{G}(\mu,\rho), \mathcal{G}(\gamma,\sigma), \theta(t)) \geq \Big\{ J(\mathfrak{h}(\mu), \mathfrak{h}(\mu), u, t) \odot J(\mathfrak{h}(\mu), \mathfrak{h}(\mu), F(\mu,\rho), t) \odot J(\mathfrak{h}(\gamma), \mathfrak{h}(\gamma), F(\gamma,\sigma), t) \Big\},$$

$$S(\mathcal{G}(\mu,\rho), \mathcal{G}(\mu,\rho), \mathcal{G}(\gamma,\sigma), \theta(t)) \leq \Big\{ S(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \gamma, t) \oslash S(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu,\rho), t) \otimes S(\mathfrak{h}(\gamma), \mathfrak{h}(\gamma), \mathcal{G}(\gamma,\sigma), t) \Big\},$$

$$F(\mathcal{G}(\mu,\rho), \mathcal{G}(\mu,\rho), \mathcal{G}(\gamma,\sigma), \theta(t)) \leq \Big\{ F(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \gamma, t) \otimes F(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu,\rho), t) \oslash F(\mathfrak{h}(\gamma), \mathfrak{h}(\gamma), \mathcal{G}(\gamma,\sigma), t) \Big\},$$

for all $\mu, \rho, \gamma, \sigma \in \mathcal{A}$ and $t > 0$ with $\mathfrak{h}(\mu) \leq \mathfrak{h}(\gamma),\ \mathfrak{h}(\rho) \geq \mathfrak{h}(\sigma)$ or $\mathfrak{h}(\mu) \geq \mathfrak{h}(\gamma),\ \mathfrak{h}(\rho) \leq \mathfrak{h}(\sigma)$.

Lemma 2.15. *Let $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ be a JNMS and*

$$\Lambda_\kappa(\mu,\rho,\upsilon) = inf\{t > 0 : J(\mu,\rho,\upsilon,t) > 1-\kappa, S(\mu,\rho,\upsilon,t) < \kappa, F(\mu,\rho,\upsilon,t) < \kappa\}$$

for all $\mu, \rho, \upsilon \in \mathcal{A}$, $\kappa \in$(0,1] and $t > 0$. Then for each $\kappa \in$(0,1], there exists $\mu \in$(0,1] such that $\Lambda_\kappa(\mu_1, \mu_1, \mu_{\mathrm{q}}) \leq \sum_{\mathrm{q}=1}^{n-1} \Lambda_\mu(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu_{\mathrm{q}+1})$

Proof. For $\kappa \in$(0,1], choose $\mu \in$(0,1] such that $\odot^{(n-1)}(1-\mu) > 1-\kappa$, $\oslash^{(n-1)}\mu < \kappa$, $\otimes^{(n-1)}\mu < \kappa$. Let $\epsilon > 0$. Then

$$\begin{aligned}
&J(\mu_1, \mu_1, \mu_{\mathrm{q}}, \Lambda_\mu(\mu_{\mathrm{q}}, \mu_{\mathrm{q}}, \mu_{\mathrm{q}+1}) + (n-1)\epsilon) \\
&\geq J(\mu_1, \mu_1, \mu_2, \Lambda_\mu(\mu_1, \mu_1, \mu_2) + \epsilon) \odot \cdots \odot J(\mu_{\mathrm{q}-1}, \mu_{\mathrm{q}-1}, \mu_{\mathrm{q}}, \Lambda_\mu(\mu_{\mathrm{q}-1}, \mu_{\mathrm{q}-1}, \mu_{\mathrm{q}}) + \epsilon) \\
&> \odot^{(n-1)}(1-\mu) \\
&> 1-\kappa
\end{aligned}$$

$$\begin{aligned}
&S(\mu_1, \mu_1, \mu_{\rm q}, \Lambda_\mu(\mu_{\rm q}, \mu_{\rm q}, \mu_{\rm q+1}) + (n-1)\epsilon) \\
&\leq S(\mu_1, \mu_1, \mu_2, \Lambda_\mu(\mu_1, \mu_1, \mu_2) + \epsilon) \oslash \cdots \oslash S(\mu_{\rm q-1}, \mu_{\rm q-1}, \mu_{\rm q}, \Lambda_\mu(\mu_{\rm q-1}, \mu_{\rm q-1}, \mu_{\rm q}) + \epsilon) \\
&< \oslash^{(n-1)}\mu \\
&< \kappa
\end{aligned}$$

$$\begin{aligned}
&F(\mu_1, \mu_1, \mu_{\rm q}, \Lambda_\mu(\mu_{\rm q}, \mu_{\rm q}, \mu_{\rm q+1}) + (n-1)\epsilon) \\
&\leq S(\mu_1, \mu_1, \mu_2, \Lambda_\mu(\mu_1, \mu_1, \mu_2) + \epsilon) \otimes \cdots \otimes S(\mu_{\rm q-1}, \mu_{\rm q-1}, \mu_{\rm q}, \Lambda_\mu(\mu_{\rm q-1}, \mu_{\rm q-1}, \mu_{\rm q}) + \epsilon) \\
&< \otimes^{(n-1)}\mu \\
&< \kappa
\end{aligned}$$

Therefore, we have $\Lambda_\mu(\mu_1, \mu_1, \mu_{\rm q}) \leq \Lambda_\mu(\mu_1, \mu_1, \mu_2) + \cdots + \Lambda_\mu(\mu_{\rm q-1}, \mu_{\rm q-1}, \mu_{\rm q}) + (n-1)\epsilon$. As $\epsilon > 0$ is arbitrary, $\Lambda_\mu(\mu_1, \mu_1, \mu_{\rm q}) \leq \Lambda_\mu(\mu_1, \mu_1, \mu_2) + \cdots + \Lambda_\mu(\mu_{\rm q-1}, \mu_{\rm q-1}, \mu_{\rm q})$. □

Lemma 2.16. *A sequence $\{\mu_{\rm q}\}$ in a JNMS $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ is JN-Cauchy if for some $\theta \in \Theta$,*

$$\begin{aligned}
J(\mu_{\rm q}, \mu_{\rm q}, \mu_{\rm q+1}, \theta(t)) &\geq J(\mu_{\rm q-1}, \mu_{\rm q-1}, \mu_{\rm q}, t) \odot J(\mu_{\rm q}, \mu_{\rm q}, \mu_{\rm q+1}, t), \\
S(\mu_{\rm q}, \mu_{\rm q}, \mu_{\rm q+1}, \theta(t)) &\leq S(\mu_{\rm q-1}, \mu_{\rm q-1}, \mu_{\rm q}, t) \oslash S(\mu_{\rm q}, \mu_{\rm q}, \mu_{\rm q+1}, t), \\
F(\mu_{\rm q}, \mu_{\rm q}, \mu_{\rm q+1}, \theta(t)) &\leq F(\mu_{\rm q-1}, \mu_{\rm q-1}, \mu_{\rm q}, t) \otimes F(\mu_{\rm q}, \mu_{\rm q}, \mu_{\rm q+1}, t),
\end{aligned}$$

for all $t > 0$.

Proof. Denote $\Lambda_\mu(\mu_{\rm q-1}, \mu_{\rm q-1}, \mu_{\rm q})$ by $a_{\rm q}$. For given $\epsilon > 0$ and each $a_{\rm q}$, we can find $\mathsf{m}_{\rm q} > a_{\rm q}$ such that $\theta(\mathsf{m}_{\rm q}) < \theta(a_{\rm q}) + \epsilon$. Now,

$$\begin{aligned}
J(\mu_{\rm q}, \mu_{\rm q}, \mu_{\rm q+1}, \mathsf{m}_{\rm q}) &> 1 - \kappa, \\
S(\mu_{\rm q}, \mu_{\rm q}, \mu_{\rm q+1}, \mathsf{m}_{\rm q}) &< \kappa, \\
F(\mu_{\rm q}, \mu_{\rm q}, \mu_{\rm q+1}, \mathsf{m}_{\rm q}) &< \kappa.
\end{aligned}$$

Take $M_{\rm q} = max\{\mathsf{m}_{\rm q}, \mathsf{m}_{\rm q+1}\}$, then

$$\begin{aligned}
J(\mu_{\rm q}, \mu_{\rm q}, \mu_{\rm q+1}, \theta(M_{\rm q})) &\geq J(\mu_{\rm q-1}, \mu_{\rm q-1}, \mu_{\rm q}, M_{\rm q}) \odot J(\mu_{\rm q}, \mu_{\rm q}, \mu_{\rm q+1}, M_{\rm q}) \\
&\geq J(\mu_{\rm q-1}, \mu_{\rm q-1}, \mu_{\rm q}, \mathsf{m}_{\rm q}) \odot J(\mu_{\rm q}, \mu_{\rm q}, \mu_{\rm q+1}, \mathsf{m}_{\rm q+1}) \\
&> 1 - \kappa.
\end{aligned}$$

$$\begin{aligned}
S(\mu_{\rm q}, \mu_{\rm q}, \mu_{\rm q+1}, \theta(M_{\rm q})) &\leq S(\mu_{\rm q-1}, \mu_{\rm q-1}, \mu_{\rm q}, M_{\rm q}) \oslash S(\mu_{\rm q}, \mu_{\rm q}, \mu_{\rm q+1}, M_{\rm q}) \\
&\leq S(\mu_{\rm q-1}, \mu_{\rm q-1}, \mu_{\rm q}, \mathsf{m}_{\rm q}) \oslash S(\mu_{\rm q}, \mu_{\rm q}, \mu_{\rm q+1}, \mathsf{m}_{\rm q+1}) \\
&< \kappa.
\end{aligned}$$

$$\begin{aligned}F(\mu_{\mathrm{q}},\mu_{\mathrm{q}},\mu_{\mathrm{q}+1},\theta(M_{\mathrm{q}})) &\leq F(\mu_{\mathrm{q}-1},\mu_{\mathrm{q}-1},\mu_{\mathrm{q}},M_{\mathrm{q}}\otimes F(\mu_{\mathrm{q}},\mu_{\mathrm{q}},\mu_{\mathrm{q}+1},M_{\mathrm{q}})\\ &\leq F(\mu_{\mathrm{q}-1},\mu_{\mathrm{q}-1},\mu_{\mathrm{q}},\mathsf{m}_{\mathrm{q}})\otimes F(\mu_{\mathrm{q}},\mu_{\mathrm{q}},\mu_{\mathrm{q}+1},\mathsf{m}_{\mathrm{q}+1})\\ &< \kappa.\end{aligned}$$

From Lemma 2.15,

$$\Lambda_{\kappa}(\mu_{\mathrm{q}},\mu_{\mathrm{q}},\mu_{\mathrm{q}+1}) \leq \theta(M_{\mathrm{q}}) = max\{\theta(\mathsf{m}_{\mathrm{q}}),\theta(\mathsf{m}_{\mathrm{q}+1})\} \leq max\{\theta(a_{\mathrm{q}}),\theta(a_{\mathrm{q}+1})\}+\epsilon.$$

From the choice of ϵ, $a_{\mathrm{q}+1} \leq max\{\theta(a_{\mathrm{q}}),\theta(a_{\mathrm{q}+1})\}$. Hence, for $\epsilon > 0$, we can find n_0 for which $J(\mu_{\mathrm{q}},\mu_{\mathrm{q}},\mu_m,\epsilon) > 1-\kappa$, $S(\mu_{\mathrm{q}},\mu_{\mathrm{q}},\mu_m,\epsilon) < \kappa$ and $F(\mu_{\mathrm{q}},\mu_{\mathrm{q}},\mu_m,\epsilon) < \kappa$ for all $n,m \geq n_0$. Therefore $\{\mu_{\mathrm{q}}\}$ is a JN-Cauchy sequence. □

3. **Main Results**

The following theorem exhibits the existence of coupled coincidence point of two continuous, compatible functions $\mathcal{G}:\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ and $\mathfrak{h}:\mathcal{A}\to\mathcal{A}$ where $\mathcal{G}$ has the mixed $\mathfrak{h}$-monotone property.

Theorem 3.1. *Consider a complete JNMS $(\mathcal{A},J,S,F,\odot,\oslash,\otimes)$ where $\mathcal{A}$ is a partially ordered set. Consider the mappings $\mathcal{G}:\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ and $\mathfrak{h}:\mathcal{A}\to\mathcal{A}$ where*

(a) *$\mathcal{G}(\mathcal{A}\times\mathcal{A})\subseteq\mathfrak{h}(\mathcal{A})$,*
(b) *$\mathcal{G}$ and $\mathfrak{h}$ are continuous,*
(c) *$\mathcal{G}$ and $\mathfrak{h}$ are compatible,*
(d) *$\mathcal{G}$ has mixed $\mathfrak{h}$-monotone property,*
(e) *$\mathcal{G}$ and $\mathfrak{h}$ are $\theta_{JN}-$coupled for some $\theta\in\Theta$.*

If there exist $\mu_0,\rho_0\in\mathcal{A}$ for which $\mathfrak{h}(\mu_0)\leq\mathcal{G}(\mu_0,\rho_0)$ and $\mathfrak{h}(\rho_0)\geq\mathcal{G}(\rho_0,\mu_0)$, then $\mathcal{G}$ and $\mathfrak{h}$ have a coupled coincidence point.

Proof. Define sequences $\{\mu_{\mathrm{q}}\}$ and $\{\rho_{\mathrm{q}}\}$ by $\mathfrak{h}(\mu_{\mathrm{q}+1})=\mathcal{G}(\mu_{\mathrm{q}},\rho_{\mathrm{q}})$ and $\mathfrak{h}(\rho_{\mathrm{q}+1})=\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}})$, $n\geq 0$. Let us prove by induction that

$$\mathfrak{h}(\mu_{\mathrm{q}})\leq\mathfrak{h}(\mu_{\mathrm{q}+1}),\ \mathfrak{h}(\rho_{\mathrm{q}})\geq\mathfrak{h}(\rho_{\mathrm{q}+1}) \text{ for all } n\geq 0. \tag{1}$$

The choice of μ_0,ρ_0 gives that $\mathfrak{h}(\mu_0)\leq\mathfrak{h}(\mu_1)$, $\mathfrak{h}(\rho_0)\geq\mathfrak{h}(\rho_1)$. Suppose (1) is true for $n=m$. Then by the mixed $\mathfrak{h}$-monotone property of $\mathcal{G}$, we have that

$$\mathfrak{h}(\mu_{\mathrm{q}+1})=\mathcal{G}(\mu_{\mathrm{q}},\rho_{\mathrm{q}})\leq\mathcal{G}(\mu_{\mathrm{q}+1},\rho_{\mathrm{q}}),\ \mathfrak{h}(\rho_{\mathrm{q}+1})=\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}})\geq\mathcal{G}(\rho_{\mathrm{q}+1},\mu_{\mathrm{q}})$$

which gives that

$$\mathfrak{h}(\mu_{\mathrm{q}+2})=\mathcal{G}(\mu_{\mathrm{q}+1},\rho_{\mathrm{q}+1})\geq\mathcal{G}(\mu_{\mathrm{q}+1},\rho_{\mathrm{q}}),\ \mathfrak{h}(\rho_{\mathrm{q}+2})=\mathcal{G}(\rho_{\mathrm{q}+1},\mu_{\mathrm{q}+1})\leq\mathcal{G}(\rho_{\mathrm{q}+1},\mu_{\mathrm{q}}).$$

Hence $\mathfrak{h}(\mu_{q+1}) \leq \mathfrak{h}(\mu_{q+2})$, $\mathfrak{h}(\rho_{q+1}) \geq \mathfrak{h}(\rho_{q+2})$ and 1 follows.

If $\mu = \mu_{q-1}, \rho = \rho_{q-1}, \gamma = \mu_q, \sigma = \rho_q$, then from (e) and (1), we have that

$$J(\mathcal{G}(\mu_{q-1},\rho_{q-1}),\mathcal{G}(\mu_{q-1},\rho_{q-1}),\mathcal{G}(\mu_q,\rho_q),\theta(t)) \geq \Big\{J(\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_q),t) \odot J(\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_{q-1}),\mathcal{G}(\mu_{q-1},\rho_{q-1}),t) \odot J(\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_q),\mathcal{G}(\mu_q,\rho_q),t)\Big\},$$

$$S(\mathcal{G}(\mu_{q-1},\rho_{q-1}),\mathcal{G}(\mu_{q-1},\rho_{q-1}),\mathcal{G}(\mu_q,\rho_q),\theta(t)) \leq \Big\{S(\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_q),t) \oslash S(\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_{q-1}),\mathcal{G}(\mu_{q-1},\rho_{q-1}),t) \oslash S(\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_q),\mathcal{G}(\mu_q,\rho_q),t)\Big\},$$

$$F(\mathcal{G}(\mu_{q-1},\rho_{q-1}),\mathcal{G}(\mu_{q-1},\rho_{q-1}),\mathcal{G}(\mu_q,\rho_q),\theta(t)) \leq \Big\{J(\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_q),t) \otimes F(\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_{q-1}),F(\mu_{q-1},\rho_{q-1}),t) \otimes F(\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_q),\mathcal{G}(\mu_q,\rho_q),t)\Big\}.$$

These inequalities imply that

$$\begin{aligned}J(\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_{q+1}),\theta(t)) &\geq \Big\{J(\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_q),t) \odot J(\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_q),t)\\ &\odot J(\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_{q+1}),t)\Big\}\\ &= J(\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_q),t) \odot J(\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_{q+1}),t).\end{aligned}$$

$$\begin{aligned}S(\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_{q+1}),\theta(t)) &\leq \Big\{S(\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_q),t) \oslash S(\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_q),t)\\ &\oslash S(\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_{q+1}),t)\Big\}\\ &= S(\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_q),t) \oslash S(\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_{q+1}),t).\end{aligned}$$

$$\begin{aligned}F(\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_{q+1}),\theta(t)) &\leq \Big\{F(\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_q),t) \otimes F(\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_q),t)\\ &\otimes F(\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_{q+1}),t)\Big\}\\ &= F(\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_{q-1}),\mathfrak{h}(\mu_q),t) \otimes F(\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_q),\mathfrak{h}(\mu_{q+1}),t).\end{aligned}$$

By lemma 2.16, $\{\mathfrak{h}(\mu_q)\}$ is JN-Cauchy.

If $x = \rho_q, y = \mu_q, u = \rho_{q-1}, v = \mu_{q-1}$, (1) gives that

$$\begin{aligned}J(\mathcal{G}(\rho_{q-1},\mu_{q-1}),\mathcal{G}(\rho_{q-1},\mu_{q-1}),\mathcal{G}(\rho_q,\mu_q),\theta(t)) \geq \Big\{&J(\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_q),t)\\ &\odot J(\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_{q-1}),\mathcal{G}(\rho_{q-1},\mu_{q-1}),t)\\ &\odot J(\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_q),\mathcal{G}(\rho_q,\mu_q),t)\Big\}.\end{aligned}$$

$$\begin{aligned}S(\mathcal{G}(\rho_{q-1},\mu_{q-1}),\mathcal{G}(\rho_{q-1},\mu_{q-1}),\mathcal{G}(\rho_q,\mu_q),\theta(t)) \leq \Big\{&S(\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_q),t)\\ &\oslash S(\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_{q-1}),\mathcal{G}(\rho_{q-1},\mu_{q-1}),t)\\ &\oslash S(\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_q),\mathcal{G}(\rho_q,\mu_q),t)\Big\}.\end{aligned}$$

$$F(\mathcal{G}(\rho_{q-1},\mu_{q-1}),\mathcal{G}(\rho_{q-1},\mu_{q-1}),\mathcal{G}(\rho_q,\mu_q),\theta(t)) \leq \Big\{F(\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_q),t) \otimes F(\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_{q-1}),\mathcal{G}(\rho_{q-1},\mu_{q-1}),t) \otimes F(\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_q),\mathcal{G}(\rho_q,\mu_q),t)\Big\}.$$

These inequalities give that

$$J(\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_{q+1}),\theta(t)) \geq \Big\{J(\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_q),t) \odot J(\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_q),t) \odot J(\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_{q+1}),t)\Big\}$$
$$= J(\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_q),t) \odot J(\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_{q+1}),t).$$

$$S(\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_{q+1}),\theta(t)) \leq \Big\{S(\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_q),t) \oslash S(\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_q),t) \oslash S(\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_{q+1}),t)\Big\}$$
$$= S(\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_q),t) \oslash S(\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_{q+1}),t).$$

$$F(\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_{q+1}),\theta(t)) \leq \Big\{F(\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_q),t) \otimes F(\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_q),t) \otimes F(\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_{q+1}),t)\Big\}$$
$$= F(\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_{q-1}),\mathfrak{h}(\rho_q),t) \otimes F(\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_q),\mathfrak{h}(\rho_{q+1}),t).$$

By lemma 2.16 $\{\mathfrak{h}(\mu_q)\}$ is JN-Cauchy.
The completeness of $\mathcal{A}$ gives $\mu,\rho \in \mathcal{A}$ such that

$$\lim_{q\to+\infty}\mathcal{G}(\mu_q,\rho_q) = \lim_{q\to+\infty}\mathfrak{h}(\mu_q) = \mu,\ \lim_{q\to+\infty}\mathcal{G}(\rho_q,\mu_q) = \lim_{q\to+\infty}\mathfrak{h}(\rho_q) = \rho.$$

$\mathcal{G}$ and $\mathfrak{h}$ are JN-compatible. Hence, for all $t > 0$, we have that

$$\lim_{q\to+\infty} J(\mathfrak{h}(\mathcal{G}(\mu_q,\rho_q)),\mathfrak{h}(\mathcal{G}(\mu_q,\rho_q)),\mathcal{G}(\mathfrak{h}(\mu_q),\mathfrak{h}(\rho_q)),t) = 1,$$
$$\lim_{q\to+\infty} S(\mathfrak{h}(\mathcal{G}(\mu_q,\rho_q)),\mathfrak{h}(\mathcal{G}(\mu_q,\rho_q)),\mathcal{G}(\mathfrak{h}(\mu_q),\mathfrak{h}(\rho_q)),t) = 0,$$
$$\lim_{q\to+\infty} F(\mathfrak{h}(\mathcal{G}(\mu_q,\rho_q)),\mathfrak{h}(\mathcal{G}(\mu_q,\rho_q)),\mathcal{G}(\mathfrak{h}(\mu_q),\mathfrak{h}(\rho_q)),t) = 0,$$
$$\lim_{q\to+\infty} J(\mathfrak{h}(\mathcal{G}(\rho_q,\mu_q)),\mathfrak{h}(\mathcal{G}(\rho_q,\mu_q)),\mathcal{G}(\mathfrak{h}(\rho_q),\mathfrak{h}(\mu_q)),t) = 1,$$
$$\lim_{q\to+\infty} S(\mathfrak{h}(\mathcal{G}(\rho_q,\mu_q)),\mathfrak{h}(\mathcal{G}(\rho_q,\mu_q)),\mathcal{G}(\mathfrak{h}(\rho_q),\mathfrak{h}(\mu_q)),t) = 0,$$
$$\lim_{q\to+\infty} F(\mathfrak{h}(\mathcal{G}(\rho_q,\mu_q)),\mathfrak{h}(\mathcal{G}(\rho_q,\mu_q)),\mathcal{G}(\mathfrak{h}(\rho_q),\mathfrak{h}(\mu_q)),t) = 0.$$

Since $\mathcal{G}$ and $\mathfrak{h}$ are continuous, we have, for all $t > 0$, that

$$J(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu, \rho), t) = 1,\ J(\mathfrak{h}(\rho), \mathfrak{h}(\rho), \mathcal{G}(\rho, \mu), t) = 1$$
$$S(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu, \rho), t) = 0,\ S(\mathfrak{h}(\rho), \mathfrak{h}(\rho), \mathcal{G}(\rho, \mu), t) = 0$$
$$F(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu, \rho), t) = 0,\ F(\mathfrak{h}(\rho), \mathfrak{h}(\rho), \mathcal{G}(\rho, \mu), t) = 0.$$

Thus we can conclude that $\mathcal{G}(\mu, \rho) = \mathfrak{h}(\mu)$ and $\mathcal{G}(\rho, \mu) = \mathfrak{h}(\rho)$. □

In theorem 3.1, if we take $\mathcal{A}$ to be regular, then $\mathcal{G}$ need not be continuous to get the results. The next theorem proves the same.

Theorem 3.2. *Consider a complete JNMS $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ where $\mathcal{A}$ is regular and partially ordered. Consider the mappings $\mathcal{G} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and $\mathfrak{h} : \mathcal{A} \to \mathcal{A}$ where*

(a) $\mathcal{G}(\mathcal{A} \times \mathcal{A}) \subseteq \mathfrak{h}(\mathcal{A})$,

(b) $\mathfrak{h}$ *is continuous,*

(c) $\mathcal{G}$ *and* $\mathfrak{h}$ *are compatible,*

(d) $\mathcal{G}$ *and* $\mathfrak{h}$ *are* $\theta_{JN} -$ *coupled,*

(e) $\mathcal{G}$ *and* $\mathfrak{h}$ *are* $\theta_{JN}-$*coupled for some* $\theta \in \Theta$.

If there exist $\mu_0, \rho_0 \in \mathcal{A}$ for which $\mathfrak{h}(\mu_0) \leq \mathcal{G}(\mu_0, \rho_0)$ and $\mathfrak{h}(\rho_0) \geq F(\rho_0, \mu_0)$, then $\mathcal{G}$ and $\mathfrak{h}$ have a coupled coincidence point.

Proof. Since $\mathcal{A}$ is regular, $\mathfrak{h}(\mu_{\mathrm{q}}) \leq \mu$ and $\mathfrak{h}(\rho_{\mathrm{q}}) \geq \rho$, where $\mu_{\mathrm{q}}) \to \mu$, $\rho_{\mathrm{q}}) \to \rho$ as $q \to +\infty$. As $\mathcal{G}$ and $\mathfrak{h}$ are compatible and $\mathfrak{h}$ is continuous,

$$\lim_{\mathrm{q}\to+\infty} \mathfrak{h}(\mathfrak{h}(\mu_{\mathrm{q}})) = \mathfrak{h}(\mu) = \lim_{\mathrm{q}\to+\infty} \mathfrak{h}(\mathcal{G}(\mu_{\mathrm{q}}, \rho_{\mathrm{q}})) = \lim_{\mathrm{q}\to+\infty} \mathcal{G}(\mathfrak{h}(\mu_{\mathrm{q}}), \mathfrak{h}(\rho_{\mathrm{q}})),$$
$$\lim_{\mathrm{q}\to+\infty} \mathfrak{h}(\mathfrak{h}(\rho_{\mathrm{q}})) = \mathfrak{h}(\rho) = \lim_{\mathrm{q}\to+\infty} \mathfrak{h}(\mathcal{G}(\rho_{\mathrm{q}}, \mu_{\mathrm{q}})) = \lim_{\mathrm{q}\to+\infty} \mathcal{G}(\mathfrak{h}(\rho_{\mathrm{q}}), \mathfrak{h}(\mu_{\mathrm{q}})).$$

For all $0 \leq k < 1$, we have that

$$J(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu, \rho), \theta(t)) \geq \Big\{ J(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathfrak{h}(\mathfrak{h}(\mu_{\mathrm{q}+1}), \theta(t) - \theta(kt)) \odot J(\mathfrak{h}(\mathfrak{h}(\mu_{\mathrm{q}+1}), \mathfrak{h}(\mathfrak{h}(\mu_{\mathrm{q}+1}), \mathcal{G}(\mu, \rho), \theta(kt)) \Big\}$$
$$S(\mathfrak{h}(\mu), \mathfrak{h}(\mu), F(\mu, \rho), \theta(t)) \leq \Big\{ S(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathfrak{h}(\mathfrak{h}(\mu_{\mathrm{q}+1}), \theta(t) - \theta(kt)) \oslash S(\mathfrak{h}(\mathfrak{h}(\mu_{\mathrm{q}+1}), \mathfrak{h}(\mathfrak{h}(\mu_{\mathrm{q}+1}), \mathcal{G}(\mu, \rho), \theta(kt)) \Big\}$$
$$F(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu, \rho), \theta(t)) \leq \Big\{ F(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathfrak{h}(\mathfrak{h}(\mu_{\mathrm{q}+1}), \theta(t) - \theta(kt)) \otimes F(\mathfrak{h}(\mathfrak{h}(\mu_{\mathrm{q}+1}), \mathfrak{h}(\mathfrak{h}(\mu_{\mathrm{q}+1}), \mathcal{G}(\mu, \rho), \theta(kt)) \Big\}.$$

Letting $n \to +\infty$ in the above inequalities, we get that

$$\begin{aligned} J(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu,\rho).\theta(t)) &\geq \lim_{q\to+\infty} \Big\{ J(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathfrak{h}(\mathfrak{h}(\mu_{q+1}), \theta(t)-\theta(kt)) \\ &\odot J(\mathfrak{h}(\mathfrak{h}(\mu_{q+1}), \mathfrak{h}(\mathfrak{h}(\mu_{q+1}), \mathcal{G}(\mu,\rho), \theta(kt)) \Big\} \\ &\geq \lim_{q\to+\infty} J(\mathcal{G}(\mathfrak{h}(\mu_q), \mathfrak{h}(\rho_q)), \mathcal{G}(\mathfrak{h}(\mu_q), \mathfrak{h}(\rho_q)), \mathcal{G}(\mu,\rho), \theta(kt)), \\ S(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu,\rho).\theta(t)) &\leq \lim_{q\to+\infty} \Big\{ S(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathfrak{h}(\mathfrak{h}(\mu_{q+1}), \theta(t)-\theta(kt)) \\ &\oslash S(\mathfrak{h}(\mathfrak{h}(\mu_{q+1}), \mathfrak{h}(\mathfrak{h}(\mu_{q+1}), \mathcal{G}(\mu,\rho), \theta(kt)) \Big\} \\ &\leq \lim_{q\to+\infty} S(\mathcal{G}(\mathfrak{h}(\mu_q), \mathfrak{h}(\rho_q)), \mathcal{G}(\mathfrak{h}(\mu_q), \mathfrak{h}(\rho_q)), F(\mu,\rho), \theta(kt)), \\ F(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu,\rho).\theta(t)) &\leq \lim_{q\to+\infty} \Big\{ J(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathfrak{h}(\mathfrak{h}(\mu_{q+1}), \theta(t)-\theta(kt)) \\ &\otimes F(\mathfrak{h}(\mathfrak{h}(\mu_{q+1}), \mathfrak{h}(\mathfrak{h}(\mu_{q+1}), \mathcal{G}(\mu,\rho), \theta(kt)) \Big\} \\ &\leq \lim_{q\to+\infty} F(\mathcal{G}(\mathfrak{h}(\mu_q), \mathfrak{h}(\rho_q)), \mathcal{G}(\mathfrak{h}(\mu_q), \mathfrak{h}(\rho_q)), \mathcal{G}(\mu,\rho), \theta(kt)). \end{aligned}$$

Since $\mathcal{G}$ and $\mathfrak{h}$ are $\theta_{JN}-$coupled, from the above inequalities, we obtain that

$$\begin{aligned} J(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu,\rho).\theta(t)) &\geq \Big\{ J(\mathfrak{h}(\mathfrak{h}(\mu_q), \mathfrak{h}(\mathfrak{h}(\mu_q), \mathfrak{h}(\mu), kt) \odot J(\mathfrak{h}(\mathfrak{h}(\mu_q), \mathfrak{h}(\mathfrak{h}(\mu_q), \mathcal{G}(\mathfrak{h}(\mu_q), \mathfrak{h}(\rho_q)), kt) \\ &\odot J(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu,\rho), kt) \Big\} \\ &\geq J(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu,\rho), kt), \\ S(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu,\rho).\theta(t)) &\geq \Big\{ S(\mathfrak{h}(\mathfrak{h}(\mu_q), \mathfrak{h}(\mathfrak{h}(\mu_q), \mathfrak{h}(\mu), kt) \oslash S(\mathfrak{h}(\mathfrak{h}(\mu_q), \mathfrak{h}(\mathfrak{h}(\mu_q), \mathcal{G}(\mathfrak{h}(\mu_q), \mathfrak{h}(\rho_q)), kt) \\ &\oslash S(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu,\rho), kt) \Big\} \\ &\geq S(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu,\rho), kt), \\ F(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu,\rho).\theta(t)) &\geq \Big\{ F(\mathfrak{h}(\mathfrak{h}(\mu_q), \mathfrak{h}(\mathfrak{h}(\mu_q), \mathfrak{h}(\mu), kt) \otimes F(\mathfrak{h}(\mathfrak{h}(\mu_q), \mathfrak{h}(\mathfrak{h}(\mu_q), \mathcal{G}(\mathfrak{h}(\mu_q), \mathfrak{h}(\rho_q)), kt) \\ &\otimes F(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu,\rho), kt) \Big\} \\ &\geq F(\mathfrak{h}(\mu), \mathfrak{h}(\mu), \mathcal{G}(\mu,\rho), kt) \end{aligned}$$

Allowing k tending to 1, we obtain that $\mathcal{G}(\mu,\rho) = \mathfrak{h}(\mu)$. In a similar way, we can obtain that $\mathcal{G}(\rho,\mu) = \mathfrak{h}(\rho)$. □

If we take $\mathfrak{h}$ to be the identity mapping in the above theorems, then it leads to the following corollary.

Corollary 3.3. *Consider a complete JNMS* $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ *where* $\mathcal{A}$ *is a partially ordered set. Let* $\mathcal{G} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ *and* $\mathfrak{h} : \mathcal{A} \to \mathcal{A}$*. Assume that*

(a) *either* $\mathcal{A}$ *is regular or* $\mathcal{G}$ *is continuous,*

(b) $\mathfrak{h}$ *is continuous,*

(c) $\mathcal{G}$ *has mixed monotone property,*

(d) $\mathcal{G}$ *is self* $\theta_{JN}-$*coupled for some* $\theta \in \Theta$.

If there exist $\mu_0, \rho_0 \in \mathcal{A}$ *for which* $\mathfrak{h}(\mu_0) \leq \mathcal{G}(\mu_0, \rho_0)$ *and* $\mathfrak{h}(\rho_0) \geq \mathcal{G}(\rho_0, \mu_0)$*, then* $\mathcal{G}$ *has a coupled fixed point.*

Example 3.4. Consider the JNMS $(\mathcal{A}, J, S, F, \odot, \oslash, \otimes)$ as in Example 2.5 where $\mathcal{A} = [0,1]$ is with natural ordering and $G(\mu, \rho, \upsilon) =| \mu - \rho | + | \rho - \upsilon | + | \upsilon - \mu |$ for all $\mu, \rho, \in \mathcal{A}$. Let $\theta(t) = \frac{2t}{8}$, for $t \in [0, +\infty)$. Let us consider the functions $\mathcal{G} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and $\mathfrak{h} : \mathcal{A} \to \mathcal{A}$ defined by

$$\mathcal{G}(\mu, \rho) = \begin{cases} \frac{\mu^3 - 2\rho^3}{8}, & \text{if } \mu \geq \rho, \\ 0 & \text{otherwise,} \end{cases}$$
$$\mathfrak{h}(\mu) = \mu^3.$$

Consider sequences $\{\mu_{\mathrm{q}}\}$ and $\{\rho_{\mathrm{q}}\}$ in $\mathcal{A}$ such that

$$\lim_{\mathrm{q}\to+\infty} \mathcal{G}(\mu_{\mathrm{q}}, \rho_{\mathrm{q}}) = \lim_{\mathrm{q}\to+\infty} \mathfrak{h}(\mu_{\mathrm{q}}) \text{ and } \lim_{\mathrm{q}\to+\infty} \mathcal{G}(\rho_{\mathrm{q}}, \mu_{\mathrm{q}}) = \lim_{\mathrm{q}\to+\infty} \mathfrak{h}(\rho_{\mathrm{q}}).$$

It is then obvious that all these limit values must be zero. Let us show that $\mathcal{G}$ and $\mathfrak{h}$ are compatible.

$$\begin{aligned} J(\mathfrak{h}(\mathcal{G}(\mu_{\mathrm{q}}, \rho_{\mathrm{q}})), \mathfrak{h}(\mathcal{G}(\mu_{\mathrm{q}}, \rho_{\mathrm{q}})), \mathcal{G}(\mathfrak{h}(\mu_{\mathrm{q}}), \mathfrak{h}(\rho_{\mathrm{q}})), t) &= J(\mathfrak{h}(\frac{\mu_{\mathrm{q}}^3 - 2\rho_{\mathrm{q}}^3}{8})), \mathfrak{h}(\frac{\mu_{\mathrm{q}}^3 - 2\rho_{\mathrm{q}}^3}{8}), \mathcal{G}(\mu_{\mathrm{q}}^3, \rho_{\mathrm{q}}^3), t) \\ &= J((\frac{\mu_{\mathrm{q}}^3 - 2\rho_{\mathrm{q}}^3}{8})^3, (\frac{\mu_{\mathrm{q}}^3 - 2\rho_{\mathrm{q}}^3}{8})^3, \frac{\mu_{\mathrm{q}}^9 - 2\rho_{\mathrm{q}}^9}{8}, t) \\ &= \frac{t}{t + 2 \mid (\frac{\mu_{\mathrm{q}}^3 - 2\rho_{\mathrm{q}}^3}{8})^3 - \frac{\mu_{\mathrm{q}}^9 - 2\rho_{\mathrm{q}}^9}{8} \mid} \\ &\to 1 \text{ as } n \to +\infty. \end{aligned}$$

$$\begin{aligned} S(\mathfrak{h}(\mathcal{G}(\mu_{\mathrm{q}}, \rho_{\mathrm{q}})), \mathfrak{h}(\mathcal{G}(\mu_{\mathrm{q}}, \rho_{\mathrm{q}})), \mathcal{G}(\mathfrak{h}(\mu_{\mathrm{q}}), \mathfrak{h}(\rho_{\mathrm{q}})), t) &= S(\mathfrak{h}(\frac{\mu_{\mathrm{q}}^3 - 2\rho_{\mathrm{q}}^3}{8})), \mathfrak{h}(\frac{\mu_{\mathrm{q}}^3 - 2\rho_{\mathrm{q}}^3}{8}), \mathcal{G}(\mu_{\mathrm{q}}^3, \rho_{\mathrm{q}}^3), t) \\ &= S((\frac{\mu_{\mathrm{q}}^3 - 2\rho_{\mathrm{q}}^3}{8})^3, (\frac{\mu_{\mathrm{q}}^3 - 2\rho_{\mathrm{q}}^3}{8})^3, \frac{\mu_{\mathrm{q}}^9 - 2\rho_{\mathrm{q}}^9}{8}, t) \\ &= \frac{2 \mid (\frac{\mu_{\mathrm{q}}^3 - 2\rho_{\mathrm{q}}^3}{8})^3 - \frac{\mu_{\mathrm{q}}^9 - 2\rho_{\mathrm{q}}^9}{8} \mid}{t + 2 \mid (\frac{\mu_{\mathrm{q}}^3 - 2\rho_{\mathrm{q}}^3}{8})^3 - \frac{\mu_{\mathrm{q}}^9 - 2\rho_{\mathrm{q}}^9}{8} \mid} \\ &\to 0 \text{ as } n \to +\infty. \end{aligned}$$

$$\begin{aligned}F(\mathfrak{h}(\mathcal{G}(\mu_{\mathrm{q}},\rho_{\mathrm{q}})),\mathfrak{h}(\mathcal{G}(\mu_{\mathrm{q}},\rho_{\mathrm{q}})),\mathcal{G}(\mathfrak{h}(\mu_{\mathrm{q}}),\mathfrak{h}(\rho_{\mathrm{q}})),t) &= F(\mathfrak{h}(\frac{\mu_{\mathrm{q}}^3-2\rho_{\mathrm{q}}^3}{8})),\mathfrak{h}(\frac{\mu_{\mathrm{q}}^3-2\rho_{\mathrm{q}}^3}{8}),\mathcal{G}(\mu_{\mathrm{q}}^3,\rho_{\mathrm{q}}^3),t)\\ &= F((\frac{\mu_{\mathrm{q}}^3-2\rho_{\mathrm{q}}^3}{8})^3,(\frac{\mu_{\mathrm{q}}^3-2\rho_{\mathrm{q}}^3}{8})^3,\frac{\mu_{\mathrm{q}}^9-2\rho_{\mathrm{q}}^9}{8},t)\\ &= \frac{2\mid(\frac{\mu_{\mathrm{q}}^3-2\rho_{\mathrm{q}}^3}{8})^3-\frac{\mu_{\mathrm{q}}^9-2\rho_{\mathrm{q}}^9}{8}\mid}{t}\\ &\to 0 \text{ as } n\to+\infty.\end{aligned}$$

In a similar way, we can deduce that

$$\begin{aligned}J(\mathfrak{h}(\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}})),\mathfrak{h}(\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}})),\mathcal{G}(\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\mu_{\mathrm{q}})),t)&\to 1,\\ S(\mathfrak{h}(\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}})),\mathfrak{h}(\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}})),\mathcal{G}(\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\mu_{\mathrm{q}})),t)&\to 0,\\ F(\mathfrak{h}(\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}})),\mathfrak{h}(\mathcal{G}(\rho_{\mathrm{q}},\mu_{\mathrm{q}})),\mathcal{G}(\mathfrak{h}(\rho_{\mathrm{q}}),\mathfrak{h}(\mu_{\mathrm{q}})),t)&\to 0.\end{aligned}$$

Therefore $\mathcal{G}$ and $\mathfrak{h}$ are compatible. Take $\mu_0 = 0, \rho_0 = \alpha and \mu, \rho,$ in $\mathcal{A}$ such that $\mathfrak{h}(\mu_0) = \mathcal{G}(\mu_0,\rho_0)$, $\mathfrak{h}(\rho_0) = \mathcal{G}(\rho_0,\mu_0)$ and $\mathfrak{h}(\mu) \leq \mathfrak{h}(\gamma), \mathfrak{h}(\rho) \geq \mathfrak{h}(\sigma)$. Let us consider the following cases to verify 3.1(e).
case(i) $\mu \geq \rho$, $\gamma \geq \sigma$.

$$\begin{aligned}J(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) &= J(\frac{\mu^3-2\rho^3}{8},\frac{\mu^3-2\rho^3}{8},\frac{\gamma^3-2\sigma^3}{8},\frac{2t}{8})\\ &= \frac{t}{t+\mid(\mu^3-2\rho^3)-(\gamma^3-2\sigma^3)\mid}\\ &\geq \frac{t}{t+2\mid\gamma^3-\frac{\gamma^3-2\sigma^3}{8}\mid}\\ &\geq J(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t)\\ &\geq \Big\{J(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathfrak{h}(\gamma),t)\odot J(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho),t)\\ &\qquad\odot J(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t)\Big\}\end{aligned}$$

$$\begin{aligned}
S(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),F(\gamma,\sigma),\theta(t)) &= S(\frac{\mu^3-2\rho^3}{8},\frac{\mu^3-2\rho^3}{8},\frac{\gamma^3-2\sigma^3}{8},\frac{2t}{8}) \\
&= \frac{\mid(\mu^3-2\rho^3)-(\gamma^3-2\sigma^3)\mid}{t+\mid(\mu^3-2\rho^3)-(\gamma^3-2\sigma^3)\mid} \\
&\leq \frac{2\mid\gamma^3-\frac{\gamma^3-2\sigma^3}{8}\mid}{t+2\mid\gamma^3-\frac{\gamma^3-2\sigma^3}{8}\mid} \\
&\leq S(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t) \\
&\leq \Big\{S(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathfrak{h}(\gamma),t)\oslash S(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho),t) \\
&\quad \oslash S(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t)\Big\} \\
F(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) &= F(\frac{\mu^3-2\rho^3}{8},\frac{\mu^3-2\rho^ho3}{8},\frac{\gamma^3-2\sigma^3}{8},\frac{2t}{8}) \\
&= \frac{\mid(\mu^3-2\rho^3)-(\gamma^3-2\sigma^3)\mid}{t} \\
&\leq \frac{2\mid\gamma^3-\frac{\gamma^3-2\sigma^3}{8}\mid}{t} \\
&\leq F(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t) \\
&\leq \Big\{F(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathfrak{h}(\gamma),t)\otimes F(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho),t) \\
&\quad \otimes F(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t)\Big\}
\end{aligned}$$

case(ii) $\mu<\rho$, $\gamma\geq\sigma$.

$$\begin{aligned}
J(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) &= J(0,0,\frac{\gamma^3-2\sigma^3}{8},\frac{2t}{8}) \\
&= \frac{t}{t+\mid\frac{\gamma^3-2\sigma^3}{8}\mid} \geq \frac{t}{t+8\mid\mu^3-\gamma^3\mid} \\
&\geq J(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t) \\
&\geq \Big\{J(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathfrak{h}(\gamma),t)\odot J(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho),t) \\
&\quad \odot J(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t)\Big\} \\
S(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) &= S(0,0,\frac{\gamma^3-2\sigma^3}{8},\frac{2t}{8}) \\
&= \frac{\mid\frac{\gamma^3-2\sigma^3}{8}\mid}{t+\mid\frac{\gamma^3-2\sigma^3}{8}\mid} \\
&\leq \frac{8\mid\mu^3-\gamma^3\mid}{t+8\mid\mu^3-\gamma^3\mid} \\
&\leq S(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t) \\
&\leq \Big\{S(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathfrak{h}(\gamma),t)\oslash S(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho),t) \\
&\quad \oslash S(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t)\Big\}
\end{aligned}$$

$$\begin{aligned}F(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) &= F(0,0,\frac{\gamma^3-2\sigma^3}{8},\frac{2t}{8})\\ &= \frac{\mid \frac{\gamma^3-2\sigma^3}{8} \mid}{t}\\ &\leq \frac{8\mid \mu^3-\gamma^3 \mid}{t}\\ &\leq F(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t)\\ &\leq \Big\{F(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathfrak{h}(\gamma),t)F(\mathfrak{h}(\mu),\mathfrak{h}(\mu),\mathcal{G}(\mu,\rho),t)\\ &\quad \otimes F(\mathfrak{h}(\gamma),\mathfrak{h}(\gamma),\mathcal{G}(\gamma,\sigma),t)\Big\}\end{aligned}$$

case(iii) $\mu < \rho$, $\gamma < \sigma$.

This case is obvious, since we have that

$$J(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) = 1,$$
$$S(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) = 0,$$
$$F(\mathcal{G}(\mu,\rho),\mathcal{G}(\mu,\rho),\mathcal{G}(\gamma,\sigma),\theta(t)) = 0.$$

We have thus shown that $\mathcal{G}$ and $\mathfrak{h}$ fit into the theorem 3.1. Therefore $\mathcal{G}$ and $\mathfrak{h}$ must have a coupled coincidence point and it is the point $(0,0)$.

4. **Conclusion**

This work built a generalized neutrosophic metric space, called J-Neutrosophic metric space, based on the concept of neutrosophy. We proved coupled coincidence point results for JN-compatible mappings satisfying certain conditions. As the space introduced here considers the indeterminacy along with the degree of nearness and the degree of non-nearness and generalizes the ideas of intuitionistic sets, fuzzy sets, classical sets, paraconsistent sets and dialetheist sets, this work has the scope of further extension and analysis.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Attanssov, K., Intuitionistic fuzzy sets. Fuzzy sets and systems 1986, 20, 87-96.
2. Bhaskar, T.G.; Lakshmikantam, V., Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. 2006, TMA 65, 1379-1397.
3. Cantor, G., Ueber eine Eigenschaft des Begriffes aller reelen algenraishen Zahlen. Journal fiir die reine und angewandte Mathematic 1874, 77, 258-262.
4. Das, R., Smarandache, F., Tripathy, B.C., Neutrosophic fuzzy matrices and some algebraic operations. Neutrosophic Sets and Systems, 2020, 32,401-409.
5. Das, R.; Tripathy, B.C., Neutrosophic multi-set topological space. Neutrosophic Sets and Systems, 2020, 35, 142-152.

6. Das, S.; Das, R.; Tripathy, B.C., Multi-criteria group decision making model using single-valued neutrosophic set. LogForum, 16(3), 421-429.
7. Eser; Derya, A new edge detection approach via neutrosophy based on maximum norm entropy, Expert Systems with Applications, 115(2019), 499-511.
8. Johnson Philip, A history of set theory, Prindle, Weber & Schmidt, 1972, ISBN 0-87150-154-6.
9. Klaua, D., Uber einen Ansatz zur mehrwertigen Mengenlehre. Monatsb, Deutsh Akad. Wiss. Berlin 1965, 7, 859-876.
10. Lakshmikantham, V.; Ciric, L., Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Analysis: Theory, Methods and Appl. 2009, 70(12),4341-4349.
11. Leyva-Vázquez, M.; Smarandache, F.; Ricardo, J.E., Artificial intelligence: challenges, perspectives and neutrosophy role (Master Conference). Dilemas Contemporáneos: Educación, Política y Valore 2018, 6(Special).
12. Mohiuddine, S.A.; Alotaibi, A., Coupled coincidence point theorems for compatible mappings in partially ordered intuitionistic generalized fuzzy metric spaces. Fixed point theory and applications 2013, 2013(1), 265.
13. Mukherjee, A.; Das, R., Neutrosophic bipolar vague soft set and its application to decision making problems. Neutrosophic Sets and Systems, 2020, 32,410-424.
14. Mustafa, Z.; Sims, B., A new approach to generalized metric spaces. J. Nonlinear convex Anal. 2006, 7(2), 289-297.
15. Rashno, E.; Akbari, A.; Nasersharif, B., A Convoloutional Neural Network model based on Neutrosophy for Noisy Speech Recognition. Infinite Study 2019.
16. Schweizer, B.; Sklar, A., Statistical metric spaces. Pac. J. Math. 1960, 10, 313-334.
17. Smarandache, F., Neutrosophy / Neutrosophic Probability, Set, and Logic. American Research Press, 1998.
18. Smarandache, F., Neutrosophy, a new Branch of Philosophy. Infinite Study, 2002.
19. Smarandache, F., Neutrosophic set, a generalization of intuitionistic fuzzy sets. Int. J. Pure Appl. Math. 2005, 24, 287-297.
20. Sowndrarajan, S.; Jeyaraman, M.; Smarandache, F., Fixed Point Results for Contraction Theorems in Neutrosophic Metric Spaces. Neutrosophic Sets and Systems 2020, 36, 308-318.
21. Sun, G.; Yang, K.; Generalized fuzzy metric spaces with properties. Res. J. Appl. Sci. 2010, 2, 673-678.
22. Zadeh, L.A., Fuzzy sets. Inform and control, 1965, 8, 338-353.

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