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Abstract. In this manuscript, We investigate new outcomes in the field of neutrosophic metric space due to Kirisci and Simsek. We analyse weakly commuting and R-weakly commuting in the setting of neutrosophic metric space and prove some fixed point results. We develop the results to obtain common fixed point theorem in neutrosophic version. We validate our results by suitable examples.

Keywords: Fixed point; Neutrosophic metric Space; Banach contraction; Weakly commuting; R-Weakly commuting.

1. Introduction

Fuzzy Sets(FSs) was presented by Zadeh [22] as a class of elements with a grade of membership. Kramosil and Michalek [10] defined new notion called Fuzzy Metric Space (FMS). George and Veeramani [6] redefined the concept of FMS with the assistance of triangular norms. Afterward, numerous researchers have analyzed the characteristics of FMS and proved many fixed point results. FMS has wide range of applications in applied science fields such as fixed point theory, decision making, medical imaging and signal processing. In 1986, Atanassov [1] defined Intuitionistic Fuzzy Sets(IFSs) by adding non - membership to FSs. Park [15] defined Intuitionistic Fuzzy Metric Space (IFMS) from the concept of IFSs and given some fixed point results. Fixed point theorems related to FMS and IFMS given by Alaca et al [2] and numerous researchers [5, 12, 17]. In 1998, Smarandache [20] characterized the new idea called neutrosophic set. In general the notion of fuzzy set and IFS deal with degree of membership

and non - membership respectively. Neutrosophic set is a generalized state of Fuzzy and Intuitionistic Fuzzy Set by incorporating degree of indeterminacy. In addition, several researchers contributed significantly to develop the neutrosophic theory. Recently, Baset et al. [3, 4] explored the neutrosophic applications in different fields such as model for sustainable supply chain risk management, resource levelling problem in construction projects, Decision Making, financial performance and evaluation of manufacturing industries. In 2019, Kirisci et al [11] defined neutrosophic metric space as a generalization of IFMS and brings about fixed point theorems in complete neutrosophic metric space. In 2020, Sowndrarajan and Jeyaraman et al [21] proved some fixed point results in neutrosophic metric spaces.
In this paper, we define the concept of weakly commuting and R-weakly commuting mappings in the setting of neutrosophic metric space and prove common fixed point theorems with the help of Pant's theorem. [14].

2. Preliminaries

Definition 2.1 [20] Let Σ be a non-empty fixed set. A Neutrosophic Set (NS for short) N in Σ is an object having the form $N = \{\langle a, \xi_N(a), \varrho_N(a), \nu_N(a)\rangle : a \in \Sigma\}$ where the functions $\xi_N(a), \varrho_N(a)$ and $\nu_N(a)$ represent the degree of membership, degree of indeterminacy and the degree of non-membership respectively of each element a $\in$ N to the set Σ.
A neutrosophic set $N = \{\langle a, \xi_N(a), \varrho_N(a), \nu_N(a)\rangle : a \in \Sigma\}$ is expressed as an ordered triple $N = \langle a, \xi_N(a), \varrho_N(a), \nu_N(a)\rangle$ in Σ. In NS, there is no restriction on $(\xi_N(a), \varrho_N(a), \nu_N(a))$ other than they are subsets of $]^-0, 1^+[$.
Triangular Norms (TNs) were initiated by menger. Triangular Co norms(TCs) knowns as dual operations of triangular norms.

Definition 2.2 [7] A binary operation $\star : [0, 1] \times [0, 1] \to [0, 1]$ is called continuous t - norm (CTN) if it satisfies the following conditions;
For all $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in [0,\ 1]$
(i) $\zeta_1 \star 1 = \zeta_1$;
(ii) If $\zeta_1 \leq \zeta_3$ and $\zeta_2 \leq \zeta_4$ then $\zeta_1 \star \zeta_2 \leq \zeta_3 \star \zeta_4$;
(iii) $\star$ is continuous;
(iv) $\star$ is commutative and associative.

Definition 2.3 [7] A binary operation $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ is called continuous t - co norm (CTC) if it satisfies the following conditions;
For all $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in [0,\ 1]$
(i) $\zeta_1 \diamond 0 = \zeta_1$;

(ii) If $\zeta_1 \leq \zeta_3$ and $\zeta_2 \leq \zeta_4$ then $\zeta_1 \diamond \zeta_2 \leq \zeta_3 \diamond \zeta_4$;

(iii) $\diamond$ is continuous;

(iv) $\diamond$ is commutative and associative.

Remark 2.4 [11] From the definitions of CTN and CTC, we note that if we take $0 < \zeta_1, \zeta_2 < 1$ for $\zeta_1 < \zeta_2$ then there exist $0 < \zeta_3, \zeta_4 < 1$ such that $\zeta_1 \star \zeta_3 \geq \zeta_2$ and $\zeta_1 \geq \zeta_2 \diamond \zeta_4$. Further we choose $\zeta_5 \in (0,1)$ then there exists $\zeta_6, \zeta_7 \in (0,1)$ such that $\zeta_6 \star \zeta_6 \geq \zeta_5$ and $\zeta_7 \diamond \zeta_7 \leq \zeta_5$.

3. Neutrosophic Metric Spaces

In this section, we apply neutrosophic theory to generalize the intuitionistic fuzzy metric space. we also discuss some properties and examples in it.

Definition 3.1 [21] A 6 - tuple $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$is called Neutrosophic Metric Space(NMS), if Σ is an arbitrary non empty set, $\star$ is a neutrosophic CTN and $\diamond$ is a neutrosophic CTC and Ξ, Θ, Υ are neutrosophic sets on $\Sigma^2 \times \mathbb{R}^+$ satisfying the following conditions:

For all $\zeta, \eta, \omega \in \Sigma, \lambda \in \mathbb{R}^+$

(i) $0 \leq \Xi(\zeta, \eta, \lambda) \leq 1;\ 0 \leq \Theta(\zeta, \eta, \lambda) \leq 1;\ 0 \leq \Upsilon(\zeta, \eta, \lambda) \leq 1;$

(ii) $\Xi(\zeta, \eta, \lambda) + \Theta(\zeta, \eta, \lambda) + \Upsilon(\zeta, \eta, \lambda) \leq 3;$

(iii) $\Xi(\zeta, \eta, \lambda) = 1$ if and only if $\zeta = \eta$;

(iv) $\Xi(\zeta, \eta, \lambda) = \Xi(\eta, \zeta, \lambda)$ for $\lambda > 0$;

(v) $\Xi(\zeta, \eta, \lambda) \star \Xi(\eta, \zeta, \mu) \leq \Xi(\zeta, \omega, \lambda + \mu)$, for all $\lambda, \mu > 0$;

(vi) $\Xi(\zeta, \eta, .) : [0, \infty) \to [0, 1]$ is neutrosophic continuous ;

(vii) $lim_{\lambda \to \infty} \Xi(\zeta, \eta, \lambda) = 1$ for all $\lambda > 0$;

(viii) $\Theta(\zeta, \eta, \lambda) = 0$ if and only if $\zeta = \eta$;

(ix) $\Theta(\zeta, \eta, \lambda) = \Theta(\eta, \zeta, \lambda)$ for $\lambda > 0$;

(x) $\Theta(\zeta, \eta, \lambda) \diamond \Theta(\zeta, \omega, \mu) \geq \Theta(\zeta, \omega, \lambda + \mu)$, for all $\lambda, \mu > 0$;

(xi) $\Theta(\zeta, \eta, .) : [0, \infty) \to [0, 1]$ is neutrosophic continuous ;

(xii) $lim_{\lambda \to \infty} \Theta(\zeta, \eta, \lambda) = 0$ for all $\lambda > 0$;

(xiii) $\Upsilon(\zeta, \eta, \lambda) = 0$ if and only if $\zeta = \eta$;

(xiv) $\Upsilon(\zeta, \eta, \lambda) = \Upsilon(\eta, \zeta, \lambda)$ for $\lambda > 0$;

(xv) $\Upsilon(\zeta, \eta, \lambda) \diamond \Upsilon(\zeta, \omega, \mu) \geq \Upsilon(\zeta, \omega, \lambda + \mu)$, for all $\lambda, \mu > 0$;

(xvi) $\Upsilon(\zeta, \eta, .) : [0, \infty) \to [0, 1]$ is neutrosophic continuous ;

(xvii) $lim_{\lambda \to \infty} \Upsilon(\zeta, \eta, \lambda) = 0$ for all $\lambda > 0$;

(xviii) If $\lambda > 0$ then $\Xi(\zeta, \eta, \lambda) = 0, \Theta(\zeta, \eta, \lambda) = 1, \Upsilon(\zeta, \eta, \lambda) = 1$.

Then (Ξ, Θ, Υ) is called Neutrosophic Metric on Σ. The functons Ξ, Θ and Υ denote degree of closedness, neturalness and non - closedness between ζ and η with respect to λ respectively.

Example 3.2 [21] Let (Σ, d) be a metric space. Define $\zeta \star \eta = min\{\zeta, \eta\}$ and $\zeta \diamond \eta = max\{\zeta, \eta\}$, and $\Xi, \Theta, \Upsilon : \Sigma^2 \times \mathbb{R}^+ \to [0,1]$ defined by

$$\Xi(\zeta, \eta, \lambda) = \frac{\lambda}{\lambda + d(\zeta, \eta)}; \quad \Theta(\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda + d(\zeta, \eta)}; \quad \Upsilon(\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda}$$

for all $\zeta, \eta \in \Sigma$ and $\lambda > 0$. Then $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ is called neutrosophic metric space induced by a metric d the standard neutrosophic metric.

Remark 3.3 [11] In neutrosophic metric space Ξ is non - decreasing , Θ is a non - increasing, Υ is decreasing function for all $\zeta, \eta \in \Sigma$.

Definition 3.4 Let $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ be neutrosophic metric space . Then

(a) $\{\zeta_n\}$ in Σ is converging to a point $\zeta \in \Sigma$ if for each $\lambda > 0$

$$lim_{n\to\infty}\Xi(\zeta_n, \zeta, \lambda) = 1; \quad lim_{n\to\infty}\Theta(\zeta_n, \zeta, \lambda) = 0; \quad lim_{n\to\infty}\Upsilon(\zeta_n, \zeta, \lambda) = 0.$$

(b) ζ_n in Σ is called a Cauchy if for each $\epsilon > 0$ and $\lambda > 0$ there exist $N \in \mathbb{N}$ such that $\Xi(\zeta_{n+p}, \zeta_n, \lambda) = 1$; $\Theta(\zeta_{n+p}, \zeta_n, \lambda) = 0$; $\Upsilon(\zeta_{n+p}, \zeta_n, \lambda) = 0$.

(c) $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ is said to be complete neutrosophic metric space if every Cauchy sequence is convergence in it.

4. Main Results

In this section, we present some interesting concepts such as weakly commuting and R-weakly commuting as an extensive work from Banach's contraction principle with suitable examples.

Theorem 4.1 Let $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ be a complete neutrosophic metric space. Let $\varphi, \varrho : \Sigma \to \Sigma$ be functions satisfying the following conditions:

(i) $\varphi(\Sigma) \subseteq \varrho(\Sigma)$;

(ii) ϱ is continuous;

(iii) there exists $0 \le k \le 1$ such that, for all $\zeta, \eta, \omega \in \Sigma$

$$\Xi(\varphi(\zeta), \varphi(\eta), k\lambda) \ge \Xi(\varrho(\zeta), \varrho(\eta), \lambda),$$
$$\Theta(\varphi(\zeta), \varphi(\eta), k\lambda) \le \Theta(\varrho(\zeta), \varrho(\eta), \lambda),$$
$$\Upsilon(\varphi(\zeta), \varphi(\eta), k\lambda) \le \Upsilon(\varrho(\zeta), \varrho(\eta), \lambda).$$

Then ϱ and φ have a unique common unique fixed point in Σ provided ϱ and φ commute on Σ.

Proof: Let $\zeta_0 \in \Sigma$, from (i) we can get ζ_1 such that $\varrho(\zeta_1) = \varphi(\zeta_0)$. By mathematical induction, we define ζ_n in Σ such that $\varrho(\zeta_n) = \varphi(\zeta_{n-1})$. Again by induction

$$\Xi(\varrho(\zeta_n),\varrho(\zeta_{n+1}),\lambda)=\Xi(\varphi(\zeta_{n-1}),\varphi(\zeta_n),\lambda)\quad\geq\Xi(\varrho(\zeta_{n-1}),\varrho(\zeta_n),\frac{\lambda}{k})\cdots\quad\geq\Xi(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{k^n}),$$
$$\Theta(\varrho(\zeta_n),\varrho(\zeta_{n+1}),\lambda)=\Theta(\varphi(\zeta_{n-1}),\varphi(\zeta_n),\lambda)\quad\leq\Theta(\varrho(\zeta_{n-1}),\varrho(\zeta_n),\frac{\lambda}{k})\cdots\quad\leq\Theta(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{k^n}),$$
$$\Upsilon(\varrho(\zeta_n),\varrho(\zeta_{n+1}),\lambda)=\Upsilon(\varphi(\zeta_{n-1}),\varphi(\zeta_n),\lambda)\quad\leq\Upsilon(\varrho(\zeta_{n-1}),\varrho(\zeta_n),\frac{\lambda}{k})\cdots\quad\leq\Upsilon(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{k^n}),$$

for all $n>0$ and $\lambda>0$. Thus, for any non-negative integer p, we have

$$\begin{aligned}\Xi(\varrho(\zeta_n),\varrho(\zeta_{n+p}),\lambda)&\geq\Xi\varrho(\zeta_n),\varrho(\zeta_{n+1}),\frac{\lambda}{k})\star\cdots^{(p-times)}\cdots\star\Xi(\varrho(\zeta_{n+p-1}),\varrho(\zeta_{n+p}),\frac{\lambda}{k})\\&\geq\Xi(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{pk^n})\star\cdots^{(p-times)}\cdots\star\Xi(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{pk^{n+p-1}}),\\\Theta(\varrho(\zeta_n),\varrho(\zeta_{n+p}),\lambda)&\leq\Theta(\varrho(\zeta_n),\varrho(\zeta_{n+1}),\frac{\lambda}{k})\diamond\cdots^{(p-times)}\cdots\diamond\Theta(\varrho(\zeta_{n+p-1}),\varrho(\zeta_{n+p}),\frac{\lambda}{k})\\&\leq\Theta(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{pk^n})\diamond\cdots^{(p-times)}\cdots\diamond\Theta(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{pk^{n+p-1}}),\\\Upsilon(\varrho(\zeta_n),\varrho(\zeta_{n+p}),\lambda)&\leq\Upsilon(\varrho(\zeta_n),\varrho(\zeta_{n+1}),\frac{\lambda}{k})\diamond\cdots^{(p-times)}\cdots\diamond\Upsilon(\varrho(\zeta_{n+p-1}),\varrho(\zeta_{n+p}),\frac{\lambda}{k})\\&\leq\Upsilon(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{pk^n})\diamond\cdots^{(p-times)}\cdots\diamond\Upsilon(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{pk^{n+p-1}}).\end{aligned}$$

by conditions (vii), (xii) and (xvii) of definition (3.1), we get

$$\begin{aligned}lim_{n\to\infty}\Xi(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{pk^n})&=1,\\lim_{n\to\infty}\Theta(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{pk^n})&=0,\\lim_{n\to\infty}\Upsilon(\varrho(\zeta_0),\varrho(\zeta_1),\frac{\lambda}{pk^n})&=0.\end{aligned}$$

It follows that

$$\begin{aligned}lim_{n\to\infty}\Xi(\varrho(\zeta_n),\varrho(\zeta_{n+p}),\lambda)&\geq 1\star\cdots^{(p-times)}\cdots\star 1=1,\\lim_{n\to\infty}\Theta(\varrho(\zeta_n),\varrho(\zeta_{n+p}),\lambda)&\leq 0\diamond\cdots^{(p-times)}\cdots\diamond 0=0,\\lim_{n\to\infty}\Upsilon(\varrho(\zeta_n),\varrho(\zeta_{n+p}),\lambda)&\leq 0\diamond\cdots^{(p-times)}\cdots\diamond 0=0.\end{aligned}$$

Since Σ is complete NMS, $\{\varrho(\zeta_n)\}$ is a Cauchy sequence that converges to a point η and $\varphi(\zeta_{n-1})=\varrho(\zeta_n)$ converges to the same point η. From (iii), it is shown that continuity of ϱ implies continuity of φ. Hence, $\{\varphi(\varrho(\zeta_n))\}$ converges to $\varphi(\eta)$. However, ϱ and φ are commute on Σ, $\varphi(\varrho(\zeta_n))=\varrho(\varphi(\zeta_n))$ and so $\varrho(\varphi(\zeta_n))$ converges to $\varrho(\eta)$. Thus $\varrho(\eta)=\varphi(\eta)$, which implies $\varrho(\varrho(\eta))=\varrho(\varphi(\eta))$. Thus, we get

$$\begin{aligned}
\Xi(\varphi(\eta),\varphi(\varphi(\eta)),\lambda) &\geq \Xi(\varrho(\eta),\varrho(\varphi(\eta)),\frac{\lambda}{k})\\
&\geq \Xi(\varphi(\eta),\varphi(\varphi(\eta)),\frac{\lambda}{k})\\
&\geq \cdots\\
&\geq \Xi(\varphi(\eta),\varphi(\varphi(\eta)),\frac{\lambda}{k^n}),\\
\Theta(\varphi(\eta),\varphi(\varphi(\eta)),\lambda) &\leq \Theta(\varrho(\eta),\varrho(\varphi(\eta)),\frac{\lambda}{k})\\
&\leq \Theta(\varphi(\eta),\varphi(\varphi(\eta)),\frac{\lambda}{k})\\
&\leq \cdots\\
&\leq \Theta(\varphi(\eta),\varphi(\varphi(\eta)),\frac{\lambda}{k^n}),\\
\Upsilon(\varphi(\eta),\varphi(\varphi(\eta)),\lambda) &\leq \Upsilon(\varrho(\eta),\varrho(\varphi(\eta)),\frac{\lambda}{k})\\
&\leq \Upsilon(\varphi(\eta),\varphi(\varphi(\eta)),\frac{\lambda}{k})\\
&\leq \cdots\\
&\leq \Upsilon(\varphi(\eta)),\varphi(\varphi(\eta)),\frac{\lambda}{k^n}).
\end{aligned}$$

Therefore from the definition of (3.1), it follows that $\varphi(\eta) = \varphi(\varphi(\eta))$. Thus $\varphi(\eta) = \varphi(\varphi(\eta)) = \varrho(\varphi(\eta))$. Hence $\varphi(\eta)$ is a common fixed point of the mappings ϱ and φ.

To prove uniqueness, let us assume η and ω are two fixed points of ϱ and φ, then

$$\begin{aligned}
&1 \geq \Xi(\zeta,\omega,\lambda) = \Xi(\varphi(\eta),\varphi(\omega),\lambda) \geq \Xi(\varrho(\zeta),\varrho(\omega),\frac{\lambda}{k})\\
&\quad = \Xi(\eta,\omega,\frac{\lambda}{k}) \geq \cdots \geq \Xi(\zeta,\omega,\frac{\lambda}{k^n}),\\
&0 \leq \Theta(\zeta,\omega,\lambda) = \Theta(\varphi(\eta),\varphi(\omega),\lambda) \leq \Theta(\varrho(\zeta),\varrho(\omega),\frac{\lambda}{k})\\
&\quad = \Theta(\eta,\omega,\frac{\lambda}{k}) \leq \cdots \leq \Theta(\zeta,\omega,\frac{\lambda}{k^n}),\\
&0 \leq \Upsilon(\zeta,\omega,\lambda) = \Upsilon(\varphi(\eta),\varphi(\omega),\lambda) \leq \Upsilon(\varrho(\zeta),\varrho(\omega),\frac{\lambda}{k})\\
&\quad = \Upsilon(\eta,\omega,\frac{\lambda}{k}) \leq \cdots \leq \Upsilon(\zeta,\omega,\frac{\lambda}{k^n}).
\end{aligned}$$

From the definition(3.1), we get

$$\lim_{n\to\infty} \Xi(\eta,\omega,\frac{\lambda}{k^n}) = 1, \quad \lim_{n\to\infty} \Theta(\eta,\omega,\frac{\lambda}{k^n}) = 0, \quad \lim_{n\to\infty} \Upsilon(\eta,\omega,\frac{\lambda}{k^n}) = 0.$$

It follows that

$$1 \geq \Xi(\eta,\omega,\lambda) \geq 1,\ 0 \leq \Theta(\eta,\omega,\lambda) \leq 0,\ 0 \leq \Upsilon(\eta,\omega,\lambda) \leq 0,$$

which states that $\eta = \omega$. Hence, we obtain a unique common fixed point of both φ and ϱ.

Example 4.2 Let $\Sigma = \{\frac{1}{n}; n \in \mathbb{N}\}$ with the standard metric $d(\zeta, \eta) = |\zeta - \eta|$. For all $\zeta, \eta \in \Sigma$ and $\lambda \in [0, \infty)$, define

$$\Xi(\zeta, \eta, \lambda) = \begin{cases} 0, & if \quad \lambda = 0 \\ \frac{\lambda}{\lambda + d(\zeta,\eta)}, & if \quad \lambda > 0 \end{cases}$$

$$\Theta(\zeta, \eta, \lambda) = \begin{cases} 1 & if \quad \lambda = 0 \\ \frac{d(\zeta,\eta)}{k\lambda + d(\zeta,\eta)} & if \quad k > 0,\ \lambda > 0 \end{cases}$$

$$\Upsilon(\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda} \quad if \quad \lambda > 0.$$

for all $\zeta, \eta \in \Sigma$ and $\lambda > 0$. Then $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ is called complete neutrosophic metric space on Σ, Here $\star$ is defined by $\zeta \star \eta = \zeta\eta$ and $\diamond$ is defined as $\zeta \diamond \eta = min\{1,\ \zeta + \eta\}$. Define $\varphi(\zeta) = \frac{\zeta}{9}; \varrho(\zeta) = \frac{\zeta}{3}$. Clearly $\varphi(\Sigma) \subseteq \varrho(\Sigma)$, Also for $k = \frac{1}{3}$, we get

$$\Xi(\varphi(\zeta), \varphi(\eta), \frac{\lambda}{3}) = \frac{\frac{\lambda}{3}}{\frac{\lambda}{3} + d(\varphi(\zeta), \varphi(\eta))} \geq \frac{\lambda}{\lambda + \frac{d(\zeta,\eta)}{3}} = \Xi(\varrho(\zeta), \varrho(\eta), \lambda),$$

Similarly, we get

$$\Theta(\varphi(\zeta), \varphi(\eta), \frac{\lambda}{3}) \leq \Theta(\varrho(\zeta)\varrho(\eta), \lambda),$$
$$\Upsilon(\varphi(\zeta), \varphi(\eta), \frac{\lambda}{3}) \leq \Upsilon(\varrho(\zeta), \varrho(\eta), \lambda).$$

Hence the conditions in Theorem (4.1) are satisfied and so ϱ and φ have common fixed point 0.

Definition 4.3 [18] Let ϱ and φ be two self mappings from neutrosophic metric space $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ into itself. The mappings ϱ and φ is called weakly commuting if for all $\zeta \in \Sigma$

$$\Xi(\varrho\varphi(\zeta)) \geq \Xi(\varphi\varrho(\zeta)), \quad \Theta(\varrho\varphi(\zeta)) \leq \Theta(\varphi\varrho(\zeta)), \quad \Upsilon(\varrho\varphi(\zeta)) \leq \Upsilon(\varphi\varrho(\zeta)).$$

Definition 4.4 Let ϱ and φ be two self mappings from neutrosophic metric space $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ into itself. The mappings ϱ and φ is called R-weakly commuting if there exist a positive real number R such that for all $\zeta \in \Sigma$.

$$\Xi(\varrho\varphi(\zeta), \varphi\varrho(\zeta), \lambda) \geq \Xi(\varphi(\zeta), \varrho(\zeta), \frac{\lambda}{R}),$$
$$\Theta(\varrho\varphi(\zeta), \varphi\varrho(\zeta), \lambda) \leq \Theta(\varphi(\zeta), \varrho(\zeta), \frac{\lambda}{R}),$$
$$\Upsilon(\varrho\varphi(\zeta), \varphi\varrho(\zeta), \lambda) \leq \Upsilon(\varphi(\zeta), \varrho(\zeta), \frac{\lambda}{R}).$$

Remark 4.5 In Neutrosophic metric spaces, Weak commutativity implies R-weak commutativity, but weak commutativity can be derived from R-weakly commuting only when $R \leq 1$.

Example 4.6 Let $\Sigma = \mathbb{R}$ be set of all real numbers. $\star$ and $\diamond$ defined by $a \star b = ab, \quad a \diamond b = min\{1, a+b\}$, define $d(\zeta,\eta) = |\zeta - \eta|$

$$\Xi(\zeta,\eta,\lambda) = \left(exp\left(\frac{d(\zeta,\eta)}{\lambda}\right)\right)^{-1},$$

$$\Theta(\zeta,\eta,\lambda) = \frac{exp\left(\frac{d(\zeta,\eta)}{\lambda}\right) - 1}{exp\left(\frac{d(\zeta,\eta)}{\lambda}\right)},$$

$$\Upsilon(\zeta,\eta,\lambda) = exp\left(\frac{d(\zeta,\eta)}{\lambda}\right).$$

for all $\zeta, \eta \in \Sigma$ and $\lambda > 0$. Then $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ is a neutrosophic metric space. We define $\varrho(\zeta) = 2\zeta - 1$ and $g(\zeta) = \zeta^2$. Then , we have

$$\Xi(\varrho\varphi(\zeta), \varphi\varrho(\eta), \lambda) = \left(exp\left(2\frac{d(\zeta,\eta)^2}{\lambda}\right)\right)^{-1},$$

$$\Theta(\varrho\varphi(\zeta), \varphi\varrho(\eta), \lambda) = \frac{exp\left(2\frac{d(\zeta,\eta)^2}{\lambda}\right) - 1}{exp\left(2\frac{d(\zeta,\eta)^2}{\lambda}\right)},$$

$$\Upsilon(\varrho\varphi(\zeta), \varphi\varrho(\eta), \lambda) = exp\left(2\frac{d(\zeta,\eta)^2}{\lambda}\right).$$

Also, we have,

$$\Xi(\varrho(\zeta), \varphi(\eta), \frac{\lambda}{2}) = \left(exp\left(2\frac{d(\zeta,\eta)^2}{\lambda}\right)\right)^{-1},$$

$$\Theta(\varrho(\zeta), \varphi(\eta), \frac{\lambda}{2}) = \frac{exp\left(2\frac{d(\zeta,\eta)^2}{\lambda}\right) - 1}{exp\left(2\frac{d(\zeta,\eta)^2}{\lambda}\right)},$$

$$\Upsilon(\varrho(\zeta), \varphi(\eta), \frac{\lambda}{2}) = exp\left(2\frac{d(\zeta,\eta)^2}{\lambda}\right).$$

Therefore, the self mappings ϱ and φ are R-weakly commuting only for R = 2, but converse is not true since the exponential function is non-decreasing.

Now, we define R-weakly commuting on Σ and prove the neutrosophic version of Pant's theorem.

Definition 4.7 Let $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ is a neutrosophic metric space and ϱ and φ be R-weakly commuting self-mappings of Σ satisfying the following condition:

$$\Xi(\varrho(\zeta), \varrho(\eta), \lambda) \geq r \ \Xi(\varphi(\zeta), \varphi(\eta), \lambda)$$

$$\Theta(\varrho(\zeta), \varrho(\eta), \lambda) \leq r \ \Theta(\varphi(\zeta), \varphi(\eta), \lambda)$$

$$\Upsilon(\varrho(\zeta), \varrho(\eta), \lambda) \leq r \ \Upsilon(\varphi(\zeta), \varphi(\eta), \lambda)$$

for all $\zeta, \eta \in \Sigma$. where $r : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $r(\lambda) < \lambda$ for all $\lambda > 0$. By hypothesis of theorem, ϱ and φ have a unique common fixed point in Σ.

Now, we prove the neutrosophic version of Pant's theorem.

Theorem 4.8 Let $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ is a complete neutrosophic metric space and ϱ *and* φ be R-weakly commuting self-mappings of Σ satisfying the following condition:

(i) $\varphi(\Sigma) \subseteq \varrho(\Sigma)$;

(ii) ϱ or φ is continuous;

(iii) There exists $0 \leq \lambda \leq 1$ such that, for all $\zeta, \eta, \omega \in \Sigma$

$$\Xi(\varrho(\zeta), \varrho(\eta), \lambda) \geq \gamma(\Xi(\varphi(\zeta), \varphi(\eta), \lambda)),$$

$$\Theta(\varrho(\zeta), \varrho(\eta), \lambda) \leq \gamma^{'}(\Theta(\varphi(\zeta), \varphi(\eta), \lambda)),$$

$$\Upsilon(\varrho(\zeta), \varrho(\eta), \lambda) \leq \gamma^{''}(\Upsilon(\varphi(\zeta), \varphi(\eta), \lambda)),$$

where $\gamma, \gamma^{'}$ and $\gamma^{''} : [0\ 1] \to [0\ 1]$ are continuous function such that $\gamma(\lambda) > \lambda$, $\gamma^{'}(\lambda) < \lambda$ and $\gamma^{''}(\lambda) < \lambda$.

(iv) If the sequence $\{\zeta_n\}$ and $\{\eta_n\}$ in Σ are such that, for all $\zeta, \eta \in \Sigma$ and $\lambda > 0$, $lim_{n\to\infty}\zeta_n = \zeta$ and $lim_{n\to\infty}\eta_n = \eta$ implies,

$$lim_{n\to\infty}\Xi(\zeta_n, \eta_n, \lambda) = \Xi(\zeta, \eta, \lambda),$$

$$lim_{n\to\infty}\Theta(\zeta_n, \eta_n, \lambda) = \Theta(\zeta, \eta, \lambda),$$

$$lim_{n\to\infty}\Upsilon(\zeta_n, \eta_n, \lambda) = \Upsilon(\zeta, \eta, \lambda).$$

Then ϱ and φ have a unique common unique fixed point in Σ.

Proof. Let ζ_0 be an arbitrary point in Σ. By the condition (i), Let $\zeta_1 \in \Sigma$ such that $\varrho(\zeta_0) = \varphi(\zeta_1)$. So we choose ζ_{n+1} such that $\varrho(\zeta_n) = \varphi(\zeta_{n+1})$ for all $n \geq 0$. Then, for all $\lambda > 0$,

$$\begin{aligned}\Xi(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \lambda) &\geq \gamma(\Xi(\varphi(\zeta_n), \varphi(\zeta_{n+1}), \lambda))\\ &= \gamma(\Xi(\varrho(\zeta_{n-1}), \varrho(\zeta_n), \lambda)\\ &> \Xi(\varrho(\zeta_{n-1}), \varrho(\zeta_n), \lambda) \qquad (4.8.1)\\ \Theta(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \lambda) &\leq \gamma^{'}(\Theta(\varphi(\zeta_n), \varphi(\zeta_{n+1}), \lambda))\\ &= \gamma^{'}(\Theta(\varrho(\zeta_{n-1}), \varrho(\zeta_n), \lambda)\\ &< \Theta(\varrho(\zeta_{n-1}), \varrho(\zeta_n), \lambda) \qquad (4.8.2)\\ \Upsilon(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \lambda) &\leq \gamma^{''}(\Upsilon(\varphi(\zeta_n), \varphi(\zeta_{n+1}), \lambda))\\ &= \gamma^{''}(\Upsilon(\varrho(\zeta_{n-1}), \varrho(\zeta_n), \lambda)\\ &< \Upsilon(\varrho(\zeta_{n-1}), \varrho(\zeta_n), \lambda) \qquad (4.8.3)\end{aligned}$$

since $\gamma(\lambda) > \lambda$, $\gamma^{'}(\lambda) < \lambda$ and $\gamma^{''}(\lambda) < \lambda$ for all $0 < \lambda < 1$. Thus

$\{\Xi(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \lambda)\}$ is an increasing sequence of positive real numbers in [0, 1] and $\{\Theta(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \lambda)\}$, $\{\Upsilon(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \lambda)\}$ is a decreasing sequence of positive real number in [0, 1]. Therefore, they converge to the limits $S \leq 1$, $S^{'} < 0$ and $S^{''} < 0$, respectively.
Now, we claim that $S = 1$, $S^{'} = 0$ and $S^{''} = 0$. For, let $S < 1$. Letting $n \to \infty$ in (4.8.1), we have $S \geq \gamma(S) > S$, which is a contradiction and so $S = 1$. Similarly, let $S^{'} > 0$ and $S^{''} > 0$. Letting $n \to \infty$ in (4.8.2) and (4.8.3), we have $S^{'} \geq \gamma(S^{'}) > S^{'}$ and $S^{''} \geq \gamma(S^{''}) > S^{''}$. which is a contradiction and so $S^{'} = 0$ and $S^{''} = 0$.
Now for any positive integer p and $\lambda > 0$, we get

$$\begin{aligned}
\Xi(\varrho(\zeta_n), \varrho(\zeta_{n+p}), \lambda) &\geq \Xi(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}) \star \cdots \star \Xi(\varrho(\zeta_{n+p-1}), \varrho(\zeta_{n+p}), \frac{\lambda}{p}) \\
&\geq \Xi(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}) \star \cdots \star \Xi(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}), \\
\Theta(\varrho(\zeta_n), \varrho(\zeta_{n+p}), \lambda) &\leq \Theta(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}) \diamond \cdots \diamond \Theta(\varrho(\zeta_{n+p-1}), \varrho(\zeta_{n+p}), \frac{\lambda}{p}) \\
&\leq \Theta(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}) \diamond \cdots \diamond \Theta(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}), \\
\Upsilon(\varrho(\zeta_n), \varrho(\zeta_{n+p}), \lambda) &\leq \Upsilon(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}) \diamond \cdots \diamond \Upsilon(\varrho(\zeta_{n+p-1}), \varrho(\zeta_{n+p}), \frac{\lambda}{p}) \\
&\leq \Upsilon(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}) \diamond \cdots \diamond \Upsilon(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}).
\end{aligned}$$

Since, we have

$$\begin{aligned}
lim_{n\to\infty}\Xi(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}) &= 1, \\
lim_{n\to\infty}\Theta(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}) &= 0, \\
lim_{n\to\infty}\Upsilon(\varrho(\zeta_n), \varrho(\zeta_{n+1}), \frac{\lambda}{p}) &= 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
lim_{n\to\infty}\Xi(\varrho(\zeta_n), \varrho(\zeta_{n+p}), \frac{\lambda}{p}) &\geq 1 \star \cdots \star 1 \geq 1, \\
lim_{n\to\infty}\Theta(\varrho(\zeta_n), \varrho(\zeta_{n+p}), \frac{\lambda}{p}) &\leq 0 \diamond \cdots \diamond \leq 0, \\
lim_{n\to\infty}\Upsilon(\varrho(\zeta_n), \varrho(\zeta_{n+p}), \frac{\lambda}{p}) &\leq 0 \diamond \cdots \diamond \leq 0.
\end{aligned}$$

Thus, by definition (3.4), $\{\varrho(\zeta_n)\}$ is a Cauchy sequence and by the completeness of Σ, $\{\varrho(\zeta_n)\}$converges to a point $\omega \in \Sigma$. Also, $\{\varphi(\zeta_n)\}$ converges to the point ω.
Suppose that, by (ii) the mapping ϱ is continuous. Then $lim_{n\to\infty}\varrho\varrho(\zeta_n) = \varrho(\omega)$ and $lim_{n\to\infty}\varrho\varphi(\zeta_n) = \varrho(\omega)$. Further, since ϱ and φ are R-weakly commuting, we have

$$\Xi(\varrho\varphi(\zeta_n), \varphi\varrho(\zeta_n), \lambda) \geq \Xi(\varrho(\zeta_n), \varphi(\zeta_n), \frac{\lambda}{R}),$$
$$\Theta(\varphi\varrho(\zeta_n), \varrho\varphi(\zeta_n), \lambda) \leq \Theta(\varrho(\zeta_n), \varphi(\zeta_n), \frac{\lambda}{R}),$$
$$\Upsilon(\varphi\varrho(\zeta_n), \varrho\varphi(\zeta_n), \lambda) \leq \Upsilon(\varrho(\zeta_n), \varphi(\zeta_n), \frac{\lambda}{R}).$$

Letting $n \to \infty$ by the definition of NMS, we have $lim_{n\to\infty}\varphi\varrho(\zeta_n) = \varrho(\omega.)$
Now, we show that $\omega = \varrho(\omega)$. Suppose $\omega \neq \varrho(\omega)$. Then there exists $\lambda > 0$ such that

$$\Xi(\omega, \varrho(\omega), \lambda) < 1, \quad \Theta(\omega, \varrho(\omega), \lambda) > 1, \quad \Upsilon(\omega, \varrho(\omega), \lambda) > 1.$$

By (iii), we have

$$\Xi(\varrho(\zeta_n), \varrho\varrho(\zeta_n), \lambda) \geq \gamma(\Xi(\varphi(\zeta_n), \varphi\varrho(\zeta_n), \lambda)),$$
$$\Theta(\varrho(\zeta_n), \varrho\varrho(\zeta_n), \lambda) \leq \gamma^{'}(\Theta(\varphi(\zeta_n), \varphi\varrho(\zeta_n), \lambda)),$$
$$\Upsilon(\varrho(\zeta_n), \varrho\varrho(\zeta_n), \lambda) \leq \gamma^{''}(\Upsilon(\varphi(\zeta_n), \varphi\varrho(\zeta_n), \lambda)).$$

Letting $n \to \infty$ in the above inequalities, we get

$$\Xi(\omega, \varrho(\omega), \lambda) \geq \gamma(\Xi(\omega, \varrho(\omega), \lambda)) > \Xi(\omega, \varrho(\omega), \lambda),$$
$$\Theta(\omega, \varrho(\omega), \lambda) \leq \gamma^{'}(\Theta(\omega, \varrho(\omega), \lambda)) < \Theta(\omega, \varrho(\omega), \lambda),$$
$$\Upsilon(\omega, \varrho(\omega), \lambda) \leq \gamma^{''}(\Upsilon(\omega, \varrho(\omega), \lambda)) < \Upsilon(\omega, \varrho(\omega), \lambda).$$

Which are contradiction. Therefore, $\omega = \varrho(\omega)$. By condition (i), we can find a point $\omega_1 \in \Sigma$ such that $\omega = \varrho(\omega) = \varphi(\omega_1)$. Now, it follows that,

$$\Xi(\varrho\varrho(\zeta_n), \varrho(\zeta_1), \lambda) \geq \gamma(\Xi(\varphi\varrho(\zeta_n), \varphi(\zeta_1), \lambda)),$$
$$\Theta(\varrho\varrho(\zeta_n), \varrho(\zeta_1), \lambda) \leq \gamma^{'}(\Theta(\varphi\varrho(\zeta_n), \varphi(\zeta_1), \lambda)),$$
$$\Upsilon(\varrho\varrho(\zeta_n), \varrho(\zeta_1), \lambda) \leq \gamma^{''}(\Upsilon(\varphi\varrho(\zeta_n), \varphi(\zeta_1), \lambda)).$$

Letting $n \to \infty$ in the above inequalities, we have

$$\Xi(\varrho(\omega), \varrho(\omega_1), \lambda) \geq \gamma(\Xi(\varrho(\omega), \varphi(\omega_1), \lambda)) = 1,$$
$$\Theta(\varrho(\omega), \varrho(\omega_1), \lambda) \leq \gamma^{'}(\Theta(\varrho(\omega), \varphi(\omega_1), \lambda)) = 0,$$
$$\Upsilon(\varrho(\omega), \varrho(\omega_1), \lambda) \leq \gamma^{''}(\Upsilon(\varrho(\omega), \varphi(\omega_1), \lambda)) = 0.$$

which implies that $\varrho(\omega) = \varrho(\omega_1)$ since $\gamma(\lambda) = 1$, $\gamma^{'}(\lambda) = 0$ and $\gamma^{''}(\lambda) = 0$ for $\lambda = 1$. So, we get $\omega = \varrho(\omega) = \varrho(\omega_1) = \varphi(\omega_1)$. For any $\lambda > 0$,

$$\Xi(\varrho(\zeta), \varphi(\zeta), \lambda) = \Xi(\varrho\varphi(\zeta_1), \varphi\varrho(\zeta_1), \lambda)) \geq \Xi(\varrho(\zeta_1), \varphi(\zeta)1), \frac{\lambda}{R}) = 1,$$
$$\Theta(\varrho(\zeta), \varphi(\zeta), \lambda) = \Theta(\varrho\varphi(\zeta_1), \varphi\varrho(\zeta_1), \lambda)) \leq \Theta(\varrho(\zeta_1), \varphi(\zeta)1), \frac{\lambda}{R}) = 0,$$
$$\Upsilon(\varrho(\zeta), \varphi(\zeta), \lambda) = \Upsilon(\varrho\varphi(\zeta_1), \varphi\varrho(\zeta_1), \lambda)) \leq \Upsilon(\varrho(\zeta_1), \varphi(\zeta)1), \frac{\lambda}{R}) = 0.$$

Which again implies that $\varrho(\omega) = \varphi(\omega)$. Hence ω is a common fixed point of ϱ and φ. Next, we prove the uniqueness, let η ($\eta \neq \omega$) be another common fixed point of ϱ and φ. Then there exists $\lambda > 0$ such that,$\Xi(\omega,\eta,\lambda) < 1, \Theta(\omega,\eta,\lambda) > 0, \Upsilon(\omega,\eta,\lambda) > 0$ and

$$\Xi(\omega,\eta,\lambda) = \Xi(\varrho(\zeta),\varphi(\zeta),\lambda) \geq \gamma(\Xi(\varphi(\omega),\varphi(\eta),\lambda)) = \gamma(\Xi(\omega,\eta,\lambda) > \Xi(\omega,\eta,\lambda),$$
$$\Theta(\omega,\eta,\lambda) = \Theta(\varrho(\zeta),\varphi(\zeta),\lambda) \leq \gamma^{'}(\Theta(\varphi(\omega),\varphi(\eta),\lambda)) = \gamma^{'}(\Theta(\omega,\eta,\lambda) < \Theta(\omega,\eta,\lambda),$$
$$\Upsilon(\omega,\eta,\lambda) = \Upsilon(\varrho(\zeta),\varphi(\zeta),\lambda) \leq \gamma^{''}(\Upsilon(\varphi(\omega),\varphi(\eta),\lambda)) = \gamma^{''}(\Upsilon(\omega,\eta,\lambda) < \Upsilon(\omega,\eta,\lambda).$$

Which is a contradiction. Since $\gamma(\lambda) > \lambda$, $\gamma^{'}(\lambda) < \lambda$ and $\gamma^{''}(\lambda) < \lambda$ for any $0 < \lambda < 1$. Therefore $\eta = \omega$. Hence η is the only common fixed point of ϱ and φ. Hence Proved.
Now, we prove an example to validate the above theorem.
Example 4.9 Let $\Sigma = \{\frac{1}{n}; n \in \mathbb{N}\} \cup \{0\}$ with metric d defined by $d(\zeta,\eta) = |\zeta - \eta|$. For all $\zeta,\eta \in \Sigma$ and $\lambda \in (0,\infty)$, define

$$\Xi(\zeta,\eta,\lambda) = \frac{\lambda}{\lambda + |\zeta - \eta|}; \quad \Theta(\zeta,\eta,\lambda) = \frac{|\zeta - \eta|}{\lambda + |\zeta - \eta|}; \quad \Upsilon(\zeta,\eta,\lambda) = \frac{|\zeta - \eta|}{\lambda}$$

Clearly $(\Sigma,\Xi,\Theta,\Upsilon,\star,\diamond)$ is a complete neutrosophic metric space on Σ. Here $\star$ is defined by $\zeta \star \eta = \zeta\eta$ and $\diamond$ is defined as $\zeta \diamond \eta = min\{1,\ \zeta + \eta\}$.
Define

$$\varrho(\zeta) = 1, \quad \varphi(\zeta) = \begin{cases} 1, & if \quad \zeta \quad is\ a\ rational\ number \\ 0, & if \quad \zeta \quad is\ an\ irrational\ number. \end{cases}$$

It is evident that $\varrho \subset \varphi$, also ϱ is continuous and φ is discontinuous. Define a function $\gamma : [0,1] \to [0,1]$ by $\gamma(\lambda) = \sqrt{\lambda}$ for any $0 < \lambda < 1$ and $\gamma(\lambda) = 1$ for $\lambda = 1, \gamma^{'} : [0,1] \to [0,1]$ by $\gamma^{'}(\lambda) = \lambda^2$ for any $0 < \lambda < 1$ and $\gamma^{'}(\lambda) = 0$, for $\lambda = 0$. Next, we define $\gamma^{''} : [0,1] \to [0,1]$ by $\gamma^{''}(\lambda) = \lambda^2$ for any $0 < \lambda < 1$ and $\gamma^{''}(\lambda) = 0$, for $\lambda = 0$. Then $\gamma(\lambda) > \lambda \quad \gamma^{'}(\lambda) < \lambda \quad \gamma^{''}(\lambda) < \lambda$ for any $0 < \lambda < 1$, we have

$$\Xi(\varrho(\zeta),\varrho(\eta),\lambda) \geq \gamma\Xi(\varphi(\zeta),\varphi(\eta),\lambda),$$
$$\Theta(\varrho(\zeta),\varrho(\eta),\lambda) \leq \gamma^{'}\Theta(\varphi(\zeta),\varphi(\eta),\lambda),$$
$$\Upsilon(\varrho(\zeta),\varrho(\eta),\lambda) \leq \gamma^{''}\Upsilon(\varphi(\zeta),\varphi(\eta),\lambda).$$

for all $\zeta,\eta \in \Sigma$. Also ϱ *and* φ are R- weakly commuting. Thus, all the conditions of Theorem (4.8) are satisfied and so ϱ and φ have 1 as a common fixed point.

Conclusion: In this manuscript, we explored new results in the notion of neutrosophic metric spaces (NMS) due to Kirisci, Simsek. We first formulated the definition of weakly commuting and R-weakly commuting mappings in NMS and proved the neutrosophic version of Pant's theorem. Also, we have given some examples to validate our results.

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