

4-20-2021

Some New Structures in Neutrosophic Metric Spaces

M. Jeyaraman

V. Jeyanthi

A.N. Mangayarkkarasi

Florentin Smarandache

Follow this and additional works at: https://digitalrepository.unm.edu/nss_journal

Recommended Citation

Jeyaraman, M.; V. Jeyanthi; A.N. Mangayarkkarasi; and Florentin Smarandache. "Some New Structures in Neutrosophic Metric Spaces." *Neutrosophic Sets and Systems* 42, 1 (2021).

https://digitalrepository.unm.edu/nss_journal/vol42/iss1/4

This Article is brought to you for free and open access by UNM Digital Repository. It has been accepted for inclusion in Neutrosophic Sets and Systems by an authorized editor of UNM Digital Repository. For more information, please contact disc@unm.edu.



Some New Structures in Neutrosophic Metric Spaces

M. Jeyaraman¹, V. Jeyanthi², A.N. Mangayarkkarasi³ and Florentin Smarandache⁴

¹P.G. and Research Department of Mathematics,
Raja Doraisingam Government Arts College, Sivagangai.
Affiliated to Alagappa University, Karaikudi, Tamilnadu, India.
E-mail: jeyaraman.maths@rdgcollege.in

ORCID: <https://orcid.org/0000-0002-0364-1845>
²Government Arts College for Women, Sivagangai.
Affiliated to Alagappa University, Karaikudi, Tamilnadu, India.
E-mail: jeykaliappa@gmail.com.

³Department of Mathematics,
Nachiappa Swamigal Arts & Science College, Karaikudi.
Affiliated to Alagappa University, Karaikudi, Tamilnadu, India.
E-mail: murugappan.mangai@gmail.com

⁴Department of Mathematics,
University of New Mexico,
705 Gurley Avenue, Gallup, NM 87301, USA.

Abstract: Neutrosophic sets deals with inconsistent, indeterminate and imprecise datas. The concept of Neutrosophic Metric Space (NMS) uses the idea of continuous t - norm and continuous t - conorm in intuitionistic fuzzy metric spaces. In this paper, we introduce the definition of subcompatible maps of types (J-1 and J-2). We extend the structure of weak non-Archimedean with the help of subcompatible maps of types (J-1 and J-2) in NMS. Finally, we obtain common fixed point theorems for four subcompatible maps of type (J-1) in weak non-Archimedean NMS.

Keywords: Weak non-Archimedean, NMS, Compatible map, Sub compatible, Subcompatible maps of types (J-1) and (J-2).

1. Introduction

Fuzzy set was presented by Zadeh [22] as a class of elements with a grade of membership. Kramosil and Michalek [8] defined new notion called Fuzzy Metric Space (FMS). Later, many authors have examined the concept of fuzzy metric in various aspects. In 2013, Muthuraj and Pandiselvi [17] introduced the concept of compatible mappings of type (P-1) and type (P-2) in generalized fuzzy metric spaces and obtains common fixed point theorems are obtained for compatible maps of type (P-1) and type (P-2). Since then, many authors have obtained fixed point results in fuzzy metric space using these compatible notions.

Atanassov [1] introduced and studied the notion of intuitionistic fuzzy set by generalizing the notion of fuzzy set. Park [9] defined the notion of intuitionistic fuzzy metric space as a

generalization of fuzzy metric space. In 1998, Smarandache [14-16] characterized the new concept called neutrosophic logic and neutrosophic set and explored many results in it. In the idea of neutrosophic sets, there is T degree of membership, I degree of indeterminacy and F degree of non-membership. Baset et al. [2] Explored the neutrosophic applications in dif and only iferent fields such as model for sustainable supply chain risk management, resource levelling problem in construction projects, Decision Making.

In 2019, Kirisci et al [9] defined NMS as a generalization of IFMS and brings about fixed point theorems in complete NMS. Erduran et.al.[13] introduced the concept of weak non-Archimedean intuitionistic fuzzy metric space and proved a common fixed point theorem for a pair of generalized (φ, Ψ) – contractive mappings. Later Jeyaraman at el [19,20] proved Fixed point results in non-Archimedean generalized intuitionistic fuzzy metric spaces. In 2020, Sowndrarajan Jeyaraman and Florentin Smarandache [18] proved some fixed point results for contraction theorems in neutrosophic metric spaces.

In this paper, we introduce the definition of sub compatible maps and sub compatible maps of types (J-1) and (J-2) in weak non-Archimedean NMS and give some examples and relationship between these definitions. We extend the structure of weak non-Archimedean with the help of subcompatible maps of types (J-1 and J-2) in NMS. Thereafter, we prove common fixed point theorems for four subcompatible maps of type (J-1) in weak non-Archimedean NMS.

2. Preliminaries

Definition: 2.1

A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm [CTN] if it satisfies the following conditions :

- (i) $*$ is commutative and associative,
- (ii) $*$ is continuous,
- (iii) $\varepsilon_1 * 1 = \varepsilon_1$ for all $\varepsilon_1 \in [0, 1]$,
- (iv) $\varepsilon_1 * \varepsilon_2 \leq \varepsilon_3 * \varepsilon_4$ whenever $\varepsilon_1 \leq \varepsilon_3$ and $\varepsilon_2 \leq \varepsilon_4$, for each $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0, 1]$.

Definition: 2.2

A binary operation \diamond : $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-conorm [CTC] if it satisfies the following conditions:

- (i) \diamond is commutative and associative,
- (ii) \diamond is continuous,
- (iii) $\varepsilon_1 \diamond 0 = \varepsilon_1$ for all $\varepsilon_1 \in [0, 1]$,
- (iv) $\varepsilon_1 \diamond \varepsilon_2 \leq \varepsilon_3 \diamond \varepsilon_4$ whenever $\varepsilon_1 \leq \varepsilon_3$ and $\varepsilon_2 \leq \varepsilon_4$, for each $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and $\varepsilon_4 \in [0, 1]$.

Definition: 2.3

A 6-tuple $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is said to be an NMS (shortly NMS), if Σ is an arbitrary non empty set, $*$ is a neutrosophic CTN, \diamond is a neutrosophic CTC and Ξ, Θ and Υ are neutrosophic on $\Sigma^3 \times \mathbb{R}^+$ satisfying the following conditions:

For all $\zeta, \eta, \delta, \omega \in \Sigma, \lambda \in \mathbb{R}^+$.

1. $0 \leq \Xi (\zeta, \eta, \delta, \lambda) \leq 1; 0 \leq \Theta (\zeta, \eta, \delta, \lambda) \leq 1; 0 \leq \Upsilon (\zeta, \eta, \delta, \lambda) \leq 1;$
2. $\Xi (\zeta, \eta, \delta, \lambda) + \Theta (\zeta, \eta, \delta, \lambda) + \Upsilon (\zeta, \eta, \delta, \lambda) \leq 3;$
3. $\Xi (\zeta, \eta, \delta, \lambda) = 1$ if and only if $\zeta = \eta = \delta;$
4. $\Xi (\zeta, \eta, \delta, \lambda) = \Xi (\rho (\zeta, \eta, \delta, \lambda))$, when ρ is the permutation function;
5. $\Xi (\zeta, \eta, \omega, \lambda) * \Xi (\omega, \delta, \delta, \mu) \leq \Xi (\zeta, \eta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0;$

6. $\Xi (\zeta, \eta, \delta, \cdot) : [0, \infty) \rightarrow [0, 1]$ is neutrosophic continuous ;
7. $\lim_{\lambda \rightarrow \infty} \Xi (\zeta, \eta, \delta, \lambda) = 1$ for all $\lambda > 0$;
8. $\Theta (\zeta, \eta, \delta, \lambda) = 0$ if and only if $\zeta = \eta = \delta$;
9. $\Theta (\zeta, \eta, \delta, \lambda) = \Theta (\rho (\zeta, \eta, \delta, \lambda))$, when ρ is the permutation function;
10. $\Theta (\zeta, \eta, \omega, \lambda) \diamond \Theta (\omega, \delta, \delta, \mu) \geq \Theta (\zeta, \eta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
11. $\Theta (\zeta, \eta, \delta, \cdot) : [0, \infty) \rightarrow [0, 1]$ is neutrosophic continuous;
12. $\lim_{\lambda \rightarrow \infty} \Theta (\zeta, \eta, \delta, \lambda) = 0$ for all $\lambda > 0$;
13. $\Upsilon (\zeta, \eta, \delta, \lambda) = 0$ if and only if $\zeta = \eta = \delta$;
14. $\Upsilon (\zeta, \eta, \delta, \lambda) = \Upsilon (\rho (\zeta, \eta, \delta, \lambda))$, when ρ is the permutation function;
15. $\Upsilon (\zeta, \eta, \omega, \lambda) \diamond \Upsilon (\omega, \delta, \delta, \mu) \geq \Upsilon (\zeta, \eta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
16. $\Upsilon (\zeta, \eta, \delta, \cdot) : [0, \infty) \rightarrow [0, 1]$ is neutrosophic continuous;
17. $\lim_{\lambda \rightarrow \infty} \Upsilon (\zeta, \eta, \delta, \lambda) = 0$ for all $\lambda > 0$;
18. If $\lambda > 0$ then $\Xi (\zeta, \eta, \delta, \lambda) = 0$; $\Theta (\zeta, \eta, \delta, \lambda) = 1$; $\Upsilon (\zeta, \eta, \delta, \lambda) = 1$.

Then, (Ξ, Θ, Υ) is called an NMS on Σ . The functions Ξ, Θ and Υ denote degree of closedness, naturalness and non-closedness between ζ, η and δ with respect to λ respectively.

Example: 2.4

Let (Σ, D) be a metric space. Define $\omega * \tau = \min \{ \omega, \tau \}$ and $\omega \diamond \tau = \max \{ \omega, \tau \}$ and $\Xi, \Theta, \Upsilon : \Sigma^3 \times \mathbb{R}^+ \rightarrow [0, 1]$ defined by, we define $\Xi (\zeta, \eta, \delta, \lambda) = \frac{\lambda}{\lambda + D(\zeta, \eta, \delta)}$; $\Theta (\zeta, \eta, \delta, \lambda) = \frac{D(\zeta, \eta, \delta)}{\lambda + D(\zeta, \eta, \delta)}$; $\Upsilon (\zeta, \eta, \delta, \lambda) = \frac{D(\zeta, \eta, \delta)}{\lambda}$ for all $\zeta, \eta, \delta \in \Sigma$ and $\lambda > 0$. Then $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is called NMS induced by a metric D the standard neutrosophic metric.

Remark: 2.5

In NMSE $(\zeta, \eta, \delta, \lambda, \cdot)$ is non-decreasing, $\Theta (\zeta, \eta, \delta, \cdot)$ is non-increasing and $\Upsilon (\zeta, \eta, \delta, \cdot)$ is decreasing for all $\zeta, \eta, \delta \in \Sigma$.

In the above definition, if the triangular inequality (v), (x) and (xv) are replaced by the following:

$$\begin{aligned} \Xi (\zeta, \eta, \delta, \max\{\lambda, \mu\}) &\geq \Xi (\zeta, \eta, \omega, \lambda) * \Xi (\omega, \delta, \delta, \mu), \\ \Theta (\zeta, \eta, \delta, \min\{\lambda, \mu\}) &\leq \Theta (\zeta, \eta, \omega, \lambda) \diamond \Theta (\omega, \delta, \delta, \mu), \\ \Upsilon (\zeta, \eta, \delta, \min\{\lambda, \mu\}) &\leq \Upsilon (\zeta, \eta, \omega, \lambda) \diamond \Upsilon (\omega, \delta, \delta, \mu) \end{aligned}$$

or equivalently

$$\begin{aligned} \Xi (\zeta, \eta, \delta, \lambda) &\geq \Xi (\zeta, \eta, \omega, \lambda) * \Xi (\omega, \delta, \delta, \lambda), \\ \Theta (\zeta, \eta, \delta, \lambda) &\leq \Theta (\zeta, \eta, \omega, \lambda) \diamond \Theta (\omega, \delta, \delta, \lambda), \\ \Upsilon (\zeta, \eta, \delta, \lambda) &\leq \Upsilon (\zeta, \eta, \omega, \lambda) \diamond \Upsilon (\omega, \delta, \delta, \lambda). \end{aligned}$$

Then $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is called non-Archimedean NMS. It is easy to check that the triangle inequality (NA) implies (5), (10) and (15), that is, every non-Archimedean NMS is itself an NMS.

Example: 2.6

Let Σ be a non-empty set with at least two elements. Define $\Xi (\zeta, \eta, \delta, \lambda)$ by: If we define the neutrosophic set $(\Sigma, \Xi, \Theta, \Upsilon)$ by $\Xi (\zeta, \zeta, \zeta, \lambda) = 1, \Theta (\zeta, \zeta, \zeta, \lambda) = 0$ and $\Upsilon (\zeta, \zeta, \zeta, \lambda) = 0$ for all $\zeta \in \Sigma$ and $\lambda > 0$, and $\Xi (\zeta, \eta, \delta, \lambda) = 0, \Theta (\zeta, \eta, \delta, \lambda) = 1$ and $\Upsilon (\zeta, \eta, \delta, \lambda) = 1$, for $\zeta \neq \eta \neq \delta$ and $0 < \lambda \leq 1$, and $\Xi (\zeta, \eta, \delta, \lambda) = 1, \Theta (\zeta, \eta, \delta, \lambda) = 0$ and $\Upsilon (\zeta, \eta, \delta, \lambda) = 0$, for $\zeta \neq \eta \neq \delta$ and $\lambda > 1$. Then $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is a non-Archimedean NMS with arbitrary $*$ is a neutrosophic CTN, \diamond is a neutrosophic CTC. Clearly $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is also an NMS.

Definition: 2.7

In Definition 2.3, if the triangular inequality (v), (x) and (xv) are replaced by the following:
 $\Xi(\zeta, \eta, \delta, \lambda) \geq \max \{ \Xi(\zeta, \eta, \omega, \lambda) * \Xi(\omega, \delta, \delta, \lambda/2), \Xi(\zeta, \eta, \omega, \lambda/2) * \Xi(\omega, \delta, \delta, \lambda) \},$
 $\Theta(\zeta, \eta, \delta, \lambda) \leq \min \{ \Theta(\zeta, \eta, \omega, \lambda) \circ \Theta(\omega, \delta, \delta, \lambda/2), \Theta(\zeta, \eta, \omega, \lambda/2) \circ \Theta(\omega, \delta, \delta, \lambda) \},$
 $Y(\zeta, \eta, \delta, \lambda) \leq \min \{ Y(\zeta, \eta, \omega, \lambda) \diamond Y(\omega, \delta, \delta, \lambda/2), Y(\zeta, \eta, \omega, \lambda/2) \diamond Y(\omega, \delta, \delta, \lambda) \},$
 for all $\Xi, \Theta, Y \in \Sigma$ and $\lambda > 0$, then $(\Sigma, \Xi, \Theta, Y, *, \diamond)$ is said to be a Weak Non- Archimedean (WNA) NMS.

Obviously, every non-Archimedean NMS is itself a weak non-Archimedean NMS. The inequality (WNA) does not imply that $\Xi(\zeta, \eta, \delta, \lambda, \cdot)$ is non-decreasing, $\Theta(\zeta, \eta, \delta, \cdot)$ is non-increasing and $Y(\zeta, \eta, \delta, \cdot)$ is decreasing. Thus, a weak non-Archimedean NMS is not necessarily an NMS.

Example: 2.8

Let $\Sigma = [0, \infty)$ and define $\Xi(\zeta, \eta, \delta, \lambda)$; $\Theta(\zeta, \eta, \delta, \lambda)$ and $Y(\zeta, \eta, \delta, \lambda)$ by

$$\begin{aligned} \Xi(\zeta, \eta, \delta, \lambda) &= \begin{cases} 1, & \zeta = \eta = \delta \\ \frac{\lambda}{\lambda+1}, & \zeta \neq \eta \neq \delta \end{cases} \\ \Theta(\zeta, \eta, \delta, \lambda) &= \begin{cases} 0, & \zeta = \eta = \delta \\ \frac{1}{\lambda+1}, & \zeta \neq \eta \neq \delta \end{cases} \\ Y(\zeta, \eta, \delta, \lambda) &= \begin{cases} 0, & \zeta = \eta = \delta \\ \lambda + 1, & \zeta \neq \eta \neq \delta \end{cases} \end{aligned}$$

for all $\lambda > 0$. $(\Sigma, \Xi, \Theta, Y, *, \diamond)$ is a weak non-Archimedean NMS with $\omega * \tau = \omega\tau$ and $\omega \diamond \tau = \{ \omega + \tau - \omega\tau \}$ for every $\omega, \tau \in [0, 1]$.

Definition: 2.9

Let Γ and Ω be maps from an NMS $(\Sigma, \Xi, \Theta, Y, *, \diamond)$. Then the mappings are said to be compatible if

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &= 1, \\ \lim_{n \rightarrow \infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &= 0, \text{ and} \\ \lim_{n \rightarrow \infty} Y(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &= 0, \end{aligned}$$

for all $\lambda > 0$, whenever $\{\zeta_n\}$ is a sequence in Σ such that $\lim_{n \rightarrow \infty} \Gamma\zeta_n = \lim_{n \rightarrow \infty} \Omega\zeta_n = \zeta$ for some $\zeta \in \Sigma$.

Definition: 2.10

Let Γ and Ω be self mappings of an NMS $(\Sigma, \Xi, \Theta, Y, *, \diamond)$. Then the mappings are said to be compatible of type (J-1), if

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) &= 1, \\ \lim_{n \rightarrow \infty} \Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) &= 0, \text{ and} \\ \lim_{n \rightarrow \infty} Y(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) &= 0, \end{aligned}$$

for all $\lambda > 0$, whenever $\{\zeta_n\}$ is a sequence in Σ such that $\lim_{n \rightarrow \infty} \Gamma\zeta_n = \lim_{n \rightarrow \infty} \Omega\zeta_n = \zeta$ for some $\zeta \in \Sigma$.

Definition: 2.11

Let Γ and Ω be self mappings of an NMS $(\Sigma, \Xi, \Theta, Y, *, \diamond)$. Then the mappings are said to be compatible of type (J-2), if

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &= 1, \\ \lim_{n \rightarrow \infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &= 0, \text{ and} \end{aligned}$$

$$\lim_{n \rightarrow \infty} Y(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$

for all $\lambda > 0$, whenever $\{\zeta_n\}$ is a sequence in Σ such that $\lim_{n \rightarrow \infty} \Gamma\zeta_n = \lim_{n \rightarrow \infty} \Omega\zeta_n = \zeta$ for some $\zeta \in \Sigma$.

Definition:2.12

Let Γ and Ω be maps from an NMS $(\Sigma, \Xi, \Theta, Y, *, \diamond)$ into itself. The maps Γ and Ω are said to be Occasionally Weakly Compatible (OWC) if and only if there is a point $\zeta \in \Sigma$ which is a coincidence point of Γ and Ω at which Γ and Ω commute i.e., there is a point $\zeta \in \Sigma$ such that $\Gamma\zeta = \Omega\zeta$ and $\Gamma\Omega\zeta = \Omega\Gamma\zeta$.

Definition:2.13

Let Γ and Ω be maps from an NMS $(\Sigma, \Xi, \Theta, Y, *, \diamond)$. The maps Γ and Ω are said to be reciprocally continuous if $\lim_{n \rightarrow \infty} \Gamma\Omega\zeta_n = \Gamma\zeta$, $\lim_{n \rightarrow \infty} \Omega\Gamma\zeta_n = \Omega\zeta$, whenever $\{\zeta_n\}$ is a sequence in Σ such that $\lim_{n \rightarrow \infty} \Gamma\zeta_n = \lim_{n \rightarrow \infty} \Omega\zeta_n = \zeta$ for some $\zeta \in \Sigma$.

3. Types Of Subcompatible Maps In Weak Non-Archimedean NMS.

Definition:3.1

Let $(\Sigma, \Xi, \Theta, Y, *, \diamond)$ be a weak non-Archimedean NMS. Self- maps Γ and Ω on Σ are said to be subsequently continuous if there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n \rightarrow \infty} \Gamma\zeta_n = \lim_{n \rightarrow \infty} \Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfy $\lim_{n \rightarrow \infty} \Gamma\Omega\zeta_n = \Gamma\zeta$, $\lim_{n \rightarrow \infty} \Omega\Gamma\zeta_n = \Omega\zeta$.

Clearly, if Γ and Ω are continuous or reciprocally continuous, then they are subsequentially continuous, but converse is not true in general.

Example: 3.2

Let $\Sigma = [0, \infty)$ and define, for all $\lambda > 0$, $\Xi(\zeta, \eta, \delta, \lambda)$; $\Theta(\zeta, \eta, \delta, \lambda)$ and $Y(\zeta, \eta, \delta, \lambda)$ by

$$\Xi(\zeta, \eta, \delta, \lambda) = \begin{cases} 1, & \zeta = \eta = \delta, \\ \frac{\lambda}{\lambda+1}, & \zeta \neq \eta \neq \delta, \end{cases}$$

$$\Theta(\zeta, \eta, \delta, \lambda) = \begin{cases} 0, & \zeta = \eta = \delta, \\ \frac{1}{\lambda+1}, & \zeta \neq \eta \neq \delta, \end{cases}$$

$$Y(\zeta, \eta, \delta, \lambda) = \begin{cases} 0, & \zeta = \eta = \delta, \\ \lambda + 1, & \zeta \neq \eta \neq \delta. \end{cases}$$

Then $(\Sigma, \Xi, \Theta, Y, *, \diamond)$ is a weak non-Archimedean NMS with $\omega * \tau = \omega\tau$ and $\omega \diamond \tau = \{\omega + \tau - \omega\tau\}$ for every $\omega, \tau \in [0, 1]$. Define Γ and Ω as follows:

$$\Gamma\zeta = \begin{cases} 2, & \zeta < 3 \\ \zeta, & \zeta \geq 3 \end{cases}, \Omega\zeta = \begin{cases} 2\zeta - 4, & \zeta \leq 3, \\ 3, & \zeta > 3. \end{cases}$$

Clearly Γ and Ω are discontinuous at $\zeta = 3$. Let $\{\zeta_n\}$ be a sequence in Σ defined by $\zeta_n = 3 - \frac{1}{n}$ for $n = 1, 2, \dots$, then $\lim_{n \rightarrow \infty} \Gamma\zeta_n = \lim_{n \rightarrow \infty} \Omega\zeta_n = 2$, $2 \in \Sigma$ and $\lim_{n \rightarrow \infty} \Gamma\Omega\zeta_n = 2 = \Gamma(2)$, $\lim_{n \rightarrow \infty} \Omega\Gamma\zeta_n = 0 = \Omega(2)$. Therefore, Γ and Ω are subsequentially continuous. Now, let $\{\zeta_n\}$ be a sequence in Σ defined by $\zeta_n = 3 + \frac{1}{n}$ for $n = 1, 2, \dots$, then $\lim_{n \rightarrow \infty} \Gamma\zeta_n = \lim_{n \rightarrow \infty} \Omega\zeta_n = 3$, $3 \in \Sigma$ and $\lim_{n \rightarrow \infty} \Omega\Gamma\zeta_n = 3 \neq 2 = \Omega(3)$. Hence Γ and Ω are not reciprocally continuous.

Definition: 3.3

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS. Self- maps Γ and Ω on Σ are said to be subcompatible if and only if there exist a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n \rightarrow \infty} \Gamma \zeta_n = \lim_{n \rightarrow \infty} \Omega \zeta_n = \zeta, \zeta \in \Sigma$ and satisfies

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi(\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda) &= 1, \\ \lim_{n \rightarrow \infty} \Theta(\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda) &= 0, \text{ and} \\ \lim_{n \rightarrow \infty} \Upsilon(\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda) &= 0. \end{aligned}$$

It is easy to see that two owc maps are subcompatible, however the converse is not true in general. It is also interesting to see the following one-way implication:

Commuting \Rightarrow Weakly commuting \Rightarrow Compatibility \Rightarrow Weak compatibility \Rightarrow OWC \Rightarrow Sub compatibility.

Definition:3.4

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS. Self- maps Γ and Ω on Σ are said to be subcompatible of type (J-1) if there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n \rightarrow \infty} \Gamma \zeta_n = \lim_{n \rightarrow \infty} \Omega \zeta_n = \zeta, \zeta \in \Sigma$ and satisfies

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi(\Gamma \Omega \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &= 1, \\ \lim_{n \rightarrow \infty} \Theta(\Gamma \Omega \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &= 0, \\ \lim_{n \rightarrow \infty} \Upsilon(\Gamma \Omega \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &= 0, \\ \lim_{n \rightarrow \infty} \Xi(\Omega \Gamma \zeta_n, \Gamma \Gamma \zeta_n, \Gamma \Gamma \zeta_n, \lambda) &= 1, \\ \lim_{n \rightarrow \infty} \Theta(\Omega \Gamma \zeta_n, \Gamma \Gamma \zeta_n, \Gamma \Gamma \zeta_n, \lambda) &= 0, \text{ and,} \\ \lim_{n \rightarrow \infty} \Upsilon(\Omega \Gamma \zeta_n, \Gamma \Gamma \zeta_n, \Gamma \Gamma \zeta_n, \lambda) &= 0. \end{aligned}$$

Clearly, if Γ and Ω are compatible of type (J-1), then they are subcompatible of type (J-1), but converse is not true in general.

Example: 3.5

Let $\Sigma = [0, \infty)$. Define $\Xi(\zeta, \eta, \delta, \lambda); \Theta(\zeta, \eta, \delta, \lambda)$ and $\Upsilon(\zeta, \eta, \delta, \lambda)$ by $\Xi(\zeta, \eta, \delta, \lambda) = \frac{\lambda}{\lambda + |\zeta - \eta| + |\eta - \delta| + |\delta - \zeta|}$, $\Theta(\zeta, \eta, \delta, \lambda) = \frac{|\zeta - \eta| + |\eta - \delta| + |\delta - \zeta|}{\lambda + |\zeta - \eta| + |\eta - \delta| + |\delta - \zeta|}$ and $\Upsilon(\zeta, \eta, \delta, \lambda) = \frac{|\zeta - \eta| + |\eta - \delta| + |\delta - \zeta|}{\lambda}$ for all $\lambda > 0$. Then, $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is a weak non-Archimedean NMS with $\omega * \tau = \omega \tau$ and $\omega \diamond \tau = \{\omega + \tau - \omega \tau\}$ for every $\omega, \tau \in [0, 1]$.

Define Γ and Ω as follows:

$$\Gamma x = \begin{cases} \zeta^2 + 1, & \zeta < 1 \\ 2\zeta - 1, & \zeta \geq 1 \end{cases}, \quad \Omega \zeta = \begin{cases} \zeta + 1, & \zeta < 1 \\ 3\zeta - 2, & \zeta \geq 1 \end{cases}.$$

Let $\{\zeta_n\}$ be a sequence in Σ defined by $\zeta_n = 1 + \frac{1}{n}$, for $n = 1, 2, \dots$, then $\lim_{n \rightarrow \infty} \Gamma \zeta_n = \lim_{n \rightarrow \infty} \Omega \zeta_n = 1, 1 \in \Sigma$ and

$$\begin{aligned} \Gamma \Omega \zeta_n &= \Gamma \left(1 + \frac{3}{n}\right) = 2 \left(1 + \frac{3}{n}\right) - 1 = 1 + \left(\frac{6}{n}\right), \\ \Omega \Gamma \zeta_n &= \Omega \left(1 + \frac{2}{n}\right) = 3 \left(1 + \frac{2}{n}\right) - 2 = 1 + \left(\frac{6}{n}\right), \end{aligned}$$

$$\Gamma\zeta_n = \Gamma\left(1 + \frac{2}{n}\right) = 2\left(1 + \frac{2}{n}\right) - 1 = 1 + \left(\frac{4}{n}\right),$$

$$\Omega\Omega\zeta_n = \Omega\left(1 + \frac{3}{n}\right) = 3\left(1 + \frac{3}{n}\right) - 2 = 1 + \left(\frac{9}{n}\right).$$

Therefore,

$$\lim_{n \rightarrow \infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$

$$\lim_{n \rightarrow \infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0, \text{ and}$$

$$\lim_{n \rightarrow \infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

And,

$$\lim_{n \rightarrow \infty} \Xi(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 1,$$

$$\lim_{n \rightarrow \infty} \Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0, \text{ and}$$

$$\lim_{n \rightarrow \infty} \Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0.$$

That is, Γ and Ω are subcompatible of type (J-1) but if we consider a sequence $\zeta_n = 1 - \frac{1}{n}$ for $n = 1, 2, \dots$, then $\lim_{n \rightarrow \infty} \Gamma\zeta_n = \lim_{n \rightarrow \infty} \Omega\zeta_n = 2, 2 \in \Sigma$ and

$$\Gamma\Omega\zeta_n = \Gamma\left(2 - \frac{1}{n}\right) = 2\left(2 - \frac{1}{n}\right) - 1 = 3 - \left(\frac{2}{n}\right), \Omega\Gamma\zeta_n = \Omega\left(\left(1 - \frac{1}{n}\right)^2 + 1\right) = 3\left(\left(1 - \frac{1}{n}\right)^2 + 1\right) - 2,$$

$$\Gamma\Gamma\zeta_n = \Gamma\left(\left(1 - \frac{1}{n}\right)^2 + 1\right) = \Gamma\left(1 - \frac{2}{n} + \frac{1}{n^2}\right) = \left(1 - \frac{2}{n} + \frac{1}{n^2}\right)^2 + 1,$$

$$\Omega\Omega\zeta_n = \Omega\left(2 - \frac{1}{n}\right) = 3\left(2 - \frac{1}{n}\right) - 2 = 4 - \left(\frac{3}{n}\right).$$

Therefore,

$$\lim_{n \rightarrow \infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 1,$$

$$\lim_{n \rightarrow \infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 0,$$

$$\lim_{n \rightarrow \infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 0,$$

$$\lim_{n \rightarrow \infty} \Xi(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \neq 1,$$

$$\lim_{n \rightarrow \infty} \Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \neq 0, \text{ and}$$

$$\lim_{n \rightarrow \infty} \Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \neq 0.$$

That is, Γ and Ω are not compatible of type (J-1).

Definition: 3.6

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS. Self- maps Γ and Ω on Σ are said to be subcompatible of type (J-1) if and only if there exist a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n \rightarrow \infty} \Gamma\zeta_n = \lim_{n \rightarrow \infty} \Omega\zeta_n = \zeta, \zeta \in \Sigma$ and satisfies

$$\lim_{n \rightarrow \infty} \Xi(\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$

$$\lim_{n \rightarrow \infty} \Theta(\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$

$$\lim_{n \rightarrow \infty} \Upsilon(\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

Clearly, if Γ and Ω are compatible of type (J-2), then they are subcompatible of type (J-2), but converse is not true in general.

Example: 3.7

Let $\Sigma = [0, \infty)$ and define $\Xi(\zeta, \eta, \delta, \lambda); \Theta(\zeta, \eta, \delta, \lambda)$ and $\Upsilon(\zeta, \eta, \delta, \lambda)$ by

$$\Xi(\zeta, \eta, \delta, \lambda) = \begin{cases} 1, & \zeta = \eta = \delta, \\ \frac{\lambda}{\lambda + 1}, & \zeta \neq \eta \neq \delta, \end{cases}$$

$$\Theta(\zeta, \eta, \delta, \lambda) = \begin{cases} 0, & \zeta = \eta = \delta, \\ \frac{1}{\lambda+1}, & \zeta \neq \eta \neq \delta, \end{cases}$$

$$Y(\zeta, \eta, \delta, \lambda) = \begin{cases} 0, & \zeta = \eta = \delta, \\ \lambda + 1, & \zeta \neq \eta \neq \delta. \end{cases}$$

Then, $(\Sigma, \Xi, \Theta, Y, *, \diamond)$ is a weak non-Archimedean NMS with $\omega * \tau = \omega\tau$ and $\omega \diamond \tau = \{\omega + \tau - \omega\tau\}$ for every $\omega, \tau \in [0, 1]$. Define Γ and Ω as follows:

$$\Gamma\zeta = \zeta^2, \quad \Omega\zeta = \begin{cases} \zeta + 2, & \zeta \in [0,4] \cup (5, \infty) \\ \zeta + 12, & \zeta \in (4,5] \end{cases}.$$

Let $\{\zeta_n\}$ be a sequence in Σ defined by $\zeta_n = 2 + \frac{1}{n}$ for $n = 1, 2, \dots$, then $\lim_{n \rightarrow \infty} \Gamma\zeta_n = \lim_{n \rightarrow \infty} \Omega\zeta_n = 4$, and $\Gamma\Omega\zeta_n = \Gamma\left(\left(2 + \frac{1}{n}\right)^2\right) = \left(2 + \frac{1}{n}\right)^4$, $\Omega\Omega\zeta_n = \Omega\left(4 + \frac{1}{n}\right) = 4 + \frac{1}{n} + 12 = 16 + \frac{1}{n}$.

Therefore,

$$\lim_{n \rightarrow \infty} \Xi(\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$

$$\lim_{n \rightarrow \infty} \Theta(\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0, \text{ and}$$

$$\lim_{n \rightarrow \infty} Y(\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

That is, Γ and Ω are subcompatible of type (J-2) but if we consider a sequence $\zeta_n = 2 - \frac{1}{n}$ for $n = 1, 2, \dots$, then $\lim_{n \rightarrow \infty} \Gamma\zeta_n = \lim_{n \rightarrow \infty} \Omega\zeta_n = 4$ and $\Gamma\Omega\zeta_n = \Gamma\left(\left(2 - \frac{1}{n}\right)^2\right) = \left(2 - \frac{1}{n}\right)^4$, $\Omega\Omega\zeta_n = \Omega\left(4 - \frac{1}{n}\right) = 4 - \frac{1}{n} + 2 = 6 - \frac{1}{n}$.

Therefore,

$$\lim_{n \rightarrow \infty} \Xi(\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 1,$$

$$\lim_{n \rightarrow \infty} \Theta(\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 0, \text{ and}$$

$$\lim_{n \rightarrow \infty} Y(\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 0.$$

That is, Γ and Ω are not compatible of type (J-2).

Proposition: 3.8

Let $(\Sigma, \Xi, \Theta, Y, *, \diamond)$ be a weak non-Archimedean NMS and $\Gamma, \Omega: \Sigma \rightarrow \Sigma$ are subsequentially continuous mappings. Γ and Ω are subcompatible maps if and only if they are not subcompatible of type (J-1).

Proof:

Suppose Γ and Ω are subcompatible, then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n \rightarrow \infty} \Gamma\zeta_n = \lim_{n \rightarrow \infty} \Omega\zeta_n = \zeta, \zeta \in \Sigma$ and satisfying

$$\lim_{n \rightarrow \infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma_n, \Omega\Gamma_n, \lambda) = 1,$$

$$\lim_{n \rightarrow \infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma_n, \Omega\Gamma_n, \lambda) = 0, \text{ and}$$

$$\lim_{n \rightarrow \infty} Y(\Gamma\Omega\zeta_n, \Omega\Gamma_n, \Omega\Gamma_n, \lambda) = 0.$$

Since Γ and Ω are subsequentially continuous, we have

$$\lim_{n \rightarrow \infty} \Gamma\Omega\zeta_n = \Gamma\zeta = \lim_{n \rightarrow \infty} \Gamma\Gamma\zeta_n, \quad \lim_{n \rightarrow \infty} \Omega\Gamma\zeta_n = \Omega\zeta = \lim_{n \rightarrow \infty} \Omega\Omega\zeta_n.$$

Thus, from the inequality (WNA), for all $\lambda > 0$,

$$\Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) * \Xi(\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$

$$\Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \diamond \Theta(\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$

$$Y(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq Y(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \diamond Y(\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$

and it follows that

$$\Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq 1 * 1 = 1,$$

$$\begin{aligned} \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &\leq 0 \diamond 0 = 0, \\ \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &\leq 0 \diamond 0 = 0. \end{aligned}$$

That is, for all $\lambda > 0$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &= 1, \\ \lim_{n \rightarrow \infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &= 0, \\ \lim_{n \rightarrow \infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &= 0. \end{aligned}$$

By the same way,

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) &= 1, \\ \lim_{n \rightarrow \infty} \Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) &= 0, \\ \lim_{n \rightarrow \infty} \Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) &= 0. \end{aligned}$$

Consequently, Γ and Ω are subcompatible of type (J-1).

Conversely, suppose that Γ and Ω are subcompatible of type (J-1), then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n \rightarrow \infty} \Gamma\zeta_n = \lim_{n \rightarrow \infty} \Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfying

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &= 1, \lim_{n \rightarrow \infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0 \text{ and} \\ \lim_{n \rightarrow \infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &= 0, \lim_{n \rightarrow \infty} \Xi(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 1, \\ \lim_{n \rightarrow \infty} \Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) &= 0 \text{ and } \lim_{n \rightarrow \infty} \Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0. \end{aligned}$$

Since Γ and Ω are subsequentially continuous, we have

$$\lim_{n \rightarrow \infty} \Gamma\Omega\zeta_n = \Gamma\zeta = \lim_{n \rightarrow \infty} \Gamma\Gamma\zeta_n, \lim_{n \rightarrow \infty} \Omega\Gamma\zeta_n = \Omega\zeta = \lim_{n \rightarrow \infty} \Omega\Omega\zeta_n.$$

Now, from the inequality (WNA), for all $\lambda > 0$,

$$\begin{aligned} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &\geq \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) * \Xi(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2), \\ \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &\leq \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \diamond \Theta(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2), \\ \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &\leq \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \diamond \Upsilon(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2), \end{aligned}$$

and, it follows that,

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &\geq 1 * 1 = 1, \\ \lim_{n \rightarrow \infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &\leq 0 \diamond 0 = 0, \\ \lim_{n \rightarrow \infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &\leq 0 \diamond 0 = 0, \end{aligned}$$

which implies that

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &= 1, \\ \lim_{n \rightarrow \infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &= 0, \\ \lim_{n \rightarrow \infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &= 0. \end{aligned}$$

Therefore, Γ and Ω are subcompatible.

Proposition: 3.9

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS and $\Gamma, \Omega: \Sigma \rightarrow \Sigma$ are subsequentially continuous mappings. Γ and Ω are subcompatible maps if and only if they are not subcompatible of type (J-2).

Proof:

Suppose Γ and Ω are subcompatible, then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n \rightarrow \infty} \Gamma\zeta_n = \lim_{n \rightarrow \infty} \Omega\zeta_n = \delta$, $\delta \in \Sigma$ and satisfy

$$\lim_{n \rightarrow \infty} \Xi (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda) = 1, \lim_{n \rightarrow \infty} \Theta (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda) = 0, \text{ and } \lim_{n \rightarrow \infty} \Upsilon (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda) = 0.$$

Since Γ and Ω are subsequentially continuous, we have

$$\lim_{n \rightarrow \infty} \Gamma \Omega \zeta_n = \Gamma \zeta = \lim_{n \rightarrow \infty} \Gamma \Gamma \zeta_n, \lim_{n \rightarrow \infty} \Omega \Gamma \zeta_n = \Omega \zeta = \lim_{n \rightarrow \infty} \Omega \Omega \zeta_n.$$

Thus, from the inequality (WNA),

$$\begin{aligned} \Xi (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &\geq \Xi (\Gamma \Gamma \zeta_n, \Gamma \Omega \zeta_n, \Gamma \Omega \zeta_n, \lambda) * \Xi (\Gamma \Omega \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda/2) \\ &\geq \Xi (\Gamma \Gamma \zeta_n, \Gamma \Omega \zeta_n, \Gamma \Omega \zeta_n, \lambda) * \Xi (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda/2) * \\ &\quad \Xi (\Omega \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda/4), \\ \Theta (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &\leq \Theta (\Gamma \Gamma \zeta_n, \Gamma \Omega \zeta_n, \Gamma \Omega \zeta_n, \lambda) \diamond \Theta (\Gamma \Omega \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda/2) \\ &\leq \Theta (\Gamma \Gamma \zeta_n, \Gamma \Omega \zeta_n, \Gamma \Omega \zeta_n, \lambda) \diamond \Theta (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda/2) \diamond \\ &\quad \Theta (\Omega \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda/4) \text{ and} \\ \Upsilon (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &\leq \Upsilon (\Gamma \Gamma \zeta_n, \Gamma \Omega \zeta_n, \Gamma \Omega \zeta_n, \lambda) \diamond \Upsilon (\Gamma \Omega \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda/2) \\ &\leq \Upsilon (\Gamma \Gamma \zeta_n, \Gamma \Omega \zeta_n, \Gamma \Omega \zeta_n, \lambda) \diamond \Upsilon (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda/2) \diamond \\ &\quad \Upsilon (\Omega \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda/4), \end{aligned}$$

for all $\lambda > 0$, and, it follows that, for all $\lambda > 0$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &\geq 1 * 1 = 1, \\ \lim_{n \rightarrow \infty} \Theta (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &\leq 0 \diamond 0 = 0, \\ \lim_{n \rightarrow \infty} \Upsilon (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &\leq 0 \diamond 0 = 0, \end{aligned}$$

which implies that,

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &= 1, \\ \lim_{n \rightarrow \infty} \Theta (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &= 0, \\ \lim_{n \rightarrow \infty} \Upsilon (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &= 0. \end{aligned}$$

Consequently, Γ and Ω are subcompatible of type (J-2). Conversely, suppose that Γ and Ω are subcompatible of type (J-2), then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n \rightarrow \infty} \Gamma \zeta_n = \lim_{n \rightarrow \infty} \Omega \zeta_n = \zeta, \zeta \in \Sigma$ and satisfying

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &= 1, \\ \lim_{n \rightarrow \infty} \Theta (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &= 0, \\ \lim_{n \rightarrow \infty} \Upsilon (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &= 0. \end{aligned}$$

Now, from the inequality (WNA), we have

$$\begin{aligned} \Xi (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda) &\geq \Xi (\Gamma \Omega \zeta_n, \Gamma \Gamma \zeta_n, \Gamma \Gamma \zeta_n, \lambda) * \Xi (\Gamma \Gamma \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda/2) \\ &\geq \Xi (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda) * \Xi (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Gamma \zeta_n, \lambda/2) \\ &\quad * \Xi (\Omega \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda/4), \\ \Theta (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda) &\leq \Theta (\Gamma \Omega \zeta_n, \Gamma \Gamma \zeta_n, \Gamma \Gamma \zeta_n, \lambda) \diamond \Theta (\Gamma \Gamma \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda/2) \\ &\leq \Theta (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda) \diamond \Theta (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Gamma \zeta_n, \lambda/2) \\ &\quad \diamond \Theta (\Omega \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda/4) \text{ and} \\ \Upsilon (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda) &\leq \Upsilon (\Gamma \Omega \zeta_n, \Gamma \Gamma \zeta_n, \Gamma \Gamma \zeta_n, \lambda) \diamond \Upsilon (\Gamma \Gamma \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda/2) \\ &\leq \Upsilon (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda) \diamond \Upsilon (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Gamma \zeta_n, \lambda/2) \\ &\quad \diamond \Upsilon (\Omega \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda/4), \end{aligned}$$

and, it follows that, for all $\lambda > 0$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda) &\geq 1 * 1 * 1 = 1, \\ \lim_{n \rightarrow \infty} \Theta (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda) &\leq 0 \diamond 0 \diamond 0 = 0, \\ \lim_{n \rightarrow \infty} \Upsilon (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda) &\leq 0 \diamond 0 \diamond 0 = 0, \end{aligned}$$

which implies that

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda) &= 1, \\ \lim_{n \rightarrow \infty} \Theta (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda) &= 0, \\ \lim_{n \rightarrow \infty} \Upsilon (\Gamma \Omega \zeta_n, \Omega \Gamma \zeta_n, \Omega \Gamma \zeta_n, \lambda) &= 0. \end{aligned}$$

Therefore, Γ and Ω are subcompatible.

Proposition: 3.10

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS and $\Gamma, \Omega: \Sigma \rightarrow \Sigma$ are subsequentially continuous mappings. Γ and Ω are subcompatible maps of type (J-1) if and only if they are subcompatible of type (J-2).

Proof:

Suppose Γ and Ω are subcompatible of type (J-1), then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n \rightarrow \infty} \Gamma \zeta_n = \lim_{n \rightarrow \infty} \Omega \zeta_n = \zeta, \zeta \in \Sigma$ and satisfy

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &= 1, \\ \lim_{n \rightarrow \infty} \Theta (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &= 0, \text{ and,} \\ \lim_{n \rightarrow \infty} \Upsilon (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &= 0, \\ \lim_{n \rightarrow \infty} \Xi (\Omega \Gamma \zeta_n, \Gamma \Gamma \zeta_n, \Gamma \Gamma \zeta_n, \lambda) &= 1, \\ \lim_{n \rightarrow \infty} \Theta (\Omega \Gamma \zeta_n, \Gamma \Gamma \zeta_n, \Gamma \Gamma \zeta_n, \lambda) &= 0, \text{ and,} \\ \lim_{n \rightarrow \infty} \Upsilon (\Omega \Gamma \zeta_n, \Gamma \Gamma \zeta_n, \Gamma \Gamma \zeta_n, \lambda) &= 0. \end{aligned}$$

Since Γ and Ω are subsequentially continuous, we have

$$\lim_{n \rightarrow \infty} \Gamma \Omega \zeta_n = \Gamma \zeta = \lim_{n \rightarrow \infty} \Gamma \Gamma \zeta_n, \lim_{n \rightarrow \infty} \Omega \Gamma \zeta_n = \Omega \zeta = \lim_{n \rightarrow \infty} \Omega \Omega \zeta_n.$$

Thus, from the inequality (WNA),

$$\begin{aligned} \Xi (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &\geq \Xi (\Gamma \Gamma \zeta_n, \Gamma \Omega \zeta_n, \Gamma \Omega \zeta_n, \lambda) * \Xi (\Gamma \Omega \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda/2), \\ \Theta (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &\leq \Theta (\Gamma \Gamma \zeta_n, \Gamma \Omega \zeta_n, \Gamma \Omega \zeta_n, \lambda) \diamond \Theta (\Gamma \Omega \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda/2), \\ \Upsilon (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &\leq \Upsilon (\Gamma \Gamma \zeta_n, \Gamma \Omega \zeta_n, \Gamma \Omega \zeta_n, \lambda) \diamond \Upsilon (\Gamma \Omega \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda/2), \end{aligned}$$

and, it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &\geq 1 * 1 = 1, \\ \lim_{n \rightarrow \infty} \Theta (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &\leq 0 \diamond 0 = 0, \\ \lim_{n \rightarrow \infty} \Upsilon (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &\leq 0 \diamond 0 = 0, \end{aligned}$$

which implies that

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &= 1, \\ \lim_{n \rightarrow \infty} \Theta (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &= 0, \\ \lim_{n \rightarrow \infty} \Upsilon (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &= 0. \end{aligned}$$

Therefore, Γ and Ω are subcompatible of type (J-2).

Conversely, suppose that Γ and Ω are subcompatible of type (J-2), then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n \rightarrow \infty} \Gamma \zeta_n = \lim_{n \rightarrow \infty} \Omega \zeta_n = \zeta, \zeta \in \Sigma$ and satisfying

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &= 1, \\ \lim_{n \rightarrow \infty} \Theta (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \Omega \Omega \zeta_n, \lambda) &= 0, \\ \lim_{n \rightarrow \infty} \Upsilon (\Gamma \Gamma \zeta_n, \Omega \Omega \zeta_n, \lambda) &= 0. \end{aligned}$$

Now, from the inequality (WNA), we have

$$\begin{aligned} \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &\geq \Xi(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) * \Xi(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2), \\ \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &\leq \Theta(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \diamond \Theta(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2), \\ \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &\leq \Upsilon(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \diamond \Upsilon(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2), \end{aligned}$$

and, it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} (\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &\geq 1 * 1 = 1, \\ \lim_{n \rightarrow \infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &\leq 0 \diamond 0 = 0, \\ \lim_{n \rightarrow \infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &\leq 0 \diamond 0 = 0, \end{aligned}$$

which implies that, for all $\lambda > 0$,

$$\begin{aligned} \lim_{n \rightarrow \infty} (\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &= 1, \\ \lim_{n \rightarrow \infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &= 0, \\ \lim_{n \rightarrow \infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &= 0. \end{aligned}$$

By the same way, we obtain that

$$\begin{aligned} \lim_{n \rightarrow \infty} (\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) &= 1, \\ \lim_{n \rightarrow \infty} \Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) &= 0, \\ \lim_{n \rightarrow \infty} \Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) &= 0. \end{aligned}$$

Therefore, Γ and Ω are subcompatible of type (J-1).

4. Main Theorems

Theorem: 4.1

Let Γ, Λ, Ω and H be self-maps of a weak non-Archimedean NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ and let the pairs (Γ, Ω) and (Λ, H) are subcompatible maps of type (J-1) and subsequentially continuous.

$$\Xi(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \geq \psi(\min\{\Xi(\Omega\zeta, H\eta, H\eta, \lambda), \Xi(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Xi(\Lambda\eta, H\eta, H\eta, \lambda), \frac{1}{2}[\Xi(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Xi(\Gamma\zeta, H\eta, H\eta, \lambda)]\}) \tag{4.1.1}$$

$$\Theta(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \leq \phi(\max\{\Theta(\Omega\zeta, H\eta, H\eta, \lambda), \Theta(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Theta(\Lambda\eta, H\eta, H\eta, \lambda), \frac{1}{2}[\Theta(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Theta(\Gamma\zeta, H\eta, H\eta, \lambda)]\}) \tag{4.1.2}$$

$$\Upsilon(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \leq \varphi(\max\{\Upsilon(\Omega\zeta, H\eta, H\eta, \lambda), \Upsilon(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Upsilon(\Lambda\eta, H\eta, H\eta, \lambda), \frac{1}{2}[\Upsilon(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Upsilon(\Gamma\zeta, H\eta, H\eta, \lambda)]\}) \tag{4.1.3}$$

for all $\zeta, \eta \in \Sigma, \lambda > 0$, where $\psi, \phi, \varphi : [0,1] \rightarrow [0,1]$ are continuous functions such that $\psi(s) > s, \phi(s) < s$ and $\varphi(s) < s$ for each $s \in (0,1)$. Then Γ, Λ, Ω and H have a unique common fixed point in Σ .

Proof

Since the pairs (Γ, Ω) and (Λ, H) are subcompatible maps of type (J-1) and subsequentially continuous, then there exist two sequences $\{\zeta_n\}$ and $\{\eta_n\}$ in Σ such that $\lim_{n \rightarrow \infty} \Gamma\zeta_n = \lim_{n \rightarrow \infty} \Omega\zeta_n = \delta, \delta \in \Sigma$

and satisfy

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &= \Xi(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda) = 1, \\ \lim_{n \rightarrow \infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &= \Theta(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda) = 0, \\ \lim_{n \rightarrow \infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &= \Upsilon(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda) = 0, \\ \lim_{n \rightarrow \infty} \Xi(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) &= \Xi(\Omega\delta, \Gamma\delta, \Gamma\delta, \lambda) = 1, \\ \lim_{n \rightarrow \infty} \Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) &= \Theta(\Omega\delta, \Gamma\delta, \Gamma\delta, \lambda) = 0, \\ \lim_{n \rightarrow \infty} \Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) &= \Upsilon(\Omega\delta, \Gamma\delta, \Gamma\delta, \lambda) = 0. \end{aligned}$$

$\lim_{n \rightarrow \infty} \Lambda\zeta_n = \lim_{n \rightarrow \infty} H\eta_n = \omega, \omega \in \Sigma$, and

$$\lim_{n \rightarrow \infty} \Xi(\Lambda H\eta_n, H H\eta_n, H H\eta_n, \lambda) = \Xi(\Lambda\omega, H\omega, H\omega, \lambda) = 1,$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \Theta (\Lambda H\eta_n, HH\eta_n, HH\eta_n, \lambda) &= \Theta (\Lambda\omega, H\omega, H\omega, \lambda) = 0, \\ \lim_{n \rightarrow \infty} Y (\Lambda H\eta_n, HH\eta_n, HH\eta_n, \lambda) &= Y (\Lambda\omega, H\omega, H\omega, \lambda) = 0, \\ \lim_{n \rightarrow \infty} \Xi (H\Lambda\eta_n, \Lambda\Lambda\eta_n, \Lambda\Lambda\eta_n, \lambda) &= \Xi (H\omega, \Lambda\omega, \Lambda\omega, \lambda) = 1, \\ \lim_{n \rightarrow \infty} \Theta (H\Lambda\eta_n, \Lambda\Lambda\eta_n, \Lambda\Lambda\eta_n, \lambda) &= \Theta (H\omega, \Lambda\omega, \Lambda\omega, \lambda) = 0, \\ \lim_{n \rightarrow \infty} Y (H\Lambda\eta_n, \Lambda\Lambda\eta_n, \Lambda\Lambda\eta_n, \lambda) &= Y (H\omega, \Lambda\omega, \Lambda\omega, \lambda) = 0. \end{aligned}$$

Therefore, $\Gamma\delta = \Omega\delta$ and $\Lambda\omega = H\omega$, that is δ is a coincidence point of Γ and Ω , ω is a coincidence point of Λ and H . Now, we prove that $\delta = \omega$. By using (3.1) for $\zeta = \zeta_n$ and $\eta = \eta_n$, we get

$$\begin{aligned} \Xi(\Gamma\zeta_n, \Lambda\eta_n, \Lambda\eta_n, \lambda) &\geq \psi (\min \{ \Xi(\Omega\zeta_n, H\eta_n, H\eta_n, \lambda), \Xi(\Gamma\zeta_n, \Omega\zeta_n, \Omega\zeta_n, \lambda), \Xi(\Lambda\eta_n, H\eta_n, H\eta_n, \lambda), \\ &\quad \frac{1}{2}[\Xi(\Lambda\eta_n, \Omega\zeta_n, \Omega\zeta_n, \lambda) + \Xi(\Gamma\zeta_n, H\eta_n, H\eta_n, \lambda)] \}), \\ \Theta(\Gamma\zeta_n, \Lambda\eta_n, \Lambda\eta_n, \lambda) &\leq \phi (\max \{ \Theta(\Omega\zeta_n, H\eta_n, H\eta_n, \lambda), \Theta(\Gamma\zeta_n, \Omega\zeta_n, \Omega\zeta_n, \lambda), \Theta(\Lambda\eta_n, H\eta_n, H\eta_n, \lambda), \\ &\quad \frac{1}{2}[\Theta(\Lambda\eta_n, \Omega\zeta_n, \Omega\zeta_n, \lambda) + \Theta(\Gamma\zeta_n, H\eta_n, H\eta_n, \lambda)] \}), \\ Y(\Gamma\zeta_n, \Lambda\eta_n, \Lambda\eta_n, \lambda) &\leq \varphi (\max \{ Y(\Omega\zeta_n, H\eta_n, H\eta_n, \lambda), Y(\Gamma\zeta_n, \Omega\zeta_n, \Omega\zeta_n, \lambda), Y(\Lambda\eta_n, H\eta_n, H\eta_n, \lambda), \\ &\quad \frac{1}{2}[Y(\Lambda\eta_n, \Omega\zeta_n, \Omega\zeta_n, \lambda) + Y(\Gamma\zeta_n, H\eta_n, H\eta_n, \lambda)] \}). \end{aligned}$$

Taking the limit $n \rightarrow \infty$, we have

$$\begin{aligned} \Xi(\delta, \omega, \omega, \lambda) &\geq \psi (\min \{ \Xi(\delta, \omega, \omega, \lambda), \Xi(\delta, \delta, \delta, \lambda), \Xi(\omega, \omega, \omega, \lambda), \frac{1}{2}[\Xi(\omega, \delta, \delta, \lambda) + \Xi(\delta, \omega, \omega, \lambda)] \}), \\ \Theta(\delta, \omega, \omega, \lambda) &\leq \phi (\max \{ \Theta(\delta, \omega, \omega, \lambda), \Theta(\delta, \delta, \delta, \lambda), \Theta(\omega, \omega, \omega, \lambda), \frac{1}{2}[\Theta(\omega, \delta, \delta, \lambda) + \Theta(\delta, \omega, \omega, \lambda)] \}), \\ Y(\delta, \omega, \omega, \lambda) &\leq \varphi (\max \{ Y(\delta, \omega, \omega, \lambda), Y(\delta, \delta, \delta, \lambda), Y(\omega, \omega, \omega, \lambda), \frac{1}{2}[Y(\omega, \delta, \delta, \lambda) + Y(\delta, \omega, \omega, \lambda)] \}), \end{aligned}$$

that is,

$$\begin{aligned} \Xi(\delta, \omega, \omega, \lambda) &\geq \psi (\Xi(\delta, \omega, \omega, \lambda)) > \Xi(\delta, \omega, \omega, \lambda), \\ \Theta(\delta, \omega, \omega, \lambda) &\leq \phi (\Theta(\delta, \omega, \omega, \lambda)) < \Theta(\delta, \omega, \omega, \lambda), \\ Y(\delta, \omega, \omega, \lambda) &\leq \varphi (Y(\delta, \omega, \omega, \lambda)) < Y(\delta, \omega, \omega, \lambda), \end{aligned}$$

which yield $\delta = \omega$.

Again using (3.1) for $\zeta = \delta$ and $\eta = \eta_n$, we obtain

$$\begin{aligned} \Xi(\Gamma\delta, \Lambda\eta_n, \Lambda\eta_n, \lambda) &\geq \psi (\min \{ \Xi(\Omega\delta, H\eta_n, H\eta_n, \lambda), \Xi(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda), \Xi(\Lambda\eta_n, H\eta_n, H\eta_n, \lambda), \\ &\quad \frac{1}{2}[\Xi(\Lambda\eta_n, \Omega\delta, \Omega\delta, \lambda) + \Xi(\Gamma\delta, H\eta_n, H\eta_n, \lambda)] \}), \\ \Theta(\Gamma\delta, \Lambda\eta_n, \Lambda\eta_n, \lambda) &\leq \phi (\max \{ \Theta(\Omega\delta, H\eta_n, H\eta_n, \lambda), \Theta(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda), \Theta(\Lambda\eta_n, H\eta_n, H\eta_n, \lambda), \\ &\quad \frac{1}{2}[\Theta(\Lambda\eta_n, \Omega\delta, \Omega\delta, \lambda) + \Theta(\Gamma\delta, H\eta_n, H\eta_n, \lambda)] \}), \\ Y(\Gamma\delta, \Lambda\eta_n, \Lambda\eta_n, \lambda) &\leq \varphi (\max \{ Y(\Omega\delta, H\eta_n, H\eta_n, \lambda), Y(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda), Y(\Lambda\eta_n, H\eta_n, H\eta_n, \lambda), \\ &\quad \frac{1}{2}[Y(\Lambda\eta_n, \Omega\delta, \Omega\delta, \lambda) + Y(\Gamma\delta, H\eta_n, H\eta_n, \lambda)] \}). \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we have,

$$\begin{aligned} \Xi(\Gamma\delta, \omega, \omega, \lambda) &\geq \psi (\min \{ \Xi(\Omega\delta, \omega, \omega, \lambda), \Xi(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda), \Xi(\omega, \omega, \omega, \lambda), \\ &\quad \frac{1}{2}[\Xi(\omega, \Omega\delta, \Omega\delta, \lambda) + \Xi(\Gamma\delta, \omega, \omega, \lambda)] \}), \\ \Theta(\Gamma\delta, \omega, \omega, \lambda) &\leq \phi (\max \{ \Theta(\Omega\delta, \omega, \omega, \lambda), \Theta(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda), \Theta(\omega, \omega, \omega, \lambda), \\ &\quad \frac{1}{2}[\Theta(\omega, \Omega\delta, \Omega\delta, \lambda) + \Theta(\Gamma\delta, \omega, \omega, \lambda)] \}), \\ Y(\Gamma\delta, \omega, \omega, \lambda) &\leq \varphi (\max \{ Y(\Omega\delta, \omega, \omega, \lambda), Y(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda), Y(\omega, \omega, \omega, \lambda), \\ &\quad \frac{1}{2}[Y(\omega, \Omega\delta, \Omega\delta, \lambda) + Y(\Gamma\delta, \omega, \omega, \lambda)] \}). \end{aligned}$$

That is,

$$\begin{aligned} \Xi(\Gamma\delta, \omega, \omega, \lambda) &\geq \psi (\Xi(\Gamma\delta, \omega, \omega, \lambda)) > \Xi(\Gamma\delta, \omega, \omega, \lambda), \\ \Theta(\Gamma\delta, \omega, \omega, \lambda) &\leq \phi (\Theta(\Gamma\delta, \omega, \omega, \lambda)) < \Theta(\Gamma\delta, \omega, \omega, \lambda), \\ Y(\Gamma\delta, \omega, \omega, \lambda) &\leq \varphi (Y(\Gamma\delta, \omega, \omega, \lambda)) < Y(\Gamma\delta, \omega, \omega, \lambda). \end{aligned}$$

which yield $\Gamma\delta = \omega = \delta$.

Therefore $\delta = \omega$ is a common fixed point of Γ, Λ, Ω and H .

For uniqueness, suppose that there exist another fixed point u of Γ, Λ, Ω and H .

Then from (3.1), we have

$$\begin{aligned} \Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda) &\geq \psi(\min \{\Xi(\Omega\delta, Hu, Hu, \lambda), \Xi(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda), \Xi(\Lambda u, Hu, Hu, \lambda), \\ &\quad \frac{1}{2}[\Xi(\Lambda u, \Omega\delta, \Omega\delta, \lambda) + \Xi(\Gamma\delta, Hu, Hu, \lambda)]\}) \\ &= \psi(\min \{\Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda), 1, \Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda), \\ &\quad \frac{1}{2}[\Xi(\Lambda u, \Gamma\delta, \Gamma\delta, \lambda) + \Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda)]\}) \\ &= \psi(\Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda)) \\ &> \Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda), \end{aligned}$$

$$\begin{aligned} \Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda) &\leq \phi(\max \{\Theta(\Omega\delta, Hu, Hu, \lambda), \Theta(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda), \Theta(\Lambda u, Hu, Hu, \lambda), \\ &\quad \frac{1}{2}[\Theta(\Lambda u, \Omega\delta, \Omega\delta, \lambda) + \Theta(\Gamma\delta, Hu, Hu, \lambda)]\}) \\ &= \phi(\max \{\Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda), 0, \Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda), \\ &\quad \frac{1}{2}[\Theta(\Lambda u, \Gamma\delta, \Gamma\delta, \lambda) + \Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda)]\}) \\ &= \phi(\Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda)) \\ &< \Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda), \end{aligned}$$

$$\begin{aligned} \Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda) &\leq \varphi(\max \{\Upsilon(\Omega\delta, Hu, Hu, \lambda), \Upsilon(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda), \Upsilon(\Lambda u, Hu, Hu, \lambda), \\ &\quad \frac{1}{2}[\Upsilon(\Lambda u, \Omega\delta, \Omega\delta, \lambda) + \Upsilon(\Gamma\delta, Hu, Hu, \lambda)]\}) \\ &= \varphi(\max \{\Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda), 0, \Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda), \\ &\quad \frac{1}{2}[\Upsilon(\Lambda u, \Gamma\delta, \Gamma\delta, \lambda) + \Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda)]\}) \\ &= \varphi(\Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda)) \\ &< \Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda), \end{aligned}$$

which yield $\delta = u$. Therefore, uniqueness follows.

If we put $\Omega = H$ in Theorem 3.1, we get the following result.

Corollary: 4.2

Let Γ, Λ , and Ω be self-maps of a weak non-Archimedean NMS $(\Sigma, \Xi, \theta, \Upsilon, *, \circ)$ and let the pairs (Γ, Ω) and (Λ, Ω) are subcompatible maps of type (J-1) and subsequentially continuous. If

$$\Xi(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \geq \psi(\min \{\Xi(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Xi(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Xi(\Lambda\eta, \Omega\eta, \Omega\eta, \lambda), \\ \frac{1}{2}[\Xi(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Xi(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}) \tag{4.2.1}$$

$$\Theta(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \leq \phi(\max \{\Theta(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Theta(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Theta(\Lambda\eta, \Omega\eta, \Omega\eta, \lambda), \\ \frac{1}{2}[\Theta(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Theta(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}) \tag{4.2.2}$$

$$\Upsilon(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \leq \varphi(\max \{\Upsilon(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Upsilon(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Upsilon(\Lambda\eta, \Omega\eta, \Omega\eta, \lambda), \\ \frac{1}{2}[\Upsilon(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Upsilon(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}) \tag{4.2.3}$$

for all $\zeta, \eta \in \Sigma, \lambda > 0$, where $\psi, \phi, \varphi : [0,1] \rightarrow [0,1]$ are continuous functions such that $\psi(s) > s, \phi(s) < s$ and $\varphi(s) < s$ for each $s \in (0,1)$. Then Γ, Λ and Ω have a unique common fixed point in Σ .

If we put $\Gamma = \Lambda$ and $\Omega = H$ in Theorem 4.1, we get the following result.

Corollary: 4.3

Let Γ and Ω be self-maps of a weak non-Archimedean NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ and let the pairs (Γ, Ω) is subcompatible maps of type (J-1) and subsequentially continuous. If

$$\Xi(\Gamma\zeta, \Gamma\eta, \Gamma\eta, \lambda) \geq \psi (\min \{\Xi(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Xi(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Xi(\Gamma\eta, \Omega\eta, \Omega\eta, \lambda), \frac{1}{2}[\Xi(\Gamma\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Xi(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}), \quad (4.3.1)$$

$$\Theta(\Gamma\zeta, \Gamma\eta, \Gamma\eta, \lambda) \leq \phi (\max \{\Theta(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Theta(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Theta(\Gamma\eta, \Omega\eta, \Omega\eta, \lambda), \frac{1}{2}[\Theta(\Gamma\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Theta(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}), \quad (4.3.2)$$

$$\Upsilon(\Gamma\zeta, \Gamma\eta, \Gamma\eta, \lambda) \leq \varphi (\max \{\Upsilon(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Upsilon(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Upsilon(\Gamma\eta, \Omega\eta, \Omega\eta, \lambda), \frac{1}{2}[\Upsilon(\Gamma\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Upsilon(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}), \quad (4.3.3)$$

for all $\zeta, \eta \in \Sigma$, $\lambda > 0$, where $\psi, \phi, \varphi : [0,1] \rightarrow [0,1]$ are continuous functions such that $\psi(s) > s$, $\phi(s) < s$ and $\varphi(s) < s$ for each $s \in (0,1)$. Then Γ and Ω have a unique common fixed point in Σ .

5. Conclusion

In this work, we obtained new structure of weak non-Archimedean with the help of subcompatible maps of types (J-1) and (J-2) in NMS. Also, we proved common fixed point theorems for four subcompatible maps of type (J-1) in weak non-Archimedean NMS.

References:

- [1] Abdel-Basset, M., et al. (2019). "An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number." *Applied Soft Computing* 77: 438-452.
- [2] Abdel-Basset, M., et al. (2019). "A group decision making framework based on neutrosophic TOPSIS approach for smart medical device selection." *Journal of medical systems* 43(2): 38.
- [3] Abdel-Basset, M., Gamal, A., Chakraborty, R.K., Ryan, M., 2021. A new hybrid multi-criteria decision-making approach for location selection of sustainable offshore wind energy stations: A case study. *J. Clean. Prod.* 280. <https://doi.org/10.1016/j.jclepro.2020.124462>
- [4] Abdel-Basset, Mohamed, Gamal, A., Chakraborty, R.K., Ryan, M.J., 2020a. Evaluation of sustainable hydrogen production options using an advanced hybrid MCDM approach: A case study. *Int. J. Hydrogen Energy.* <https://doi.org/https://doi.org/10.1016/j.ijhydene.2020.10.232>.
- [5] Abdel-Basset, M, Mohamed, R, Sallam, K, and Elhoseny, M, A novel decision-making model for sustainable supply chain maintenance under uncertainty environment. *Journal of Cleaner Production*, 122324, (2020).
- [6] Alaca, C, Altun, I, and Turkoglu, D, On compatible mappings of type (I) and (II) in intuitionistic fuzzy metric spaces, *Comm. Korean Math. Soci.*, 23(3)(2008), 427– 446.
- [7] Atanassov, K, Intuitionistic fuzzy sets, *Fuzzy, Sets and Systems* 20, 87- 96, (1986).
- [8] George. A, Veeramani. P , On some results in fuzzy metric spaces, *Fuzzy sets and Systems*, 395 – 399, (1994).
- [9] Grabiec. M, Fixed Points in Fuzzy Metric Spaces, *Fuzzy Sets and Systems*, 27 a. 385 – 389, (1988).
- [10] Jeyaraman, M, Poovaragavan, D, Fixed Point Results in Non- Archimedean Generalized Fuzzy Metric Spaces, *International Journal of Pure and Applied Mathematics*, 119(9), 213-222, 2018.

- [11]Jeyaraman, M, Suganthi, M, Sowndrarajan, S, Fixed point results in non-Archimedean generalized intuitionistic fuzzy metric spaces, Notes on Intuitionistic Fuzzy Sets,25(4), 48-58, 2019.
- [12]Jungck, G, Compatible mappings and common fixed points, Int. J. Math. Math. Sci. 9, 771 – 779, (1986),
- [13]Jungck, G, Murthy, P.P, and Cho, Y.J, Compatible mappings of type (A) and common fixed points, Math. Japan., 38(2), 381 – 390, (1993).
- [14]Kramosil, I, and Michalek, J, Fuzzy metric and statistical metric space, Kybernetika, 11, 326 – 334, (1975).
- [15]Kirisci, M, Simsek, N, Neutrosophic metric spaces, Mathematical Sciences (2020) 14, 241–248.
- [16]Muthuraj. R and Pandiselvi.R, Common fixed point theorems for compatible mappings of type (P-1) and type (P-2) in M-fuzzy metric spaces, Arab journal of Mathematics, and Mathematical Sciences, 1-10, (2013).
- [17]Park. J. H, Intuitionistic fuzzy metric spaces, Chaos Solit. Fract., 22(2004), 1039 - 1046.
- [18]Pathak. H. K, Cho. Y. J, Kang. S. M and Madharia. B, Compatible mappings of type (C) and common fixed point theorems of Gregus type, Demon. Math.,1(3), 499 – 518, (1998)
- [19]Singh. M. R, and Singh, Y. M, Compatible mappings of type (E) and common fixed point theorems of Meir-Keeler type, Internat. J. Math. Sci. & Eng. Appl., 1 (2), 299 – 315, (2007).
- [20]Smarandache, F. A Unifying Field in Logics, Neutrosophy: Neutrosophic Probability, Set and Logic; American Research Press: Rehoboth, MA, USA, 1998.
- [21]Smarandache, F. Neutrosophy: Neutrosophic Probability, Set and Logic; ProQuest Information and Learning: Ann Arbor, MI, USA, 1998; pp. 105.
- [22]Smarandache, F. Neutrosophic set, a generalisation of the intuitionistic fuzzy sets, Inter J Pure Appl Math. Vol 24, pp 287-297, 2005.
- [23]Sola Erduran.F , Yildiz. C Kutukcu .S, A Common fixed point theorem in weak non-Archimedean intuitionistic fuzzy metric spaces, International journal of Open Compt. Math., 7, 1.1 – 1.7, (2014).
- [24]Sowndrarajan, S, Jeyaraman, M, and Florentin Smarandache, Fixed Point Results for Contraction Theorems in Neutrosophic Metric Spaces, Neutrosophic Sets and Systems.,36, 308-310, 2020.
- [25]Wadei F. Al-Omeri, Saeid Jafari and Florentin Smarandache, Neutrosophic Fixed Point Theorems and Cone Metric Spaces, Neutrosophic Sets and Systems, Volume 31, pp 250-264, 2020.
- [26]Zadeh. L. A, Fuzzy sets, Infor. and Control.,8338 – 353, (1965).

Received: Jan. 7, 2021. Accepted: April 4, 2021.