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Abstract. In this paper, we introduce the notion of Pythagorean neutrosophic ideals, Pythagorean neutrosophic bi-ideal, Pythagorean neutrosophic interior ideal, Pythagorean neutrosophic (1,2) ideal of semigroups and some of them interesting properties.

Keywords: Pythagorean fuzzy set; Neutrosophic set; fuzzy ideals; semigroup.

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1. **Introduction**

After the introduction of the fuzzy set by Zadeh [11], several researchers conducted experiments on the generalizations of the notion of a fuzzy set. The concept of the intuitionistic fuzzy set was introduced by Atanassov [1,2] as a generalization of the fuzzy set. Jun et al. [4,5] considered the fuzzification of interior ideals in semigroups and the notion of an intuitionistic fuzzy interior ideal of a semigroup S, and its properties were investigated. Kuroki [8] discussed some properties of fuzzy ideals and fuzzy bi-ideals in the semigroup. Jun et al. [6] considered the fuzzification of (1,2)-ideals in semigroups and investigated its properties. Yager [9, 10] introduced the Pythagorean fuzzy set as a generalization of the fuzzy set. After its existence, several researchers also studied the properties of fuzzy ideals of the semigroup. Yager and Abbasov [37] initiated the notion of Pythagorean fuzzy set and this concept could be considered as a successful generalization of intuitionistic fuzzy sets. The main difference between intuitionistic fuzzy sets and Pythagorean fuzzy sets is that, in the latter case, the sum of membership and non-membership grades is greater than 1, however, the sum of their squares belongs to the unit interval [0,1]. Analogously, in this novel pattern, the associated uncertainty of membership grade and non-membership grade can be explained in a valuable method that than of intuitionistic fuzzy set. Gun et al. [7] introduced the new concept of spherical fuzzy set and discuss the new operations. Smarandache [13] introduced the new concept of neutrosophic set. Khan et.al [12] introduced the Neutrosophic N-Structures and their application in semigroups. The neutrosophic theories have received greater attention in recent years [14]-[32]. Abdel-Basset et al. [33] proposed a new hybrid multi-criteria decision-making (MCDM) using Analytical Hierarchy Process(AHP) and Preference Ranking Organization Method for Enrichment Evaluations (PROMETHEE)-II approach for optimal offshore wind power station location selection. Abdel-Basset et al. [34] Provided a neutrosophic PROMETHEE technique for MCDM problems to describe fuzzy information efficiently. Abdel-Basset et al. [35] discussed how smart internet of things technology can assist medical staff in monitoring the spread of COVID-19. Abdel-Basset et al. [36] studied a comprehensive evaluation of the sustainability of hydrogen production options through the use of a MCDM model.

In this paper, we discuss the properties of Pythagorean neutrosophic ideals in semigroups.

2. Preliminaries

Definition 2.1. [3] Let S be a semigroup. M and N be subsets of S, the product of M and N is defined as $MN = \{mn \in S \mid m \in M \ and \ n \in N\}$ A non- empty subset M of S is called a sub-semigroup of S if $MM \subseteq M$. A non-empty subset M of S is called a left (resp. right) ideal of S if $SM \subseteq M$ (resp.$MS \subseteq M$). A is called a two sided ideal of S if it is both a left ideal and right ideal of S. A sub- semigroup M of S is called a bi-ideal of S if $MSM \subseteq M$. A sub-semigroup M of S is called a (1,2) ideal of S if $MSM^2 \subseteq M$. A semigroup S is said to be (2,2)- regular if $m \in m^2Sm^2$ for any $m \in S$. A semigroup S is called regular if for each element $m \in S$ there exists $x \in S$ such that $m = mxm$. A semigroup S is said to be completely regular if, for any $m \in S$, there exists $x \in S$ such that $m = mxm$ and $mx = xm$. For a semigroup S, is completely regular if and only if(iff) S is a union of groups iff S is (2,2)-regular. By a fuzzy set μ in a non-empty set S we mean a function $\mu : S \to [0,1]$, and the complement of μ, denoted by $\overline{\mu}$, is the fuzzy set in S given by $\overline{\mu}(x) = 1 - \mu(x)$ for all $x \in S$.

Definition 2.2. [9] Let X be a universe of discourse, A **Pythagorean fuzzy set** (PFS) $P = \{z, \vartheta_p(x), \omega_p(x)/z \in X\}$ where $\vartheta : X \to [0,1]$ and $\omega : X \to [0,1]$ represent the degree of membership and non-membership of the object $z \in X$ to the set P subset to the condition $0 \leq (\vartheta_p(z))^2 + (\omega_p(z))^2 \leq 1$ for all $z \in X$. For the sake of simplicity a PFS is denoted as $P = (\vartheta_p(z), \omega_p(z))$.

Definition 2.3. [13] Let X be a universe of discourse, A **Neutrosophic set** (NS) $N = \{z, \vartheta_N(z), \omega_N(z), \psi_N(z)/z \in X\}$ where $\vartheta : X \to [0,1]$, $\omega : X \to [0,1]$ and $\psi : X \to [0,1]$ represent the degree of truth membership, indeterminacy-membership and false-membership of the object $z \in X$ to the set N subset to the condition $0 \leq (\vartheta_N(z)) + (\omega_N(z)) + (\psi_N(z)) \leq 3$ for all $z \in X$. For the sake of simplicity a NS is denoted as $N = (\vartheta_N(z), \omega_N(z), \psi_N(z))$.

3. Pythagorean neutrosophic set

Definition 3.1. Let X be a universe of discourse, A **Pythagorean neutrosophic set** (PNS) $P_N = \{z, \mu_p(z), \zeta_p(z), \psi_p(z)/z \in X\}$ where $\mu : X \to [0,1]$, $\zeta : X \to [0,1]$ and $\psi : X \to [0,1]$ represent the degree of membership, non-membership and inderminancy of the object $z \in X$ to the set P_N subset to the condition $0 \leq (\mu_p(z))^2 + (\zeta_p(z))^2 + (\psi_p(z))^2 \leq 2$ for all $z \in X$. For the sake of simplicity a PNS is denoted as $P_N = (\mu_p(z), \zeta_p(z), \psi_p(z))$.

Definition 3.2. Let X be a nonempty set and I the unit interval [0, 1]. A Pythagorean neutrosophic set with neutrosophic components [PNS] P_{N_1} and P_{N_2} of the form $P_{N_1} = (z, \mu_{p_1}(z), \zeta_{p_1}(z), \psi_{p_1}(z)/z \in X)$ *and* $P_{N_2} = (z, \mu_{p_2}(z), \zeta_{p_2}(z), \psi_{p_2}(z)/z \in X)$. Then

1)$P_N^c = (z, \psi_{p_1}(z), \zeta_{p_1}(z), \mu_{p_1}(z)/z \in X)$

2)$P_{N_1} \cup P_{N_2} = \{z, max(\mu_{P_1}(z), \mu_{P_2}(z)), max(\zeta_{P_1}(z), \zeta_{P_2}(z)), min(\psi_{P_1}(z), \psi_{P_2}(z))/z \in X\}$

3)$P_{N_1} \cap P_{N_2} = \{z, min(\mu_{P_1}(z), \mu_{P_2}(z)), min(\zeta_{P_1}(z), \zeta_{P_2}(z)), max(\psi_{P_1}(z), \psi_{P_2}(z))/z \in X\}$

4. Pythagorean neutrosophic ideals in semigroups

In this section, let S denote a semigroup unless otherwise specified. We discuss the details of Pythagorean neutrosophic ideals in semigroups.

Definition 4.1. A Pythagorean neutrosophic (PNS) $P_N = (\mu_p, \zeta_p, \psi_p)$ in S is called an Pythagorean neutrosophic sub-semigroup of S, if

(i) $\mu_p(x_1x_2) \leq max\{\mu_p(x_1), \mu_p(x_2)\}$

(ii) $\zeta_p(x_1x_2) \geq max\{\zeta_p(x_1), \zeta_p(x_2)\}$

(iii) $\psi_p(x_1x_2) \leq max\{\psi_p(x_1), \psi_p(x_2)\}$ for all $x_1, x_2 \in S$.

Definition 4.2. A PNS $P = (\mu_p, \zeta_p, \psi_p)$ in S is called an Pythagorean neutrosophic left ideal of S, if

(i) $\mu_p(x_1x_2) \leq \mu_p(x_2)$

(ii) $\zeta_p(x_1x_2) \geq \zeta_p(x_2)$

(iii) $\psi_p(x_1x_2) \leq \psi_p(x_2)$ for all $x_1, x_2 \in S$.

A Pythagorean neutrosophic right ideal of S is defined in an analogous way.An PNS $P_N = (\mu_p, \zeta_p, \psi_p)$ in S is called an Pythagorean neutrosophic ideal of S, if it is both a Pythagorean neutrosophic left and Pythagorean neutrosophic right ideal of S.It is clear that any Pythagorean neutrosophic left(resp. right) ideal of S is a Pythagorean neutrosophic sub-semigroup of S.

Definition 4.3. A Pythagorean neutrosophic sub-semigroup $P_N = (\mu_p, \zeta_p, \psi_p)$ of S is called an Pythagorean neutrosophic bi-ideal(PNBI) of S.

(i) $\mu_p(x_1ux_2) \leq max\{\mu_p(x_1), \mu_p(x_2)\}$

(ii) $\zeta_p(x_1ux_2) \geq max\{\zeta_p(x_1), \zeta_p(x_2)\}$

(ii) $\psi_p(x_1ux_2) \leq max\{\psi_p(x_1), \psi_p(x_2)\}$ for all $u, x_1, x_2 \in S$.

Theorem 4.4. *If $\{P_i\}_{i\in I}$ is a family of PNBI of S, then $\cap P_i$ is an PNBI of S. Where $\cap P_i = (\vee\mu_{p_i}, \vee\zeta_{p_i}, \vee\psi_{p_i})$ and $\vee\mu_{p_i} = sup\{\mu_{p_i}(x_1)|i \in I, x_1 \in S\}$, $\vee\zeta_{p_i} = sup\{\zeta_{p_i}(x_1)|i \in I, x_1 \in S\}$, $\vee\psi_{p_i} = sup\{\psi_{p_i}(x_1)|i \in I, x_1 \in S\}$.*

Proof. Let $x_1, x_2 \in S$. Then we have

$$\begin{aligned}
\vee\mu_{p_i}(x_1x_2) &\leq \vee\{max\{\mu_{p_i}(x_1), \mu_{p_i}(x_2)\}\}\\
&= max\{max\{\mu_{p_i}(x_1), \mu_{p_i}(x_2)\}\}\\
&= max\{max\{\mu_{p_i}(x_1)\}, max\{\mu_{p_i}(x_2)\}\}\\
&= max\{\vee\mu_{p_i}(x_1), \vee\mu_{p_i}(x_2)\}\\
\vee\zeta_{p_i}(x_1x_2) &\geq \vee\{max\{\zeta_{p_i}(x_1), \zeta_{p_i}(x_2)\}\}\\
&= max\{max\{\zeta_{p_i}(x_1), \zeta_{p_i}(x_2)\}\}\\
&= max\{max\{\zeta_{p_i}(x_1)\}, max\{\zeta_{p_i}(x_2)\}\}\\
&= max\{\wedge\zeta_{p_i}(x_1), \wedge\zeta_{p_i}(x_2)\}\\
\vee\psi_{p_i}(x_1x_2) &\leq \vee\{max\{\psi_{p_i}(x_1), \psi_{p_i}(x_2)\}\}\\
&= max\{max\{\psi_{p_i}(x_1), \psi_{p_i}(x_2)\}\}\\
&= max\{max\{\psi_{p_i}(x_1)\}, max\{\psi_{p_i}(x_2)\}\}\\
&= max\{\vee\psi_{p_i}(x_1), \vee\psi_{p_i}(x_2)\}.
\end{aligned}$$

Hence $\cap P_i$ is an Pythagorean neutrosophic sub-semigroup of S.

Next for $u, x_1, x_2 \in S$, we obtain

$$\begin{aligned}
\vee\mu_{p_i}(x_1ux_2) &\leq \vee\{min\{\mu_{p_i}(x_1), \mu_{p_i}(x_2)\}\}\\
&= max\{max\{\mu_{p_i}(x_1), \mu_{p_i}(x_2)\}\}\\
&= max\{max\{\mu_{p_i}(x_1)\}, max\{\mu_{p_i}(x_2)\}\}\\
&= max\{\vee\mu_{p_i}(x_1), \vee\mu_{p_i}(x_2)\}\\
\vee\zeta_{p_i}(x_1ux_2) &\geq \vee\{min\{\zeta_{p_i}(x_1), \zeta_{p_i}(x_2)\}\}\\
&= max\{max\{\zeta_{p_i}(x_1), \zeta_{p_i}(x_2)\}\}\\
&= max\{max\{\zeta_{p_i}(x_1)\}, max\{\zeta_{p_i}(x_2)\}\}\\
&= max\{\vee\zeta_{p_i}(x_1), \vee\zeta_{p_i}(x_2)\}\\
\vee\psi_{p_i}(x_1ux_2) &\leq \vee\{max\{\psi_{p_i}(x_1), \psi_{p_i}(x_2)\}\}\\
&= max\{max\{\psi_{p_i}(x_1), \psi_{p_i}(x_2)\}\}\\
&= max\{max\{\psi_{p_i}(x_1)\}, max\{\psi_{p_i}(x_2)\}\}\\
&= max\{\vee\psi_{p_i}(x_1), \vee\psi_{p_i}(x_2)\}.
\end{aligned}$$

Hence $\cap P_i$ is an PNBI of S.

This completes the proof. □

Theorem 4.5. *Every Pythagorean neutrosophic left(right) ideal of S is an Pythagorean neutrosophic bi-ideal of S.*

Proof. Let $P_N = (\mu_p, \zeta_p, \psi_p)$ is a Pythagorean neutrosophic left ideal of S and $u, x_1, x_2 \in S$. Then

$$\begin{aligned} \mu_p(x_1ux_2) &= \mu_p(x_1ux_2) \\ &\leq \mu_p(x_2) \end{aligned}$$
$$\mu_p(x_1ux_2) \leq max\{\mu_p(x_1, \mu_p(x_2))\}$$
$$\begin{aligned} \zeta_p(x_1ux_2) &= \zeta_p(x_1ux_2) \\ &\geq \zeta_p(x_2) \end{aligned}$$
$$\zeta_p(x_1ux_2) \geq max\{\zeta_p(x_1, \zeta_p(x_2))\}$$
$$\begin{aligned} \psi_p(x_1ux_2) &= \psi_p(x_1ux_2) \\ &\leq \psi_p(x_2) \end{aligned}$$
$$\psi_p(x_1ux_2) \leq max\{\psi_p(x_1, \psi_p(x_2))\}$$

Thus $P_N = (\mu_p, \zeta_p, \psi_p)$ is PNBI of S.

The right case is provided in an analogous way. □

Theorem 4.6. *Every Pythagorean neutrosophic bi-ideal of a group S is constant.*

Proof. Let $P_N = (\mu_p, \zeta_p, \psi_p)$ be an PNBI of a group S and let x_1 be any element of S. Then

$$\begin{aligned} \mu_p(x_1) &= \mu_p(ex_1e) \\ &\leq max\{\mu_p(e), \mu_p(e)\} \\ &= \mu_p(e) \\ &= \mu_p(ee) \\ &= \mu_p(x_1x_1^{-1})(x_1^{-1}x_1) \\ &= \mu_p(x_1(x_1^{-1}x_1^{-1})x_1) \\ &\leq max\{\mu_p(x_1, \mu_p(x_1))\} \\ &= \mu_p(x_1) \end{aligned}$$
$$\begin{aligned} \zeta_p(x_1) &= \zeta_p(ex_1e) \\ &\geq max\{\zeta_p(e), \zeta_p(e)\} \\ &= \zeta_p(e) \\ &= \zeta_p(ee) \\ &= \zeta_p(x_1x_1^{-1})(x_1^{-1}x_1) \\ &= \zeta_p(x_1(x_1^{-1}x_1^{-1})x_1) \\ &\geq max\{\zeta_p(x_1, \zeta_p(x_1))\} \\ &= \zeta_p(x_1) \end{aligned}$$

and

$$\begin{aligned}\psi_p(x_1) &= \psi_p(ex_1e)\\ &\leq max\{\psi_p(e), \psi_p(e)\}\\ &= \psi_p(e)\\ &= \psi_p(ee)\\ &= \psi_p(x_1x_1^{-1})(x_1^{-1}x_1)\\ &= \psi_p(x_1(x_1^{-1}x_1^{-1})x_1)\\ &\leq max\{\psi_p(x_1, \psi_p(x_1))\}\\ &= \psi_p(x_1).\end{aligned}$$

Where e is the identity of S. It follows that $\mu_p(x_1) = \mu_p(e)$, $\zeta_p(x_1) = \zeta_p(e)$ and $\psi_p(x_1) = \psi_p(e)$ which means that $P_N = (\mu_p, \zeta_p, \psi_p)$ is constant. □

Theorem 4.7. *If an PNS $P_N = (\mu_p, \zeta_p, \psi_p)$ in S is an PNBI of S, then so is $\Box P_N = \left(\mu_p, \zeta_p, \overline{\psi}_p\right)$.*

Proof. It is sufficient to show that $\overline{\psi}_p$ satisfies the conditions in Definition 3.1 and Definition 3.4. For any $u, x_1, x_2 \in S$, we have

$$\begin{aligned}\overline{\psi}_p(x_1x_2) &= 1 - \psi_p(x_1x_2)\\ &\leq 1 - min\{\psi_p(x_1), \psi_p(x_2)\}\\ &= max\{1 - \psi_p(x_1), 1 - \psi_p(x_2)\}\\ &= max\left\{\overline{\psi}_p(x_1), \overline{\psi}_p(x_2)\right\}\end{aligned}$$

and

$$\begin{aligned}\overline{\psi}_p(x_1ux_2) &= 1 - \psi_p(x_1ux_2)\\ &\leq 1 - min\{\psi_p(x_1), \psi_p(x_2)\}\\ &= max\{1 - \psi_p(x_1), 1 - \psi_p(x_2)\}\\ &= max\left\{\overline{\psi}_p(x_1), \overline{\psi}_p(x_2)\right\}.\end{aligned}$$

Therefore $\Box P_N$ is an PNBI of S. □

Definition 4.8. A Pythagorean neutrosophic sub-semigroup $P_N = (\mu_p, \zeta_p, \psi_p)$ of S is called a Pythagorean neutrosophic (1,2) ideal of S. If

(i) $\mu_p(x_1u(x_2x_3)) \leq max\{\mu_p(x_1), \mu_p(x_2), \mu_p(x_3)\}$

(ii) $\zeta_p(x_1u(x_2x_3)) \geq max\{\zeta_p(x_1), \zeta_p(x_2), \zeta_p(x_3)\}$

(iii) $\psi_p(x_1u(x_2x_3)) \leq max\{\psi_p(x_1), \psi_p(x_2), \psi_p(x_3)\}$ $u, x_1, x_2, x_3 \in S$.

Theorem 4.9. *Every PNBI is a Pythagorean neutrosophic (1,2) ideal of S.*

Proof. Let PNS $P_N = (\mu_p, \zeta_p, \psi_p)$ be an PNBI of S and let $u, x_1, x_2, x_3 \in S$.
Then

$\mu_p(x_1u(x_2x_3)) = \mu_p((x_1ux_2)x_3)$

$$\begin{aligned}
&\leq max\left\{\mu_p(x_1ux_2),\mu_p(x_3)\right\}\\
&\leq max\left\{max\left\{\mu_p(x_1),\mu_p(x_2)\right\},\mu_p(x_3)\right\}\\
&= max\left\{\mu_p(x_1),\mu_p(x_2),\mu_p(x_3)\right\}
\end{aligned}$$

$$\begin{aligned}
\zeta_p(x_1u(x_2x_3)) &= \zeta_p((x_1ux_2)x_3)\\
&\geq max\left\{\zeta_p(x_1ux_2),\zeta_p(x_3)\right\}\\
&\geq max\left\{max\left\{\zeta_p(x_1),\zeta_p(x_2)\right\},\zeta_p(x_3)\right\}\\
&= max\left\{\zeta_p(x_1),\zeta_p(x_2),\zeta_p(x_3)\right\}
\end{aligned}$$

and

$$\begin{aligned}
\psi_p(x_1u(x_2x_3)) &= \psi_p((x_1ux_2)x_3)\\
&\leq max\left\{\psi_p(x_1ux_2),\psi_p(x_3)\right\}\\
&\leq max\left\{max\left\{\psi_p(x_1),\psi_p(x_2)\right\},\psi_p(x_3)\right\}\\
&= max\left\{\psi_p(x_1),\psi_p(x_2),\psi_p(x_3)\right\}.
\end{aligned}$$

Hence $P_N = (\mu_p, \zeta_p, \psi_p)$ is a Pythagorean neutrosophic (1,2) ideal of S. □

To consider the converse of theorem next theorem, we need to strengthen the condition of a semigroup S.

Theorem 4.10. *If S is a regular semigroup, then every Pythagorean neutrosophic (1,2) ideal of S is an PNBI of S.*

Proof. Assume that a semigroup S is regular and let $P_N = (\mu_p, \zeta_p, \psi_p)$ be an Pythagorean neutrosophic (1,2) ideal of S. Let $u, x_1, x_2, x_3 \in S$. Since S is regular, we have $x_1u \in (x_1Sx_1)S \subseteq x_1Sx_1$, which implies that $x_1u = x_1Sx_1$ for some $s \in S$.

Thus

$$\begin{aligned}
\mu_p(x_1ux_2) &= \mu_p((x_1sx_1)x_2)\\
&= \mu_p(x_1s(x_1x_2))\\
&\leq max\left\{\mu_p(x_1),\mu_p(x_1),\mu_p(x_2)\right\}\\
&= max\left\{\mu_p(x_1),\mu_p(x_2)\right\}
\end{aligned}$$

$$\begin{aligned}
\zeta_p(x_1ux_2) &= \zeta_p((x_1sx_1)x_2)\\
&= \zeta_p(x_1s(x_1x_2))\\
&\geq max\left\{\zeta_p(x_1),\zeta_p(x_1),\zeta_p(x_2)\right\}\\
&= max\left\{\zeta_p(x_1),\zeta_p(x_2)\right\}
\end{aligned}$$

and

$$\begin{aligned}
\psi_p(x_1ux_2) &= \psi_p((x_1sx_1)x_2)\\
&= \psi_p(x_1s(x_1x_2))\\
&\leq max\left\{\psi_p(x_1),\psi_p(x_1),\psi_p(x_2)\right\}
\end{aligned}$$

$= max\{\psi_p(x_1), \psi_p(x_2)\}$.
Therefore $P_N = (\zeta_p, \psi_p)$ is PNBI of S. □

Theorem 4.11. *A PNS $P_N = (\mu_p, \zeta_p, \psi_p)$ is an PNBI of S if and only if μ_p, ζ_p and $\overline{\psi_p}$ are FBI of S.*

Proof. Let $P_N = (\mu_p, \zeta_p, \psi_p)$ be an PNBI of S. Then clearly μ_p is a FBI of S. Let $u, x_1, x_2 \in S$. Then

$$\begin{aligned}
\overline{\psi_p}(x_1x_2) &= 1 - \psi_p(x_1x_2)\\
&\geq 1 - max\{\psi_p(x_1), \psi_p(x_2)\}\\
&= min\{(1-\psi_p(x_1)), (1-\psi_p(x_2))\}\\
&= min\{\overline{\psi_p}(x_1), \overline{\psi_p}(x_2)\}\\
\overline{\psi_p}(x_1ux_2) &= 1 - \psi_p(x_1ux_2)\\
&\geq 1 - max\{\psi_p(x_1), \psi_p(x_2)\}\\
&= min\{(1-\psi_p(x_1)), (1-\psi_p(x_2))\}\\
&= min\{\overline{\psi_p}(x_1), \overline{\psi_p}(x_2)\}.
\end{aligned}$$

Hence $\overline{\psi_p}$ is a fuzzy bi-ideal of S.
Conversely, suppose that ζ_p and $\overline{\psi_p}$ are FBI of S. Let $u, x_1, x_2 \in S$.
Then

$$\begin{aligned}
1 - \psi_p(x_1x_2) &= \overline{\psi_p}(x_1x_2)\\
&\leq min\{\overline{\psi_p}(x_1), \overline{\psi_p}(x_2)\}\\
&= min\{(1-\psi_p(x_1)), (1-\psi_p(x_2))\}\\
&= max\{\psi_p(x_1), \psi_p(x_2)\}\\
1 - \psi_p(x_1ux_2) &= \overline{\psi_p}(x_1ux_2)\\
&\geq min\{\overline{\psi_p}(x_1), \overline{\psi_p}(x_2)\}\\
&= 1 - max\{\psi_p(x_1), \psi_p(x_2)\}.
\end{aligned}$$

Which implies that $\psi_p(x_1x_2) \leq max\{\psi_p(x_1), \psi_p(x_2)\}$ and $\psi_p(x_1ux_2) \leq max\{\psi_p(x_1), \psi_p(x_2)\}$
This completes the proof. □

Definition 4.12. A PNS $P_N = (\mu_p, \zeta_p, \psi_p)$ in S is called an Pythagorean neutrosophic interior ideal(PNII) of S if it satisfies

(i) $\mu_p(x_1ux_2) \leq \mu_p(u)$

(ii) $\zeta_p(x_1ux_2) \geq \zeta_p(u)$

(iii) $\psi_p(x_1ux_2) \leq \psi_p(u)$ $u, x_1, x_2 \in S$.

Theorem 4.13. *If $\{P_i\}_{i\in I}$ is a family of PNII of S, then $\cap P_i$ is a PNII of S. Where $\cap P_i = (\vee\mu_{p_i}, \vee\zeta_{p_i}, \vee\psi_{p_i})$ and $\vee\mu_{p_i}(x_1) = sup\{\mu_{p_i}(x_1) | i \in I, x_1 \in S\}$,*
$\vee\zeta_{p_i}(x_1) = sup\{\zeta_{p_i}(x_1) | i \in I, x_1 \in S\}$, $\vee\psi_{p_i}(x_1) = sup\{\psi_{p_i}(x_1) | i \in I, x_1 \in S\}$.

Proof. Let $u, x_1, x_2 \in S$.
Then
$\vee\mu_{p_i}(x_1x_2) \leq max\{max\{\mu_{p_i}(x_1), \mu_{p_i}(x_2)\}\}$
$= (\vee\mu_{p_i}(x_1)) \vee (\vee\mu_{p_i}(x_2))$
$\vee\zeta_{p_i}(x_1x_2) \geq max\{max\{\zeta_{p_i}(x_1), \zeta_{p_i}(x_2)\}\}$
$= (\vee\zeta_{p_i}(x_1)) \vee (\vee\zeta_{p_i}(x_2))$
and
$\vee\psi_{p_i}(x_1x_2) \leq max\{max\{\psi_{p_i}(x_1), \psi_{p_i}(x_2)\}\}$
$= (\vee\psi_{p_i}(x_1)) \vee (\vee\psi_{p_i}(x_2))$
$\vee\mu_{p_i}(x_1ux_2) \leq \vee\mu_{p_i}(u)$
$\vee\zeta_{p_i}(x_1ux_2) \geq \vee\zeta_{p_i}(u)$
and
$\vee\psi_{p_i}(x_1ux_2) \leq \vee\psi_{p_i}(u)$.
Hence $\cap P_i$ is an PNII of S. □

Definition 4.14. Let $P_N = (\mu_p, \zeta_p, \psi_p)$ is a PNS of S and let $\alpha \in [0, 1]$ then the sets. $\mu_{p,\alpha} = \{x_1 \in S : \mu_p(x_1)\alpha\}$, $\zeta_{p,\alpha} = \{x_1 \in S : \zeta_p(x_1)\alpha\}$ and $\psi_{p,\alpha} = \{x_1 \in S : \psi_p(x_1)\alpha\}$ are called a μ_p-level α-cut, ζ_p-level α-cut and ψ_p-level α-cut of K respectively.

Theorem 4.15. *If an PNS $P_N = (\mu_p, \zeta_p, \psi_p)$ in S is an PNII of S, then the μ-level α-cut $\mu_{p,\alpha}$, ζ-level α-cut $\zeta_{p,\alpha}$and ψ-level α-cut $\psi_{p,\alpha}$ of P_N are interior ideal of S, for every $\alpha \in Im(\mu_p) \cap Im(\zeta_p) \cap Im(\psi_p) \subseteq [0, 1]$.*

Proof. Let $\alpha \in Im(\mu_p) \cap Im(\zeta_p) \cap Im(\psi_p) \subseteq [0, 1]$.
let $x_1, x_2 \in \mu_{p,\alpha}$ then $\mu_p(x_1) \leq \alpha$ and $\mu_p(x_2) \leq \alpha$. It follows from that
$\mu_p(x_1x_2) \leq \mu_p(x_1) \vee \mu_p(x_2) \leq \alpha$. So that $x_1, x_2 \in \mu_{p,\alpha}$.
If $x_1, x_2 \in \zeta_{p,\alpha}$ then $\zeta_p(x_1) \geq \alpha$ and $\zeta_p(x_2) \geq \alpha$. It follows from that.
$\zeta_p(x_1x_2) \geq \zeta_p(x_1) \vee \zeta_p(x_2) \geq \alpha$. So that $x_1, x_2 \in \zeta_{p,\alpha}$.
If $x_1, x_2 \in \psi_{p,\alpha}$, then $\psi_p(x_1) \leq \alpha$ and $\psi_p(x_2) \leq \alpha$ and so $\psi_p(x_1x_2) \leq \psi_p(x_1) \vee \psi_p(x_2) \leq \alpha$, that is $x_1, x_2 \in \psi_{p,\alpha}$.
Hence $\mu_{p,\alpha}$, $\zeta_{p,\alpha}$ and $\psi_{p,\alpha}$ are sub-semigroup of S. Now let $x_1x_2 \in S$ and $u \in \mu_{p,\alpha}$. Then $\mu_p(x_1ux_2) \leq \mu_p(u) \leq \alpha$ and so $x_1ux_2 \in \mu_{p,\alpha}$.
If $u \in \zeta_{p,\alpha}$. Then $\zeta_p(x_1ux_2) \geq \zeta_p(u) \geq \alpha$ and so $x_1ux_2 \in \zeta_{p,\alpha}$.
If $u \in \psi_{p,\alpha}$. Then $\psi_p(x_1ux_2) \leq \psi_p(u) \leq \alpha$ thus $x_1ux_2 \in \psi_{p,\alpha}$.
Therefore $\mu_{p,\alpha}$,$\zeta_{p,\alpha}$ and $\psi_{p,\alpha}$ are interior ideal of S. □

Theorem 4.16. *A PNS $P_N = (\mu_p, \zeta_p, \psi_p)$ is and PNII of S if and only if $\mu_p, \zeta_p, \overline{\psi}_p$ are fuzzy interior ideal (FII) of S.*

Proof. Let $P_N = (\mu_p, \zeta_p, \psi_p)$ be an PNII of S. Then clearly μ_p is FII of S. Let $u, x_1, x_2 \in S$. Then

$$\begin{aligned}
\overline{\psi_p}(x_1x_2) &= 1 - \psi_p(x_1x_2) \\
&\geq 1 - (\psi_p(x_1)) \vee \psi_p(x_2) \\
&= (1 - \psi_p(x_1)) \wedge (1 - \psi_p(x_2)) \\
&= \overline{\psi_k}(x_1) \wedge \overline{\psi_p}(x_2)
\end{aligned}$$

$$\begin{aligned}
\overline{\psi_p}(x_1ux_2) &= 1 - \psi_p(x_1ux_2) \\
&\geq 1 - (\psi_p(u)) \\
&= \overline{\psi_p}(u)
\end{aligned}$$

$\overline{\psi_k}$ is a FII of S.

Conversely.

Suppose that ζ_p and $\overline{\psi_p}$ are FII of S. Let $u, x_1, x_2 \in S$.

$$\begin{aligned}
1 - \psi_p(x_1x_2) &= \overline{\psi_p}(x_1x_2) \\
&\geq \overline{\psi_p}(x_1) \wedge \overline{\psi_p}(x_2) \\
&= (1 - \psi_p(x_1)) \wedge (1 - \psi_p(x_2)) \\
&= 1 - \psi_p(x_1) \vee \psi_p(x_2) \\
&= 1 - \psi_p(x_1ux_2) = \overline{\psi_p}(x_1ux_2) \\
&\geq \overline{\psi_p}(u) = 1 - \psi_p(u)
\end{aligned}$$

which implies $\psi_p(x_1x_2) \leq \psi_p(x_1) \vee \psi_p(x_2)$

and

$\psi_p(x_1ux_2) \leq \psi_p(u)$

This completes the proof. □

5. **Conclusions**

In this paper Pythagorean neutrosophic sub-semigroup, Pythagorean neutrosophic left(resp.right) ideal, Pythagorean neutrosophic ideal, Pythagorean neutrosophic bi-ideal, Pythagorean neutrosophic interior ideal and investigated some properties.

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