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Neutrosophic Vague Binary $G$ – subalgebra of $G$ - algebra

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Abstract: Nowadays, human society is using artificial intelligence in a large manner so as to upgrade the present existing applicational criteria’s and tools. Logic is the underlying principle to these works. Algebra is inevitably inter-connected with logic. Hence its achievements to the scientific research outputs have to be addressed. For these reasons, nowadays, research on various algebraic structures are going on wide. Crisp set has also got developed in a parallel way in the forms as fuzzy, intuitionistic fuzzy, rough, vague, neutrosophic, plithogenic etc. Sets with one or more algebraic operations will form different new algebraic structures for giving assistance to these logics, which in turn acts to as, a support to artificial intelligence. BCH/BCI/BCK- are some algebras developed in the first phase of algebraic development output. After that, so many outputs got flashed out, individually and in combinations in no time. Q- algebra and QS – algebra are some of these and could be showed as such kind of productions. G- algebra is considered as an extension to QS – algebra. In this paper neutrosophic vague binary G – subalgebra of G – algebra is generated with example. Notions like, 0 – commutative G - subalgebra, minimal element, normal subset etc. are investigated. Conditions to define derivation and regular derivation for this novel concept are clearly presented with example. Constant of G – algebra can’t be treated as the identity element, generally. In this paper, it is well explained with example. Cosets for neutrosophic vague binary G – subalgebra of G - algebra is developed with proper explanation. Homomorphism for this new concept has been also got commented. Its kernel, monomorphism and isomorphism are also have discussed with proper attention.


Notations: NVBS – neutrosophic vague binary set, NVBSS – neutrosophic vague binary subset. In this paper NVB is used to indicate neutrosophic vague binary and NV is used to indicate neutrosophic vague and N is used to indicate neutrosophic.

1. Introduction

Without mathematics mobility in human-life even became an unthinkable process. But when get into the mathematical world, one faces with, versatile facets of maths, which again get take diversions. The thing is that, dry subject is less get commented on or even less get touched with!
Algebra can also be considered so. But the entry of artificial intelligence made things different. Human world can simply neither ignore nor reject ‘robots and computers’ from their presently existing life pattern, due to their high impact in changing life style. So the question is that, what is the importance of algebra to these new scenario? Is it really useful for this robotic framed world? Answer is, yes! Since artificial intelligence is the raw material to robotics and to all the other newly developing phenomenon’s. Logic is a foundation to artificial intelligence. Here a rapport activity can be seen in the picture. For logical calculations, algebra is very important. So these mixed works of algebra and sets is needed for the future research works in higher level. Chaotic and turbulences in human life situations, made data mining more difficult. To handle these crisis, new kind of extensions to cantor set have also got arose. Human – life is going to get controlled by chips in next step of evolution. So hereafter, have to think on, ‘what algebra can do? ’ in these kind of cross- breed structures in a ‘chip oriented human life’. In this point, some debates are necessary. Whether is it good or bad? If bad, how these bad impact can convert into good, by taming these research works? Definitely these robotic effect made human life much easier both in ‘profit and labour’ level. Some bad outputs are also there and have to think of removing such negatives! From our washing machines to rocket technology, one can found this logic and algebraic illuminations. So giving some applications to algebra is irrelevant in one sense. But can think of the other part, in a little bit humorously. Where algebra is ‘not showed off, its face ‘in this modern world? Following will give an idea to the newly developed algebraic structures in the family of algebra.


In 2020, P.B. Remya and A. Francina Shalini [10] developed BCK/BCI – algebra for neutrosophic vague binary sets. Authors proposed a new suggestion of ‘inclusion of new set’ in the structure in addition to the ‘underlying universal set’, for avoiding more confusions in theoretical calculations. In this paper, authors further modified that structure and proposed a new approach in the structure mentioned in [10], by presenting a single set in the structure instead of the above mentioned two sets. This will give a combined effect of those two sets discussed above. New structure, convey the same effect of the structure used in paper [10], with a single set outlook and by skipping the two set pattern from structure. So here authors tried to present a one more modified form to the structure discussed in paper [10]. This one more refined pattern can be used in the all existing algebraic structures of various sets like fuzzy, vague, neutrosophic, etc., and for their hybrid forms in future works. This new pattern will be helpful to get more clarity and stability in these works.

This paper focuses on the development of $G$ – algebraic structure to neutrosophic vague binary set. Discussions on $G$ – algebra need some more attention while comparing to other algebraic structures. Its axioms are very simple and can be handled in a very clear manner. Neutrosophic ideas and Neutrosophic Vague ideas in $G$ – algebra deserve more attention due to its easily accessible practical applications. This paper concentrates on neutrosophic vague binary $G$ – subalgebra and its theoretical implementations.

Follwing are the newly introduced concepts in this paper.

- **Neutrosophic Vague Binary $G$ – subalgebra [Section 3]**
  - Neutrosophic Vague Binary $G$ – subalgebra [Definition 3.1]
- **Different notions of Neutrosophic Vague Binary $G$ – subalgebra [Section 4]**
  - Neutrosophic Vague Binary $G$ - G part [Definition 4.1 (i)]
  - Neutrosophic Vague Binary $G$ - p radical [Definition 4.1 (ii)]

In this section some preliminaries are given.

**Definition 2.1** [9] (Neutrosophic Vague Binary Set)
A neutrosophic vague binary set (NVBS in short) $M_{NVB}$ over a common universe
$$\{U_1 = \{x_j \mid 1 \leq j \leq n\}; U_2 = \{y_k \mid 1 \leq k \leq p\}\}$$
is an object of the form
$$M_{NVB} = \left\{ \left( \text{f}_{M_{NVB}(x_j)}, \text{i}_{M_{NVB}(x_j)}, \text{f}_{M_{NVB}(y_k)}, \text{i}_{M_{NVB}(y_k)} \right); \forall x_j \in U_1 \right\}$$
is defined as
$$\text{f}_{M_{NVB}(x_j)} = \left[ T^+(x_j), T^+(x_j) \right], \quad \text{i}_{M_{NVB}(x_j)} = \left[ I^+(x_j), I^+(x_j) \right]$$
and
$$\text{f}_{M_{NVB}(y_k)} = \left[ F^+(y_k), F^+(y_k) \right], \quad \text{i}_{M_{NVB}(y_k)} = \left[ I^+(y_k), I^+(y_k) \right]$$
where
(1) $T^+(x_j) = 1 - F^-(x_j); F^+(x_j) = 1 - T^-(x_j) \quad \forall x_j \in U_1$ and
$$T^+(y_k) = 1 - F^-(y_k); F^+(y_k) = 1 - T^-(y_k) \quad \forall y_k \in U_2$$
(2) $-0 \leq T^-(x_j) + I^+(x_j) + F^-(x_j) \leq 2^+$
and
$$-0 \leq T^-(y_k) + I^+(y_k) + F^-(y_k) \leq 2^+$$
or
$$-0 \leq T^-(x_j) + I^+(x_j) + F^-(x_j) + T^-(y_k) + I^+(y_k) + F^-(y_k) \leq 4^+$$
and
\[-0 \leq T^+(x_i) + I^+(x_i) + F^+(x_i) \leq 2^+; \quad -0 \leq T^+(y_k) + I^+(y_k) + F^+(y_k) \leq 2^+ \]
or
\[-0 \leq T^+(x_i) + I^+(x_i) + F^+(x_i) + T^+(y_k) + I^+(y_k) + F^+(y_k) \leq 4^+ \]

(3) \( T^-(x_i), I^-(x_i), F^-(x_i) : V(U_1) \rightarrow [0, 1] \) and \( T^-(y_k), I^-(y_k), F^-(y_k) : V(U_2) \rightarrow [0, 1] \)

Here \( V(U_1), V(U_2) \) denotes power set of vague sets on \( U_1, U_2 \) respectively.

**Definition 2.2 [14] (G – algebra)**
A G-algebra is a non-empty set \( A \) with a constant 0 and a binary operation \( * \) satisfying axioms:

\((B_2) \) \( (x \ast x) = 0 \) \( (B_{12}) \) \( x \ast (x \ast y) = y \) for all \( x, y \in A \);

A G-algebra is denoted by \( \langle A, *, 0 \rangle \)

**Proposition 2.3 [14]**
Any G-algebra \( X \) satisfies the following axioms: for all \( x, y, z \in X \),

(i) \( x \ast 0 = x \)  \( (ii) \) \( (x \ast (x \ast y)) \ast y = 0 \)  \( (iii) \) \( 0 \ast (0 \ast x) = x \)  \( (iv) \) \( x \ast y = 0 \) implies \( x = y \)

(v) \( 0 \ast x = 0 \ast y \) implies \( x = y \)

**Definition 2.4 [14] (G - subalgebra)**
A non-empty subset \( S \) of a G-algebra \( X \), is called a G-subalgebra of \( X \) if \( (x \ast y) \in S \), for all \( x, y \in S \)

**Definition 2.5 [14] (0 – commutative G - algebra)**
A G-algebra \( \langle A, *, 0 \rangle \) is said to be 0 – commutative if: \( x \ast (0 \ast y) = y \ast (0 \ast x) \), for any \( x, y \in A \)

**Theorem 2.6 [14] (G-part, p-radical, p – semisimple)**
Let \( A \) be a G-algebra. For any subset \( S \) of \( A \), we define \( G(S) = \{ x \in S \mid 0 \ast x = x \} \). In particular, if \( S = A \) then we say that \( G(A) \) is the G-part of a G-algebra. For any G-algebra \( A \), the set \( B(A) = \{ x \in A \mid 0 \ast x = x \} \) is called a p-radical of \( A \). A G-algebra is said to be p-semisimple if \( B(A) = \emptyset \).

The following property is obvious: \( G(A) \cap B(A) = \{ 0 \} \)

**Proposition 2.7 [14]**
Let \( \langle U, *, 0 \rangle \) be a G-algebra. Then, the following conditions hold for any \( x, y \in X \)

1. \( (x \ast (x \ast y)) \ast y = 0 \)
2. \( (x \ast y) = 0 \Rightarrow x = y \)
3. \( (0 \ast x) = (0 \ast y) \Rightarrow x = y \)

**Definition 2.8 [17] (Fuzzy G – subalgebra)**
Let \( A = \{(x, \alpha_A(x)) / x \in X \} \) be a fuzzy set in \( X \), where \( X \) is a G-subalgebra. Then the set \( A \) is a fuzzy G-subalgebra over the binary operator \( * \) if it satisfies the condition \( \alpha_A(x \ast y) \geq \min(\alpha_A(x), \alpha_A(y)) \)

for all \( x, y \in X \)

**Definition 2.9 [3] (Intuitionistic Fuzzy G – subalgebra)**
An IFS \( A = (\alpha_A, \beta_A) \) in \( X \) is called an intuitionistic fuzzy G-subalgebra of \( X \) if for all \( x, y \in X \) it satisfies:

\[ GS1 \quad \alpha_A(x \ast y) \geq \min(\alpha_A(x), \alpha_A(y)) \quad ; \quad GS2 \quad \beta_A(x \ast y) \leq \max(\beta_A(x), \beta_A(y)) \]

**Definition 2.10 [18] (Normal Subalgebra of a G - algebra)**
Let \( N \) be a non-empty subset of a G-algebra \( X \). We say that \( N \) is a normal subset of \( X \) if for all \( x, y, z \in X \) such that \( (x \ast y) \in N \) and \( (z \ast t) \in N \), we have \( ((x \ast z) \ast (y \ast t)) \in N \)

**Definition 2.11 [19] (Fuzzy normal subset of a B - algebra)**
Let \( (X, *, 0) \) be a B-algebra and let a fuzzy set \( \mu \) in \( X \) is said to be fuzzy normal if it satisfies the inequality \( \mu((x \ast a) \ast (y \ast b)) \geq \min\{\mu(x \ast y), \mu(a \ast b)\} \) for all \( a, b, x, y \in X \).
Definition 2.12 [19] (Fuzzy normal subalgebra of B – algebra)
A fuzzy set μ in a B – algebra X is called a fuzzy normal B - algebra if it is a fuzzy B – algebra which is fuzzy normal

Definition 2.13 [5] (Derivation of G – algebra)
Let X be a G – algebra and d is a self-map on X. We say that, d is (l, r)- derivation of X if d(x * y) = (d(x) * y) ∧ (x * d(y))
That is, for all x, y ∈ X: d(x * y) = d(x) * y and d(x * y) = x * d(y), respectively.

Remark 2.14 [5]
In a G – algebra, \( (x ∨ y) = x \)

Definition 2.15 [5] (Modified definition of G -derivation)
Let X be a G – algebra and d a self – map on X. We say that d is a derivation of X if, d is (l, r)- derivation of X and (r, l)- derivation of X.
That is, for all x, y ∈ X: d(x * y) = d(x) * y and d(x * y) = x * d(y), respectively.

Definition 2.16 [2] (Vague Coset)
Let A be a vague group of a group \( (G, .) \). For any \( a ∈ G \).
(i) A vague left coset of A is denoted by aA and defined by \( V_{aA}(x) = V_A(a^{-1}x) \).
\( i.e., t_{aA}(x) = t_A(a^{-1}x) \) and \( f_{aA}(x) = f_A(a^{-1}x) \)
(ii) A vague right coset of A is denoted by Aa and defined by \( V_{Aa}(x) = V_A(ax^{-1}) \).
\( i.e., t_{Aa}(x) = t_A(ax^{-1}) \) and \( f_{Aa}(x) = f_A(ax^{-1}) \)

Definition 2.17 [5] (Homomorphism, Epimorphism, Endomorphism of G - algebra)
Let X and Y be G-algebras. A mapping \( φ: X → Y \) is called a homomorphism if \( φ(x * y) = φ(x) * φ(y) \), \( ∀ x, y ∈ X \). The homomorphism \( φ \) is said to be a monomorphism (resp., an epimorphism) if it is injective (resp., surjective). If the map \( φ \) is both injective and surjective then X and Y are said to be isomorphic, written X \( ≅ \) Y. For any homomorphism \( φ: X → Y \), the set \( \{ x ∈ X | φ(x) = 0_Y \} \) is called the kernel of \( φ \) and denoted by Ker \( φ \)

3. Neutrosophic vague binary G - subalgebra

In this section neutrosophic vague binary G - subalgebra is developed with its properties and with some theorems.

Definition 3.1 (Neutrosophic vague binary G - subalgebra)
Let \( M_{NVB} \) be a neutrosophic vague binary set (in short, NVBS) with two universes \( U_1 \) and \( U_2 \).
A neutrosophic vague binary G - subalgebra is a structure \( G_{M_{NVB}} = (U^{φ_M_{NVB}}, *, 0) \) which satisfies, the following \( G_{M_{NVB}} \) inequality:
\[ G_{M_{NVB}} \textrm{ inequality :} \]
\[ \text{NVB}_{M_{NVB}}(x * y) ≥ r \min \{ \text{NVB}_{M_{NVB}}(x), \text{NVB}_{M_{NVB}}(y) \} ; \ ∀ \ x, y ∈ U \]
That is, \( ∀ \ x, y ∈ U \)
\[ t_{M_{NVB}}(x * y) ≥ \min \{ t_{M_{NVB}}(x), t_{M_{NVB}}(y) \} ; \ i_{M_{NVB}}(x * y) ≤ \max \{ i_{M_{NVB}}(x), i_{M_{NVB}}(y) \} ; \ f_{M_{NVB}}(x * y) ≤ \max \{ f_{M_{NVB}}(x), f_{M_{NVB}}(y) \} \]
\[ * \ \text{and} \ 0 \ \text{are as in} \ U^{φ_M_{NVB}} \ \text{and} \ T = [T^-, T^+] ; \ I = [I^-, I^+] ; \ F = [F^-, F^+] \]
Here,

- \( M_{NVB} \) is a neutrosophic vague binary set with two universes \( U_1 \) and \( U_2 \).

- \( U^{NVB} = (U = \{U_1 \cup U_2\}, \ast, 0) \) is a \( G \)-algebraic structure with a binary operation \( \ast \) & a constant \( 0 \), which satisfies following axioms: \( \forall x, y \in U \), (i) \( (x \ast x) = 0 \); (ii) \( x \ast (x \ast y) = y \).

**Remark 3.2**
(i) Neutrosophic vague binary \( G \)-subalgebra is written in short as NVB \( G \)-subalgebra.
(ii) In NVB \( G \)-subalgebra universal set \( U \) is taken as “union” of elements of \( U_1 \) and \( U_2 \).
(iii) Before applying \( \Phi_{NVB} \) condition, neutrosophic vague binary union [Here, (max, min, min)] have to take for common elements of \( U_1 \) and \( U_2 \). Combined neutrosophic vague binary membership grades will draw and implement combined effect to neutrosophic vague binary values of \( U_1 \) and \( U_2 \). This will fulfill the binary effect in neutrosophic vague sets in the practical point of view.

**Example 3.3**

Let \( U_1 = \{0, a, b\} \), \( U_2 = \{0, b, c\} \) be two universes. Combined universe \( U = \{U_1 \cup U_2\} = \{0, a, b, c\} \).

Binary operation \( \ast \) is defined as given by the Cayley table given below:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>c</td>
<td>b</td>
<td>a</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Clearly, \( (U, \ast, 0) \) is a \( G \)-algebra. Consider a NVBS formed based on \( U_1 \) & \( U_2 \).

\( M_{NVB} = \begin{pmatrix} 0.9 & 0.9 & 0.9 & 0.9 \\ 0.9 & 0.9 & 0.9 & 0.9 \\ 0.9 & 0.9 & 0.9 & 0.9 \\ 0.9 & 0.9 & 0.9 & 0.9 \end{pmatrix} \)

Combined neutrosophic vague binary membership grade is given by

\( M_{NVB} (s) = \begin{cases} 0.9 & 0.2 & 0.6 & 0.1 & 0.1 \\ 0.8 & 0.9 & 0.3 & 0.7 & 0.1 & 0.2 \\ 0.8 & 0.9 & 0.2 & 0.6 & 0.1 & 0.2 \\ 0.8 & 0.9 & 0.3 & 0.7 & 0.1 & 0.2 \end{cases} \)

Calculations shows that \( M_{NVB} \) is a NVB \( G \)-subalgebra.

**Remark 3.4**

In a NVB \( G \)-algebra, construction of the underlying \( G \)-algebraic structure, using a binary operation \( \ast \) deserves prime importance. Instead of \( \ast \) different symbols like \( +, -, \times, +_4 \) etc can be applied. Binary operation can be formed in different ways. Construction of \( G \)-algebra using the following points always defines a \( G \)-algebra. In the Binary Operation,
(i) If “first operand = second operand” then the output will be zero.

[Using definition of \( G \)-algebra, \( (x \ast x) = 0 \Rightarrow \) principal diagonal elements should occupy with constant 0, in the Cayley table of a \( G \)-algebra.]

(ii) If “first operand \( \neq \) second operand” with “first operand \( \neq 0 \) & second operand = 0”, then output will be first operand

(iii) If “first operand \( \neq \) second operand” with “first operand \( \neq 0 \) & the second operand \( \neq 0 \),

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then output will be second operand

Following Cayley Table will make idea clear. \( U = \{0, a_1 \neq 0, a_2 \neq 0, \ldots, a_n \neq 0\} \).

From above, numbers in the square brackets indicates specific points used to frame the output.

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>( a_1 \neq 0 )</th>
<th>( a_2 \neq 0 )</th>
<th>( \ldots \neq 0 )</th>
<th>( a_n \neq 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 [i]</td>
<td>( a_1 [ii] )</td>
<td>( a_2 [ii] )</td>
<td>( \ldots [ii] )</td>
<td>( a_n [ii] )</td>
</tr>
<tr>
<td>( a_1 \neq 0 )</td>
<td>( a_1 [ii] )</td>
<td>0 [i]</td>
<td>( a_2 [ii] )</td>
<td>( \ldots [ii] )</td>
<td>( a_n [ii] )</td>
</tr>
<tr>
<td>( a_2 \neq 0 )</td>
<td>( a_2 [ii] )</td>
<td>( a_1 [iii] )</td>
<td>0 [i]</td>
<td>( \ldots [ii] )</td>
<td>( a_n [ii] )</td>
</tr>
<tr>
<td>( \ldots \neq 0 )</td>
<td>( \ldots [ii] )</td>
<td>( \ldots [iii] )</td>
<td>( \ldots [iii] )</td>
<td>( \ldots [iii] )</td>
<td>( \ldots [iii] )</td>
</tr>
<tr>
<td>( a_n \neq 0 )</td>
<td>( a_n [ii] )</td>
<td>( a_1 [iii] )</td>
<td>( a_2 [iii] )</td>
<td>( \ldots [ii] )</td>
<td>0 [i]</td>
</tr>
</tbody>
</table>

**Remark 3.5**

A neutrosophic vague G - subalgebra is a structure \( G_{NV} = (U^{6_{NV}}, *, 0) \) which satisfies,

\[ NV_{NV}(x * y) \geq r \min [NV_{NV}(x), NV_{NV}(y)] \]  

known as, \( G_{NV} \) condition. That is,

\[ f_{NV}(x * y) \geq \min [f_{NV}(x), f_{NV}(y)]; f_{NV}(x * y) \leq \max [f_{NV}(x), f_{NV}(y)]; f_{NV}(x * y) \leq \max [f_{NV}(x), f_{NV}(y)] \]

\[ * \text{ and } 0 \text{ are as in } U^{6_{NV}} \cup \bar{T} = [T^-, T^+]; \bar{1} = [1^-, 1^+]; \bar{0} = [F^-, F^+] \]

Here,

- \( M_{NV} \) is a neutrosophic vague set with a single universe \( U \)
- \( U^{6_{NV}} = (U, *, 0) \) is a G - algebraic structure with a binary operation \( * \) & a constant 0, which satisfies following axioms: \( \forall x, y \in U \) \( (i) \ x * x = 0 \) ; \( (ii) \ x * (x * y) = y \)

**Remark 3.6**

It is straight forward to check that, intersection of neutrosophic vague binary G - subalgebras produce a neutrosophic vague binary G - subalgebra itself. But union may not be!

**4. Different notions of Neutrosophic Vague Binary G - subalgebra**

In this section following notions to a NVB – G subalgebra are discussed.

- G – part of a Neutrosophic Vague Binary G – subalgebra
- G – p radical of a Neutrosophic Vague Binary G – subalgebra
- G - p semi simple of a Neutrosophic Vague Binary G – subalgebra
- G – minimal element of a Neutrosophic Vague Binary G – subalgebra

**Definition 4.1**

Let \( M_{NV} \) be a NVB G -subalgebra with structure \( G_{M_{NV}} = (U^{6_{NV}}, *, 0) \)

i. G – part of a Neutrosophic Vague Binary G – subalgebra

Let \( S_{NV} \) be any NVBSS of \( M_{NV} \). Define, \( G(S_{NV}) = \{x \in S_{NV} / NVB_{NV}(0 * x) = NVB_{NV}(x)\} \).

In particular, if \( S_{NV} = M_{NV} \) then \( G(M_{NV}) \) is called the **neutrosophic vague binary G – G part** (in short, NVB G – G part) of the NVB G – subalgebra.

ii. p – radical of a Neutrosophic Vague Binary G – subalgebra
B(M_NV) = \{ x \in U / NVB_{M_NV}(0 * x) = NVB_{M_NV}(0) \} is called, a neutrosophic vague binary G – p radical (in short, NVB G – p radical) of the NVB G -subalgebra M_NV

iii. p- semi simple of a Neutrosophic Vague Binary G – subalgebra

M_NV is called neutrosophic vague binary G - p semi simple (in short, NVB G – p semi simple), if B(M_NV) = \{ x \in U / NVB_{M_NV}(0 * x) = NVB_{M_NV}(0) \} = \{0\}


Any element x ∈ U is neutrosophic vague binary G – minimal element (in short, NVB G – minimal element), if NVB_{M_NV}(x * y) = NVB_{M_NV}(0) ⇒ NVB_{M_NV}(y) = NVB_{M_NV}(x)

Remark 4.2

It is clear that, G(M_NV) ∩ B(M_NV) = \{0\}

Theorem 4.3

Let M_NV be a NVB G - subalgebra with structure G_{M_NV} = (U^{BM_NV}, *, 0). Then, x ∈ G(M_NV) if and only if NVB_{M_NV}(0 * x) ∈ G(M_NV)

Proof

\( x \in G(M_NV) \Rightarrow NVB_{M_NV}(0 * x) = NVB_{M_NV}(x) \)

\( ⇒ NVB_{M_NV}(0 * x) = NVB_{M_NV}(0 * (0 * x)), \) [from proposition 3.8]

\( ⇒ (0 * x) \in G(M_NV), \) [by definition 6.1].

Conversely, if \( (0 * x) \in G(M_NV) \), then \( NVB_{M_NV}(0 * (0 * x)) = NVB_{M_NV}(0 * x) \)

\( ⇒ NVB_{M_NV}(x) = NVB_{M_NV}(0 * x) \Rightarrow x \in G(M_NV) \)

Theorem 4.4

Let M_NV be a NVB G - subalgebra with structure G_{M_NV} = (U^{BM_NV}, *, 0).

(i) M_NV is NVB G - p semi simple. (ii) Every element in U is a NVB G - minimal element.

Proof

(i) From definition 6.1(iii), B(M_NV) = \{ x \in U / NVB_{M_NV}(0 * x) = NVB_{M_NV}(0) \}

\( ⇒ B(M_NV) = \{ x \in U / NVB_{M_NV}(0) = NVB_{M_NV}(x) \} = \{0\} \)

(ii) Assume (ii). Let x be an arbitrary element in U such that y ≤ x for some y ∈ U

\( ⇒ (x * y) = 0 \Rightarrow NVB_{M_NV}(x * y) = NVB_{M_NV}(0) \Rightarrow NVB_{M_NV}(x) = NVB_{M_NV}(y) \)

Theorem 4.5

If G(M_NV) = G_{M_NV}, then M_NV is NVB G - p semi simple.

That is, if a NVB G – subalgebra coincides with its G – part then it is NVB G – p semi simple

Proof

Let M_NV be a NVB G - subalgebra with structure G_{M_NV} = (U^{BM_NV}, *, 0).

From definition 6.1(ii), G(M_NV) = \{ x ∈ M_NV / NVB_{M_NV}(0 * x) = NVB_{M_NV}(x) \}

If G(M_NV) = G_{M_NV} then B(M_NV) = \{0\} ⇒ M_NV is NVB G - p semi simple

Remark 4.6

(1) In any NVB G – subalgebra: NVB_{M_NV}(x) ≤ NVB_{M_NV}(y) ⇒ NVB_{M_NV}(y * x) = NVB_{M_NV}(0)

(2) Denote NVB_{M_NV}(x * y) by NVB_{M_NV}(x ∧ y) for all x, y ∈ U. From definition 3.1 (i) (2).

Theorem 4.7
Let $M_{NVB}$ be a NVB $G$ - subalgebra with structure $0_{M_{NVB}} = (U^{0_{M_{NVB}}}, *, 0)$. Then for any $x, y, z \in U$,

1. For $x \neq y$, $NVB_{M_{NVB}}(x \land y) \neq NVB_{M_{NVB}}(y \land x)$
2. $NVB_{M_{NVB}}(x \land (y \land z)) = NVB_{M_{NVB}}((x \land y) \land z)$
3. $NVB_{M_{NVB}}((x \land 0) = NVB_{M_{NVB}}(x)$ and $NVB_{M_{NVB}}(0 \land x) = NVB_{M_{NVB}}(0)$
4. For $x \neq 0$, $NVB_{M_{NVB}}[x \land (y \land z)] \neq NVB_{M_{NVB}}[(x \land y) \land (x \land z)]$

**Proof**

1. For a NVB $G$ – subalgebra, $NVB_{M_{NVB}}(x \land y) = NVB_{M_{NVB}}(x) \land NVB_{M_{NVB}}(y)$
2. $NVB_{M_{NVB}}((x \land y) \land z) = NVB_{M_{NVB}}(x \land (y \land z))$
3. $NVB_{M_{NVB}}((x \land y) \land z) = NVB_{M_{NVB}}(x \land (y \land z))$
4. It is clear that, for a NVB $G$ – subalgebra, $NVB_{M_{NVB}}(x) \neq NVB_{M_{NVB}}(0) \Rightarrow NVB_{M_{NVB}}[x \land (y \land z)] \neq NVB_{M_{NVB}}[(x \land y) \land (x \land z)]$

**Theorem 4.8**

Every NVB $G$ – subalgebra satisfies the inequality, $NVB_{M_{NVB}}(0) \succ NVB_{M_{NVB}}(x)$; $\forall x \in U$

**Proof**

$NVB_{M_{NVB}}(0) = NVB_{M_{NVB}}(x \land x) \succ r \min \{NVB_{M_{NVB}}(x), NVB_{M_{NVB}}(x)\} = NVB_{M_{NVB}}(x)$

**Theorem 4.9**

Let $0_{M_{NVB}} = (U^{0_{M_{NVB}}}, *, 0)$ be a NVB $G$ - subalgebra. Then the following conditions hold:
(i) $NVB_{M_{NVB}}(x \land 0) = NVB_{M_{NVB}}(x), \forall x \in U$ (ii) $NVB_{M_{NVB}}(0 \land (0 \land x)) = NVB_{M_{NVB}}(x), \forall x \in U$

**Proof**

Let $0_{M_{NVB}} = (U^{0_{M_{NVB}}}, *, 0)$ be a NVB $G$ – subalgebra and $x, y \in U^{0_{M_{NVB}}}$. Then,

(i) $NVB_{M_{NVB}}(x \land 0) = NVB_{M_{NVB}}(x \land (x \land x))$

[Using first condition in the $G$ – algebraic structure of NVB $G$ – subalgebra]

$= NVB_{M_{NVB}}(x)$ [Using second condition in the $G$ – algebraic structure of NVB $G$ – subalgebra]

(ii) Since $0_{M_{NVB}}$ is a NVB $G$ – subalgebra, $NVB_{M_{NVB}}(x \land (x \land y)) = NVB_{M_{NVB}}(y)$

[Using second condition in the $G$ – algebraic structure of NVB $G$ – subalgebra] Put $x = 0$ and $y = x$ in the above then (ii) follows

**Theorem 4.10**

Let $0_{M_{NVB}} = (U^{0_{M_{NVB}}}, *, 0)$ be a NVB $G$ - subalgebra. Then following conditions hold: $\forall x, y \in U$,

(i) $NVB_{M_{NVB}}((x \land (x \land y)) \land y) = NVB_{M_{NVB}}(0)$

(ii) $NVB_{M_{NVB}}(x \land y) = NVB_{M_{NVB}}(0) \Rightarrow NVB_{M_{NVB}}(x) = NVB_{M_{NVB}}(y)$

(iii) $NVB_{M_{NVB}}(0 \land x) = NVB_{M_{NVB}}(0 \land y) \Rightarrow NVB_{M_{NVB}}(x) = NVB_{M_{NVB}}(y)$

**Proof**

(i) \( \text{NVB}_{MNVB}(x \ast (x \ast y)) \ast y = \text{NVB}_{MNVB}((y \ast (y \ast y)) \ast y) \) by putting \( x = y \)

\[ = \text{NVB}_{MNVB}((y \ast 0) \ast y) = \text{NVB}_{MNVB}(y \ast y) = \text{NVB}_{MNVB}(0) \]

(ii) Assume \( \text{NVB}_{MNVB}(x \ast y) = \text{NVB}_{MNVB}(0) \)

\[ \therefore \text{NVB}_{MNVB}(x) = \text{NVB}_{MNVB}(x \ast 0) = \text{NVB}_{MNVB}(x \ast (x \ast y)), \text{ by assumption} \]

\[ = \text{NVB}_{MNVB}(y) \]

(iii) Assume \( \text{NVB}_{MNVB}(0 \ast x) = \text{NVB}_{MNVB}(0 \ast y) \)

\[ \therefore \text{NVB}_{MNVB}(x) = \text{NVB}_{MNVB}(0 \ast (0 \ast x)) = \text{NVB}_{MNVB}(0 \ast (0 \ast y)), \text{ by assumption} \]

\[ = \text{NVB}_{MNVB}(y) \]

**Theorem 4.11**

Let \( \mathcal{G}_{MNVB} = (U @_{MNVB}, \ast, 0) \) be a NVB \( G \) – subalgebra. Then,

\[ \text{NVB}_{MNVB}(a \ast x) = \text{NVB}_{MNVB}(a \ast y) \Rightarrow \text{NVB}_{MNVB}(x) = \text{NVB}_{MNVB}(y), \text{ for any } a, x, y \in U \]

**Proof**

If \( \mathcal{G}_{MNVB} = (U @_{MNVB}, \ast, 0) \) be a NVB \( G \) – subalgebra satisfying,

\[ \text{NVB}_{MNVB}(a \ast x) = \text{NVB}_{MNVB}(a \ast y), \text{ for any } a, x, y \in U \]

Then,

\[ \text{NVB}_{MNVB}(x) = \text{NVB}_{MNVB}(a \ast (a \ast x)) = \text{NVB}_{MNVB}(a \ast (a \ast y)) = \text{NVB}_{MNVB}(y) \]

**Theorem 4.12**

Let \( \mathcal{G}_{MNVB} = (U @_{MNVB}, \ast, 0) \) be a NVB \( G \) – subalgebra. Then the following are equivalent:

(1) \( \text{NVB}_{MNVB}((x \ast y) \ast (x \ast z)) = \text{NVB}_{MNVB}(z \ast y); \; \forall \; x, y, z \in U \)

(2) \( \text{NVB}_{MNVB}((x \ast z) \ast (y \ast z)) = \text{NVB}_{MNVB}(x \ast y); \; \forall \; x, y, z \in U \)

**Proof**

(i) \( \Rightarrow \) (ii)

Assume (i). i.e., \( \text{NVB}_{MNVB}((x \ast y) \ast (x \ast z)) = \text{NVB}_{MNVB}(z \ast y); \; \forall \; x, y, z \in U \)

\[ \therefore \text{NVB}_{MNVB}((x \ast z) \ast (x \ast y)) = \text{NVB}_{MNVB}(y \ast z) \]

Consider, \( \text{NVB}_{MNVB}((x \ast z) \ast (y \ast z)) \)

\[ = \text{NVB}_{MNVB}((x \ast z) \ast ((x \ast z) \ast (x \ast y))) \]

\[ = \text{NVB}_{MNVB}(x \ast y), \text{ since } \text{NVB}_{MNVB}(x \ast (x \ast y)) = \text{NVB}_{MNVB}(y) \]

(ii) \( \Rightarrow \) (i)

Assume (ii). i.e., \( \text{NVB}_{MNVB}((x \ast z) \ast (y \ast z)) = \text{NVB}_{MNVB}(x \ast y) \)

\[ \therefore \text{NVB}_{MNVB}((x \ast y) \ast (z \ast y)) = \text{NVB}_{MNVB}(x \ast z) \]

Consider, \( \text{NVB}_{MNVB}((x \ast y) \ast (x \ast z)) \)

\[ = \text{NVB}_{MNVB}((x \ast y) \ast ((x \ast y) \ast (z \ast y))) \]

\[ = \text{NVB}_{MNVB}(z \ast y), \text{ since } \text{NVB}_{MNVB}(x \ast (x \ast y)) = \text{NVB}_{MNVB}(y) \]

5. Neutrosophic Vague Binary \( G \) – normal subalgebra

In this section neutrosophic vague binary \( G \) – normal subalgebra is introduced
Definition 5.1 (Neutrosophic vague binary G – normal subalgebra)
Let \( M_{NVB} \) be a neutrosophic vague binary set (in short, NVBS) with two universes \( U_1 \) and \( U_2 \). Neutrosophic vague binary G – normal subalgebra is a structure \( \Theta_{NVB}^N = (U^{\Theta_{NVB}^N}, *, 0) \) which satisfies, the following 2 conditions known as \( \Theta_{NVB}^N \) inequalities:

\( \Theta_{NVB}^N \) inequality (1):
\[
NVB_{M_{NVB}}(x * y) \geq r \min \{NVB_{M_{NVB}}(x), NVB_{M_{NVB}}(y)\}; \forall x, y \in U
\]
That is, \( \forall x, y \in U \), \( T_{min}(x * y) \geq \min \{T_{max}(x), T_{min}(y)\} \)
\[
\Theta_{NVB}^N \) inequality (2):
\[
NVB_{M_{NVB}}(x * a) * (y * b)) \geq r \min \{NVB_{M_{NVB}}(x * y), NVB_{M_{NVB}}(a * b)\}; \forall a, b, x, y \in U
\]
That is, \( \forall a, b, x, y \in U \), \( T_{max}(x * a) * (y * b)) \geq \min \{T_{max}(x * y), T_{min}(a * b)\} \)

Here,
- \( M_{NVB} \) is a neutrosophic vague binary set with two universes \( U_1 \) and \( U_2 \)
- \( U^{\Theta_{NVB}^N} = (U = \{U_1 \cup U_2\}, *, 0) \) is a G - algebraic structure with a binary operation \( * \) & a constant 0,
which satisfies following axioms : \( \forall x, y \in U \), (i) \( x * x = 0 \); (ii) \( x * (x * y) = y \)

Definition 5.2 (Neutrosophic vague binary G – normal set)
Let \( M_{NVB} \) be a NVBS with two universes \( U_1, U_2 \). Take \( U = \{U_1 \cup U_2\} \).
A NVBS \( M_{NVB} \) in \( U \) is said to be NVB G – normal set if it satisfies the inequality
\[
\{NVB_{M_{NVB}}((x * a) * (y * b)) \geq r \min \{NVB_{M_{NVB}}(x * y), NVB_{M_{NVB}}(a * b)\} \} \forall x, y, a, b \in U
\]
That is,
\[
\begin{align*}
\{T_{M_{NVB}}((x * a) * (y * b)) \geq \min \{T_{M_{NVB}}(x * y), T_{M_{NVB}}(a * b)\}\} \\
\{I_{M_{NVB}}((x * a) * (y * b)) \geq \max \{I_{M_{NVB}}(x * y), I_{M_{NVB}}(a * b)\}\} \\
\{F_{M_{NVB}}((x * a) * (y * b)) \geq \max \{F_{M_{NVB}}(x * y), F_{M_{NVB}}(a * b)\}\}
\end{align*}
\]

Remark 5.3
(i) Neutrosophic vague binary G - normal subalgebra is written in short as NVB G – normal subalgebra. It is denoted by \( \Theta_{M_{NVB}}^N \).
(ii) In other words, a NVBS \( M_{NVB} \) in a G – algebra \( U \) is called a NVB G – normal subalgebra if it is a NVB G - subalgebra which is NVB G - normal set.

Theorem 5.4
Every NVB G - normal set \( M_{NVB} \) in \( U \) is a NVB G – subalgebra of \( U \).
That is, every NVB G – normal set \( M_{NVB} \) is a \( \Theta_{M_{NVB}}^N \).

Proof
Let \( M_{NVB} \) be a NVB G – normal set in \( U \) \( \Rightarrow \) \( NVB_{M_{NVB}}((x * a) * (y * b)) \geq r \min \{NVB_{M_{NVB}}(x * y), NVB_{M_{NVB}}(a * b)\} \)
Consider, \( NVB_{M_{NVB}}(x * y) = NVB_{M_{NVB}}((x * y) * (0 * 0)) \) \( \Rightarrow \) \( r \min \{NVB_{M_{NVB}}(x * y), NVB_{M_{NVB}}(y * 0)\} \)

---

\[ r \min \{ \text{NVB}_{M_{\text{NVB}}}(x), \text{NVB}_{M_{\text{NVB}}}(y) \} \Rightarrow \text{NVB}_{M_{\text{NVB}}}(x \ast y) \geq r \min \{ \text{NVB}_{M_{\text{NVB}}}(x), \text{NVB}_{M_{\text{NVB}}}(y) \} \quad \forall \ x, y \in U \Rightarrow M_{M_{\text{NVB}}} \text{ is a NVB G – subalgebra} \]

**Remark 5.5**
Converse of theorem 5.4 is not true.
That is, a NVB G – subalgebra \( M_{\text{NVB}} \) in \( U \) is not a NVB G – normal set, generally.

**Proof**
Consider example 4.3, in which \( M_{\text{NVB}} \) is a NVB G – subalgebra. In this example, \( \text{NVB}_{M_{\text{NVB}}}((a \ast a) \ast (b \ast a)) \not\geq r \min \{ \text{NVB}_{M_{\text{NVB}}}(a \ast b), \text{NVB}_{M_{\text{NVB}}}(a \ast a) \} \Rightarrow M_{M_{\text{NVB}}} \) is not a NVB G – normal set.

**Theorem 5.6**
If a neutrosophic vague binary set \( M_{\text{NVB}} \) in \( U \) is a NVB G – normal subalgebra, then \( \text{NVB}_{M_{\text{NVB}}}(x \ast y) = \text{NVB}_{M_{\text{NVB}}}(y \ast x) ; \forall \ x, y \in U \)

**Proof**
Let \( x, y \in U \). Then, \( \text{NVB}_{M_{\text{NVB}}}(x \ast y) = \text{NVB}_{M_{\text{NVB}}}(x \ast (y \ast x)) \) [From theorem 4.8] \( = \text{NVB}_{M_{\text{NVB}}}(x \ast y \ast (x \ast x)) \) \( = r \min \{ \text{NVB}_{M_{\text{NVB}}}(x \ast x), \text{NVB}_{M_{\text{NVB}}}(y \ast x) \} \) \[ \geq r \min \{ \text{NVB}_{M_{\text{NVB}}}(x \ast x), \text{NVB}_{M_{\text{NVB}}}(y \ast x) \} = \text{NVB}_{M_{\text{NVB}}}(y \ast x) \] [From theorem 4.7]. \( \ast \text{NVB}_{M_{\text{NVB}}}(x \ast y) \geq \text{NVB}_{M_{\text{NVB}}}(y \ast x) \). Similarly, \( \text{NVB}_{M_{\text{NVB}}}(y \ast x) \geq \text{NVB}_{M_{\text{NVB}}}(x \ast y) \) \( \Rightarrow \text{NVB}_{M_{\text{NVB}}}(x \ast y) = \text{NVB}_{M_{\text{NVB}}}(y \ast x) \)

### 6. 0 – commutative neutrosophic vague binary G – subalgebra

In this section 0 – commutative neutrosophic vague binary G – subalgebra with its properties are introduced.

**Definition 6.1** (0 - commutative of a NVB G - subalgebra)

Let \( M_{\text{NVB}} \) be a neutrosophic vague binary set (in short, NVBS) with two universes \( U_1 \) and \( U_2 \). 0 - commutative neutrosophic vague binary G - subalgebra is a structure \( \mathcal{G}^0_{M_{\text{NVB}}} = (U^0_{M_{\text{NVB}}}, \ast, 0) \) which satisfies the following \( \mathcal{G}^0_{M_{\text{NVB}}} \) inequality:

\[
\text{NVB}_{M_{\text{NVB}}}(x \ast y) \geq r \min \{ \text{NVB}_{M_{\text{NVB}}}(x), \text{NVB}_{M_{\text{NVB}}}(y) \} ; \forall \ x, y \in U
\]
i.e., \( \tilde{v}_{\text{min}}(x \ast y) \geq \min (\tilde{v}_{\text{min}}(x), \tilde{v}_{\text{min}}(y)) \); \( \tilde{v}_{\text{min}}(x \ast y) \leq \max (\tilde{v}_{\text{min}}(x), \tilde{v}_{\text{min}}(y)) \); \( \tilde{v}_{\text{max}}(x \ast y) \leq \max (\tilde{v}_{\text{max}}(x), \tilde{v}_{\text{max}}(y)) \)

\[
\ast \text{ and } 0 \text{ are as in } U^0_{M_{\text{NVB}}} \quad \& \quad \tilde{T} = [\tilde{T}^-, \tilde{T}^+] ; \quad \tilde{I} = [\tilde{I}^-, \tilde{I}^+] ; \quad \tilde{F} = [\tilde{F}^-, \tilde{F}^+] \]

Here,

- \( M_{\text{NVB}} \) is a neutrosophic vague binary set with two universes \( U_1 \) and \( U_2 \)
- \( U^0_{M_{\text{NVB}}} = (U = (U_1 \cup U_2), \ast, 0) \) is a 0 – commutative G - algebraic structure with a binary operation \( \ast \) & a constant 0, which satisfies the following axioms:
  
  \( \forall \ x, y \in U \), (i) \( x \ast x = 0 \) ; (ii) \( x \ast (x \ast y) = y \); (iii) \( x \ast (0 \ast y) = y \ast (0 \ast x) \)

0 - commutative neutrosophic vague binary G - subalgebra is denoted by \( \mathcal{G}^0_{M_{\text{NVB}}} \)

---

Example 6.2
Consider example 4.3. with a different binary operation defined as given in following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

\( U^0_{\text{NVB}} \) will become 0 - commutative \( \mathcal{G}_{\text{MNVB}}^0 \) only if \( \mathcal{G}_{\text{MNVB}}^0 \) inequality got satisfied. Verifications showed that \( M_{\text{NVB}} \) given in example 3.3 is not only a \( \mathcal{G}_{\text{MNVB}} \) but it is clearly a \( \mathcal{G}_{\text{MNVB}}^0 \) too!

Theorem 6.3
Let \( \mathcal{G}_{\text{MNVB}} = (U^0_{\text{MNVB}}, \ast, 0) \) be a \( \mathcal{G}_{\text{MNVB}}^0 \).
Then, \( \text{NVB}_{\text{MNVB}}( (0 \ast x) \ast (0 \ast y) ) = \text{NVB}_{\text{MNVB}}(y \ast x) \) for any \( x, y \in U \)

Proof
Let \( x, y \in U \) where \( U \in U^0_{\text{MNVB}} \). Then, \( (0 \ast x) \ast (0 \ast y) = (y \ast (0 \ast (0 \ast x))) = (y \ast x) \)
\( \Rightarrow x, y \in U \) where \( U \in U^0_{\text{MNVB}} \Rightarrow U^0_{\text{MNVB}} \) becomes \( U^0_{\text{MNVB}} \Rightarrow \mathcal{G}_{\text{MNVB}} \) becomes \( \mathcal{G}_{\text{MNVB}}^0 \)

7. Derivations of Neutrosophic Vague Binary G – subalgebra

In this section following points are developed

i. neutrosophic vague binary \( G \) – derivation

ii. neutrosophic vague binary \( G \) – regular derivation

Definition 7.1 \( \text{(G – derivation of neutrosophic vague binary G – subalgebra)} \)

Let \( M_{\text{NVB}} \) be a neutrosophic vague binary set in short, \( \text{NVBS} \) with two universes \( U_1 \) and \( U_2 \). Also let considered \( M_{\text{NVB}} \) is a \( \text{NVB G} \) - subalgebra with structure \( \mathcal{G}_{\text{MNVB}} = (U^0_{\text{MNVB}}, \ast, 0) \) and with a self – map \( d : U \rightarrow U \) on \( M_{\text{NVB}} \) with \( U = \{ U_1 \cup U_2 \} \). Then,
(i) \( d \) is \( (l, r) \) neutrosophic vague binary \( G \) - derivation of \( M_{\text{NVB}} \) if,
\( \text{NVB}_{\text{MNVB}}[d(x \ast y)] = \text{NVB}_{\text{MNVB}}((d(x) \ast y) \land (x \ast d(y))) \)
(ii) \( d \) is \( (r, l) \) neutrosophic vague binary \( G \) - derivation of \( M_{\text{NVB}} \) if,
\( \text{NVB}_{\text{MNVB}}[d(x \ast y)] = \text{NVB}_{\text{MNVB}}((x \ast d(y)) \land (d(x) \ast y)) \)

\( d \) is a neutrosophic vague binary \( G \) – derivation (in short, \( \text{NVB G} \) - derivation) of \( M_{\text{NVB}} \) only if
\( d \) is both \( (l, r) \) neutrosophic vague binary \( G \) – derivation [in short, \( (l, r) \text{ NVB G} \) - derivation]
& \( (r, l) \) neutrosophic vague binary \( G \) – derivation [in short, \( (r, l) \text{ NVB G} \) - derivation] of \( M_{\text{NVB}} \).
In this definition, \( (l, r) \) indicates left-right and \( (r, l) \) indicates right-left

Remark 7.2
For a \( \text{NVB G} \) - subalgebra, \( \text{NVB}_{\text{MNVB}}(x \land y) = \text{NVB}_{\text{MNVB}}(x) \)

(i) \( \therefore \) To check, \( d \) is \((l, r)\) \( \text{NVB G} \) - derivation of \( M_{\text{NVB}} \), it is enough to check that,

\[
\text{NVB}_{\text{MNVB}}(d(x \ast y)) = \text{NVB}_{\text{MNVB}}(d(x \ast y)); \quad [\text{Using definition 2.13 & by remark 2.14}]
\]

(ii) \( \therefore \) To check, \( d_{(r,l)} \) \( \text{NVB G} \) - derivation of \( M_{\text{NVB}} \), it is enough to check that,

\[
\text{NVB}_{\text{MNVB}}(d(x \ast y)) = \text{NVB}_{\text{MNVB}}(x \ast d(y)) \quad [\text{Using definition 2.13 & by remark 2.14}]
\]

\( \therefore \) Definition 7.1, can be re-written as, definition 7.3

**Definition 7.3**

Let \( \mathcal{G}_{\text{MNVB}} \) be a \( \text{NVB G} \) – subalgebra and \( d \) be a self – map on \( U \).
\( d \) is a neutrosophic vague binary \( G \) - derivation of \( U \) if

(i) \( d \) is \((l, r)\) – neutrosophic vague binary \( G \) - derivation of \( U \)

\( \text{i.e.,} \quad \text{NVB}_{\text{MNVB}}(d(x \ast y)) = \text{NVB}_{\text{MNVB}}(d(x) \ast y) \); for all \( x, y \in U \) & it is denoted by \( d_{(lr)}^{\text{MNVB}} \)

(ii) \( d \) is \((r, l)\) – neutrosophic vague binary \( G \) - derivation of \( U \)

\( \text{i.e.,} \quad \text{NVB}_{\text{MNVB}}(d(x \ast y)) = \text{NVB}_{\text{MNVB}}(x \ast d(y)) \); for all \( x, y \in U \) & it is denoted by \( d_{(rl)}^{\text{MNVB}} \).

\( d \) is a \( \text{NVB G} \) – derivation on \( \mathcal{G}_{\text{MNVB}} \) if \( d \) is both \( d_{(lr)}^{\text{MNVB}} \) and \( d_{(rl)}^{\text{MNVB}} \). It is denoted by \( d_{r}^{\text{MNVB}} \).

**Definition 7.4** (Regular derivation of a Neutrosophic Vague Binary \( G \) – subalgebra)

A derivation \( d_{r}^{\text{MNVB}} \) of a \( \text{NVB G} \) – subalgebra is said to be regular if,

\[
\text{NVB}_{\text{MNVB}}(d(0)) = \text{NVB}_{\text{MNVB}}(0). \quad \text{It is denoted by} \quad d_{r}^{\text{MNVB}}.
\]

**Example 7.5**

From example 3.3, \( M_{\text{NVB}} \) is a \( \mathcal{G}_{\text{MNVB}} \).

Case (i) Define a self – map, \( d : \ U = \{0, a, b\} \rightarrow U = \{0, a, b\} \) by \( d(s) = \begin{cases} 0 & \text{if } s = 0 \\ a & \text{if } s = a \\ b & \text{if } s = b \end{cases} \)

Here the given self – map is an identity map.

From calculations, \( d \) is a \( d_{(lr)}^{\text{MNVB}} \) & \( d_{(rl)}^{\text{MNVB}} \Rightarrow d \) is a \( d_{r}^{\text{MNVB}} \)

Case (ii) Define a self-map, \( d : \ U = \{0, a, b\} \rightarrow U = \{0, a, b\} \) by \( d(s) = \begin{cases} a & \text{if } s = 0 \\ 0 & \text{if } s = a \\ b & \text{if } s = b \end{cases} \)

\( d \) is not a \( \text{NVB G} \) – derivation on \( M_{\text{NVB}} \). One violation is attached below.

\[
\text{NVB}_{\text{MNVB}}(d(b \ast a)) = \text{NVB}_{\text{MNVB}}(d(a)) = \text{NVB}_{\text{MNVB}}(0) = [0.9, 0.9]\{0, 0.1, 1\}\{0, 0.1, 1\}
\]

\[
\text{NVB}_{\text{MNVB}}(d(b) \ast a) = \text{NVB}_{\text{MNVB}}(b \ast a) = \text{NVB}_{\text{MNVB}}(a) = [0.7, 0.9]\{0, 0.3, 1\}\{0, 0.1, 1\}
\]

\( d_{(lr)}^{\text{MNVB}}(b \ast a) \) does not exist, since \( \text{NVB}_{\text{MNVB}}(d(b) \ast a) \neq \text{NVB}_{\text{MNVB}}(d(b) \ast a) \)

\( d_{(rl)}^{\text{MNVB}}(b \ast d(a)) = \text{NVB}_{\text{MNVB}}(b \ast d(a)) = \text{NVB}_{\text{MNVB}}(b) = [0.2, 0.6]\{0, 0.1, 2\}\{0.4, 0.8\}

\( d_{(rl)}^{\text{MNVB}}(b \ast a) \) does not exist, since \( \text{NVB}_{\text{MNVB}}(d(b) \ast a) \neq \text{NVB}_{\text{MNVB}}(b \ast d(a)) \)

\( \Rightarrow d \) is not a \( d_{r}^{\text{MNVB}} \).

**Theorem 7.6**

In a \( \mathcal{G}_{\text{MNVB}} \), the identity map \( d \) on \( U \) is a \( d_{r}^{\text{MNVB}} \). Converse not true in general. But if \( d_{r}^{\text{MNVB}} \) is a \( d_{r}^{\text{MNVB}} \), then converse hold good. That is, if \( d_{r}^{\text{MNVB}} \) is a \( d_{r}^{\text{MNVB}} \), then \( d \) is the identity map on \( U \)

**Proof**

(i) Let \( x, y \in U \) & also let \( d \) is an identity map on \( U \)

Case (i) \( x = y ; y \neq 0 \)

\[
\text{NVB}_{\text{MNVB}}(d(x \ast y)) = \text{NVB}_{\text{MNVB}}(d(x \ast x)) = \text{NVB}_{\text{MNVB}}(d(0)) = \text{NVB}_{\text{MNVB}}(0).
\]

\[
\text{NVB}_{\text{MNVB}}(d(x) \ast y) = \text{NVB}_{\text{MNVB}}(d(x) \ast x) = \text{NVB}_{\text{MNVB}}(x \ast x) = \text{NVB}_{\text{MNVB}}(0).
\]

\[
\text{NVB}_{\text{MNVB}}(x \ast d(y)) = \text{NVB}_{\text{MNVB}}(x \ast d(x)) = \text{NVB}_{\text{MNVB}}(x \ast x) = \text{NVB}_{\text{MNVB}}(0).
\]

\( \therefore \) \( \text{NVB}_{\text{MNVB}}(d(x \ast y)) = \text{NVB}_{\text{MNVB}}(d(x) \ast y) = \text{NVB}_{\text{MNVB}}(x \ast d(y)) \)
Case (ii) : \( x \neq y \); \( y \neq 0 \)

Either \( \text{NVB}_{\text{MNVB}}(d(x \ast y)) = \text{NVB}_{\text{MNVB}}(d(x)) \ast \text{NVB}_{\text{MNVB}}(d(y)) \) or \( \text{NVB}_{\text{MNVB}}(d(x \ast y)) = \text{NVB}_{\text{MNVB}}(d(y)) \ast \text{NVB}_{\text{MNVB}}(d(x)) \)

\[ \Rightarrow d(x \ast y) = d(x) \text{ or } d(x \ast y) = d(y) \Rightarrow \text{ either } (x \ast y) = x \text{ or } (x \ast y) = y, \text{ since } d \text{ is identity map} \]

\[ \Rightarrow \text{ either } y = 0 \text{ or } y \neq 0. \]

Consider \( y \neq 0, \text{ i.e., } \ d(x \ast y) = d(y), \text{ i.e., } (x \ast y) = y. \)

\[ \text{NVB}_{\text{MNVB}}(d(x \ast y)) = \text{NVB}_{\text{MNVB}}(d(y)) = \text{NVB}_{\text{MNVB}}(y) \cdot \text{NVB}_{\text{MNVB}}(d(x)) = \text{NVB}_{\text{MNVB}}(x \ast d(y)) = \text{NVB}_{\text{MNVB}}(x \ast y) = \text{NVB}_{\text{MNVB}}(y). \]

Case (iii) : \( x \neq y \); \( y = 0 \)

Either \( \text{NVB}_{\text{MNVB}}(d(x \ast y)) = \text{NVB}_{\text{MNVB}}(d(x)) - \text{NVB}_{\text{MNVB}}(d(y)) \text{ or } \text{NVB}_{\text{MNVB}}(d(x \ast y)) = \text{NVB}_{\text{MNVB}}(d(y)) - \text{NVB}_{\text{MNVB}}(d(x)) \)

\[ \Rightarrow d(x \ast y) = d(x) \text{ or } d(x \ast y) = d(y) \Rightarrow \text{ either } (x \ast y) = x \text{ or } (x \ast y) = y, \text{ since } d \text{ is identity map} \]

\[ \Rightarrow \text{ either } y = 0 \text{ or } y \neq 0. \]

Consider \( y = 0, \text{ i.e., } \ d(x \ast y) = d(x), \text{ i.e., } (x \ast y) = x. \)

\[ \text{NVB}_{\text{MNVB}}(d(x \ast y)) = \text{NVB}_{\text{MNVB}}(d(x)) = \text{NVB}_{\text{MNVB}}(x) \cdot \text{NVB}_{\text{MNVB}}(d(x \ast y)) = \text{NVB}_{\text{MNVB}}(x \ast d(y)) = \text{NVB}_{\text{MNVB}}(x \ast y) = \text{NVB}_{\text{MNVB}}(x \ast d(y)) \]

\[ \Rightarrow \text{ d is both } d^\ast_{\text{MNVB}} \text{ & } d^\ast_{\text{MNVB}} \text{. Hence d is a } d^\ast_{\text{MNVB}} \]

Converse

\[ d^\ast_{\text{MNVB}} \text{ is a } d_{\text{MNVB}} \Rightarrow \text{NVB}_{\text{MNVB}}(d(x)) = \text{NVB}_{\text{MNVB}}(0) \Rightarrow \text{NVB}_{\text{MNVB}}(d(x \ast x)) = \text{NVB}_{\text{MNVB}}(0) \]

\[ \Rightarrow \text{NVB}_{\text{MNVB}}(d(x \ast x) = \text{NVB}_{\text{MNVB}}(0)) \Rightarrow d(x) = x \text{ [By proposition 3.11 (ii)]} \]

\[ \Rightarrow d \text{ is the identity map on } U \]

Remark 7.7

Let \( M_{\text{MNVB}} \) be a NVB G - subalgebra with structure \( e_{\text{MNVB}} = (u^\ast, *, 0). \) A NVB G - derivation on \( M_{\text{MNVB}} \) is a mapping \( d : U \rightarrow U \) such that \( \text{NVB}_{\text{MNVB}}(d(x \ast y)) = \text{NVB}_{\text{MNVB}}(d(x) \ast y) = \text{NVB}_{\text{MNVB}}(x \ast d(y)), \forall x, y \in U. \) Set of all neutrosophic vague binary G -derivations on \( M_{\text{MNVB}} \) is denoted as \( \Gamma d_{\text{MNVB}} \)

8. Neutrosophic vague binary G – Coset

General properties that are true for abstract algebra and G – algebra may not be true in the case of neutrosophic G – subalgebra/neutrosophic vague G – subalgebra/ neutrosophic vague binary G – subalgebra. In this section coset for neutrosophic vague binary G – subalgebra is developed. Neutrosophic vague binary G – Coset is considered as a shifted (or translated) neutrosophic vague binary G – subalgebra. Existence of identity element and inverse element can’t be assured in every neutrosophic vague binary G – subalgebra. In generalization process, this will become a crisis. As a result, generalization is confined to a particular area. It will lead to the formation of different concepts like Lagrange neutrosophic vague binary G – subalgebra etc.


Let \( M_{\text{MNVB}} \) be a neutrosophic vague binary set (in short, NVBS) with two universes \( U_1 \) and \( U_2. \) and also let the considered \( M_{\text{MNVB}} \) is a NVB G – subalgebra of a G – algebra with algebraic structure \( e_{\text{MNVB}} = (U^\ast, *, 0) \text{ where } U^\ast_{\text{MNVB}} = (U, *, 0_{\text{MNVB}}). \) Also, \( \bar{T} = [T^+, T^-], I = [-I, I], \bar{P} = [F^-, F^+] \) and \( U = \{U_1 \cup U_2\} \)

Case (i) (Neutrosophic Vague Binary G – Right Coset)
Let a ∈ U₁ and b ∈ U₂ be fixed elements. Then define, for every c ∈ U₁ and for every d ∈ U₂ a neutrosophic vague binary G – right coset of M_{NVB} is denoted by M_{NVB}(a, b) and defined by,

\[(M_{NVB}(a, b)(c, d) = \text{NVB}_{(a,b)}(c, d) = \{(\text{NVB}_{(a,b)}(c, d) : \forall c \in U₁ \}\{\text{NVB}_{(a,b)}(d, b') \mid \forall d \in U₂}\}\]

i.e., \[\{(\text{NVB}_{(a,b)}(c, d) : \text{NVB}_{(a,b)}(c, d) : \forall c \in U₁ \}\{(\text{NVB}_{(a,b)}(b, d) : \text{NVB}_{(a,b)}(b, d) : \forall d \in U₂)\}\]

Then M_{NVB}(a, b) is called a neutrosophic vague binary G -Right Coset (in short NVBG – Right Coset) determined by M_{NVB} and (a, b).

Case (ii) (Neutrosophic vague binary G – Left Coset)

Let a ∈ U₁ and b ∈ U₂ be fixed elements. Then define, for every c ∈ U₁ and for every d ∈ U₂ a neutrosophic vague binary G – right coset of M_{NVB} is denoted by (a, b) M_{NVB} and defined by,

\[(a, b) M_{NVB}(c, d) = \text{NVB}_{(a,b)}(c, d) = \{(\text{NVB}_{(a,b)}(c, d) : \forall c \in U₁ \}\{\text{NVB}_{(b,a)}(d, b') \mid \forall d \in U₂)\}

\[\{(a, b) M_{NVB}(c, d) : \text{NVB}_{(a,b)}(c, d) : \forall c \in U₁ \}\{(a, b) M_{NVB}(b, d) : \text{NVB}_{(b,a)}(b, d) : \forall d \in U₂)\}\]

Then M_{NVB}(a, b) is called a neutrosophic vague binary left coset (in short NVBG – left coset) determined by M_{NVB} and (a, b).

Remark 8.2

NVBG – right coset is a NVBS. Similarly, a NVBG – left coset is a NVBS.

Definition 8.3 (Neutrosophic Vague Binary G – Coset)

Let the neutrosophic vague binary set M_{NVB} be a neutrosophic vague binary G – subalgebra of a G – algebra. If M_{NVB} is both neutrosophic vague binary G – right coset and neutrosophic vague binary G – left coset then M_{NVB} is called a Neutrosophic Vague Binary G – Coset

Example 8.4

Let U₁ = {0, u₁, u₃} and U₂ = {0, u₂, u₄, u₅} be two universes.

Let M_{NVB} = \[
\begin{bmatrix}
0.7,0.8 & 0.3,0.4 & 0.2,0.3 & 0.2,0.7 & 0.5,0.7 & 0.3,0.8 & 0.6,0.7 & 0.1,0.4 & 0.3,0.4 \\
0.2,0.9 & 0.1,0.8 & 0.3,0.5 & 0.6,0.7 & 0.5,0.7 & 0.2,0.8 & 0.4,0.7 & 0.2,0.8 & 0.6,0.9 & 0.3,0.7 & 0.1,0.4
\end{bmatrix}
\]

be a NVBS.

Here, combined universe U = {0, u₁, u₂, u₃, u₄, u₅} & combined NVB membership grades are,

\[\text{NVB}_{M_{NVB}}(s) = \begin{bmatrix}
0.7,0.9 & [0.1,0.4] & [0.1,0.3] & s = 0 \\
0.2,0.7 & [0.5,0.7] & [0.3,0.8] & s = u₁ \\
0.3,0.5 & [0.6,0.7] & [0.5,0.7] & s = u₂ \\
0.6,0.7 & [0.1,0.4] & [0.3,0.4] & s = u₃ \\
0.2,0.8 & [0.4,0.7] & [0.2,0.8] & s = u₄ \\
0.6,0.9 & [0.3,0.7] & [0.1,0.4] & s = u₅
\end{bmatrix}\]

Corresponding Cayley table is:

\[
\begin{array}{cccccc}
* & 0 & u₁ & u₂ & u₃ & u₄ & u₅ \\
0 & 0 & u₁ & u₂ & u₃ & u₄ & u₅ \\
u₁ & u₁ & 0 & u₂ & u₃ & u₄ & u₅ \\
u₂ & u₂ & u₁ & 0 & u₃ & u₄ & u₅ \\
u₃ & u₃ & u₂ & u₁ & 0 & u₄ & u₅ \\
u₄ & u₄ & u₃ & u₂ & u₁ & 0 & u₅ \\
u₅ & u₅ & u₄ & u₃ & u₂ & u₁ & 0
\end{array}
\]
Obviously, $M_{NVB}$ is a NVB G–subalgebra
In every G – algebra 0 may not be the identity element. But in the present case it is clear that 0 acts as an identity element. Hence inverses got as :
\[(0)^{-1} = 0 ; \ (u_1)^{-1} = u_1 ; \ (u_2)^{-1} = u_2 ; \ (u_3)^{-1} = u_3 ; \ (u_4)^{-1} = u_4 ; \ (u_5)^{-1} = u_5\]

To construct a NVB G -right coset:

Let $u_a = u_1 \in U_1$ and $\forall \ u_c \in U_1 = \{0, u_1, u_3\}$

\[
\begin{align*}
NVB_{M_{NVB}} u_1 \ (0) &= NVB_{M_{NVB}}(0 * (u_1^{-1})) = NVB_{M_{NVB}}(0 * u_1) = NVB_{M_{NVB}}(u_1) = [0.2, 0.7][0.5, 0.7][0.3, 0.8] \\
NVB_{M_{NVB}} u_1 \ (u_1) &= NVB_{M_{NVB}}(u_1 * (u_1^{-1})) = NVB_{M_{NVB}}(u_1 * u_1) = NVB_{M_{NVB}}(0) = [0.7, 0.9][0.1, 0.4][0.1, 0.3] \\
NVB_{M_{NVB}} u_1 \ (u_3) &= NVB_{M_{NVB}}(u_3 * (u_1^{-1})) = NVB_{M_{NVB}}(u_3 * u_1) = NVB_{M_{NVB}}(u_1) = [0.2, 0.7][0.5, 0.7][0.3, 0.8] \\
\end{align*}
\]

&

Let $u_b = u_2 \in U_2$ and $\forall \ u_d \in U_2 = \{0, u_2, u_4, u_5\}$

\[
\begin{align*}
NVB_{M_{NVB}} u_2 \ (0) &= NVB_{M_{NVB}}(0 * (u_2^{-1})) = NVB_{M_{NVB}}(0 * u_2) = NVB_{M_{NVB}}(u_2) = [0.3, 0.5][0.6, 0.7][0.5, 0.7] \\
NVB_{M_{NVB}} u_2 \ (u_2) &= NVB_{M_{NVB}}(u_2 * (u_2^{-1})) = NVB_{M_{NVB}}(u_2 * u_2) = NVB_{M_{NVB}}(0) = [0.7, 0.9][0.1, 0.4][0.1, 0.3] \\
NVB_{M_{NVB}} u_2 \ (u_4) &= NVB_{M_{NVB}}(u_4 * (u_2^{-1})) = NVB_{M_{NVB}}(u_4 * u_2) = NVB_{M_{NVB}}(u_2) = [0.3, 0.5][0.6, 0.7][0.5, 0.7] \\
NVB_{M_{NVB}} u_2 \ (u_5) &= NVB_{M_{NVB}}(u_5 * (u_2^{-1})) = NVB_{M_{NVB}}(u_5 * u_2) = NVB_{M_{NVB}}(u_2) = [0.3, 0.5][0.6, 0.7][0.5, 0.7] \\
\end{align*}
\]

To construct a NVB G -left coset:

Let $a = u_1 \in U_1$ and $\forall \ c \in U_1 = \{0, u_1, u_3\}$

\[
\begin{align*}
NVB \ u_1 \ M_{NVB} \ (0) &= NVB_{M_{NVB}}((u_1^{-1}) * 0) = NVB_{M_{NVB}}(u_1 * 0) = NVB_{M_{NVB}}(u_1) = [0.2, 0.7][0.5, 0.7][0.3, 0.8] \\
NVB \ u_1 \ M_{NVB} \ (u_1) &= NVB_{M_{NVB}}(u_1 * (u_1^{-1})) = NVB_{M_{NVB}}(u_1 * u_1) = NVB_{M_{NVB}}(u_1) = [0.3, 0.5][0.6, 0.7][0.5, 0.7] \\
NVB \ u_1 \ M_{NVB} \ (u_3) &= NVB_{M_{NVB}}((u_1^{-1}) * u_3) = NVB_{M_{NVB}}(u_1 * u_3) = NVB_{M_{NVB}}(u_3) = [0.6, 0.7][0.1, 0.4][0.3, 0.4] \\
\end{align*}
\]

Let $b = u_2 \in U_2$ and $\forall \ d \in U_2 = \{0, u_2, u_4, u_5\}$

\[
\begin{align*}
NVB \ u_2 \ M_{NVB} \ (0) &= NVB_{M_{NVB}}((u_2^{-1}) * 0) = NVB_{M_{NVB}}(u_2 * 0) = NVB_{M_{NVB}}(u_2) = [0.3, 0.5][0.6, 0.7][0.5, 0.7] \\
NVB \ u_2 \ M_{NVB} \ (u_2) &= NVB_{M_{NVB}}((u_2^{-1}) * u_2) = NVB_{M_{NVB}}(u_2 * u_2) = NVB_{M_{NVB}}(0) = [0.7, 0.9][0.1, 0.4][0.1, 0.3] \\
NVB \ u_2 \ M_{NVB} \ (u_4) &= NVB_{M_{NVB}}((u_2^{-1}) * u_4) = NVB_{M_{NVB}}(u_2 * u_4) = NVB_{M_{NVB}}(u_4) = [0.2, 0.8][0.4, 0.7][0.2, 0.8] \\
NVB \ u_2 \ M_{NVB} \ (u_5) &= NVB_{M_{NVB}}((u_2^{-1}) * u_5) = NVB_{M_{NVB}}(u_2 * u_5) = NVB_{M_{NVB}}(u_5) = [0.6, 0.9][0.3, 0.7][0.1, 0.4] \\
\end{align*}
\]

Remark 8.5

(i) In example 8.4, $M_{NVB} \ u_1; u_2 \neq u_1; u_2 \ M_{NVB}$
(ii) Constant 0 is not an identity element in G – algebra. For example, let $X = \{0, u_1, u_2, u_3, u_4, u_5\}$. ($X, * , 0$) is a G – algebra, with binary operation * is defined by the following Cayley table:
It is clear that X is a G – algebra without an identity element. And hence inverse does not exist. So neutrosophic vague binary G - cosets cannot construct in this case. This construction is possible, only for those cases where identity element exists in the basic G – algebraic structure.

(ii) If the basic G – algebraic structure is formed using the following rules, then definitely there exist identity element and hence can construct a NVB G – right coset & NVG G – left coset.

Rules in Cayley table:
(i) Principal diagonal elements = 0
(ii) Column occupied with constant 0 is a copy of column of operands
(iii) Fill each of the remaining columns (except principal diagonal entries) with the element given in the column head (i.e., elements from row of operands)

Definition 8.6 (Neutrosophic Vague G – Right Coset & Neutrosophic Vague G- Left Coset)

Let $M_{NV}$ be a neutrosophic vague set (in short, NV Set) with a single universe U and also let $M_{NV}$ be a neutrosophic vague G – subalgebra (in short, NV G – subalgebra) of a G – algebra. Algebraic structure of $M_{NV}$ is given by $\Theta_{M_{NV}} = (U_{M_{NV}} \ast, 0)$ where $U_{M_{NV}} = (U, \ast, 0)$. Also $\bar{T} = [T^-, T^+]$ ; $\bar{I} = [I^-, I^+]$ ; $\bar{F} = [F^-, F^+]$

Case (i) (Neutrosophic Vague G – Right coset)

Let $a \in U$ be a fixed element. Then define, for every $c \in U$ a neutrosophic vague G – right coset of $M_{NV}$ which is denoted by $M_{NV} a$ and defined by,

\[
(M_{NV}(a))c = NV_{M_{NV}}(a)(c) = \{NV_{M_{NV}}(c \ast (a)^{-1}) \ / \ \forall c \in U\}
\]

i.e. $\bar{M} = \langle \bar{I}_{a M_{NV}}(c), \bar{J}_{a M_{NV}}(c), \bar{F}_{a M_{NV}}(c) \rangle / \forall c \in U$ ;

Then $M_{NV} a \ast$ is called a neutrosophic vague G -right coset (in short NV G – right coset) determined by $M_{NV}$ and a.

Case (ii) (Neutrosophic Vague G – Left Coset)

Let $a \in U$ be a fixed element. Then define, for every $c \in U$ a neutrosophic vague G – right coset of $M_{NV}$ is denoted by $M_{NV} a$ and defined by,

\[
((a) M_{NV})(c) = NV_{a M_{NV}}(c) = \{NV_{M_{NV}}((a)^{-1} \ast c)) \ / \ \forall c \in U\}
\]

= $\{\bar{T}_{a M_{NV}}(c) \ast \bar{I}_{a M_{NV}}(c) \ast \bar{F}_{a M_{NV}}(c) \} / \forall c \in U$ ;

= $\{\bar{T}_{M_{NV}}((a)^{-1} \ast c), \bar{I}_{M_{NV}}((a)^{-1} \ast c), \bar{F}_{M_{NV}}((a)^{-1} \ast c) \} / \forall c \in U$
Then a $M_{NV}$ is called a neutrosophic vague left coset (in short $NVG$ – left coset) determined by $M_{NV}$ and $a$.

**Definition 8.7 (Neutrosophic Vague $G$ – Coset)**

Let the neutrosophic vague set $M_{NV}$ be a neutrosophic vague $G$ – subalgebra of a $G$ – algebra. If $M_{NV}$ is both neutrosophic vague $G$ – Right Coset and neutrosophic vague $G$ – Left Coset then $M_{NV}$ is called as a Neutrosophic Vague $G$ – Coset

**Definition 8.8 (Neutrosophic $G$ – Right Coset & Neutrosophic $G$ – Left Coset)**

Let $M_N$ be a neutrosophic set (in short, $N$ set) with single universe $U$ and also let $M_N$ be a neutrosophic $G$ – subalgebra (in short, $NG$ – subalgebra) of a $G$ – algebra. Algebraic structure of $M_N$ is given by $\mathcal{G}_{M_N} = (U^{\mathcal{G}_{M_N}}, \ast, 0)$ where $U^{\mathcal{G}_{M_N}} = (U, \ast, 0)$.

Case (ii) (Neutrosophic $G$ – Right Coset)

Let $a \in U$ be a fixed element. Then define, for every $c \in U$ a neutrosophic $G$ – right coset of $M_N$ which is denoted by $aM_N$ and defined by,

$$(M_N(a))(c) = N_{M_N}(a(c)) = [N_{M_N}(c \ast (a)^{-1}) / \forall c \in U]$$

i.e.,

$$\left\{ (T_{aM_N}(c), I_{aM_N}(c), F_{aM_N}(c)) / \forall c \in U \right\}$$

$$= \left\{ (T_{M_N}(c \ast (a)^{-1}), I_{M_N}(c \ast (a)^{-1}), F_{M_N}(c \ast (a)^{-1})) / \forall c \in U \right\}$$

Then $aM_N$ is called a neutrosophic $G$ -right coset (in short $NG$ – right coset) determined by $M_N$ and $a$.

Case (ii) (Neutrosophic $G$ – Left Coset)

Let $a \in U$ be a fixed element. Then define, for every $c \in U$ a neutrosophic $G$ – right coset of $M_N$ is denoted by $aM_N$ and defined by,

$$(a)M_N(c) = N_{M_N}(a(c)) = [(N_{M_N}((a)^{-1} \ast c)) / \forall c \in U]$$

$$= \left\{ (T_{aM_N}(c), I_{aM_N}(c), F_{aM_N}(c)) / \forall c \in U \right\}$$

$$= \left\{ (T_{M_N}((a)^{-1} \ast c), I_{M_N}((a)^{-1} \ast c), F_{M_N}((a)^{-1} \ast c)) / \forall c \in U \right\}$$

Then $aM_N$ is called a neutrosophic left coset (in short $NG$ – left coset) determined by $M_N$ and $a$.

**Definition 8.9 (Neutrosophic $G$ – Coset)**

Let the neutrosophic set $M_N$ be a neutrosophic $G$ – subalgebra of a $G$ – algebra. If $M_N$ is both neutrosophic $G$ – Right Coset and neutrosophic $G$ – Left Coset then $M_N$ is called as a Neutrosophic $G$ – Coset


In this section homomorphism on $NVB G$ – subalgebra is presented with some related theorems.

**Definition 9.1**

Let $\mathcal{G}_{M_{NV}} = (U^{\mathcal{G}_{M_{NV}}} , \ast , 0_{M_{NV}})$ and $\mathcal{G}_{P_{NV}} = (U^{\mathcal{G}_{P_{NV}}} , \ast , 0_{P_{NV}})$ be two NVB $G$ – subalgebras based on common universe $\{U, U_2\}$.

A mapping $\Psi^G : \mathcal{G}_{M_{NV}} = (U^{\mathcal{G}_{M_{NV}}} , \ast , 0_{M_{NV}}) \rightarrow \mathcal{G}_{P_{NV}} = (U^{\mathcal{G}_{P_{NV}}} , \ast , 0_{P_{NV}})$ is called a neutrosophic vague binary $G$ - homomorphism if, $\Psi^G(u_x \ast u_y) = \Psi^G(u_x) \ast \Psi^G(u_y)$, $\forall$ $u_x, u_y \in U$.

**Remark 9.2**

(i) The $NVB G$ - homomorphism $\Psi^G$ is said to be a neutrosophic vague binary $G$ - monomorphism (resp., a neutrosophic vague binary $G$ - epimorphism) if it is injective (resp., surjective).

(ii) If the map $\Psi^G$ is both injective and surjective then $\mathcal{G}_{M_{NV}}$ and $\mathcal{G}_{P_{NV}}$ are said to be isomorphic.

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written \( \mathcal{G}_{MNVB} \equiv \mathfrak{G}_{PNVB} \). For any NVB \( G \) - homomorphism \( \Psi^G : \mathcal{G}_{MNVB} \rightarrow \mathfrak{G}_{PNVB} \), the set \( \{ x \in U \mid \Psi^G(x) = 0_{PNVB} \} \) is called the kernel of \( \Psi^G \) and denoted by \( \text{Ker} \Psi^G \).

**Theorem 9.3**

Let \( \Psi^G : \mathcal{G}_{MNVB} \rightarrow (U^{\mathfrak{G}_{PNVB}} \ast 0_{MNVB}) \rightarrow \mathfrak{G}_{PNVB} = (U^{\mathfrak{G}_{PNVB}} \ast, 0_{PNVB}) \) be a neutrosophic vague binary homomorphism of NVB \( G \) - subalgebras, then:

(i) \( \Psi^G(0_{MNVB}) = 0_{PNVB} \)

(ii) \( \text{Ker} \Psi^G \) is a normal neutrosophic vague binary \( G \) - subalgebra of \( U \)

(iii) \( \text{Im} \Psi^G = \{ y \in \mathfrak{G}_{PNVB} / y = \Psi^G(x) \text{ for some } x \in \mathcal{G}_{MNVB} \} \) is a NVB \( G \) - subalgebra

**Proof**

(i) \( \Psi^G(0_{MNVB}) = \Psi^G(0_{MNVB} \ast 0_{MNVB}) = \Psi^G(0_{MNVB}) \ast \Psi^G(0_{MNVB}) = 0_{PNVB} \ast 0_{PNVB} = 0_{PNVB} \)

(ii) \( 0_{MNVB} \in \text{Ker} \Psi^G \Rightarrow \text{Ker} \Psi^G \neq \emptyset \)

Let \( (x \ast y), (a \ast b) \in \text{Ker} \Psi^G \Rightarrow \Psi^G(x \ast y) = 0_{PNVB} = \Psi^G(a \ast b) \)

\( \Rightarrow \Psi^G(x) \ast \Psi^G(y) = 0_{PNVB} = \Psi^G(a) \ast \Psi^G(b) \Rightarrow \Psi^G(x) = \Psi^G(y) \) & \( \Psi^G(a) = \Psi^G(b) \)

[By proposition 3.9(ii)]

\( \Rightarrow \Psi^G((x \ast a) \ast (y \ast b)) = \Psi^G(x \ast a) \ast \Psi^G(y \ast b) \)

\( = \left( \Psi^G(x) \ast \Psi^G(a) \ast \Psi^G(y) \right)^* = \left( \Psi^G(x) \ast \Psi^G(y) \ast \Psi^G(a) \right)^* \ast \left( \Psi^G(x) \ast \Psi^G(a) \right)^* = 0_{PNVB} \)

[From definition of NVB \( G \) - subalgebra]

\( \Rightarrow \Psi^G((x \ast a) \ast (x \ast a)) = 0_{PNVB} \) [since \( \Psi^G \) is a NVB \( G \) - homomorphism]

\( \Rightarrow \Psi^G((x \ast a) \ast (x \ast a)) = 0_{PNVB} \) [since \( \Psi^G \) is a NVB \( G \) - homomorphism]

\( \Rightarrow (x \ast a) \ast (y \ast b) \in \text{Ker} \Psi^G \Rightarrow \text{Ker} \Psi^G \) is a N NVB \( G \)-subalgebra of \( U \)

(iii) Let \( y, z \in \mathfrak{G}_{PNVB} \Rightarrow y = \Psi^G(a) \) & \( z = \Psi^G(b) \) for some \( a, b \in \mathcal{G}_{MNVB} \)

\( \Psi^G(a) \ast \Psi^G(b) = \Psi^G(a \ast b) \geq r \min \{ \Psi^G(a), \Psi^G(b) \} \). Hence the proof.

**Theorem 9.4**

A neutrosophic vague binary \( G \) - homomorphism \( \chi^G : \mathfrak{G}_{TNVB} = (U^{\mathfrak{G}_{PNVB}} \ast, 0_{TNVB}) \rightarrow \mathfrak{G}_{LNVB} = (U^{\mathfrak{G}_{PNVB}} \ast, 0_{LNVB}) \) is a neutrosophic vague binary \( G \) - monomorphism \( \iff \ker(\chi^G) = \{ 0 \} \)

**Proof**

Let \( x \in \text{Ker} (\chi^G) \Rightarrow \chi^G(x) = 0_{LNVB} = \chi^G(0_{TNVB}) \).

\( \chi^G \) is a neutrosophic vague binary \( G \) - monomorphism, then it is clearly got that, \( \text{Ker} (\chi^G) = \{ 0 \} \).

Conversely, let \( \ker (\chi^G) = \{ 0 \} \) and also let \( \chi^G(x) = \chi^G(y), \forall x, y \in U \)

\( \Rightarrow \chi^G(x) \ast \chi^G(y) = 0_{LNVB} \Rightarrow \chi^G(x \ast y) = 0_{TNVB}, \text{ since } \chi^G \) is a NVB \( G \) - homomorphism

\( \Rightarrow (x \ast y) \in \text{Ker} (\chi^G) = \{ 0_{TNVB} \} \Rightarrow (x \ast y) = 0_{TNVB} \). Hence, \( x = y \Rightarrow \chi^G \) is a neutrosophic vague binary \( G \) - monomorphism.

**10. Conclusion**

In this paper, \( \text{NVB} \ G \) – subalgebraic structure is developed with its properties for \( \text{NVBS’s} \). Some basic ideas as \( \text{NVB} \ G \) - normal set of a \( G \) - algebra, \( \text{NVB} \ G \) – normal subalgebra and \( 0 \) – commutative \( \text{NVB} \ G \) – subalgebra are illustrated with examples and basic properties. Notions like \( G \) – part, \( p \) radical and \( p \) semi simple are defined in \( \text{NVB} \ G \) - subalgebra with characterizations. \( \text{NVB} \ G \) – minimal element, Derivations of \( \text{NVB} \ G \) – subalgebra, Regular Derivation of a \( \text{NVB} \ G \) – subalgebra, \( \text{NVB} \ G \) – homomorphism are also explained. Formation of Cosets is a basic idea in any algebraic structure. Coset for neutrosophic vague binary \( G \) – subalgebra is also got developed. Based on this, present work can be extended to \( \text{NVB} \) vague binary \( G \) -groups, \( \text{NVB} \ G \) – rings, \( \text{NVB} \ G \) – product, \( \text{NVB} \ G \) – factor group, Lagrange \( \text{NVB} \ G \) – subalgebra etc. As a future scope neutrosophic vague binary models can be tried to use in hazard detection, especially in switching circuits. Another application can be given in geographical area. Development of a neutrosophic vague binary spatial
algebra could be more helpful in this area than the already existing crisp spatial algebraic concepts. Since the already existing pattern got failed to provide an accurate output when collected data becomes vague.

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References


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