Neutrosophic Sets and Systems

Volume 38 Article 38

11-25-2020

Neutrosophic Vague Binary G – subalgebra of G - algebra

P. B. Remya

A. Francina Shalini

Neutrosophic Sets and Systems, Vol. 38, 2020
University of New Mexico

Neutrosophic Vague Binary G – subalgebra of G - algebra

P. B. Remya [1,*] and A. Francina Shalini [2]

[1] Ph. D Research Scholar, Department of Mathematics, Nirmala College for Women, Affiliated to Bharathiar University, Red Fields, Coimbatore – 18, Tamil Nadu, India, krish3thulasi@gmail.com.

[2] Assistant Professor, Department of Mathematics, Nirmala College for Women, Affiliated to Bharathiar University, Red Fields, Coimbatore – 18, Tamil Nadu, India, francshalu@gmail.com.

* Correspondence: krish3thulasi@gmail.com ; Tel.: (91 - 9751335441)

Abstract: Nowadays, human society is using artificial intelligence in a large manner so as to upgrade the present existing applicational criteria's and tools. Logic is the underlying principle to these works. Algebra is inevitably inter-connected with logic. Hence its achievements to the scientific research outputs have to be addressed. For these reasons, nowadays, research on various algebraic structures are going on wide. Crisp set has also got developed in a parallel way in the forms as fuzzy, intuitionistic fuzzy, rough, vague, neutrosophic, plithogenic etc. Sets with one or more algebraic operations will form different new algebraic structures for giving assistance to these logics, which in turn acts to as, a support to artificial intelligence. BCH/BCI/BCK- are some algebras developed in the first phase of algebraic development output. After that, so many outputs got flashed out, individually and in combinations in no time. Q- algebra and QS –algebra are some of these and could be showed as such kind of productions. G- algebra is considered as an extension to QS – algebra. In this paper neutrosophic vague binary G – subalgebra of G – algebra is generated with example. Notions like, 0 – commutative G - subalgebra, minimal element, normal subset etc. are investigated. Conditions to define derivation and regular derivation for this novel concept are clearly presented with example. Constant of G – algebra can't be treated as the identity element, generally. In this paper, it is well explained with example. Cosets for neutrosophic vague binary G – subalgebra of G - algebra is developed with proper explanation. Homomorphism for this new concept has been also got commented. Its kernel, monomorphism and isomorphism are also have discussed with proper attention.

Keywords: neutrosophic vague binary G - subalgebra, neutrosophic vague binary G - normal set, neutrosophic vague binary G – normal subalgebra, neutrosophic vague binary G G - part, neutrosophic vague binary G - p radical , neutrosophic vague binary G - p semisimple, neutrosophic vague binary G - minimal element, 0- commutative neutrosophic vague binary G – subalgebra, neutrosophic vague binary G – Derivation, neutrosophic vague binary G – Regular Derivation, neutrosophic vague binary G - Coset, Kernel of neutrosophic vague binary G – Homomorphism.

Notations: NVBS – neutrosophic vague binary set, NVBSS – neutrosophic vague binary subset. In this paper NVB is used to indicate neutrosophic vague binary and NV is used to indicate neutrosophic vague and N is used to indicate neutrosophic.

1.Introduction

Without mathematics mobility in human-life even became an unthinkable process. But when get into the mathematical world, one faces with, versatile facets of maths, which again get take diversions. The thing is that, dry subject is less get commented on or even less get touched with!

Algebra can also be considered so. But the entry of artificial intelligence made things different. Human world can simply neither ignore nor reject 'robots and computers' from their presently existing life pattern, due to their high impact in changing life style. So the question is that, what is the importance of algebra to these new scenario? Is it really useful for this robotic framed world? Answer is, yes! Since artificial intelligence is the raw material to robotics and to all the other newly developing phenomenon's. Logic is a foundation to artificial intelligence. Here a rapport activity can be seen in the picture. For logical calculations, algebra is very important. So these mixed works of algebra and sets is needed for the future research works in higher level. Chatoic and turbulances in human life situations, made data mining more difficult. To handle these crisis, new kind of extensions to cantor set have also got arosed. Human – life is going to get controlled by chips in next step of evolution. So hereafter, have to think on, ' what algebra can do ? ' in these kind of cross- breed structures in a 'chip oriented human life'. In this point, some debates are necessary. Whether is it good or bad? If bad, how these bad impact can convert into good, by taming these research works? Definitely these robotic effect made human life much easier both in 'profit and labour' level. Some bad outputs are also there and have to think of removing such negatives! From our washing machines to rocket technology, one can found this logic and algebraic illuminations. So giving some applications to algebra is irrelevant in one sense. But can think of the other part, in a little bit humorously. Where algebra is 'not showed off, its face 'in this modern world ? Following will give an idea to the newly developed algebraic structures in the family of algebra.

In 1966, Yasuyuki Imai and Kiyoshi Iseki [13] introduced BCK/BCI – algebra based on two different ways. One approach is based on set theory and the other one is based on classical and non-classical propositional calculi. In 1983, followed by these works another wide class of algebra namely, BCH – algebra is introduced by Q. P. Hu and X. Li [12]. In 2001, Q- algebra is introduced by Joseph Neggers, Sun Shin Ahn, and Hee Sik Kim [14] as an extension to BCH/BCI/BCK – algebras. In 1999, Sun Shin Ahn and Hee Sik Kim [24] introduced QS – algebra and investigated some relations between QS – algebra and BCK/BCI – algebra. They also investigated G – part in QS – algebra. In 2012, Ravi Kumar Bandaru and Rafi. N [22] introduced G -algebra as a generalization of QS – algebra. Necessary and sufficient condition for a G – algebra to become a QS – algebra is presented. It has shown that every associative G – algebra is a group. Concepts like 0 - commutative, G – part and medial of a G – algebra have explained. In 1984, Mukheriee. N.P and Prabir Bhattacharya [21] introduced fuzzy cosets and fuzzy normal subgroups. They have proven several interesting elementary properties related to the context and finally presented fuzzy interpretation of Lagrange theorem. In 2002, Young Bae Jun, Eun Hwan Roh, Chinju, and Hee Sik Kim, Seoul [27] discussed on fuzzy B – algebra. They addressed fuzzy normal B – algebra and fuzzy normal set in B – algebra. In 2016, Ch. Mallika, N. Ramakrishna, G. Anandha Rao [2] studied Vague Cosets. They also presented notions like vague symmetric, vague invariant, vague normal and some related properties. In 2015, Chiranjibe Jana, Tapan Senapati, Monoranjan Bhowmik, Madhumangal Pal [3] applied the concept of intuitionistic fuzzy set to G – subalgebra. In 2015, Chiranjibe Jana and Tapan Senapati [4] introduced Cubic G - subalgebra with properties. They also defined homomorphism of cubic G – subalgebra and verified various theorems. In 2015, Tapan Senapati, Chiranjibe Jana, Monoranjan Bhowmik, Madhumangal Pal [25] introduced L-fuzzy– set G-subalgebra and gave a characterization of L - fuzzy G – subalgebra. They also discussed some related characterizations like group, homomorphism etc. In 2017, Deena Al-Kadi, and Rodyna Hosny [5] generated a G-algebra from a non-empty set. They obtained quotient G-algebra using a normal subalgebra and proved fundamental theorem on G-algebra homomorphism. They showed that every BP-algebra is a G-algebra and provided an additional condition for the existence of the converse part. They further contributed notions of left-right and right-left derivation of G – algebra. They proved that G-algebra satisfying associative law is a 2-group. In 2018, Apurba Das [1] explored homotopy G – algebraic structure on the cochain complex of hom-type algebras. In 2020, Wahiba Messirdi and Ahlam Fallatah [26] presented several results on derivations of G – algebra. In 2019, Muhammad Abdul Basit Khan, Junaid Alam Khan, Muhammad Ahsan Binyamin [20] introduced SAGBI bases in G – algebra. They constructed an algorithm to compute SAGBI bases from a subset of polynomials contained in a subalgebra of a G – algebra. In 2019, Mohsin Khalid, Rakib Iqbal and Said Broumi [19] introduced Neutrosophic soft cubic G - subalgebras of G - algebras. Some basic operations like P- union, P – intersection, R-union, R – intersection are introduced. In 2020, Mohamed Abdel-Basset, Abduallah Gamal, Le Hoang Son and Florentin Smarandache [15] gave a case study in recruitment process based on bipolar neutrosophic set and bipolar neutrosophic number. In 2019, Mohamed Abdel-Basset, Rehab Mohamed, Abd El-Nasser H. Zaied and Florentin Smarandache [16] gave a hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics and conducted a study on thailand's sugar industry. In 2020, Mohamed Abdel-Basst, Rehab Mohamed, Mohamed Elhoseny [17] suggested a novel framework to evaluate innovation value proposition for smart product–service systems. In 2020, Mohamed Abdel-Basst, Rehab Mohamed, Mohamed Elhoseny [18] developed a model for the effective COVID-19 identification in uncertainty environment using primary symptoms and CT scans.

In 1993, Gau. W. L and Buehrer. D. J [11] presented vague sets. In 2005, Florentin Smarandache [6] introduced neutrosophic set and its basic ideas. In 2005, Florentin Smarandache [7] illustrated the difference between neutrosophic set and intuitionistic fuzzy set with proper explanations and examples. In 2011, Florentin Smarandache [8] gave a geometric interpretation of neutrosophic set using a neutrosophic cube. In 2015, Shawkat Alkhazaleh [23], introduced a mixed form of neutrosophic and vague known as neutrosophic vague sets. In 2019, P. B. Remya and A. Francina Shalini [9] applied binary concept to neutrosophic vague set and developed neutrosophic vague binary set with its basic operations.

In 2020, P.B. Remya and A. Francina Shalini [10] developed BCK/BCI – algebra for neutrosophic vague binary sets. Authors proposed a new suggestion of 'inclusion of new set' in the structure in addition to the 'underlying universal set', for avoiding more confusions in theoretical calculations. In this paper, authors further modified that structure and proposed a new approach in the structure mentioned in [10], by presenting a single set in the structure instead of the above mentioned two sets. This will give a combined effect of those two sets discussed above. New structure, convey the same effect of the structure used in paper [10], with a single set outlook and by skipping the two set pattern from structure. So here authors tried to present a one more modified form to the structure discussed in paper [10]. This one more refined pattern can be used in the all existing algebraic structures of various sets like fuzzy, vague, neutrosophic, etc., and for their hybrid forms in future works. This new pattern will be helpful to get more clarity and stability in these works.

This paper focuses on the development of G – algebraic structure to neutrosophic vague binary set. Discussions on G – algebra need some more attention while comparing to other algebraic structures. Its axioms are very simple and can be handled in a very clear manner. Neutrosophic ideas and Neutrosophic Vague ideas in G – algebra deserve more attention due to its easily accessible practical applications. This paper concentrates on neutrosophic vague binary G – subalgebra and its theoretical implementations.

Folllwing are the newly introduced concepts in this paper.

- Neutrosophic Vague Binary G – subalgebra [Section 3]
    - ✓ Neutrosophic Vague Binary G – subalgebra [Definition 3.1]
- Different notions of Neutrosophic Vague Binary G – subalgebra [Section 4]
    - ✓ Neutrosophic Vague Binary G - G part [Definition 4.1 (i)]
    - ✓ Neutrosophic Vague Binary G - p radical [Definition 4.1 (ii)]

- ✓ Neutrosophic Vague Binary G – p semi simple [Definition 4.1 (iii)]
- ✓ Neutrosophic Vague Binary G - minimal element [Definition 4.1 (iv)]

- Neutrosophic Vague Binary G – normal subalgebra [Section 5]
  - ✓ Neutrosophic Vague Binary G – normal subalgebra [Definition 5.1]
  - ✓ Neutrosophic Vague Binary G – normal set [Definition 5.2]
- 0 – commutative Neutrosophic Vague Binary G – subalgebra [Section 6]
  - ✓ 0 – commutative Neutrosophic Vague Binary G – subalgebra [Definition 6.1]
- Derivations of Neutrosophic Vague Binary G – subalgebra [Section 7]
  - ✓ Neutrosophic Vague Binary G – derivation [Definition 7.1]
  - ✓ Neutrosophic Vague Binary G – regular derivation [Definition 7.4]
- Neutrosophic Vague Binary G – Coset [Section 8]
  - ✓ Neutrosophic Vague Binary G – Right Coset [Definition 8.1 (i)]
  - ✓ Neutrosophic Vague Binary G – Left Coset [Definition 8.1(ii)]
  - ✓ Neutrosophic Vague Binary G – Coset [Definition 8.3]
- Neutrosophic Vague Binary G – homomorphism [Section 9]
  - ✓ Neutrosophic Vague Binary G – homomorphism [Definition 9.1]
  - ✓ Neutrosophic Vague Binary G- Isomorphism [Definition 9.2]

2. Preliminaries

In this section some preliminaries are given.

Definition 2.1 [9] (Neutrosophic Vague Binary Set)

A neutrosophic vague binary set (NVBS in short) M_{NVB} over a common universe $\{U_1 = \{x_j / 1 \leq j \leq n\}; U_2 = \{y_k/1 \leq k \leq p\}\}$ is an object of the form

$$M_{NVB} = \left\{\langle\frac{\hat{T}_{M_{NVB}}(x_j),\ \hat{I}_{M_{NVB}}(x_j),\ \hat{F}_{M_{NVB}}(x_j)}{x_j};\ \forall\, x_j \in U_1\rangle\ \langle\frac{\hat{T}_{M_{NVB}}(y_k),\ \hat{I}_{M_{NVB}}(y_k),\ \hat{F}_{M_{NVB}}(y_k)}{y_k};\ \forall\, y_k \in U_2\rangle\right\}$$

is defined as

$\hat{T}_{M_{NVB}}(x_j) = [T^-(x_j), T^+(x_j)]$, $\hat{I}_{M_{NVB}}(x_j) = [I^-(x_j), I^+(x_j)]$ and $\hat{F}_{M_{NVB}}(x_j) = [F^-(x_j), F^+(x_j)]$; $x_j \in U_1$ and
$\hat{T}_{M_{NVB}}(y_k) = [T^-(y_k), T^+(y_k)]$, $\hat{I}_{M_{NVB}}(y_k) = [I^-(y_k), I^+(y_k)]$ and $\hat{F}_{M_{NVB}}(y_k) = [F^-(y_k), F^+(y_k)]$; $y_k \in U_2$
where

(1) $T^+(x_j) = 1 - F^-(x_j)$; $F^+(x_j) = 1 - T^-(x_j)$; $\forall\, x_j \in U_1$ and
$T^+(y_k) = 1 - F^-(y_k)$; $F^+(y_k) = 1 - T^-(y_k)$; $\forall\, y_k \in U_2$

(2) $^-0 \leq T^-(x_j) + I^-(x_j) + F^-(x_j) \leq 2^+$; $^-0 \leq T^-(y_k) + I^-(y_k) + F^-(y_k) \leq 2^+$

or

$^-0 \leq T^-(x_j) + I^-(x_j) + F^-(x_j) + T^-(y_k) + I^-(y_k) + F^-(y_k) \leq 4^+$

and

$$^{-}0 \leq T^{+}(x_j) + I^{+}(x_j) + F^{+}(x_j) \leq 2^{+} ; \quad ^{-}0 \leq T^{+}(y_k) + I^{+}(y_k) + F^{+}(y_k) \leq 2^{+}$$

or

$$^{-}0 \leq T^{+}(x_j) + I^{+}(x_j) + F^{+}(x_j) + T^{+}(y_k) + I^{+}(y_k) + F^{+}(y_k) \leq 4^{+}$$

(3) $T^{-}(x_j), I^{-}(x_j), F^{-}(x_j) : V(U_1) \longrightarrow [0,1]$ and $T^{-}(y_k), I^{-}(y_k), F^{-}(y_k) : V(U_2) \longrightarrow [0,1]$

$T^{+}(x_j), I^{+}(x_j), F^{+}(x_j) : V(U_1) \longrightarrow [0,1]$ and $T^{+}(y_k), I^{+}(y_k), F^{+}(y_k) : V(U_2) \longrightarrow [0,1]$

Here $V(U_1), V(U_2)$ denotes power set of vague sets on U_1, U_2 respectively.

Definition 2.2 [14] (G – algebra)

A G- algebra is a non-empty set A with a constant 0 and a binary operation $*$ satisfying axioms: (B_3) $(x * x) = 0$ (B_{12}) $x * (x * y) = y$ for all $x, y \in A$;

A G-algebra is denoted by $(A, *, 0)$

Proposition 2.3 [14]

Any G -algebra X satisfies the following axioms: for all x, y, z ∈ X,

(i) $x * 0 = x$ (ii) $(x * (x * y)) * y = 0$ (iii) $0 * (0 * x) = x$ (iv) $x * y = 0$ implies $x = y$

(v) $0 * x = 0 * y$ implies $x = y$

Definition 2.4 [14] (G - subalgebra)

A non-empty subset S of a G-algebra X, is called a G-subalgebra of X if $(x * y) \in S$, for all $x, y \in S$

Definition 2.5 [14] (0 – commutative G - algebra)

A G - algebra $(A, *, 0)$ is said to be 0 – commutative if: $x * (0 * y) = y * (0 * x)$, for any $x, y \in A$

Theorem 2.6 [14] (G -part, p -radical, p – semisimple)

Let A be a G – algebra. For any subset S of A, we define $G(S) = \{x \in S / 0 * x = x\}$. In particular, if S = A then we say that G(A) is the G – part of a G – algebra. For any G – algebra A, the set B(A) = $\{x \in A / 0 * x = x\}$ is called a p – radical of A. A G – algebra is said to be p -semisimple if B(A) = {0}. The following property is obvious: $G(A) \cap B(A) = \{0\}$

Proposition 2.7 [14]

Let $(U, *, 0)$ be a G - algebra. Then, the following conditions hold for any $x, y \in X$

(1) $\big(x * (x * y)\big) * y = 0$ (2) $(x * y) = 0 \Rightarrow x = y$ (3) $(0 * x) = (0 * y) \Rightarrow x = y$

Definition 2.8 [17] (Fuzzy G – subalgebra)

Let $A = \{\langle x, \alpha_A(x)\rangle / x \in X\}$ be a fuzzy set in X, where X is a G-subalgebra. Then the set A is a fuzzy G - subalgebra over the binary operator $*$ if it satisfies the condition $\alpha_A(x * y) \geq \min\{\alpha_A(x), \alpha_A(y)\}$ for all $x, y \in X$

Definition 2.9 [3] (Intuitionistic Fuzzy G – subalgebra)

An IFS $A = (\alpha_A, \beta_A)$ in X is called an intuitionistic fuzzy G - subalgebra of X if for all $x, y \in X$ it satisfies:

GS1 $\alpha_A(x * y) \geq \min\{\alpha_A(x), \alpha_A(y)\}$; GS2 $\beta_A(x * y) \leq \max\{\beta_A(x), \beta_A(y)\}$

Definition 2.10 [18] (Normal Subalgebra of a G - algebra)

Let N be a non – empty subset of a G – algebra X. We say that N is a normal subset of X if for all x, y, z and t in X such that $(x * y) \in N$ and $(z * t) \in N$, we have $\big((x * z) * (y * t)\big) \in N$

Definition 2.11 [19] (Fuzzy normal subset of a B - algebra)

Let $(X, *, 0)$ be a B – algebra and let a fuzzy set μ in X is said to be fuzzy normal if it satisfies the inequality $\mu\big((x * a) * (y * b)\big) \geq \min\{\mu(x * y), \mu(a * b)\}$ for all $a, b\, x, y \in X$.

Definition 2.12 [19] (Fuzzy normal subalgebra of B – algebra)
A fuzzy set μ in a B – algebra X is called a fuzzy normal B - algebra if it is a fuzzy B – algebra which is fuzzy normal

Definition 2.13 [5] (Derivation of G – algebra)
Let X be a G – algebra and d is a self-map on X. We say that,
d is (l, r)- derivation of X if $d(x * y) = (d(x) * y) \wedge (x * d(y))$
d is (r, l)- derivation of X if $d(x * y) = (x * d(y)) \wedge (d(x) * y)$. If d is both (l, r)- derivation and (r, l)-derivation of X then we say that d is a derivation of X. (l, r) indicates left-right and (r, l) indicates right-left

Remark 2.14 [5]
In a G – algebra, $(x \wedge y) = x$

Definition 2.15 [5] (Modified definition of G -derivation)
Let X be a G – algebra and d a self – map on X. We say that d is a derivation of X if,
d is (l, r)- derivation of X and (r, l)- derivation of X.
That is, for all x, y $\in$ X : $d(x * y) = d(x) * y$ and $d(x * y) = x * d(y)$, respectively.

Definition 2.16 [2] (Vague Coset)
Let A be a vague group of a group $(G, .)$. For any a$\in$ G.
(i) A vague left coset of A is denoted by aA and defined by $V_{aA}(x) = V_A(a^{-1}x)$.
i.e., $t_{aA}(x) = t_A(a^{-1}x)$ and $f_{aA}(x) = f_A(a^{-1}x)$
(ii) A vague right coset of A is denoted by Aa and defined by $V_{Aa}(x) = V_A(xa^{-1})$.
i.e., $t_{Aa}(x) = t_A(xa^{-1})$ and $f_{Aa}(x) = f_A(xa^{-1})$

Definition 2.17 [5] (Homomorphism, Epimorphism, Endomorphism of G - algebra)
Let X and Y be G-algebras. A mapping $\varphi: X \to Y$ is called a homomorphism if $\varphi(x * y) = \varphi(x) * \varphi(y)$, $\forall\ x, y \in X$. The homomorphism φ is said to be a monomorphism (resp., an epimorphism) if it is injective (resp., surjective). If the map φ is both injective and surjective then X and Y are said to be isomorphic, written $X \cong Y$. For any homomorphism $\varphi: X \to Y$, the set $\{x \in X / \varphi(x) = 0_Y\}$ is called the kernel of φ and denoted by Ker φ

3. Neutrosophic vague binary G - subalgebra

In this section neutrosophic vague binary G - subalgebra is developed with its properties and with some theorems.

Definition 3.1 (Neutrosophic vague binary G - subalgebra)

Let $\mathbf{M_{NVB}}$ be a neutrosophic vague binary set (in short, NVBS) with two universes $\mathbf{U_1}$ and $\mathbf{U_2}$.
A neutrosophic vague binary G - subalgebra is a structure $\mathfrak{G}_{\mathbf{M_{NVB}}} = (\mathbf{U}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}, *, \mathbf{0})$ which satisfies, the following $\mathfrak{G}_{\mathbf{M_{NVB}}}$ inequality:

$\mathfrak{G}_{M_{NVB}}$ *inequality :*

$\mathbf{NVB_{M_{NVB}}(x * y)} \succcurlyeq \mathrm{r\,min}\ \{\mathbf{NVB_{M_{NVB}}(x), NVB_{M_{NVB}}(y)}\}$; $\forall\ \mathbf{x, y} \in \mathbf{U}$
That is, $\forall\ \mathbf{x, y} \in \mathbf{U}$
$\hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{x * y}) \geq \mathbf{min}\,\{\hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{x}), \hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{y})\}$; $\hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{x * y}) \leq \mathbf{max}\,\{\hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{x}), \hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{y})\}$; $\hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{x * y}) \leq \mathbf{max}\,\{\hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{x}), \hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{y})\}$
$\left[* \text{ and } \mathbf{0} \text{ are as in } \mathbf{U}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}\ \&\ \hat{\mathbf{T}} = [\mathbf{T}^-, \mathbf{T}^+] ;\ \hat{\mathbf{I}} = [\mathbf{I}^-, \mathbf{I}^+] ;\ \hat{\mathbf{F}} = [\mathbf{F}^-, \mathbf{F}^+]\right]$.

Here,

- $\mathbf{M_{NVB}}$ is a neutrosophic vague binary set with two universes $\mathbf{U_1}$ and $\mathbf{U_2}$
- $\mathbf{U}^{\mathfrak{G}_{\mathbf{M_{NVB}}}} = (\mathbf{U} = \{\mathbf{U_1} \cup \mathbf{U_2}\},\ *,\ \mathbf{0})$ is a G – algebraic structure with a binary operation $*$ & a constant 0, which satisfies following axioms : $\forall\ \mathbf{x}, \mathbf{y} \in \mathbf{U},\ (\mathbf{i})\ (\mathbf{x} * \mathbf{x}) = \mathbf{0}$; $(\mathbf{ii})\ \mathbf{x} * (\mathbf{x} * \mathbf{y}) = \mathbf{y}$

Remark 3.2

(i) Neutrosophic vague binary G – subalgebra is written in short as NVB G - subalgebra.
(ii) In NVB G – subalgebra universal set $\mathbf{U}$ is taken as "union" of elements of $\mathbf{U_1}$ and $\mathbf{U_2}$.
(iii) Before applying $\mathfrak{G}_{\mathbf{M_{NVB}}}$ condition, neutrosophic vague binary union [Here, (max, min, min)] have to take for common elements of $\mathbf{U_1}$ and $\mathbf{U_2}$. Combined neutrosophic vague binary membership grades will draw and implement combined effect to neutrosophic vague binary values of $\mathbf{U_1}$ and $\mathbf{U_2}$. This will fulfill the binary effect in neutrosophic vague sets in the practical point of view.

Example 3.3

Let $\mathbf{U_1} = \{\mathbf{0}, \mathbf{a}, \mathbf{b}\}$, $\mathbf{U_2} = \{\mathbf{0}, \mathbf{b}, \mathbf{c}\}$ be two universes. Combined universe $\mathbf{U} = \{\mathbf{U_1} \cup \mathbf{U_2}\} = \{\mathbf{0}, \mathbf{a}, \mathbf{b}, \mathbf{c}\}$. Binary operation $*$ is defined as given by the Cayley table given below:

$*$	0	**a**	**b**	**c**
0	0	**c**	**b**	**a**
a	**a**	0	**c**	**b**
b	**b**	**a**	0	**c**
c	**c**	**b**	**a**	0

Clearly, $(\mathbf{U},\ *,\ \mathbf{0})$ is a G–algebra. Consider a NVBS formed based on $\mathbf{U_1}$ & $\mathbf{U_2}$.

$$\mathbf{M_{NVB}} = \left\{ \left\langle \frac{[0.9,0.9][0.2,0.8][0.1,0.1]}{\mathbf{0}}, \frac{[0.8,0.9][0.3,0.7][0.1,0.2]}{a}, \frac{[0.6,0.9][0.4,0.6][0.1,0.4]}{\mathbf{b}} \right\rangle \left\langle \frac{[0.6,0.9][0.3,0.6][0.1,0.4]}{\mathbf{0}}, \frac{[0.8,0.8][0.2,0.7][0.2,0.2]}{\mathbf{b}}, \frac{[0.8,0.9][0.3,0.7][0.1,0.2]}{\mathbf{c}} \right\rangle \right\}$$

Combined neutrosophic vague binary membership grade is given by

$$\mathbf{M_{NVB}}\ (\mathbf{s}) = \begin{cases} [\mathbf{0.9}, \mathbf{0.9}][\mathbf{0.2}, \mathbf{0.6}][\mathbf{0.1}, \mathbf{0.1}] & ; \quad \mathbf{if} \quad \mathbf{s} = \mathbf{0} \\ [\mathbf{0.8}, \mathbf{0.9}][\mathbf{0.3}, \mathbf{0.7}][\mathbf{0.1}, \mathbf{0.2}] & ; \quad \mathbf{if} \quad \mathbf{s} = \boldsymbol{a} \\ [\mathbf{0.8}, \mathbf{0.9}][\mathbf{0.2}, \mathbf{0.6}][\mathbf{0.1}, \mathbf{0.2}] & ; \quad \mathbf{if} \quad \mathbf{s} = \mathbf{b} \\ [\mathbf{0.8}, \mathbf{0.9}][\mathbf{0.3}, \mathbf{0.7}][\mathbf{0.1}, \mathbf{0.2}] & ; \quad \mathbf{if} \quad \mathbf{s} = \mathbf{c} \end{cases}$$

Calculations shows that $\mathbf{M_{NVB}}$ is a NVB G – subalgebra.

Remark 3.4

In a NVB G – algebra, construction of the underlying G – algebraic structure, using a binary operation $*$ deserves prime importance. Instead of $*$ different symbols like $+$, $-$, $\times$, $+_4$ etc can be applied. Binary operation can be formed in different ways. Construction of G – algebra using the following points always defines a G -algebra. In the Binary Operation,

(i) If "first operand = second operand" then the output will be zero.

[Using definition of G – algebra, $(x * x) = 0 \Rightarrow$ principal diagonal elements should occupy with constant 0, in the Cayley table of a G – algebra].

(ii) If "first operand $\neq$ second operand" with "first operand $\neq$ 0 & second operand = 0 ", then output will be first operand

(iii) If "first operand $\neq$ second operand" with "first operand $\neq$ 0 & the second operand $\neq$ 0 ", then output will be second operand

Following Cayley Table will make idea clear. $U = \{0,\ \mathbf{a_1} \neq \mathbf{0},\ \mathbf{a_2} \neq \mathbf{0},\ --,--,\ \mathbf{a_n} \neq \mathbf{0}\}$.
From above, numbers in the square brackets indicates specific points used to frame the output.

$*$	0	$\mathbf{a_1} \neq 0$	$\mathbf{a_2} \neq 0$	---- $\neq 0$	---- $\neq 0$	$\mathbf{a_n} \neq 0$
0	0 [i]	$\mathbf{a_1}$[iii]	$\mathbf{a_2}$ [iii]	----	----	$\mathbf{a_n}$ [iii]
$\mathbf{a_1} \neq 0$	$\mathbf{a_1}$[ii]	0 [i]	$\mathbf{a_2}$ [iii]	----	----	$\mathbf{a_n}$ [iii]
$\mathbf{a_2} \neq 0$	$\mathbf{a_2}$[ii]	$\mathbf{a_1}$ [iii]	0 [i]	----	----	$\mathbf{a_n}$ [iii]
----- $\neq 0$	---[ii]	--- [iii]	--- [iii]	----	----	--- [iii]
----- $\neq 0$	---[ii]	--- [iii]	--- [iii]	----	----	$\mathbf{a_n}$ [iii]
$\mathbf{a_n} \neq 0$	$\mathbf{a_n}$[ii]	$\mathbf{a_1}$ [iii]	$\mathbf{a_2}$ [iii]	----	----	0 [i]

Remark 3.5

A neutrosophic vague G - subalgebra is a structure $\mathfrak{G}_{M_{NV}} = (U^{\mathfrak{G}_{M_{NV}}}, *, 0)$ which satisfies,
$NV_{M_{NV}}(x * y) \succcurlyeq r\min\{NV_{M_{NV}}(x),\ NV_{M_{NV}}(y)$ [known as, $\mathfrak{G}_{M_{NV}}$ condition]. That is,
$\hat{T}_{M_{NV}}(x * y) \geq \min\{\hat{T}_{M_{NV}}(x), \hat{T}_{M_{NV}}(y)\}$; $\hat{I}_{M_{NV}}(x * y) \leq \max\{\hat{I}_{M_{NV}}(x), \hat{I}_{M_{NV}}(y)\}$; $\hat{F}_{M_{NV}}(x * y) \leq \max\{\hat{F}_{M_{NV}}(x), \hat{F}_{M_{NV}}(y)\}$
$\left[* \text{ and } 0 \text{ are as in } U^{\mathfrak{G}_{M_{NV}}} \ \&\ \hat{T} = [T^-, T^+];\ \hat{I} = [I^-, I^+];\ \hat{F} = [F^-, F^+]\right]$.

Here,

- $\mathbf{M_{NV}}$ is a neutrosophic vague set with a single universe $\mathbf{U}$
- $\mathbf{U}^{\mathfrak{G}_{M_{NV}}} = (\mathbf{U}, *, \mathbf{0})$ is a G – algebraic structure with a binary operation $*$ & a constant 0, which satisfies following axioms : $\forall\ x, \mathbf{y} \in \mathbf{U}$, $(\mathbf{i})\ (\mathbf{x} * \mathbf{x}) = \mathbf{0}$; $(\mathbf{ii})\ \mathbf{x} * (\mathbf{x} * \mathbf{y}) = \mathbf{y}$

Remark 3.6

It is straight forward to check that, intersection of neutrosophic vague binary G – subalgebras produce a neutrosophic vague binary G - subalgebra itself. But union may not be!

4. Different notions of Neutrosophic Vague Binary G - subalgebra

In this section following notions to a NVB – G subalgebra are discussed.

- ✓ *G – part of a Neutrosophic Vague Binary G – subalgebra*
- ✓ *G – p radical of a Neutrosophic Vague Binary G – subalgebra*
- ✓ *G - p semi simple of a Neutrosophic Vague Binary G – subalgebra*
- ✓ *G – minimal element of a Neutrosophic Vague Binary G – subalgebra*

Definition 4.1

Let $\mathbf{M_{NVB}}$ be a NVB G -subalgebra with structure $\mathfrak{G}_{\mathbf{M_{NVB}}} = (\mathbf{U}^{\mathfrak{G}_{M_{NVB}}}, *, \mathbf{0})$

i. G – part of a Neutrosophic Vague Binary G – subalgebra

Let $\mathbf{S_{NVB}}$ be any NVBSS of $\mathbf{M_{NVB}}$. Define, $\mathbf{G}(\mathbf{S_{NVB}}) = \{\mathbf{x} \in \mathbf{S_{NVB}} /\ \mathbf{NVB}_{\mathbf{S_{NVB}}}(0 * \mathbf{x}) = \mathbf{NVB}_{\mathbf{S_{NVB}}}(\mathbf{x})\}$.
In particular, if $\mathbf{S_{NVB}} = \mathbf{M_{NVB}}$ then $\mathbf{G}(\mathbf{M_{NVB}})$ is called the *neutrosophic vague binary G – G part* (in short, *NVB G – G part*) of the NVB G – subalgebra.

ii. p – radical of a Neutrosophic Vague Binary G – subalgebra

B($\mathbf{M_{NVB}}$) = {$\mathbf{x} \in \mathbf{U}$/ $\mathbf{NVB_{M_{NVB}}}(0 * \mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(0)$} is called, a *neutrosophic vague binary G – p radical* (in short, *NVB G – p radical*) of the NVB G -subalgebra $\mathbf{M_{NVB}}$

iii. p- semi simple of a Neutrosophic Vague Binary G – subalgebra

$\mathbf{M_{NVB}}$ is called *neutrosophic vague binary G - p semi simple* (in short, *NVB G – p semi simple*), if B($\mathbf{M_{NVB}}$) = {$\mathbf{x} \in \mathbf{U}$/ $\mathbf{NVB_{M_{NVB}}}(0 * \mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(0)$} = {0}

iv. Minimal Element for Neutrosophic Vague Binary G – subalgebra

Any element $\mathbf{x} \in \mathbf{U}$ is *neutrosophic vague binary G – minimal element* (in short, *NVB G – minimal element*), if $\mathbf{NVB_{M_{NVB}}}(\mathbf{x} * \mathbf{y}) = \mathbf{NVB_{M_{NVB}}}(0) \Rightarrow \mathbf{NVB_{M_{NVB}}}(\mathbf{y}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x})$

Remark 4.2
It is clear that, $\mathbf{G}(\mathbf{M_{NVB}}) \cap$ B($\mathbf{M_{NVB}}$) = {0}

Theorem 4.3
Let $\mathbf{M_{NVB}}$ be a NVB G - subalgebra with structure $\mathfrak{G}_{\mathbf{M_{NVB}}} = (\mathbf{U}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}, *, \mathbf{0})$. Then, $\mathbf{x} \in \mathbf{G}(\mathbf{M_{NVB}})$ if and only if $\mathbf{NVB_{M_{NVB}}}(\mathbf{0} * \mathbf{x}) \in \mathbf{G}(\mathbf{M_{NVB}})$

Proof
$\mathbf{x} \in \mathbf{G}(\mathbf{M_{NVB}}) \Rightarrow \mathbf{NVB_{M_{NVB}}}(\mathbf{0} * \mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x})$
$\Rightarrow \mathbf{NVB_{M_{NVB}}}(\mathbf{0} * \mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{0} * (\mathbf{0} * \mathbf{x}))$, [from proposition 3.8]
$\Rightarrow (\mathbf{0} * \mathbf{x}) \in$ G ($\mathbf{M_{NVB}}$), [**By definition 6.1**].
Conversely, if $(\mathbf{0} * \mathbf{x}) \in \mathbf{G}(\mathbf{M_{NVB}})$, then $\mathbf{NVB_{M_{NVB}}}\big(\mathbf{0} * (\mathbf{0} * \mathbf{x})\big) = \mathbf{NVB_{M_{NVB}}}(\mathbf{0} * \mathbf{x})$
$\Rightarrow \mathbf{NVB_{M_{NVB}}}(\mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{0} * \mathbf{x}) \Rightarrow \mathbf{x} \in \mathbf{G}(\mathbf{M_{NVB}})$

Theorem 4.4
Let $\mathbf{M_{NVB}}$ be a NVB G - subalgebra with structure $\mathfrak{G}_{\mathbf{M_{NVB}}} = (\mathbf{U}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}, *, \mathbf{0})$.
(i) $\mathbf{M_{NVB}}$ is NVB G - p semi simple. (ii) Every element in $\mathbf{U}$ is a NVB G - minimal element.

Proof
(i) From definition 6.1(iii), B($\mathbf{M_{NVB}}$) = {$\mathbf{x} \in \mathbf{U}$ / $\mathbf{NVB_{M_{NVB}}}(0 * \mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(0)$}
$\Rightarrow$ B($\mathbf{M_{NVB}}$) = {$\mathbf{x} \in \mathbf{U}$/ $\mathbf{NVB_{M_{NVB}}}(0) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x})$} = {0}
(ii) Assume (ii). Let $\mathbf{x}$ be an arbitrary element in $\mathbf{U}$ such that $\mathbf{y} \leq \mathbf{x}$ for some $\mathbf{y} \in \mathbf{U}$
$\Rightarrow (\mathbf{x} * \mathbf{y}) = 0 \Rightarrow \mathbf{NVB_{M_{NVB}}}(\mathbf{x} * \mathbf{y}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{0}) \Rightarrow \mathbf{NVB_{M_{NVB}}}(\mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{y})$

Theorem 4.5
If G ($\mathbf{M_{NVB}}$) = $\mathfrak{G}_{\mathbf{M_{NVB}}}$, then $\mathbf{M_{NVB}}$ is NVB G - p semi simple.
That is, if a NVB G – subalgebra coincides with its G – part then it is NVB G – p semi simple

Proof
Let $\mathbf{M_{NVB}}$ be a NVB G - subalgebra with structure $\mathfrak{G}_{\mathbf{M_{NVB}}} = (\mathbf{U}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}, *, \mathbf{0})$.
From definition 6.1(ii), $\mathbf{G}(\mathbf{M_{NVB}})$ = {$\mathbf{x} \in \mathbf{M_{NVB}}$/ $\mathbf{NVB_{M_{NVB}}}(0 * \mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x})$}
If $\mathbf{G}(\mathbf{M_{NVB}}) = \mathfrak{G}_{\mathbf{M_{NVB}}}$ then $\mathbf{B}(\mathbf{M_{NVB}})$ = {0} $\Rightarrow \mathbf{M_{NVB}}$ is NVB G - p semi simple

Remark 4.6
(1) In any NVB G – subalgebra: $\mathbf{NVB_{M_{NVB}}}(\mathbf{x}) \preccurlyeq \mathbf{NVB_{M_{NVB}}}(\mathbf{y}) \Leftrightarrow \mathbf{NVB_{M_{NVB}}}(\mathbf{y} * \mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{0})$
(2) Denote $\mathbf{NVB_{M_{NVB}}}[\mathbf{y} * (\mathbf{y} * \mathbf{x})]$ by $\mathbf{NVB_{M_{NVB}}}(\mathbf{x} \wedge \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbf{U}$. From definition 3.1 (i) (2), $\mathbf{NVB_{M_{NVB}}}(\mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x} \wedge \mathbf{y})$

Theorem 4.7

Let $\mathbf{M_{NVB}}$ be a NVB G - subalgebra with structure $\mathfrak{G}_{\mathbf{M_{NVB}}}$ = $(\mathbf{U}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}, *, \mathbf{0})$.
Then for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{U}$,
(1) For $\mathbf{x} \neq \mathbf{y}$, $\mathbf{NVB_{M_{NVB}}}(\mathbf{x} \wedge \mathbf{y}) \neq \mathbf{NVB_{M_{NVB}}}(\mathbf{y} \wedge \mathbf{x})$
(2) $\mathbf{NVB_{M_{NVB}}}[\mathbf{x} \wedge (\mathbf{y} \wedge \mathbf{z})] = \mathbf{NVB_{M_{NVB}}}[(\mathbf{x} \wedge \mathbf{y}) \wedge \mathbf{z}]$
(3) $\mathbf{NVB_{M_{NVB}}}(\mathbf{x} \wedge \mathbf{0}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x})$ and $\mathbf{NVB_{M_{NVB}}}(\mathbf{0} \wedge \mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{0})$
(4) For $\mathbf{x} \neq 0$, $\mathbf{NVB_{M_{NVB}}}[\mathbf{x} \wedge (\mathbf{y} \wedge \mathbf{z})] \neq \mathbf{NVB_{M_{NVB}}}[(\mathbf{x} \wedge \mathbf{y}) * (\mathbf{x} \wedge \mathbf{z})]$

Proof
(1) For a NVB G – subalgebra, $\mathbf{NVB_{M_{NVB}}}(\mathbf{x} \wedge \mathbf{y}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x})$ & $\mathbf{NVB_{M_{NVB}}}(\mathbf{y} \wedge \mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{y})$
$\therefore$ If $\mathbf{x} \neq \mathbf{y}$, then $\mathbf{NVB_{M_{NVB}}}(\mathbf{x} \wedge \mathbf{y}) \neq \mathbf{NVB_{M_{NVB}}}(\mathbf{y} \wedge \mathbf{x})$
(2) $\mathbf{NVB_{M_{NVB}}}[\mathbf{x} \wedge (\mathbf{y} \wedge \mathbf{z})] = \mathbf{NVB_{M_{NVB}}}(\mathbf{x} \wedge \mathbf{y}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x})$ &
$\mathbf{NVB_{M_{NVB}}}[(\mathbf{x} \wedge \mathbf{y}) \wedge \mathbf{z}] = \mathbf{NVB_{M_{NVB}}}(\mathbf{x} \wedge \mathbf{z}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x})$. $\therefore$ $\mathbf{NVB_{M_{NVB}}}[\mathbf{x} \wedge (\mathbf{y} \wedge \mathbf{z})] =$
$\mathbf{NVB_{M_{NVB}}}[(\mathbf{x} \wedge \mathbf{y}) \wedge \mathbf{z}]$
(3) For a NVB G – subalgebra, $\mathbf{NVB_{M_{NVB}}}(\mathbf{x} \wedge \mathbf{0}) = \mathbf{NVB_{M_{NVB}}}[\mathbf{0} * (\mathbf{0} * \mathbf{x})] = \mathbf{NVB_{M_{NVB}}}(\mathbf{x})$ &
$\mathbf{NVB_{M_{NVB}}}(\mathbf{0} \wedge \mathbf{x}) = \mathbf{NVB_{M_{NVB}}}[\mathbf{x} * (\mathbf{x} * \mathbf{0})] = \mathbf{NVB_{M_{NVB}}}(\mathbf{0})$
(4) $\mathbf{NVB_{M_{NVB}}}[\mathbf{x} \wedge (\mathbf{y} \wedge \mathbf{z})] = \mathbf{NVB_{M_{NVB}}}(\mathbf{x} \wedge \mathbf{y}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x})$, for a NVB G - subalgebra
$\mathbf{NVB_{M_{NVB}}}[(\mathbf{x} \wedge \mathbf{y}) * (\mathbf{x} \wedge \mathbf{z})] = \mathbf{NVB_{M_{NVB}}}(\mathbf{x} * \mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{0})$, for a NVB G – subalgebra
$\therefore$ it is clear that, for a NVB G – subalgebra,
$\mathbf{NVB_{M_{NVB}}}(\mathbf{x}) \neq \mathbf{NVB_{M_{NVB}}}(\mathbf{0}) \Rightarrow \mathbf{NVB_{M_{NVB}}}[\mathbf{x} \wedge (\mathbf{y} \wedge \mathbf{z})] \neq \mathbf{NVB_{M_{NVB}}}[(\mathbf{x} \wedge \mathbf{y}) * (\mathbf{x} \wedge \mathbf{z})]$

Theorem 4.8
Every **NVB G** – subalgebra satisfies the inequality, $\mathbf{NVB_{M_{NVB}}}(\mathbf{0}) \succcurlyeq \mathbf{NVB_{M_{NVB}}}(\mathbf{x})$; $\forall\ \mathbf{x} \in \mathbf{U}$

Proof
$\mathbf{NVB_{M_{NVB}}}(\mathbf{0}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x} * \mathbf{x}) \succcurlyeq \mathrm{r\,min}\,\{\mathbf{NVB_{M_{NVB}}}(\mathbf{x}), \mathbf{NVB_{M_{NVB}}}(\mathbf{x})\} = \mathbf{NVB_{M_{NVB}}}(\mathbf{x})$
$\therefore$ $\mathbf{NVB_{M_{NVB}}}(\mathbf{0}) \succcurlyeq \mathbf{NVB_{M_{NVB}}}(\mathbf{x})$; $\forall\ \mathbf{x} \in \mathbf{U}$

Theorem 4.9

Let $\mathfrak{G}_{\mathbf{M_{NVB}}}$ = $(\mathbf{U}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}, *, \mathbf{0})$ be a NVB G - subalgebra. Then the following conditions hold :
(i) $\mathbf{NVB_{M_{NVB}}}(\mathbf{x} * \mathbf{0}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x})$, $\forall\ \mathbf{x} \in \mathbf{U}$ (ii) $\mathbf{NVB_{M_{NVB}}}\big(\mathbf{0} * (\mathbf{0} * \mathbf{x})\big) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x})$, $\forall\ \mathbf{x} \in \mathbf{U}$

Proof
Let $\mathfrak{G}_{\mathbf{M_{NVB}}}$ = $(\mathbf{U}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}, *, \mathbf{0})$ be a NVB G – subalgebra and $\mathbf{x}, \mathbf{y} \in \mathbf{U}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$. Then,

(i) $\mathbf{NVB_{M_{NVB}}}(\mathbf{x} * \mathbf{0}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x} * (\mathbf{x} * \mathbf{x}))$
[**Using first condition in the G − algebraic structure of NVB G − subalgebra**]
= $\mathbf{NVB_{M_{NVB}}}(\mathbf{x})$ [**Using second condition in the G − algebraic structure of NVB G − subalgebra**]

(ii) (ii) Since $\mathfrak{G}_{\mathbf{M_{NVB}}}$ is a NVB G – subalgebra, $\mathbf{NVB_{M_{NVB}}}(\mathbf{x} * (\mathbf{x} * \mathbf{y})) = \mathbf{NVB_{M_{NVB}}}(\mathbf{y})$
[**Using second condition in the G − algebraic structure of NVB G − subalgebra**] Put $\mathbf{x}$ = 0 and $\mathbf{y} = \mathbf{x}$ in the above then (ii) follows

Theorem 4.10
Let $\mathfrak{G}_{\mathbf{M_{NVB}}}$= $(\mathbf{U}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}, *, \mathbf{0})$ be a NVB G - subalgebra. Then following conditions hold: $\forall\ \mathbf{x}, \mathbf{y} \in \mathbf{U}$,
(i) $\mathbf{NVB_{M_{NVB}}}((\mathbf{x} * (\mathbf{x} * \mathbf{y})) * \mathbf{y}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{0})$
(ii) $\mathbf{NVB_{M_{NVB}}}(\mathbf{x} * \mathbf{y}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{0}) \Rightarrow \mathbf{NVB_{M_{NVB}}}(\mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{y})$
(iii) $\mathbf{NVB_{M_{NVB}}}(\mathbf{0} * \mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{0} * \mathbf{y}) \Rightarrow \mathbf{NVB_{M_{NVB}}}(\mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{y})$

Proof

(i) $\mathbf{NVB_{M_{NVB}}}((\mathbf{x} * (\mathbf{x} * \mathbf{y})) * \boldsymbol{y}) = \mathbf{NVB_{M_{NVB}}}((\mathbf{y} * (\mathbf{y} * \mathbf{y})) * \mathbf{y})$ by putting $\mathbf{x} = \mathbf{y}$

$= \mathbf{NVB_{M_{NVB}}}\big((\mathbf{y} * \mathbf{0}) * \mathbf{y}\big) = \mathbf{NVB_{M_{NVB}}}(\mathbf{y} * \mathbf{y}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{0})$

(ii) Assume $\mathbf{NVB_{M_{NVB}}}(\mathbf{x} * \mathbf{y}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{0})$

$\therefore\ \mathbf{NVB_{M_{NVB}}}(\mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x} * \mathbf{0}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x} * (\mathbf{x} * \mathbf{y}))$, [**by assumption**]
$= \mathbf{NVB_{M_{NVB}}}(\mathbf{y})$

(iii) Assume $\mathbf{NVB_{M_{NVB}}}(\mathbf{0} * \mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{0} * \mathbf{y})$

$\therefore\ \mathbf{NVB_{M_{NVB}}}(\mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{0} * (\mathbf{0} * \mathbf{x})) = \mathbf{NVB_{M_{NVB}}}(\mathbf{0} * (\mathbf{0} * \mathbf{y}))$, [**by assumption**]
$= \mathbf{NVB_{M_{NVB}}}(\mathbf{y})$

Theorem 4.11
Let $\mathfrak{G}_{\mathbf{M_{NVB}}} = (\mathbf{U}^{\mathfrak{G}_{\mathrm{M_{NVB}}}}, *, \mathbf{0})$ be a NVB G – subalgebra. Then,
$\mathbf{NVB_{M_{NVB}}}(\mathbf{a} * \mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{a} * \mathbf{y}) \Rightarrow \mathbf{NVB_{M_{NVB}}}(\mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{y})$, for any $\mathbf{a}, \mathbf{x}, \mathbf{y} \in \mathbf{U}$

Proof
If $\mathfrak{G}_{\mathbf{M_{NVB}}} = (\mathbf{U}^{\mathfrak{G}_{\mathrm{M_{NVB}}}}, *, \mathbf{0})$ be a NVB G – subalgebra satisfying,
$\mathbf{NVB_{M_{NVB}}}(\mathbf{a} * \mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{a} * \mathbf{y})$, for any $\mathbf{a}, \mathbf{x}, \mathbf{y} \in \mathbf{U}$. Then,
$\mathbf{NVB_{M_{NVB}}}(\mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{a} * (\mathbf{a} * \mathbf{x})) = \mathbf{NVB_{M_{NVB}}}(\mathbf{a} * (\mathbf{a} * \mathbf{y})) = \mathbf{NVB_{M_{NVB}}}(\mathbf{y})$

Theorem 4.12
Let $\mathfrak{G}_{\mathbf{M_{NVB}}} = (\mathbf{U}^{\mathfrak{G}_{\mathrm{M_{NVB}}}}, *, \mathbf{0})$ be a NVB G – subalgebra. Then the following are equivalent:

(1) $\mathbf{NVB_{M_{NVB}}}((\mathbf{x} * \mathbf{y}) * (\mathbf{x} * \mathbf{z})) = \mathbf{NVB_{M_{NVB}}}(\mathrm{z} * \mathbf{y})$; $\forall\ \mathbf{x}, \mathbf{y}, \mathrm{z} \in \mathbf{U}$

(2) $\mathbf{NVB_{M_{NVB}}}((\mathbf{x} * \mathbf{z}) * (\mathbf{y} * \mathbf{z})) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x} * \mathbf{y})$; $\forall\ \mathbf{x}, \mathbf{y}, \mathrm{z} \in \mathbf{U}$

Proof

(i) ⇒ (ii)
Assume (i). i.e., $\mathbf{NVB_{M_{NVB}}}((\mathbf{x} * \mathbf{y}) * (\mathbf{x} * \mathbf{z})) = \mathbf{NVB_{M_{NVB}}}(\mathrm{z} * \mathbf{y})$; $\forall\ \mathbf{x}, \mathbf{y}, \mathrm{z} \in \mathbf{U}$

$\therefore\ \mathbf{NVB_{M_{NVB}}}((\mathbf{x} * \mathbf{z}) * (\mathbf{x} * \mathbf{y})) = \mathbf{NVB_{M_{NVB}}}(\mathbf{y} * \mathrm{z})$

Consider, $\mathrm{NVB_{M_{NVB}}}((\mathrm{x} * \mathrm{z}) * (\mathrm{y} * \mathrm{z}))$
$= \mathrm{NVB_{M_{NVB}}}\big((\mathrm{x} * \mathrm{z}) * ((\mathrm{x} * \mathrm{z}) * (\mathrm{x} * \mathrm{y}))\big)$
$= \mathrm{NVB_{M_{NVB}}}(\mathrm{x} * \mathrm{y})$, since $\mathrm{NVB_{M_{NVB}}}(\mathrm{x} * (\mathrm{x} * \mathrm{y})) = \mathrm{NVB_{M_{NVB}}}(\mathrm{y})$
(ii) ⇒ (i)
Assume (ii). i.e., $\mathrm{NVB_{M_{NVB}}}((\mathrm{x} * \mathrm{z}) * (\mathrm{y} * \mathrm{z})) = \mathrm{NVB_{M_{NVB}}}(\mathrm{x} * \mathrm{y})$
$\therefore\ \mathrm{NVB_{M_{NVB}}}((\mathrm{x} * \mathrm{y}) * (\mathrm{z} * \mathrm{y})) = \mathrm{NVB_{M_{NVB}}}(\mathrm{x} * \mathrm{z})$
Consider, $\mathrm{NVB_{M_{NVB}}}((\mathrm{x} * \mathrm{y}) * (\mathrm{x} * \mathrm{z}))$
$= \mathrm{NVB_{M_{NVB}}}\big((\mathrm{x} * \mathrm{y}) * ((\mathrm{x} * \mathrm{y}) * (\mathrm{z} * \mathrm{y}))\big)$
$= \mathrm{NVB_{M_{NVB}}}(\mathrm{z} * \mathrm{y})$, since $\mathrm{NVB_{M_{NVB}}}(\mathrm{x} * (\mathrm{x} * \mathrm{y})) = \mathrm{NVB_{M_{NVB}}}(\mathrm{y})$

5. Neutrosophic Vague Binary G – normal subalgebra

In this section neutrosophic vague binary G – normal subalgebra is introduced

Definition 5.1 (Neutrosophic vague binary G – normal subalgebra)
Let $\mathbf{M_{NVB}}$ be a neutrosophic vague binary set (in short, NVBS) with two universes $\mathbf{U_1}$ and $\mathbf{U_2}$.
Neutrosophic vague binary G – normal subalgebra is a structure $\mathfrak{G}^{\mathcal{N}}_{\mathbf{M_{NVB}}} = (\mathbf{U}^{\mathfrak{G}^{\mathcal{N}}_{\mathbf{M_{NVB}}}}, *, 0)$
which satisfies, the following 2 conditions known as $\mathfrak{G}^{\mathcal{N}}_{\mathbf{M_{NVB}}}$ inequalities:

$\mathfrak{G}^{\mathcal{N}}_{M_{NVB}}$ *inequality (1)* :
$\mathbf{NVB_{M_{NVB}}(x * y)} \geqslant \mathrm{r\,min}\ \{\mathbf{NVB_{M_{NVB}}(x), NVB_{M_{NVB}}(y)}\}$; $\forall$ $\mathbf{x}, \mathbf{y} \in \mathbf{U}$
That is, $\forall$ $\mathbf{x}, \mathbf{y} \in \mathbf{U}$
$\hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{x * y}) \geq \mathbf{min}\{\hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{x}), \hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{y})\}$; $\hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{x * y}) \leq \mathbf{max}\{\hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{x}), \hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{y})\}$; $\hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{x * y}) \leq \mathbf{max}\{\hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{x}), \hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{y})\}$

$\mathfrak{G}^{\mathcal{N}}_{M_{NVB}}$ *inequality (2)* :
$\mathbf{NVB_{M_{NVB}}}\big((\mathbf{x * a}) * (\mathbf{y * b})\big) \geqslant \mathrm{r\,min}\ \{\mathbf{NVB_{M_{NVB}}(x * y),\ NVB_{M_{NVB}}(a * b)}\}$; $\forall$ a, $\mathbf{b}, \mathbf{x}, \mathbf{y} \in \mathbf{U}$

That is, $\forall$ $\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y} \in \mathbf{U}$. $\hat{\mathbf{T}}_{\mathbf{M_{NVB}}}\big((\mathbf{x * a}) * (\mathbf{y * b})\big) \geq \mathbf{min}\{\hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{x * y}), \hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{a * b})\}$; $\hat{\mathbf{I}}_{\mathbf{M_{NVB}}}\big((\mathbf{x * a}) * (\mathbf{y * b})\big) \leq$ $\mathbf{max}\{\hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{x * y}), \hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{a * b})\}$. $\hat{\mathbf{F}}_{\mathbf{M_{NVB}}}\big((\mathbf{x * a}) * (\mathbf{y * b})\big) \leq$ $\mathbf{max}\{\hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{x * y}), \hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{a * b})\}$

$$\left[*\ \textbf{and 0 are as in}\ \mathbf{U}^{\mathfrak{G}^{\mathcal{N}}_{\mathbf{M_{NVB}}}} \quad \& \quad \hat{\mathbf{T}} = [\mathbf{T}^-,\ \mathbf{T}^+]\,;\ \hat{\mathbf{I}} = [\mathbf{I}^-,\ \mathbf{I}^+]\,;\ \hat{\mathbf{F}} = [\mathbf{F}^-,\ \mathbf{F}^+] \right]$$

Here,

- $\mathbf{M_{NVB}}$ is a neutrosophic vague binary set with two universes $\mathbf{U_1}$ and $\mathbf{U_2}$
- $\mathbf{U}^{\mathfrak{G}^{\mathcal{N}}_{\mathbf{M_{NVB}}}} = (\mathbf{U} = \{\mathbf{U_1} \cup \mathbf{U_2}\},\ *,\ \mathbf{0})$ is a G - algebraic structure with a binary operation $*$ & a constant 0, which satisfies following axioms : $\forall$ $\mathbf{x}, \mathbf{y} \in \mathbf{U}$, (i) $(\mathbf{x * x}) = \mathbf{0}$; (ii) $\mathbf{x} * (\mathbf{x * y}) = \mathbf{y}$

Definition 5.2 (Neutrosophic vague binary G – normal set)

Let $\mathbf{M_{NVB}}$ be a NVBS with two universes $\mathbf{U_1}, \mathbf{U_2}$. Take $\mathbf{U} = \{\mathbf{U_1} \cup \mathbf{U_2}\}$.
A NVBS $\mathbf{M_{NVB}}$ in $\mathbf{U}$ is said to be NVB G – normal set if it satisfies the inequality

$$\left\{\mathbf{NVB_{M_{NVB}}}\big((\mathbf{x * a}) * (\mathbf{y * b})\big) \geqslant \mathbf{r\,min}\,\{\mathbf{NVB_{M_{NVB}}}(\boldsymbol{x} * \mathbf{y}),\ \mathbf{NVB_{M_{NVB}}(a * b)}\}\ \middle|\ \forall\, \mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b} \in \mathbf{U}\right\}$$

That is,

$$\left\{\left.\begin{matrix} \hat{\mathbf{T}}_{\mathbf{M_{NVB}}}\big((\boldsymbol{x} * \mathbf{a}) * (\mathbf{y * b})\big) \geqslant \mathbf{min}\{\hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\boldsymbol{x} * \mathbf{y}),\ \hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{a * b})\} \\ \hat{\mathbf{I}}_{\mathbf{M_{NVB}}}\big((\boldsymbol{x} * \mathbf{a}) * (\mathbf{y * b})\big) \geqslant \mathbf{max}\{\hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{x * y}),\ \hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{a * b})\} \\ \hat{\mathbf{F}}_{\mathbf{M_{NVB}}}\big((\mathbf{x * a}) * (\mathbf{y * b})\big) \geqslant \mathbf{max}\,\{\hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{x * y}),\ \hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{a * b})\} \end{matrix}\right|\ \forall\, \mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b} \in \mathbf{U}\right\}$$

Remark 5.3
(i) Neutrosophic vague binary $\mathbf{G}$ - normal subalgebra is written in short as $\mathbf{NVB\ G}$ – normal subalgebra. It is denoted by $\mathfrak{G}^{\mathcal{N}}_{\mathbf{M_{NVB}}}$.
(ii) In other words, a $\mathbf{NVBS}$ $\mathbf{M_{NVB}}$ in a $\mathbf{G}$ – algebra $\mathbf{U}$ is called a $\mathbf{NVB\ G}$ – normal subalgebra if it is a $\mathbf{NVB\ G}$ - subalgebra which is $\mathbf{NVB\ G}$ - normal set.

Theorem 5.4
Every $\mathbf{NVB\ G}$ - normal set $\mathbf{M_{NVB}}$ in $\mathbf{U}$ is a $\mathbf{NVB\ G}$ – subalgebra of $\mathbf{U}$.
That is, every $\mathbf{NVB\ G}$ – normal set $\mathbf{M_{NVB}}$ is a $\mathfrak{G}_{\mathbf{M_{NVB}}}$.

Proof
Let $\mathbf{M_{NVB}}$ be a $\mathbf{NVB\ G}$– normal set in $\mathbf{U} \Rightarrow \mathbf{NVB_{M_{NVB}}}\big((\mathbf{x * a}) * (\mathbf{y * b})\big) \geqslant \mathrm{r\,min}\{\mathbf{NVB_{M_{NVB}}(x * y), NVB_{M_{NVB}}(a * b)}\}$
Consider, $\mathbf{NVB_{M_{NVB}}(x * y)} = \mathbf{NVB_{M_{NVB}}}\big((\mathbf{x * y}) * (\mathbf{0 * 0})\big) \geqslant \mathrm{r\,min}\ \{\mathbf{NVB_{M_{NVB}}(x * 0), NVB_{M_{NVB}}(y * 0)}\}$

= r min $\{\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x}), \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{y})\} \Rightarrow \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x} * \mathbf{y}) \succcurlyeq$ r min $\{\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x}), \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{y})\}$; $\forall$ $\mathbf{x}, \mathbf{y} \in \mathbf{U} \Rightarrow$ $\mathbf{M}_{\mathbf{NVB}}$ is a NVB G – subalgebra

Remark 5.5

Converse of theorem 5.4 is not true.

That is, a NVB G – subalgebra $\mathbf{M}_{\mathbf{NVB}}$ in $\mathbf{U}$ is not a **NVB G** – normal set, generally.

Proof

Consider example 4.3, in which $\mathbf{M}_{\mathbf{NVB}}$ is a **NVB G** – subalgebra. In this example, $\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}((\mathbf{a} * \mathbf{a}) * (\mathbf{b} * \mathbf{a})) \not\succcurlyeq$ r min $\{\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{a} * \mathbf{b}), \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{a} * \mathbf{a})\} \Rightarrow$ $\mathbf{M}_{\mathbf{NVB}}$ is not a **NVB G** – normal set.

Theorem 5.6

If a neutrosophic vague binary set $\mathbf{M}_{\mathbf{NVB}}$ in $\mathbf{U}$ is a **NVB G** – normal subalgebra, then $\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x} * \mathbf{y}) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{y} * \mathbf{x})$; $\forall$ $\mathbf{x}, \mathbf{y} \in \mathbf{U}$

Proof

Let $\mathbf{x}, \mathbf{y} \in \mathbf{U}$. Then, $\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x} * \mathbf{y}) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}((\mathbf{x} * \mathbf{y}) * \mathbf{0})$ [**From theorem 4.8**] $= \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}((\mathbf{x} * \mathbf{y}) * (\mathbf{x} * \mathbf{x})) \succcurlyeq$ r min $\{\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x} * \mathbf{x}), \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{y} * \mathbf{x})\}$ = r min $\{\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{0}), \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{y} * \mathbf{x})\}$ = $\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{y} * \mathbf{x})$ [**From theorem 4.7**]. $\therefore$ $\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x} * \mathbf{y}) \succcurlyeq \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{y} * \mathbf{x})$. Similarly, $\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{y} * \mathbf{x}) \succcurlyeq \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x} * \mathbf{y}) \Rightarrow$ $\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x} * \mathbf{y}) = \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{y} * \mathbf{x})$

6. 0 – commutative neutrosophic vague binary G – subalgebra

In this section 0 – commutative neutrosophic vague binary G – subalgebra with its properties are introduced

Definition 6.1 (0 - commutative of a NVB G - subalgebra)

Let $\mathbf{M}_{\mathbf{NVB}}$ be a neutrosophic vague binary set (in short, **NVBS**) with two universes $\mathbf{U_1}$ and $\mathbf{U_2}$.

0 - commutative neutrosophic vague binary G - subalgebra is a structure $\mathfrak{G}^{\mathbf{0}}_{\mathbf{M}_{\mathbf{NVB}}} = (\mathbf{U}^{\mathfrak{G}^{\mathbf{0}}_{\mathbf{M}_{\mathbf{NVB}}}}, *, 0)$ which satisfies the following $\mathfrak{G}^{\mathbf{0}}_{\mathbf{M}_{\mathbf{NVB}}}$ inequality:

$\mathfrak{G}^{\mathbf{0}}_{M_{NVB}}$ ***inequality*** :

$\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x} * \mathbf{y}) \succcurlyeq$ r min $\{\mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x}), \mathbf{NVB}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{y})\}$; $\forall$ $\mathbf{x}, \mathbf{y} \in \mathbf{U}$

i.e., $\hat{\mathbf{T}}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x} * \mathbf{y}) \geq \mathbf{min}\{\hat{\mathbf{T}}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x}), \hat{\mathbf{T}}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{y})\}$; $\hat{\mathbf{I}}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x} * \mathbf{y}) \leq \mathbf{max}\{\hat{\mathbf{I}}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x}), \hat{\mathbf{I}}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{y})\}$; $\hat{\mathbf{F}}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x} * \mathbf{y}) \leq \mathbf{max}\{\hat{\mathbf{F}}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{x}), \hat{\mathbf{F}}_{\mathbf{M}_{\mathbf{NVB}}}(\mathbf{y})\}$

$$\left[* \textbf{ and } \mathbf{0} \textbf{ are as in } \mathbf{U}^{\mathfrak{G}^{\mathbf{0}}_{\mathbf{M}_{\mathbf{NVB}}}} \quad \& \quad \hat{\mathbf{T}} = [\mathbf{T}^-, \mathbf{T}^+] ; \ \hat{\mathbf{I}} = [\mathbf{I}^-, \mathbf{I}^+] ; \ \hat{\mathbf{F}} = [\mathbf{F}^-, \mathbf{F}^+] \right]$$

Here,

- $\mathbf{M}_{\mathbf{NVB}}$ is a neutrosophic vague binary set with two universes $\mathbf{U_1}$ and $\mathbf{U_2}$
- $\mathbf{U}^{\mathfrak{G}^{\mathbf{0}}_{\mathbf{M}_{\mathbf{NVB}}}} = (\mathbf{U} = \{\mathbf{U_1} \cup \mathbf{U_2}\}, *, \mathbf{0})$ is a *0 – commutative G - algebraic structure* with a binary operation $*$ & a constant 0, which satisfies the following axioms :

  $\forall$ $\mathbf{x}, \mathbf{y} \in \mathbf{U}$, (**i**) $(\mathbf{x} * \mathbf{x}) = \mathbf{0}$;(**ii**) $\mathbf{x} * (\mathbf{x} * \mathbf{y}) = \mathbf{y}$; (iii) $\mathbf{x} * (\mathbf{0} * \mathbf{y}) = \mathbf{y} * (\mathbf{0} * \mathbf{x})$

0 - commutative neutrosophic vague binary G - subalgebra is denoted by $\mathfrak{G}^{\mathbf{0}}_{\mathbf{M}_{\mathbf{NVB}}}$

Example 6.2

Consider example 4.3. with a different binary operation defined as given in following Cayley table:

$*$	0	**a**	**b**
0	0	**b**	**a**
a	**a**	0	**b**
b	**b**	**a**	0

$\mathbf{U}^{\mathfrak{G}^{\mathbf{0}}_{\mathbf{M_{NVB}}}}$ will become 0 - commutative $\mathfrak{G}^{\mathbf{0}}_{\mathbf{M_{NVB}}}$ only if $\mathfrak{G}^{\mathbf{0}}_{\mathbf{M_{NVB}}}$ inequality got satisfied. Verifications showed that $\mathbf{M_{NVB}}$ given in example 3.3 is not only a $\mathfrak{G}_{\mathbf{M_{NVB}}}$but it is clearly a $\mathfrak{G}^{\mathbf{0}}_{\mathbf{M_{NVB}}}$ too!

Theorem 6.3

Let $\mathfrak{G}_{\mathbf{M_{NVB}}}$ = $(\mathbf{U}^{\mathfrak{G}_{\mathbf{M_{NVB}}}},\ *,\ \mathbf{0})$ be a $\mathfrak{G}^{\mathbf{0}}_{\mathbf{M_{NVB}}}$.

Then, $\mathbf{NVB}_{\mathbf{M_{NVB}}}\big((\mathbf{0} * \mathbf{x}) * (\mathbf{0} * \mathbf{y})\big) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{y} * \mathbf{x})$ for any $\mathbf{x},\ \mathbf{y} \in \mathbf{U}$

Proof

Let $\mathbf{x},\ \mathbf{y} \in \mathbf{U}$ where $\mathbf{U} \in \mathbf{U}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$. Then, $\big((\mathbf{0} * \mathbf{x}) * (\mathbf{0} * \mathbf{y})\big) = \Big(\boldsymbol{y} * \big(\mathbf{0} * (\mathbf{0} * \mathbf{x})\big)\Big) = (\mathbf{y} * \mathbf{x})$

$\Rightarrow \mathbf{x},\ \mathbf{y} \in \mathbf{U}$ where $\mathbf{U} \in \mathbf{U}^{\mathfrak{G}^{\mathbf{0}}_{\mathbf{M_{NVB}}}} \Rightarrow \mathbf{U}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$ becomes $\mathbf{U}^{\mathfrak{G}^{\mathbf{0}}_{\mathbf{M_{NVB}}}} \Rightarrow \mathfrak{G}_{\mathbf{M_{NVB}}}$ becomes $\mathfrak{G}^{\mathbf{0}}_{\mathbf{M_{NVB}}}$

7. Derivations of Neutrosophic Vague Binary G – subalgebra

In this section following points are developed

i. neutrosophic vague binary $\mathbf{G}$ – derivation

ii. neutrosophic vague binary $\mathbf{G}$ – regular derivation

Definition 7.1 ($\mathbf{G}$ – derivation of neutrosophic vague binary $\mathbf{G}$ – subalgebra)

Let $\mathbf{M_{NVB}}$ be a neutrosophic vague binary set (in short, **NVBS**) with two universes $\mathbf{U_1}$ and $\mathbf{U_2}$. Also let considered $\mathbf{M_{NVB}}$ is a **NVB G** - subalgebra with structure $\mathfrak{G}_{\mathbf{M_{NVB}}}$ = $(\mathbf{U}^{\mathfrak{G}_{\mathbf{M_{NVB}}}},\ *,\ \mathbf{0})$ and with a self – map $\mathbf{d} : \mathbf{U} \rightarrow \mathbf{U}$ on $\mathbf{M_{NVB}}$ with $\mathbf{U} = \{\mathbf{U_1} \cup \mathbf{U_2}\}$. Then,

(i) $\mathbf{d}$ is (l, r) neutrosophic vague binary $\mathbf{G}$ - derivation of $\mathbf{M_{NVB}}$ if,

$\mathbf{NVB}_{\mathbf{M_{NVB}}}[\mathbf{d}(\mathbf{x} * \mathbf{y})] = \mathbf{NVB}_{\mathbf{M_{NVB}}}\big[(\mathbf{d}(\mathbf{x}) * \mathbf{y}) \wedge \big(\mathbf{x} * \mathbf{d}(\mathbf{y})\big)\big]$

(ii) $\mathbf{d}$ is (r, l) neutrosophic vague binary G - derivation of $\mathbf{M_{NVB}}$ if,

$\mathbf{NVB}_{\mathbf{M_{NVB}}}[\mathbf{d}(\mathbf{x} * \mathbf{y})] = \mathbf{NVB}_{\mathbf{M_{NVB}}}\big[\big(\mathbf{x} * \mathbf{d}(\mathbf{y})\big) \wedge (\mathbf{d}(\mathbf{x}) * \mathbf{y})\big]$

$\mathbf{d}$ is a neutrosophic vague binary $\mathbf{G}$ – derivation (in short, **NVB G** - derivation) of $\mathbf{M_{NVB}}$ only if $\mathbf{d}$ is both (l, r) neutrosophic vague binary $\mathbf{G}$ – derivation [in short, (l, r) **NVB G** -derivation] & (r, l) neutrosophic vague binary $\mathbf{G}$ – derivation [in short, (r, l) **NVB G** -derivation] of $\mathbf{M_{NVB}}$.

In this definition, (l, r) indicates left-right and (r, l) indicates right-left

Remark 7.2

For a **NVB G** - subalgebra, $\mathbf{NVB}_{\mathbf{M_{NVB}}}(\boldsymbol{x} \wedge \mathbf{y}) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{x})$

(i) $\therefore$ To check, $\mathbf{d}$ is (l, r) $\mathbf{NVB\,G}$ - derivation of $\mathbf{M_{NVB}}$, it is enough to check that,
$\mathbf{NVB_{M_{NVB}}}(\mathbf{d}(x * \mathbf{y})) = \mathbf{NVB_{M_{NVB}}}(\mathbf{d}(x) * y)$; [**Using definition 2.13 &** ***by remark*** **2.14**]
(ii) $\therefore$ To check, $\mathbf{d}_{\mathfrak{G}_{\mathbf{M_{NVB}}}}$ is (r, l) NVB G - derivation of $\mathbf{M_{NVB}}$, it is enough to check that,
$\mathbf{NVB_{M_{NVB}}}(\mathbf{d}(x * \mathbf{y})) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x} * \mathbf{d}(\mathbf{y}))$ [**Using definition 2.13 &** ***by remark*** **2.14**]
$\therefore$ Definition 7.1, can be re – written as, definition 7.3

Definition 7.3
Let $\mathfrak{G}_{\mathbf{M_{NVB}}}$ be a $\mathbf{NVB\,G}$ – subalgebra and $\mathbf{d}$ be a self – map on $\mathbf{U}$.
$\mathbf{d}$ is a neutrosophic vague binary G - derivation of $\mathbf{U}$ if
(i) $\mathbf{d}$ is (l, r) – neutrosophic vague binary G - derivation of $\mathbf{U}$
i.e., $\mathbf{NVB_{M_{NVB}}}(\mathbf{d}(\mathbf{x} * \mathbf{y})) = \mathbf{NVB_{M_{NVB}}}(\mathbf{d}(\mathbf{x}) * \mathbf{y})$; for all $\mathbf{x}, \mathbf{y} \in \mathbf{U}$ & it is denoted by $\mathbf{d}_{(\mathbf{l},\mathbf{r})}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$
(ii) $\mathbf{d}$ is a (r, l) – neutrosophic vague binary G - derivation of $\mathbf{U}$.
i.e., $\mathbf{NVB_{M_{NVB}}}(\mathbf{d}(\mathbf{x} * \mathbf{y})) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x} * \mathbf{d}(\mathbf{y}))$; for all $\mathbf{x}, \mathbf{y} \in \mathbf{U}$ & it is denoted by $\mathbf{d}_{(\mathbf{r},\mathbf{l})}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$.
d is a NVB G – derivation on a $\mathfrak{G}_{\mathbf{M_{NVB}}}$ if d is both $\mathbf{d}_{(\mathbf{l},\mathbf{r})}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$ and $\mathbf{d}_{(\mathbf{r},\mathbf{l})}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$. It is denoted by $\mathbf{d}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$.

Definition 7.4 (Regular derivation of a Neutrosophic Vague Binary G – subalgebra)
A derivation $\mathbf{d}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$ of a $\mathbf{NVB\,G}$ – subalgebra is said to be regular if,
$\mathbf{NVB_{M_{NVB}}}(\mathbf{d}(\mathbf{0})) = \mathbf{NVB_{M_{NVB}}}(\mathbf{0})$. It is denoted by $\mathbf{d}_{\mathbf{r}}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$.

Example 7.5
From example 3.3, $\mathbf{M_{NVB}}$ is a $\mathfrak{G}_{\mathbf{M_{NVB}}}$.
Case (i) Define a self – map, d : $\mathbf{U} = \{0, \boldsymbol{a}, \mathbf{b}\} \rightarrow \mathbf{U} = \{0, \mathbf{a}, \mathbf{b}\}$ by $\mathbf{d}(\boldsymbol{s}) = \begin{cases} \mathbf{0} \text{ if } \mathbf{s} = \mathbf{0} \\ \mathbf{a} \text{ if } \mathbf{s} = \mathbf{a} \\ \mathbf{b} \text{ if } \mathbf{s} = \mathbf{b} \end{cases}$
Here the given self – map is an identity map.
From calculations, d is a $\mathbf{d}_{(\mathbf{l},\mathbf{r})}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$ & $\mathbf{d}_{(\mathbf{r},\mathbf{l})}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$ $\Rightarrow$ d is a $\mathbf{d}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$
Case (ii) Define a self-map, $\mathbf{d} : \mathbf{U} = \{0, \mathbf{a}, \mathbf{b}\} \rightarrow \mathbf{U} = \{0, \mathbf{a}, \mathbf{b}\}$ by $\mathbf{d}(\mathbf{s}) = \begin{cases} \mathbf{a} \text{ if } \mathbf{s} = \mathbf{0} \\ \mathbf{0} \text{ if } \mathbf{s} = \mathbf{a} \\ \mathbf{b} \text{ if } \mathbf{s} = \mathbf{b} \end{cases}$
$\mathbf{d}$ is not a NVB G – derivation on $\mathbf{M_{NVB}}$. One violation is attached below.
$\mathbf{NVB_{M_{NVB}}}(\mathbf{d}(\mathbf{b} * \mathbf{a})) = \mathbf{NVB_{M_{NVB}}}(\mathbf{d}(\boldsymbol{a})) = \mathbf{NVB_{M_{NVB}}}(\mathbf{0}) = [\mathbf{0.9}, \mathbf{0.9}][\mathbf{0.1}, \mathbf{0.1}][\mathbf{0.1}, \mathbf{0.1}]$
$\mathbf{NVB_{M_{NVB}}}[\mathbf{d}(\mathbf{b}) * \mathbf{a}] = \mathbf{NVB_{M_{NVB}}}(\mathbf{b} * \mathbf{a}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{a}) = [\mathbf{0.7}, \mathbf{0.9}][\mathbf{0.3}, \mathbf{0.4}][\mathbf{0.1}, \mathbf{0.3}]$
$\mathbf{d}_{(\mathbf{l},\mathbf{r})}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$ $(\mathbf{b} * \mathbf{a})$ does not exist, since $\mathbf{NVB_{M_{NVB}}}(\mathbf{d}(\boldsymbol{b} * \mathbf{a})) \neq \mathbf{NVB_{M_{NVB}}}(\boldsymbol{d}(\boldsymbol{b}) * \mathbf{a})$
$\mathbf{NVB_{M_{NVB}}}[\mathbf{b} * \mathbf{d}(\mathbf{a})] = \mathbf{NVB_{M_{NVB}}}(\boldsymbol{b} * \mathbf{0}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{b}) = [\mathbf{0.2}, \mathbf{0.6}][\mathbf{0.1}, \mathbf{0.2}][\mathbf{0.4}, \mathbf{0.8}]$
$\mathbf{d}_{(\mathbf{r},\mathbf{l})}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$ $(\mathbf{b} * \mathbf{a})$ does not exist, since $\mathbf{NVB_{M_{NVB}}}(\mathbf{d}(\mathbf{b} * \mathbf{a})) \neq \mathbf{NVB_{M_{NVB}}}(\mathbf{b} * \mathbf{d}(\mathbf{a}))$
$\Rightarrow$ d is not a $\mathbf{d}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$.

Theorem 7.6
In a $\mathfrak{G}_{\mathbf{M_{NVB}}}$, the identity map d on $\mathbf{U}$ is a $\mathbf{d}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$. Converse not true in general. But if $\mathbf{d}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$ is a $\mathbf{d}_{\mathbf{r}}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$ then converse hold good. That is, if $\mathbf{d}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$ is a $\mathbf{d}_{\mathbf{r}}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}$ then $\mathbf{d}$ is the identity map on $\mathbf{U}$

Proof
(i) Let $\mathbf{x}, \mathbf{y} \in \mathbf{U}$ & also let d is an identity map on $\mathbf{U}$

Case (i) : $\mathbf{x} = \mathbf{y}$; $\mathbf{y} \neq 0$
$\mathbf{NVB_{M_{NVB}}}(\mathbf{d}(\mathbf{x} * \mathbf{y})) = \mathbf{NVB_{M_{NVB}}}(\mathbf{d}(\mathbf{x} * \mathbf{x})) = \mathbf{NVB_{M_{NVB}}}(\mathbf{d}(\mathbf{0})) = \mathbf{NVB_{M_{NVB}}}(\mathbf{0})$.
$\mathbf{NVB_{M_{NVB}}}(\mathbf{d}(\mathbf{x}) * \mathbf{y}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{d}(\mathbf{x}) * \mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x} * \mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{0})$.
$\mathbf{NVB_{M_{NVB}}}(\mathbf{x} * \mathbf{d}(\mathbf{y})) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x} * \mathbf{d}(\mathbf{x})) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x} * \mathbf{x}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{0})$.
$\therefore \mathbf{NVB_{M_{NVB}}}(\mathbf{d}(\mathbf{x} * \mathbf{y})) = \mathbf{NVB_{M_{NVB}}}(\mathbf{d}(\mathbf{x}) * \mathbf{y}) = \mathbf{NVB_{M_{NVB}}}(\mathbf{x} * \mathbf{d}(\mathbf{y}))$

Case (ii) : $\mathbf{x} \neq \mathbf{y}$; $\mathbf{y} \neq 0$

Either $\mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{x}*\mathbf{y})) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{x})) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{x})$ or $\mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{x}*\mathbf{y})) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{y})) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{y})$

$\Rightarrow$ $\mathbf{d}(\mathbf{x}*\mathbf{y}) = \mathbf{d}(\mathbf{x})$ or $\mathbf{d}(\mathbf{x}*\mathbf{y}) = \mathbf{d}(\mathbf{y})$ $\Rightarrow$ either $(\mathbf{x}*\mathbf{y}) = \mathbf{x}$ or $(\mathbf{x}*\mathbf{y}) = \mathbf{y}$, since d is identity map

$\Rightarrow$ either $\mathbf{y} = 0$ or $\mathbf{y} \neq 0$.

Consider $\mathbf{y} \neq 0$, i.e., $\mathbf{d}(\mathbf{x}*\mathbf{y}) = \mathbf{d}(\mathbf{y})$, i.e., $(\mathbf{x}*\mathbf{y}) = \mathbf{y}$. $\mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{x}*\mathbf{y})) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{y})) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{y})$. $\mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{x})*\mathbf{y}) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{x}*\mathbf{y}) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{y})$. $\mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{x}*\mathbf{d}(\mathbf{y})) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{x}*\mathbf{y}) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{y})$. $\therefore$ $\mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{x}*\mathbf{y})) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{x})*\mathbf{y}) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{x}*\mathbf{d}(\mathbf{y}))$

Case (iii) : $\mathbf{x} \neq \mathbf{y}$; $\mathbf{y} = 0$

Either $\mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{x}*\mathbf{y})) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{x})) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{x})$ or $\mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{x}*\mathbf{y})) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{y})) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{y})$

$\Rightarrow$ $\mathbf{d}(\mathbf{x}*\mathbf{y}) = \mathbf{d}(\mathbf{x})$ or $\mathbf{d}(\mathbf{x}*\mathbf{y}) = \mathbf{d}(\mathbf{y})$ $\Rightarrow$ either $(\mathbf{x}*\mathbf{y}) = \mathbf{x}$ or $(\mathbf{x}*\mathbf{y}) = \mathbf{y}$, since d is identity map

$\Rightarrow$ either $\mathbf{y} = 0$ or $\mathbf{y} \neq 0$.

Consider $\mathbf{y} = 0$, i.e., $\mathbf{d}(\mathbf{x}*\mathbf{y}) = \mathbf{d}(\mathbf{x})$, i.e., $(\mathbf{x}*\mathbf{y}) = \mathbf{x}$. $\mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{x}*\mathbf{y})) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{x})) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{x})$. $\mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{x})*\mathbf{y}) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{x}*\mathbf{y}) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{x})$. $\mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{x}*\mathbf{d}(\mathbf{y})) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{x}*\mathbf{y}) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{x})$. $\therefore$ $\mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{x}*\mathbf{y})) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{x})*\mathbf{y}) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{x}*\mathbf{d}(\mathbf{y}))$

$\therefore$ d is both $\mathbf{d}_{(\mathrm{l,r})}^{\mathfrak{G}_{\mathrm{M_{NVB}}}}$ & $\mathbf{d}_{(\mathrm{r,l})}^{\mathfrak{G}_{\mathrm{M_{NVB}}}}$. Hence d is a $\mathbf{d}^{\mathfrak{G}_{\mathrm{M_{NVB}}}}$

Converse

$\mathbf{d}^{\mathfrak{G}_{\mathrm{M_{NVB}}}}$ is a $\mathbf{d}_{\mathbf{r}}^{\mathfrak{G}_{\mathrm{M_{NVB}}}}$ $\Rightarrow$ $\mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{0})) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{0}) \Rightarrow \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{x}*\mathbf{x})) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{0})$

$\Rightarrow$ $\mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{x})*\mathbf{x}) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{0}) \Rightarrow \mathbf{d}(\mathbf{x}) = \mathbf{x}$ [**By proposition 3.11 (ii)**]

$\Rightarrow$ $\mathbf{d}$ is the identity map on $\mathbf{U}$

Remark 7.7

Let $\mathbf{M_{NVB}}$ be a NVB G - subalgebra with structure $\mathfrak{G}_{\mathbf{M_{NVB}}} = (\mathbf{U}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}, *, \mathbf{0})$. A NVB G – derivation on $\mathbf{M_{NVB}}$ is a mapping $\mathbf{d}$: $\mathbf{U} \rightarrow \mathbf{U}$ such that $\mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{x}*\mathbf{y})) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{d}(\mathbf{x})*\mathbf{y}) = \mathbf{NVB}_{\mathbf{M_{NVB}}}(\mathbf{x}*\mathbf{d}(\mathbf{y}))$, $\forall$ $\mathbf{x}$, $\mathbf{y} \in \mathbf{U}$. Set of all neutrosophic vague binary G -derivations on $\mathbf{M_{NVB}}$ is denoted as $\mathbf{\Gamma}^{\mathbf{d}^{\mathfrak{G}_{\mathbf{M_{NVB}}}}}$

8. Neutrosophic vague binary G – Coset

General properties that are true for abstract algebra and G – algebra may not be true in the case of neutrosophic G – subalgebra/neutrosophic vague G – subalgebra/ neutrosophic vague binary G – subalgebra. In this section coset for neutrosophic vague binary G – subalgebra is developed. Neutrosophic vague binary G – Coset is considered as a shifted (or translated) neutrosophic vague binary G – subalgebra. Existence of identity element and inverse element can't be assured in every neutrosophic vague binary G – subalgebra. In generalization process, this will become a crisis. As a result, generalization is confined to a particular area. It will lead to the formation of different concepts like Lagrange neutrosophic vague binary G – subalgebra etc.

Definition 8.1 (Neutrosophic Vague Binary G – Right Coset & Neutrosophic Vague Binary G – Left Coset)
Let $\mathrm{M_{NVB}}$ be a neutrosophic vague binary set (in short, NVBS) with two universes $\mathrm{U_1}$ and $\mathrm{U_2}$. and also let the considered $\mathrm{M_{NVB}}$ is a NVB G – subalgebra of a G – algebra with algebraic structure $\mathfrak{G}_{\mathrm{M_{NVB}}} = (\mathrm{U}^{\mathfrak{G}_{\mathrm{M_{NVB}}}}, *, 0)$where $\mathrm{U}^{\mathfrak{G}_{\mathrm{M_{NVB}}}} = (\mathrm{U}, *, 0_{\mathrm{M_{NVB}}})$. Also,$\widehat{\mathrm{T}} = [\mathrm{T}^-, \mathrm{T}^+]$;$\hat{\mathrm{I}} = [\mathrm{I}^-, \mathrm{I}^+]$; $\hat{\mathrm{F}} = [\mathrm{F}^-, \mathrm{F}^+]$ and $\mathrm{U} = \{\mathrm{U_1} \cup \mathrm{U_2}\}$

Case (i) (Neutrosophic Vague Binary G – Right Coset)

Let $a \in U_1$ and $b \in U_2$ be fixed elements. Then define, for every $c \in U_1$ and for every $d \in U_2$ a neutrosophic vague binary G – right coset of M_{NVB} is denoted by $M_{NVB}(a, b)$ and defined by,

$(M_{NVB}(a, b))(c, d) = NVB_{M_{NVB}(a,b)}(c, d) = \{\langle NVB_{M_{NVB}}(c * (a)^{-1}) | \forall c \in U_1\rangle \langle NVB_{M_{NVB}}(d * (b)^{-1}) | \forall d \in U_2\rangle\}$

i.e., $\{\langle(\hat{T}_{M_{NVB}(a)}(c), \hat{I}_{M_{NVB}(a)}(c), \hat{F}_{M_{NVB}(a)}(c) | \forall c \in U_1)\rangle \langle(\hat{T}_{M_{NVB}(b)}(d), \hat{I}_{M_{NVB}(b)}(d), \hat{F}_{M_{NVB}(b)}(d) | \forall d \in U_2)\rangle\}$

$= \{\langle(\hat{T}_{M_{NVB}}(c * (a)^{-1}), \hat{I}_{M_{NVB}}(c * (a)^{-1}), \hat{F}_{M_{NVB}}(c * (a)^{-1}) | \forall c \in U_1)\rangle \langle(\hat{T}_{M_{NVB}}(d * (b)^{-1}), \hat{I}_{M_{NVB}}(d * (b)^{-1}), \hat{F}_{M_{NVB}}(d * (b)^{-1}) | \forall d \in U_2)\rangle\}$

Then $M_{NVB}(a, b)$ is called a neutrosophic vague binary G -Right Coset (in short NVB G – Right Coset) determined by M_{NVB} and (a, b).

Case (ii) (Neutrosophic vague binary G – Left Coset)

Let $a \in U_1$ and $b \in U_2$ be fixed elements. Then define, for every $c \in U_1$ and for every $d \in U_2$ a neutrosophic vague binary G – right coset of M_{NVB} is denoted by $(a, b)\, M_{NVB}$ and defined by,

$((a, b)\, M_{NVB})(c, d) = NVB_{(a,b)\, M_{NVB}}(c, d) = \{\langle NVB_{M_{NVB}}((a)^{-1} * c) | \forall c \in U_1\rangle \langle NVB_{M_{NVB}}((b)^{-1} * d) | \forall d \in U_2\rangle\}$

$= \{\langle(\hat{T}_{(a)\, M_{NVB}}(c), \hat{I}_{(a)\, M_{NVB}}(c), \hat{F}_{(a) M_{NVB}}(c) | \forall c \in U_1)\rangle \langle(\hat{T}_{(b)\, M_{NVB}}(d), \hat{I}_{(b)\, M_{NVB}}(d), \hat{F}_{(b) M_{NVB}}(d) | \forall d \in U_2)\rangle\}$

$= \{\langle(\hat{T}_{M_{NVB}}((a)^{-1} * c), \hat{I}_{M_{NVB}}((a)^{-1} * c), \hat{F}_{M_{NVB}}((a)^{-1} * c) | \forall c \in U_1)\rangle \langle(\hat{T}_{M_{NVB}}((b)^{-1} * d), \hat{I}_{M_{NVB}}((b)^{-1} * d), \hat{F}_{M_{NVB}}((b)^{-1} * d) | \forall d \in U_2)\rangle\}$

Then $(a, b)\, M_{NVB}$ is called a neutrosophic vague binary left coset (in short NVB G – left coset) determined by M_{NVB} and (a, b).

Remark 8.2

NVB G – right coset is a NVBS. Similarly, a NVB G – left coset is a NVBS.

Definition 8.3 (Neutrosophic Vague Binary G – Coset)

Let the neutrosophic vague binary set M_{NVB} be a neutrosophic vague binary G – subalgebra of a G – algebra. If M_{NVB} is both neutrosophic vague binary G – right coset and neutrosophic vague binary G – left coset then M_{NVB} is called as a Neutrosophic Vague Binary G – Coset

Example 8.4

Let $U_1 = \{0, u_1, u_3\}$ and $U_2 = \{0, u_2, u_4, u_5\}$ be two universes.

Let $M_{NVB} = \left\{ \begin{array}{c} \langle \frac{[0.7,0.8][0.3,0.4][0.2,0.3]}{0}, \frac{[0.2,0.7][0.5,0.7][0.3,0.8]}{u_1}, \frac{[0.6,0.7][0.1,0.4][0.3,0.4]}{u_3} \rangle \\ \langle \frac{[0.2,0.9][0.1,0.7][0.1,0.8]}{0}, \frac{[0.3,0.5][0.6,0.7][0.5,0.7]}{u_2}, \frac{[0.2,0.8][0.4,0.7][0.2,0.8]}{u_4}, \frac{[0.6,0.9][0.3,0.7][0.1,0.4]}{u_5} \rangle \end{array} \right\}$ be a NVBS.

Here, combined universe $U = \{0, u_1, u_2, u_3, u_4, u_5\}$ & combined NVB membership grades are,

$$NVB_{M_{NVB}}(s) = \begin{cases} [0.7, 0.9][0.1, 0.4][0.1, 0.3] & ; \quad s = 0 \\ [0.2, 0.7][0.5, 0.7][0.3, 0.8] & ; \quad s = u_1 \\ [0.3, 0.5][0.6, 0.7][0.5, 0.7] & ; \quad s = u_2 \\ [0.6, 0.7][0.1, 0.4][0.3, 0.4] & ; \quad s = u_3 \\ [0.2, 0.8][0.4, 0.7][0.2, 0.8] & ; \quad s = u_4 \\ [0.6, 0.9][0.3, 0.7][0.1, 0.4] & ; \quad s = u_5 \end{cases}$$

Corresponding Cayley table is :

*	0	u_1	u_2	u_3	u_4	u_5
0	0	u_1	u_2	u_3	u_4	u_5
u_1	u_1	0	u_2	u_3	u_4	u_5
u_2	u_2	u_1	0	u_3	u_4	u_5
u_3	u_3	u_1	u_2	0	u_4	u_5

u_4	u_4	u_1	u_2	u_3	0	u_5
u_5	u_5	u_1	u_2	u_3	u_4	0

Obviously, M_{NVB} is a NVB G – subalgebra

In every G – algebra 0 may not be the identity element. But in the present case it is clear that 0 acts as an identity element. Hence inverses got as :

$(0)^{-1} = 0$; $(u_1)^{-1} = u_1$; $(u_2)^{-1} = u_2$; $(u_3)^{-1} = u_3$; $(u_4)^{-1} = u_4$; $(u_5)^{-1} = u_5$

To construct a NVB G -right coset:

Let $u_a = u_1 \in U_1$ and $\forall\ u_c \in U_1 = \{0, u_1, u_3\}$

$NVB_{M_{NVB}\, u_1}(0) = NVB_{M_{NVB}}(0 * (u_1{}^{-1})) = NVB_{M_{NVB}}(0 * u_1) = NVB_{M_{NVB}}(u_1) = [0.2, 0.7][0.5, 0.7][0.3, 0.8]$
$NVB_{M_{NVB}\, u_1}(u_1) = NVB_{M_{NVB}}(u_1 * (u_1{}^{-1})) = NVB_{M_{NVB}}(u_1 * u_1) = NVB_{M_{NVB}}(0) = [0.7, 0.9][0.1, 0.4][0.1, 0.3]$
$NVB_{M_{NVB}\, u_1}(u_3) = NVB_{M_{NVB}}(u_3 * (u_1{}^{-1})) = NVB_{M_{NVB}}(u_3 * u_1) = NVB_{M_{NVB}}(u_1) = [0.2, 0.7][0.5, 0.7][0.3, 0.8]$

&

Let $u_b = u_2 \in U_2$ and $\forall\ u_d \in U_2 = \{0, u_2, u_4, u_5\}$

$NVB_{M_{NVB}\, u_2}(0) = NVB_{M_{NVB}}(0 * (u_2{}^{-1})) = NVB_{M_{NVB}}(0 * u_2) = NVB_{M_{NVB}}(u_2) = [0.3, 0.5][0.6, 0.7][0.5, 0.7]$
$NVB_{M_{NVB}\, u_2}(u_2) = NVB_{M_{NVB}}(u_2 * (u_2{}^{-1})) = NVB_{M_{NVB}}(u_2 * u_2) = NVB_{M_{NVB}}(0) = [0.7, 0.9][0.1, 0.4][0.1, 0.3]$
$NVB_{M_{NVB}\, u_2}(u_4) = NVB_{M_{NVB}}(u_4 * (u_2{}^{-1})) = NVB_{M_{NVB}}(u_4 * u_2) = NVB_{M_{NVB}}(u_2) = [0.3, 0.5][0.6, 0.7][0.5, 0.7]$
$NVB_{M_{NVB}\, u_2}(u_5) = NVB_{M_{NVB}}(u_5 * (u_2{}^{-1})) = NVB_{M_{NVB}}(u_5 * u_2) = NVB_{M_{NVB}}(u_2) = [0.3, 0.5][0.6, 0.7][0.5, 0.7]$

$M_{NVB}\ u_1 : u_2 =$

$$\left\{\left\langle\frac{[0.2,0.7][0.5,0.7][0.3,0.8]}{0}, \frac{[0.7,0.9][0.1,0.4][0.1,0.3]}{u_1}, \frac{[0.2,0.7][0.5,0.7][0.3,0.8]}{u_3}\right\rangle \left\langle\frac{[0.3,0.5][0.6,0.7][0.5,0.7]}{0}, \frac{[0.7,0.9][0.1,0.4][0.1,0.3]}{u_2}, \frac{[0.3,0.5][0.6,0.7][0.5,0.7]}{u_4}, \frac{[0.3,0.5][0.6,0.7][0.5,0.7]}{u_5}\right\rangle\right\}$$

To construct a NVB G - left coset:

Let $a = u_1 \in U_1$ and $\forall\ c \in U_1 = \{0, u_1, u_3\}$

$NVB_{\,u_1 M_{NVB}}(0) = NVB_{M_{NVB}}((u_1{}^{-1}) * 0) = NVB_{M_{NVB}}(u_1 * 0) = NVB_{M_{NVB}}(u_1) = [0.2, 0.7][0.5, 0.7][0.3, 0.8]$
$NVB_{\,u_1 M_{NVB}}(u_1) = NVB_{M_{NVB}}(u_1 * (u_2{}^{-1})) = NVB_{M_{NVB}}(u_1 * u_2) = NVB_{M_{NVB}}(u_2) = [0.3, 0.5][0.6, 0.7][0.5, 0.7]$
$NVB_{\,u_1 M_{NVB}}(u_3) = NVB_{M_{NVB}}((u_1{}^{-1}) * u_3) = NVB_{M_{NVB}}(u_1 * u_3) = NVB_{M_{NVB}}(u_3) = [0.6, 0.7][0.1, 0.4][0.3, 0.4]$

Let $u_b = u_2 \in U_2$ and $\forall\ u_d \in U_2 = \{0, u_2, u_4, u_5\}$

$NVB_{\,u_2\ M_{NVB}}(0) = NVB_{M_{NVB}}((u_2{}^{-1}) * 0) = NVB_{M_{NVB}}(u_2 * 0) = NVB_{M_{NVB}}(u_2) = [0.3, 0.5][0.6, 0.7][0.5, 0.7]$
$NVB_{\,u_2\ M_{NVB}}(u_2) = NVB_{M_{NVB}}((u_2{}^{-1}) * u_2) = NVB_{M_{NVB}}(u_2 * u_2) = NVB_{M_{NVB}}(0) = [0.7, 0.9][0.1, 0.4][0.1, 0.3]$
$NVB_{\,u_2\ M_{NVB}}(u_4) = NVB_{M_{NVB}}((u_2{}^{-1}) * u_4) = NVB_{M_{NVB}}(u_2 * u_4) = NVB_{M_{NVB}}(u_4) = [0.2, 0.8][0.4, 0.7][0.2, 0.8]$
$NVB_{\,u_2\ M_{NVB}}(u_5) = NVB_{M_{NVB}}((u_2{}^{-1}) * u_5) = NVB_{M_{NVB}}(u_2 * u_5) = NVB_{M_{NVB}}(u_5) = [0.6, 0.9][0.3, 0.7][0.1, 0.4]$

$u_1 : u_2\ M_{NVB} =$

$$\left\{\left\langle\frac{[0.2,0.7][0.5,0.7][0.3,0.8]}{0}, \frac{[0.3,0.5][0.6,0.7][0.5,0.7]}{u_1}, \frac{[0.6,0.7][0.1,0.4][0.3,0.4]}{u_3}\right\rangle \left\langle\frac{[0.3,0.5][0.6,0.7][0.5,0.7]}{0}, \frac{[0.7,0.9][0.1,0.4][0.1,0.3]}{u_2}, \frac{[0.2,0.8][0.4,0.7][0.2,0.8]}{u_4}, \frac{[0.6,0.9][0.3,0.7][0.1,0.4]}{u_5}\right\rangle\right\}$$

Remark 8.5

(i) In example 8.4, $M_{NVB}\ u_1 : u_2 \neq\ u_1 : u_2\ M_{NVB}$

(ii) Constant 0 is not an identity element in G – algebra. For example, let $X = \{0,\ u_1, u_2, u_3, u_4, u_5\}$. $(X,\ *,\ 0)$ is a G – algebra, with binary operation $*$ is defined by the following Cayley table:

$*$	0	u_1	u_2	u_3	u_4	u_5
0	0	u_2	u_1	u_3	u_4	u_5
u_1	u_1	0	u_3	u_2	u_5	u_4
u_2	u_2	u_4	0	u_5	u_1	u_3
u_3	u_3	u_5	u_4	0	u_2	u_1
u_4	u_4	u_3	u_5	u_1	0	u_2
u_5	u_5	u_1	u_2	u_4	u_3	0

It is clear that X is a G – algebra without an identity element. And hence inverse does not exist. So neutrosophic vague binary G - cosets cannot construct in this case. This construction is possible, only for those cases where identity element exists in the basic G – algebraic structure.
(ii) If the basic G – algebraic structure is formed using the following rules, then definitely there exist identity element and hence can construct a NVB G – right coset & NVB G – left coset.

Rules in Cayley table:

(i) Principal diagonal elements = 0
(ii) Column occupied with constant 0 is a copy of column of operands
(iii) Fill each of the remaining columns (except principal diagonal entries) with the element given in the column head (i.e., elements from row of operands)

Definition 8.6 (Neutrosophic Vague G – Right Coset & Neutrosophic Vague G- Left Coset)

Let M_{NV} be a neutrosophic vague set (in short, NV Set) with a single universe U and also let M_{NV} be a neutrosophic vague G – subalgebra (in short, NV G – subalgebra) of a G – algebra. Algebraic structure of M_{NV} is given by $\mathfrak{G}_{M_{NV}} = \left(U^{\mathfrak{G}_{M_{NV}}}, *, 0\right)$ where $U^{\mathfrak{G}_{M_{NV}}} = (U, *, 0)$. Also $\widehat{T} = [T^-, T^+]$; $\widehat{I} = [I^-, I^+]$; $\widehat{F} = [F^-, F^+]$

Case (i) (Neutrosophic Vague G – Right coset)

Let $a \in U$ be a fixed element. Then define, for every $c \in U$ a neutrosophic vague G – right coset of M_{NV} which is denoted by $M_{NV}\,a$ and defined by,

$\left(M_{NV}(a)\right)(c) = NV_{M_{NV}(a)}(c) = \left\{NV_{M_{NV}}(c * (a)^{-1}) \ / \ \forall\, c \in U\right\}$

i.e$\left\{\left(\widehat{T}_{M_{NV}\,a}(c), \widehat{I}_{M_{NV}\,a}(c), \widehat{F}_{M_{NV}\,a}(c)\right) / \forall\, c \in U\right\}$=$\left\{\left(\widehat{T}_{M_{NV}}(c * (a)^{-1}), \widehat{I}_{M_{NV}}(c * (a)^{-1}), \widehat{F}_{M_{NV}}(c * (a)^{-1})\right) / \forall c \in U\right\}$

Then $M_{NV}\,a$ is called a neutrosophic vague G -right coset (in short NV G – right coset) determined by M_{NV} and a.

Case (ii) (Neutrosophic Vague G – Left Coset)

Let $a \in U$ be a fixed element. Then define, for every $c \in U$ a neutrosophic vague G – right coset of M_{NV} is denoted by $a\,M_{NV}$ and defined by,

$((a)\, M_{NV})(c) = NV_{a\,M_{NV}}(c) = \left\{\left\langle NV_{M_{NV}}((a)^{-1} * c) \,\middle|\, \forall\, c \in U\right\rangle\right\}$

$= \left\{\left(\widehat{T}_{a\,M_{NV}}(c), \widehat{I}_{a\,M_{NV}}(c), \widehat{F}_{a\,M_{NV}}(c)\right) / \forall\, c \in U\right\}$

$= \left\{\left(\widehat{T}_{M_{NV}}((a)^{-1} * c), \widehat{I}_{M_{NV}}((a)^{-1} * c), \widehat{F}_{M_{NV}}((a)^{-1} * c)\right) / \forall\, c \in U\right\}$

Then a M_{NV} is called a neutrosophic vague left coset (in short NV G – left coset) determined by M_{NV} and a.

Definition 8.7 (Neutrosophic Vague G – Coset)

Let the neutrosophic vague set M_{NV} be a neutrosophic vague G – subalgebra of a G – algebra. If M_{NV} is both neutrosophic vague G – Right Coset and neutrosophic vague G – Left Coset then M_{NV} is called as a Neutrosophic Vague G – Coset

Definition 8.8 (Neutrosophic G – Right Coset & Neutrosophic G – Left Coset)

Let M_N be a neutrosophic set (in short, N set) with single universe U and also let M_N be a neutrosophic G – subalgebra (in short, N G – subalgebra) of a G – algebra. Algebraic structure of M_N is given by $\mathfrak{G}_{M_N} = \left(U^{\mathfrak{G}_{M_N}},\ *,\ 0\right)$ where $U^{\mathfrak{G}_{M_N}} = (U,\ *,\ 0)$.

Case (i) (Neutrosophic G – Right Coset)

Let $a \in U$ be a fixed element. Then define, for every $c \in U$ a neutrosophic G – right coset of M_N which is denoted by $M_N\,a$ and defined by,

$\big(M_N(a)\big)(c) = N_{M_N\,(a)}(c) = \{N_{M_N}(c * (a)^{-1}) \;/\; \forall\, c \in U\}$

i.e., $\left\{\left(T_{M_N\,a}(c), I_{M_N\,a}(c), F_{M_N\,a}(c)\right) / \;\forall\, c \in U\right\}$

$= \left\{\left(T_{M_N}(c * (a)^{-1}), I_{M_N}(c * (a)^{-1}), F_{M_N}(c * (a)^{-1})\right) / \;\forall\, c \in U\right\}$

Then $M_N\,a$ is called a neutrosophic G -right coset (in short N G – right coset) determined by M_N and a.

Case (ii) (Neutrosophic G – Left Coset)

Let $a \in U$ be a fixed element. Then define, for every $c \in U$ a neutrosophic G– right coset of M_N is denoted by aM_N and defined by,

$((a)M_N)(c) = N_{a\,M_N}(c) = \left\{\left\langle N_{M_N}((a)^{-1} * c) \,\middle|\, \forall\, c \in U\right\rangle\right\}$

$= \left\{\left(T_{a\,M_N}(c), I_{a\,M_N}(c), F_{a\,M_N}(c)\right) / \;\forall\, c \in U\right\}$

$= \left\{\left(T_{M_N}((a)^{-1} * c), I_{M_N}((a)^{-1} * c), F_{M_N}((a)^{-1} * c)\right) / \;\forall\, c \in U\right\}$

Then aM_N is called a neutrosophic left coset (in short N G – left coset) determined by M_N and a.

Definition 8.9 (Neutrosophic G – Coset)

Let the neutrosophic set M_N be a neutrosophic G – subalgebra of a G – algebra. If M_N is both neutrosophic G – Right Coset and neutrosophic G – Left Coset then M_N is called as a Neutrosophic G – Coset

9. Neutrosophic Vague Binary G – homomorphism

In this section homomorphism on **NVB G** – subalgebra is presented with some related theorems.

Definition 9.1

Let $\mathfrak{G}_{M_{NVB}} = (U^{\mathfrak{G}_{M_{NVB}}}, *, 0_{M_{NVB}})$ and $\mathfrak{G}_{P_{NVB}} = (U^{\mathfrak{G}_{P_{NVB}}},\ *', 0_{P_{NVB}})$ be two NVB G – subalgebras based on common universe $\{U_1, U_2\}$.

A mapping $\Psi^G : \mathfrak{G}_{M_{NVB}} = (U^{\mathfrak{G}_{M_{NVB}}}, *, 0_{M_{NVB}}) \to \mathfrak{G}_{P_{NVB}} = (U^{\mathfrak{G}_{P_{NVB}}},\ *', 0_{P_{NVB}})$ is called a neutrosophic vague binary G - homomorphism if, $\Psi^G(u_x * u_y) = \Psi^G(u_x)\ *'\ \Psi^G(u_y),\ \forall\ u_x,\ u_y \in U$.

Remark 9.2

(i) The NVB G - homomorphism Ψ^G is said to be a neutrosophic vague binary G - monomorphism (resp., a neutrosophic vague binary G - epimorphism) if it is injective (resp., surjective). (ii) If the map Ψ^G is both injective and surjective then $\mathfrak{G}_{M_{NVB}}$ and $\mathfrak{G}_{P_{NVB}}$ are said to be isomorphic, written $\mathfrak{G}_{M_{NVB}} \cong \mathfrak{G}_{P_{NVB}}$. For any NVB G - homomorphism $\Psi^G : \mathfrak{G}_{M_{NVB}} \to \mathfrak{G}_{P_{NVB}}$, the set $\{x \in U / \Psi^G(x) = 0_{P_{NVB}}\}$ is called the kernel of Ψ^G and denoted by Ker Ψ^G

Theorem 9.3

Let $\Psi^G : \mathfrak{G}_{M_{NVB}} = (U^{\mathfrak{G}_{M_{NVB}}}, *, 0_{M_{NVB}}) \to \mathfrak{G}_{P_{NVB}} = (U^{\mathfrak{G}_{P_{NVB}}}, *', 0_{P_{NVB}})$ be a neutrosophic vague binary G - homomorphism of NVB G - subalgebras, then :

(i) $\Psi^G(0_{M_{NVB}}) = 0_{P_{NVB}}$

(ii) Ker Ψ^G is a normal neutrosophic vague binary G - subalgebra of U

(iii) Im $\Psi^G = \{y \in \mathfrak{G}_{P_{NVB}} / y = \Psi^G(x) \text{ for some } x \in \mathfrak{G}_{M_{NVB}}\}$ is a NVB G - subalgebra

Proof

(i) $\Psi^G(0_{M_{NVB}}) = \Psi^G(0_{M_{NVB}} * 0_{M_{NVB}}) = \Psi^G(0_{M_{NVB}}) *' \Psi^G(0_{M_{NVB}}) = 0_{P_{NVB}} *' 0_{P_{NVB}} = 0_{P_{NVB}}$

(ii) $0_{M_{NVB}} \in \text{Ker}\, \Psi^G \Rightarrow \text{Ker}\ \Psi^G \neq \emptyset$

Let $(x * y), (a * b) \in \text{Ker}\, \Psi^G \Rightarrow \Psi(x * y) = 0_{P_{NVB}} = \Psi^G(a * b)$

$\Rightarrow \Psi^G(x) *' \Psi^G(y) = 0_{P_{NVB}} = \Psi^G(a) *' \Psi^G(b) \Rightarrow \Psi^G(x) = \Psi^G(y)\ \&\ \Psi^G(a) = \Psi^G(b)$

[By proposition 3.9(ii)]

$\Rightarrow \Psi^G((x * a) * (y * b)) = \Psi^G(x * a) * \Psi^G(y * b)$

$= (\Psi^G(x) *' \Psi^G(a)) * (\Psi^G(y) *' \Psi^G(b)) = (\Psi^G(x) *' \Psi^G(a)) * (\Psi^G(x) *' \Psi^G(a)) = 0_{P_{NVB}}$

[From definition of NVB G – subalgebra]

$\Rightarrow \Psi^G(x * a) *' \Psi^G(x * a) = 0_{P_{NVB}}$ [since Ψ^G is a NVB G – homomorphism]

$\Rightarrow \Psi^G[(x * a) * (x * a)] = 0_{P_{NVB}}$ [since Ψ^G is a NVB G – homomorphism]

$\Rightarrow (x * a) * (y * b) \in \text{Ker}\, \Psi^G \Rightarrow$ Ker Ψ^G is a N NVB G-subalgebra of U

(iii) Let $y, z \in \mathfrak{G}_{P_{NVB}} \Rightarrow y = \Psi^G(a)\ \&\ z = \Psi^G(b)$ for some $a, b \in \mathfrak{G}_{M_{NVB}}$

$\Psi^G(a) *' \Psi^G(b) = \Psi^G(a * b) \succcurlyeq r\min\{\Psi^G(a), \Psi^G(b)\}$. Hence the proof.

Theorem 9.4

A neutrosophic vague binary G – homomorphism $\chi^G : \mathfrak{G}_{T_{NVB}} = (U^{\mathfrak{G}_{T_{NVB}}}, *, 0_{T_{NVB}}) \to \mathfrak{G}_{L_{NVB}} = (U^{\mathfrak{G}_{L_{NVB}}}, *', 0_{L_{NVB}})$ is a neutrosophic vague binary G – monomorphism $\Leftrightarrow \ker(\chi^G) = \{0\}$

Proof

Let $x \in \text{Ker}\,(\chi^G) \Rightarrow \chi^G(x) = 0_{L_{NVB}} = \chi^G(0_{T_{NVB}})$.

χ^G is a neutrosophic vague binary G – monomorphism, then it is clearly got that, Ker $(\chi^G) = \{0\}$.

Conversely, let ker $(\chi^G) = \{0\}$ and also let $\chi^G(x) = \chi^G(y),\ \forall\ x, y \in U$

$\Rightarrow \chi^G(x) *' \chi^G(y) = 0_{L_{NVB}} \Rightarrow \chi^G(x * y) = 0_{L_{NVB}}$, since χ^G is a NVB G – homomorphism

$\Rightarrow (x * y) \in \text{Ker}\,(\chi^G) = \{0_{T_{NVB}}\} \Rightarrow (x * y) = 0_{T_{NVB}}$. Hence, $x = y \Rightarrow \chi^G$ is a neutrosophic vague binary G – monomorphism.

10. Conclusion

In this paper, **NVB G** – subalgebraic structure is developed with its properties for **NVBS**'s. Some basic ideas as **NVB G** - normal set of a **G** – algebra, **NVB G** – normal subalgebra and 0 – commutative **NVB G** – subalgebra are illustrated with examples and basic properties. Notions like **G** – part, p radical and p semi simple are defined in **NVB G** - subalgebra with characterizations. **NVB G** – minimal element, Derivations of **NVB G** – subalgebra, Regular Derivation of a **NVB G** – subalgebra, **NVB G** – homomorphism are also explained. Formation of Cosets is a basic idea in any algebraic structure. Coset for neutrosophic vague binary **G** – subalgebra is also got developed. Based on this, present work can be extended to **NVB** vague binary **G** -groups, **NVB G** – rings, **NVB G** – product, **NVB G** – factor group, Lagrange **NVB G** – subalgebra etc. As a future scope neutrosophic vague binary models can be tried to use in hazard detection, especially in switching circuits. Another application can be given in geographical area. Development of a neutrosophic vague binary spatial algebra could be more helpful in this area than the already existing crisp spatial algebraic concepts. Since the already existing pattern got failed to provide an accurate output when collected data becomes vague.

Funding: This research received no external funding from any funding agencies.

Conflicts of Interest: "The authors declare no conflict of interest."

References

[1] Apurba Das, Homotopy G – algebra structure on the Cochain Complex of Hom – Type Algebras, Last revised **2018**, Accepted for publication, arXiv : 1808.01464v3

[2] Ch. Mallika, N. Ramakrishna, G. Anandha Rao, Vage Cosets, *World Journal of Research and Review (WJRR)*, **2016**, *Vol. 3*, No. 2, pp 28-31

[3] Chiranjibe Jana, Tapan Senapati, Monoranjan Bhowmik, Madhumangal Pal, On Intuitionistic Fuzzy G – subalgebras of G – algebras, *Fuzzy Inf Eng*, **2015**, *Vol. 7*, pp 195-209

[4] Chiranjibe Jana and Tapan Senapati, Cubic G-subalgebras of G-algebras, *Annals of Pure and Applied Mathematics*, **2015**, *Vol. 10, No.1*, 22, pp 105-115

[5] Deena Al-Kadi, and Rodyna Hosny, Notes on G-Algebra and its Derivations, *Math. Sci. Lett*. 6, **2017**, No. 3, pp 287-292

[6] Florentin Smarandache, A Unifying Field in Logics: *Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability* (fifth edition), **2005**, ISBN 978-1-59973 080 – 6

[7] Florentin Smarandache, Neutrosophic Set – A Generalization of the Intuitionistic Fuzzy Set, *International Journal of Pure and Applied Mathematics*, **2005**, *Vol. 24, No. 3*, pp 287-297

[8] Florentin Smarandache, A Geometric Interpretation of the Neutrosophic Set, A Generalization of the Intuitionistic Fuzzy Set, *IEEE International Conference on Granular Computing*, **2011**, pp 602-606

[9] Francina Shalini. A and Remya. P. B, Neutrosophic Vague binary sets, *Neutrosophic Sets and Systems*, **2019**, *Vol. 29*, pp 228-241

[10] Francina Shalini. A and Remya. P. B, Neutrosophic Vague Binary BCK/BCI - algebra, *Neutrosophic Sets and Systems*, **2020**, *Vol.35, Issue 1*, pp 45 - 67

[11] Gau. W. L and Buehrer. D. J, Vague Sets, *IEEE transactions on systems, man and cybernetics*, **1993**, *Vol. 23, No. 2*, pp 610-614

[12] Hu. Q. P and Li. X, On BCH – Algebras, *Mathematics Seminar Notes (Kobe University)*, **1983**, *Vol. 11, No. 2*, pp 313 - 320

[13] Imai. Y and Iseki. K, On Axiom Systems of Propositional Calculi XIV, *Proceedings of the Japan Academy*, **1966**, *Vol. 42, No. 1*, pp 19- 22, doi: 10. 3792/pja/1195522169

[14] Joseph Neggers, Sun Shin Ahn, And Hee Sik Kim, On Q-Algebras, *International Journal of Mathematics and Mathematical Sciences*, **2001**, *Vol. 27, Issue 12*, pp 749 - 757

[15] Mohamed Abdel-Basset, Abduallah Gamal, Le Hoang Son and Florentin Smarandache, A Bipolar Neutrosophic Multi Criteria Decision Making Framework for Professional Selection, *Applied Sciences*, **2020**, *Vol. 10, Issue 4*, pp 1-22, doi:10.3390/app10041202

[16] Mohamed Abdel-Basset, Rehab Mohamed, Abd El-Nasser H. Zaied and Florentin Smarandache, A Hybrid Plithogenic Decision-Making Approach with Quality Function Deployment for Selecting Supply Chain Sustainability Metrics, *Symmetry*, 2019, *Vol. 11*, pp 1-21 doi:10.3390/sym11070903

[17] Mohamed Abdel-Basst, Rehab Mohamed, Mohamed Elhoseny, A novel framework to evaluate innovation value proposition for smart product–service systems, *Environmental Technology & Innovation*, 2020, *Vol.20*, pp 1-13, https://doi.org/10.1016/j.eti.2020.101036

[18] Mohamed Abdel-Basst, Rehab Mohamed, Mohamed Elhoseny, A model for the effective COVID-19 identification in uncertainty environment using primary symptoms and CT scans, *Health Informatics Journal*, 2020, Vol. 26, Online Issue, pp 1-18, doi : 10.1177/1460458220952918 journals.sagepub.com

[19] Mohsin Khalid, Rakib Iqbal and Said Broumi, Neutrosophic soft cubic Subalgebras of G-algebras, *Neutrosophic Sets and Systems*, **2019**, *Vol.28*, pp 259 – 272

[20] Muhammad Abdul Basit Khan, Junaid Alam Khan, Muhammad Ahsan Binyamin, SAGBI Bases in G – algebras, *Symmetry*, **2019**, Vol. 11, No. 221, pp 1 - 10

[21] Mukheriee. N. P and Prabir Bhattacharya, Fuzzy Normal Subgroups and Fuzzy Cosets, *Information Sciences*, **1984**, *Vol. 34*, pp *225-239*

[22] Ravi Kumar Bandaru and Rafi. N, On G-Algebras, *Scientia Magna*, **2012**, *Vol. 8, Issue 3*, pp 1-7

[23] Shawkat Alkhazaleh, Neutrosophic Vague Set Theory, *Critical Review*, **2015**, *Volume X*, pp 29-39

[24] Sun Shin Ahn and Hee Sik Kim, On QS - algebras, *Journal of the Chungcheong Mathematical Society*, **1999**, *Vol. 12*, pp 33 - 41

[25] Tapan Senapati, Chiranjibe Jana, Monoranjan Bhowmik, Madhumangal Pal, L -fuzzy G -subalgebras of G – algebras, *Journal of the Egyptian Mathematical Society*, **2015**, *Vol. 23, Issue 2*, pp 219-223

[26] Wahiba Messirdi and Ahlam Fallatah, About derivations on G – algebras, *Journal of Taibah University for Science*, **2020**, *Vol. 14, No.1*, pp 1 – 5

[27] Young Bae Jun, Eun Hwan Roh, Chinju, and Hee Sik Kim, Seoul, On Fuzzy B-Algebras, *Czechoslovak Mathematical Journal*, 2002, Vol. 52, No. 127, pp 375–384

Received: June 10, 2020. Accepted: Nov 25, 2020