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Neutrosophic Sets and Systems, Vol. 38, 2020
University of New Mexico

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Abstract: The aim of this paper is to answer Smarandache conjectures about neutrosophic quadruple vector spaces. This paper depends on the concept of weak n-refined neutrosophic vector space to prove that an NQ vector space V defined over the field F is isomorphic to $F \times F \times F \times F$.

Keywords: n-Refined weak neutrosophic vector space , NQ vector space , vector space homomorphism.

1. Introduction

Neutrosophy as a new king of logic has an interesting effect into algebra. Algebraic studies began with Smarandache and Kandasamy in [5]. They introduced some interesting concepts such as neutrosophic group, neutrosophic ring, and neutrosophic semi group.

Many applications of neutrosophic set and related concepts into optimization, decision making and industry were proposed in [10,11,12,13,14].

Recently, more neutrosophical algebraic structures were defined and handled as a bridge between logical and algebraic concepts such as neutrosophic vector space, quadruple neutrosophic vector space, neutrosophic module, n-refined neutrosophic ring, and n-refined neutrosophic vector space. See [1,2,3,4,7,8,9].

Quadruple neutrosophic vector space is a logical space with some algebraic properties defined in [6], the most important question about this kind of spaces is the classification of them with respect to classical vector spaces.

In [6], Smarandache et al, proposed three open conjectures concerning the algebraic structure (classification structure) of neutrosophic quadruple vector spaces (NQ vector space). These conjectures can be described as follows:

Conjecture 1: Is the NQ vector space V defined over the field R isomorphic to $R \times R \times R \times R$?.

Conjecture 2: Is the NQ vector space V defined over the field C isomorphic to $C \times C \times C \times C$?.

Conjecture 3: Is the NQ vector space V defined over the field Z_p isomorphic to $Z_p \times Z_p \times Z_p \times Z_p$?.

In this work we prove that the answer of previous conjectures is yes. We use the properties of weak n-refined neutrosophic vector space $V_n(I)$ to reach our goal.

These three conjectures are called Smarandache's conjectures.

Our motivation to write this paper is to give a classification theorem for n-refined weak neutrosophic vector spaces, as well as to answer three conjectures proposed by Smarandache about classifying quadruple neutrosophic vector spaces. This work shows for the first time the algebraic connection between n-refined neutrosophic weak vector spaces and quadruple neutrosophic vector spaces.

2. Preliminares

Definition 2.1: [8]

Let (R,+,$\times$) be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define

R_n(I)={$a_0 + a_1 I + \cdots + a_n I_n$; $a_i \in R$} to be n-refined neutrosophic ring.

Definition 2.2 :[4]

Let (V , + , .) be a vector space over the field K then (V(I) , + ,.) is called a weak neutrosophic vector space over the field K , and it is called a strong neutrosophic vector space if it is a vector space over the neutrosophic field K(I).

Definition 2.3: [9]

Let (K,+,·) be a field, we say that $K_n(I) = K + KI_1 + \cdots + KI_n = \{a_0 + a_1 I_1 + \cdots + a_n I_n; a_i \in K\}$ is an n-refined neutrosophic field.

It is clear that $K_n(I)$ is an n-refined neutrosophic field, but not a field in the classical meaning.

Definition 2.4: [9]

Let (V,+,·) be a vector space over the field K. Then we say that

$V_n(I) = V + VI_1 + \cdots + VI_n = \{x_0 + x_1 I_1 + \cdots + x_n I_n; x_i \in V\}$ is a weak n-refined neutrosophic vector space over the field K. Elements of $V_n(I)$ are called n-refined neutrosophic vectors, elements of K are called scalars.

If we take scalars from the n-refined neutrosophic field $K_n(I)$, we say that $V_n(I)$ is a strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$. Elements of $K_n(I)$ are called n-refined neutrosophic scalars.

Definition 2.5: [6]

The quadruple $(a, bT, cI, dF); a, b, c, d \in R \text{ or } C \text{ or } Z_p$, with T,I,F as in classical Neutrosophic logic with a the known part and (bT, cI, dF) defined as the unknown part, denoted by NQ= $\{(a, bT, cI, dF) | a, b, c, d \in R \text{ or } C \text{ or } Z n\}$ in called the Neutrosophic set of quadruple numbers.

Remark 2.6: [6]

(NQ,+,.) is a vector space, where (.) is an external multiplication by a scalar from the same field that NQ is built over.

Open conjectures [6]

Conjecture 1: Is the NQ vector space V defined over the field R isomorphic to $R \times R \times R \times R$?.

Conjecture 2: Is the NQ vector space V defined over the field C isomorphic to $C \times C \times C \times C$?.

Conjecture 3: Is the NQ vector space V defined over the field Z_p isomorphic to $Z_p \times Z_p \times Z_p \times Z_p$?.

3. Main concepts and results

Lemma 3.1:

Let V be a vector space over the field K, $V_n(I)$, be the corresponding weak n-refined neutrosophic vector space. Then

(a) $V_{n-1}(I)$ is a homomorphic image of $V_n(I)$.

(b) $V_n(I)/W \cong V_{n-1}(I)$, where W is a subspace of $V_n(I)$ with property $W \cong V$.

(c) $V_m(I)$ is a homomorphic image of $V_n(I)$; $m \leq n$.

Proof:

(a) We define $f: V_n(I) \to V_{n-1}(I); f(a_0 + a_1 I_1 + \cdots + a_n I_n) = a_0 + a_1 I_1 + \cdots + (a_{n-1} + a_n) I_{n-1}$.

It is easy to see that f is well defined. Let $x = \sum_{i=0}^{n} a_i I_i, y = \sum_{i=0}^{n} b_i I_i$ be two arbitrary elements in $V_n(I)$, r be any element in the field K, we have:

$f(x+y) = (a_0+b_0)+(a_1+b_1)I_1+\cdots+(b_{n-1}+a_{n-1}+b_n+a_n)I_{n-1} = f(x)+f(y).$

$f(r.x) = f(\sum_{i=0}^{n} r.a_i I_i) = ra_0 + ra_1 I_a + \cdots + (ra_{n-1}+rb_{n-1}+ra_n+rb_n)I_{n-1} = r.f(x).$

Thus f is a homomorphism.

(b) $Ker(f) = \{x = \sum_{i=0}^{n} a_i I_i \in V_n(I); f(x) = 0\}$, this implies

$a_i = 0 \; for \; all \; 0 \le i \le n-2, a_{n-1} = -a_n$, so $W = Ker(f) = \{a_n(I_n - I_{n-1});\; a_n \in V\}$.

Also, we have $V_n(I)/W \cong V_{n-1}(I)$. Now define $g: W \to V; g\left(a_n(I_n - I_{n-1})\right) = a_n$, g is a well defined map and it is an isomorphism, thus $W \cong V$.

(c) According to (a), we get a series of vector space homomorphisms

$V_n(I) \to_{f_n} V_{n-1}(I) \to_{f_{n-1}} \dots \to_{f_{n-m}} V_m(I)$, $f_n o f_{n-1} o \dots o f_{n-m}$ is a homomorphism since it is a product of homomorphisms. Thus our proof is complete.

Example 3.2:

Let $V = R^2$ be a vector space over the field R, $R_2^2(I) = \{a + bI_1 + cI_2; a, b, c \in V\}$ be the corresponding weak 2-refined neutrosophic vector space over R, $R_1^2(I) = \{a + bI_1; a, b \in V\}$ be the corresponding weak 1-refined neutrosophic vector space over R.

$f: R_2^2(I) \to R_1^2(I); f(a + bI_1 + cI_2) = a + (b+c)I_1$ is a homomorphism according to the previous theorem.

$Ker(f) = \{c(I_2 - I_1); c \in V\} \cong V.$

Theorem 3.3:

Let V be a vector space over the field K, V(I)=V+VI be the corresponding weak neutrosophic vector space over the field K, $V_n(I)$ be the corresponding weak n-refined neutrosophic vector space over K. Then:

(a) $V \times V \cong V(I)$.

(b) $V_n(I) \cong V \times V \times \dots \times V \; (n+1 \; times)$.

Proof:

(a) Define $f: V \times V \to V(I); f(x,y) = x + yI; x, y \in V$, it is clear that f is well defined bijective map.

Let $(x,y),(z,t) \in V \times V, r \in K$, we have $(x,y)+(z,t)=(x+z,y+t), r.(x,y)=(r.x,r.y)$,

$f[(x,y)+(z,t)]=(x+z)+(y+t)I=(x+yI)+(z+tI)=f(x,y)+f(z,t)$.

$f[r.(x,y)]=r.x+r.yI=r.(x+yI)=r.f(x)$. Hence f is a vector space isomorphism.

(b) Define $f:V_n(I) \to V \times V \times \dots \times V; f(x_0+x_1I_1+\cdots+x_nI_n)=(x_0,x_1,\dots,x_n)$.

By similar argument we find that f is an isomorphism.

Result 3.4:
(a) Theorem 3.3 clarifies that the concept of weak neutrosophic vector space is a rediscovering of direct product of a vector space with itself, thus all results in [4] can be obtained easily according to this result.
(b) Theorem 3.3 clarifies that the concept of weak n-refined neutrosophic vector space is a rediscovering of direct product of a vector space with itself n+1 times, thus the question about defining basis of this kind of vector spaces in [9], can be answered easily.
Theorem 3.5:

Let $V=(a,bT,cI,dF)$ be a quadruple neutrosophic vector space defined over the field F, i.e

$a,b,c,d \in F$, then V is isomorphic to weak 3-refined neutrosophic vector space $F_3(I)$ over the field F.
Proof:

We define $f:V \to F_3(I); f[(a,bT,cI,dF)]=a+bI_1+cI_2+dI_3$.

f is well defined clearly. Let $m=(a,bT,cI,dF), n=(x,yT,zI,tF)$ be two arbitrary elements in V,

where $a,b,c,d,x,y,z,t \in F$, we have:

$m+n=(a+x,[b+y]T,[c+z]I,[d+t]F)$,

$f(m+n)=(a+x)+(b+y)I_1+(c+z)I_2+(d+t)I_3=f(m)+f(n)$.

Let s be an arbitrary element in the field F, then $s.m=(s.a,s.bT,s.cI,s.dF)$,

$f(s.m)=s.a+(s.b)I_1+(s.c)I_2+(s.d)I_3=s.f(m)$.

It is easy to see that f is a bijective map, thus we get the proof.

Solving conjectures

Theorem 3.6 :

Let $V = (a, bT, cI, dF)$ be a quadruple neutrosophic vector space defined over the field R, then V is isomorphic to $R \times R \times R \times R$.

Proof:

By Theorem 3.5 , we find that $V \cong R_3(I)$, by Theorem 3.3 , we find that $R_3(I) \cong R \times R \times R \times R$, thus $V \cong R \times R \times R \times R$.

Theorem 3.7:

Let $V = (a, bT, cI, dF)$ be a quadruple neutrosophic vector space defined over the field C, then V is isomorphic to $C \times C \times C \times C$.

Proof:

By Theorem 3.5 , we find that $V \cong C_3(I)$, by Theorem 3.3 , we find that $C_3(I) \cong C \times C \times C \times C$, thus $V \cong C \times C \times C \times C$.

Theorem 3.8:

Let $V = (a, bT, cI, dF)$ be a quadruple neutrosophic vector space defined over the field Z_p, then V is isomorphic to $Z_p \times Z_p \times Z_p \times Z_p$.

Proof:

By Theorem 3.5 , we find that $V \cong {Z_p}_3(I)$, by Theorem 3.3 , we find that ${Z_p}_3(I) \cong Z_p \times Z_p \times Z_p \times Z_p$, thus

$V \cong Z_p \times Z_p \times Z_p \times Z_p$.

5. Conclusion

In this paper we have answered Smarandache's conjectures in neutrosophic quadruple vector space (NQ vector space) by investigating the algebraic relations between n-refined neutrosophic vector spaces and quadruple neutrosophic vector spaces. Also, we have proved that every weak m-refined neutrosophic vector space is a homomorphic image of n-refined neutrosophic vector space, where m $\leq$ n.

Funding:: This research received no external funding

Conflicts of Interest: The authors declare no conflict of interest

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Received: June 20, 2020. Accepted: Nov 20, 2020