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Atiqe Ur Rahman

Muhammad Saeed

Florentin Smarandache

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Convex and Concave Hypersoft Sets with Some Properties

Atiqe Ur Rahman [1,*], Muhammad Saeed [2] and Florentin Smarandache [3]

[1] Department of Mathematics, University of Management and Techology, Lahore, Pakistan
aurkhb@gmail.com

[2] Department of Mathematics, University of Management and Techology, Lahore, Pakistan
muhammad.saeed@umt.edu.pk

[3] Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA
smarand@unm.edu

*Correspondence: aurkhb@gmail.com

Abstract. Convexity plays an imperative role in optimization, pattern classification and recognition, image processing and many other relating topics in different fields of mathematical sciences like operation research, numerical analysis etc. The concept of soft sets was first formulated by Molodtsov as a completely new mathematical tool for solving problems dealing with uncertainties. Smarandache conceptualized hypersoft set as a generalization of soft set (h_S, E) as it transforms the function h_S into a multi-attribute function h_{HS}. Deli introduced the concept of convexity cum concavity on soft sets to cover above topics under uncertain scenario. In this study, a theoretic and analytical approach is employed to develop a conceptual framework of convexity cum concavity on hypersoft set which is generalized and more effective concept to deal with optimization relating problems. Moreover, some generalized properties like δ-inclusion, intersection and union, are established. The novelty of this work is maintained with the help of illustrative examples and pictorial version first time in literature.

Keywords: Convex Soft Set; Concave Soft Set; hypersoft Set; convex hypersoft set; concave hypersoft set.

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1. Introduction

The theories like theory of probability, theory of fuzzy sets, and the interval mathematics, are considered as mathematical means to tackle many Intricate problems involving various uncertainties in different fields of mathematical sciences. These theories have their own complexities which restrain them to solve these problems successfully. The reason for these hurdles is, possibly, the inadequacy of the parametrization tool. A mathematical tool is needed for dealing with uncertainties which should be free of all such Impediments. In 1999, Molodtsov [1] has the honor to introduce the such mathematical tool called soft sets in literature as a new parameterized family of subsets of the universe of discourse. In 2003, Maji et al. [2] extended the concept and introduced some fundamental terminologies and operations like equality of two soft sets, subset and super set of a soft set, complement of a soft set, null soft set, absolute soft set, AND, OR and also the operations of union and intersection. They verified De Morgan's laws and a number of other results too. In 2005, Pei et al. [3] discussed the relationship between soft sets and information systems. They showed the soft sets as a class of special information systems. In 2009, Ali et al. [4] pointed several assertions in previous work of Maji et al. and defined new notions such as the restricted intersection, the restricted union, the restricted difference and the extended intersection of two soft sets. In 2010, 2011, Babitha et al. [5,6] introduced the concepts of soft set relations as a sub soft set of the Cartesian product of the soft sets and also discussed many related concepts such as equivalent soft set relation, partition, composition and function. In 2011, Sezgin et al. [7], Ge et al. [8], Fuli [9] gave some modifications in the work of Maji et al. and also established some new results. Many researchers [10]- [19] developed certain hybrids with soft sets to get more generalized results for implementation in decision making and other related disciplines.

In 2013, Deli [20] defined soft covex and soft concave sets with some properties. In 2016, Majeed [21] investigated some more properties of convex soft sets. She developed the convex hull and the cone of a soft set with their generalized results. In 2018, Salih et al. [22] defined strictly soft convex and strictly soft concave sets and they discussed their properties.

In 2018, Smarandache [23] introduced the concept of hypersoft set and in 2020, M. Saeed et al. [24] extended the concept and discussed the fundamentals of hypersoft set such as hypersoft subset, complement, not hypersoft set, aggregation operators along with hypersoft set relation, sub relation,complement relation, function, matrices and operations on hypersoft matrices.

Convexity is an essential concept in optimization, recognition and classification of certain patterns, processing and decomposition of images, antismatroids, discrete event simulation, duality problems and many other related topics in operation research, mathematical economics, numerical analysis and other mathematical sciences. Deli provided a mathematical tool to tackle all such problems under soft set environment. Hypersoft set theory is more generalized than soft set theory so it's the need of the literature to carve out a conceptual framework for solving such kind of problems under more generalized version i.e. hypersoft set. Therefore, to meet this demand, an abstract and analytical approach is utilized to develop a basic framework of convexity and concavity on hypersoft sets along with some important results. Examples and pictorial version of convexity and concavity on hypersoft sets are presented first time in literature.

The rest of this article is structured as follows: Section 2 recalls some basic definitions and terms from literature to support main results. Section 3 discusses the main results i.e. convex and concave hypersoft sets along with some generalized results. Section 4 concludes the paper and describes future directions. Throughout the paper, G , $J^{\bullet}$, II and $P(\text{II})$, will play the role of R^n, unit interval, universal set and power set respectively.

2. Preliminaries

In this section, some fundamental terms regarding soft set, hypersoft set and their convexity-cum-concavity are presented.

Definition 2.1. [1](**Soft Set**)
Let II be an initial universe set and let E be a set of parameters. A pair (h_S, E) is called a soft set over II, where h_S is a mapping given by $h_S : E \to P(U)$. In other words, a soft set (h_S, E) over II is a parameterized family of subsets of II. For $\omega \in E, h_S(\omega)$ may be considered as the set of ω-elements or ω-approximate elements of the soft set (h_S, E).

Definition 2.2. [2]
Let (f_S, A) and (g_S, B) be two soft sets over a common universe II,

(1) we say that (f_S, A) is a **soft subset** of (g_S, B) denoted by $(f_S, A) \subseteq (g_S, B)$ if
  i $A \subseteq B$, and
  ii $\forall \omega \in A, f_S(\omega)$ and $g_S(\omega)$ are identical approximations.

(2) the **union** of (f_S, A) and (g_S, B), denoted by $(f_S, A) \cup (g_S, B)$, is a soft set (h_S, C), where $C = A \cup B$ and $\omega \in C$,

$$h_S(\omega) = \begin{cases} f_S(\omega), & \omega \in A - B \\ g_S(\omega), & \omega \in B - A \\ f_S(\omega) \cup g_S(\omega), & \omega \in A \cap B \end{cases}$$

(3) the **intersection** of (f_S, A) and (g_S, B) denoted by $(f_S, A) \cap (g_S, B)$, is a soft set (h_S, C), where $C = A \cap B$ and $\omega \in C, h_S(\omega) = f_S(\omega)$ or $g_S(\omega)$ (as both are same set).

Definition 2.3. [2](**Complement of Soft Set**)
The complement of a soft set (h_S, A), denoted by $(h_S, A)^c$, is defined as $(h_S, A)^c = ({h_S}^c, \neg A)$ where

$${h_S}^c : \neg A \to P(\text{II})$$

is a mapping given by

$${h_S}^c(\omega) = \text{II} - h_S(\neg\omega) \forall \omega \in \neg A.$$

Definition 2.4. [23](**Hypersoft Set**)
Let II be a universe of discourse, $P(\text{II})$ the power set of II. Let $a_1, a_2, a_3, ..., a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $A_1, A_2, A_3, ..., A_n$, with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, 3, ..., n\}$. Then the pair

(h_{HS}, G), where $G = A_1 \times A_2 \times A_3 \times ..... \times A_n$ and $h_{HS} : G \to P(\text{II})$ is called a hypersoft Set over II.

Definition 2.5. [24](**Union of Hypersoft Sets**)
Let (Φ, G_1) and (Ψ, G_2) be two hypersoft sets over the same universal set II, then their union $(\Phi, G_1) \cup (\Psi, G_2)$ is hypersoft set (h_{HS}, C), where $C = G_1 \cup G_2$; $G_1 = A_1 \times A_2 \times A_3 \times ..... \times A_n$, $G_2 = B_1 \times B_2 \times B_3 \times ..... \times B_n$ and $\forall\, e \in C$ with

$$h_{HS}(e) = \begin{cases} \Phi(e)\ ,\ e \in G_1 - G_2 \\ \Psi(e)\, ,\ e \in G_2 - G_1 \\ \Phi(e) \cup \Psi(e)\, ,\ e \in G_2 \cap G_1 \end{cases}$$

Definition 2.6. [24](**Intersection of Hypersoft Sets**)
Let (Φ, G_1) and (Ψ, G_2) be two hypersoft sets over the same universal set II, then their intersection $(\Phi, G_1) \cap (\Psi, G_2)$ is hypersoft set (h_{HS}, C), where $C = G_1 \cap G_2$; where $G_1 = A_1 \times A_2 \times A_3 \times ..... \times A_n$, $G_2 = B_1 \times B_2 \times B_3 \times ..... \times B_n$. and $\forall\, e \in C$ with $h_{HS}(e) = \Phi(e) \cap \Psi(e)$.

For more definition and results regarding hypersoft set, see [24–27].

Definition 2.7. [20](δ**-inclusion**)
The δ-inclusion of a soft set (h_S, Λ) (where $\delta \subseteq \text{II}$) is defined by

$$(h_S, \Lambda)^{\delta} = \{\omega \in \Lambda : h_S(\omega) \supseteq \delta\}$$

Definition 2.8. [20](**Convex Soft Set**)
The soft set (h_S, Λ) on Λ is called a convex soft set if

$$h_S(\epsilon\omega + (1 - \epsilon)\mu) \supseteq h_S(\omega) \cap h_S(\mu)$$

for every $\omega, \mu \in \Lambda$ and $\epsilon \in J^{\bullet}$.

Definition 2.9. [20](**Concave Soft Set**)
The soft set (h_S, Λ) on Λ is called a concave soft set if

$$h_S(\epsilon\omega + (1 - \epsilon)\mu) \subseteq h_S(\omega) \cup h_S(\mu)$$

for every $\omega, \mu \in \Lambda$ and $\epsilon \in J^{\bullet}$.

For more about convex soft, see [20, 21].

3. **Convex and Concave hypersoft sets**

Here convex hypersoft sets and concave hypersoft sets are defined and some important results are proved.

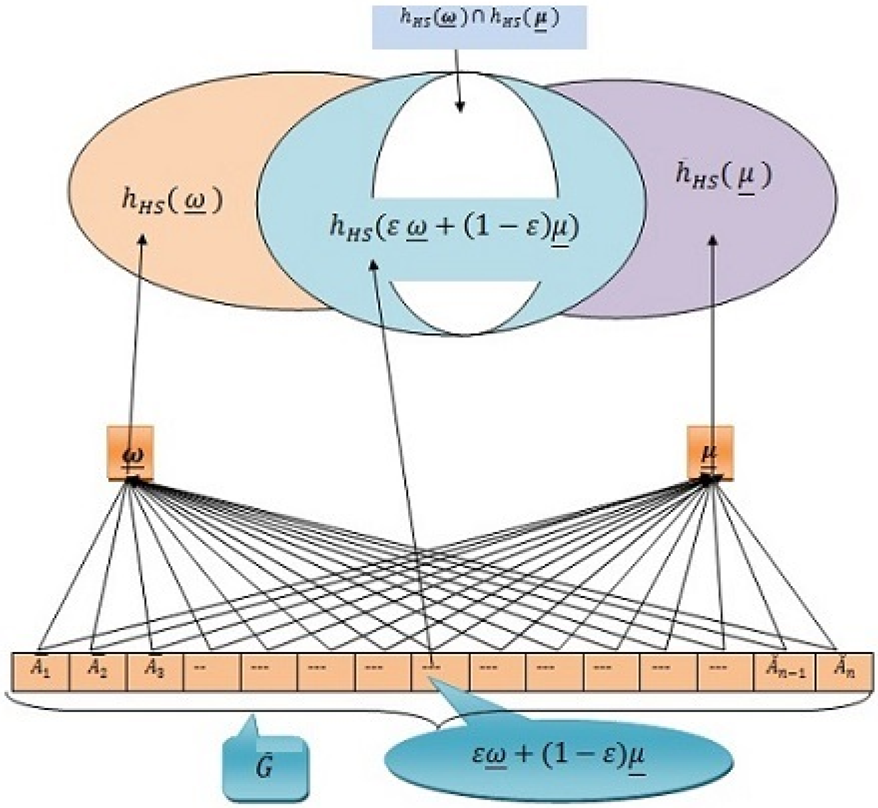

FIGURE 1. Convex hypersoft Set

Definition 3.1. δ-inclusion for hypersoft Set

The $\delta - inclusion$ of a hypersoft set (h_{HS}, G) (where $\delta \subseteq$II) is defined by

$$(h_{HS}, G)^{\delta} = \{\underline{\omega} \in G : h_{HS}(\underline{\omega}) \supseteq \delta\}$$

Definition 3.2. Convex hypersoft Set

The hypersoft set (h_{HS}, G) is called a convex hypersoft set if

$$h_{HS}\left(\epsilon\underline{\omega} + (1-\epsilon)\,\underline{\mu}\right) \supseteq h_{HS}(\underline{\omega}) \cap h_{HS}(\underline{\mu})$$

for every $\underline{\omega}$, $\underline{\mu} \in G$ where, $G = A_1 \times A_2 \times A_3 \times ..... \times A_n$ with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, 3, ..., n\}$; $h_{HS} : G \to P(\text{II})$ and $\epsilon \in J^{\bullet}$.

Example 3.3. Suppose a university wants to observe(evaluate) the characteristics of its teach ers by some defined indicators. For this purpose, consider a set of teachers as a universe of discourse II $= \{t_1, t_2, t_3, ..., t_{10}\}$. The attributes of the teachers under consideration are the set $\Lambda = \{A_1, A_2, A_3\}$, where

A_1 = Total experience in years

A_2 = Total no. of publications

$A_3 =$ Student's evaluation against each teacher
such that the attributes values against these attributes respectively are the sets given as
$A_1 = \{1year, 2years, 3years, 4years, 5years\}$
$A_2 = \{1, 2, 3, 4, 5\}$
$A_3 = \{Excellent(1), verygood(2), good(3), average(4), bad(5)\}$
For simplicity, we write
$A_1 = \{1, 2, 3, 4, 5\}$
$A_2 = \{1, 2, 3, 4, 5\}$
$A_3 = \{1, 2, 3, 4, 5\}$
The hypersoft set (h_{HS}, G) is a function defined by the mapping $h_{HS} : G \to P(\amalg)$ where $G = A_1 \times A_2 \times A_3$.
Since the cartesian product of $A_1 \times A_2 \times A_3$ is a 3-tuple. we consider $\underline{\omega} = (2, 1, 3)$, then the function becomes $h_{HS}(\underline{\omega}) = h_{HS}(2, 1, 3) = \{t_1, t_5\}$. Also, consider $\underline{\mu} = (3, 2, 2)$, then the function becomes $h_{HS}(\underline{\mu}) = h_{HS}(3, 2, 2) = \{t_1, t_3, t_4\}$
Now

$$h_{HS}(\underline{\omega}) \cap h_{HS}(\underline{\mu}) = h_{HS}(\{2, 1, 3\}) \cap h_{HS}(\{3, 2, 2\}) = \{t_1, t_5\} \cap \{t_1, t_3, t_4\} = \{t_1\} \tag{1}$$

Let $\epsilon = 0.6 \in J^\bullet$, then, we have
$\epsilon\underline{\omega} + (1 - \epsilon)\underline{\mu} = 0.6(2, 1, 3) + (10.6)(3, 2, 2) = 0.6(2, 1, 3) + 0.4(3, 2, 2)$
$= (1.2, 0.6, 1.8) + (1.2, 0.8, 0.8) = (1.2 + 1.2, 0.6 + 0.8, 1.8 + 0.8) = (2.4, 1.4, 2.6)$
which is again a 3-tuple. By using the decimal round off property, we get $(2, 1, 3)$

$$h_{HS}(\epsilon\underline{\omega} + (1 - \epsilon)\underline{\mu}) = h_{HS}(2, 1, 3) = \{t_1, t_5\} \tag{2}$$

it is vivid from equations (1) and (2), we have

$$h_{HS}\left(\epsilon\underline{\omega} + (1 - \epsilon)\,\underline{\mu}\right) \supseteq h_{HS}\left(\underline{\omega}\right) \cap h_{HS}\left(\underline{\mu}\right)$$

Theorem 3.4. *$(f_{HS}, S) \cap (g_{HS}, T)$ is a convex hypersoft set when both (f_{HS}, S) and (g_{HS}, T) are convex hypersoft sets.*

Proof. Suppose that $(f_{HS}, S) \cap (g_{HS}, T) = (h_{HS}, G)$ with $G = S \cap T$, for $\underline{\omega}_1$, $\underline{\omega}_2 \in G; \epsilon \in J^\bullet$, we have then

$$h_{HS}\left(\epsilon\underline{\omega}_1 + (1 - \epsilon)\underline{\omega}_2\right) = f_{HS}\left(\epsilon\underline{\omega}_1 + (1 - \epsilon)\underline{\omega}_2\right) \cap g_{HS}\left(\epsilon\underline{\omega}_1 + (1 - \epsilon)\underline{\omega}_2\right)$$

As (f_{HS}, S) and (g_{HS}, T) are convex hypersoft sets,

$$f_{HS}\left(\epsilon\underline{\omega}_1 + (1 - \epsilon)\underline{\omega}_2\right) \supseteq f_{HS}\left(\underline{\omega}_1\right) \cap f_{HS}\left(\underline{\omega}_2\right)$$

$$g_{HS}\left(\epsilon\underline{\omega}_1 + (1 - \epsilon)\underline{\omega}_2\right) \supseteq g_{HS}\left(\underline{\omega}_1\right) \cap g_{HS}\left(\underline{\omega}_2\right)$$

which implies

$$h_{HS}\left(\epsilon\underline{\omega}_1+(1-\epsilon)\underline{\omega}_2\right)\supseteq\left(f_{HS}\left(\underline{\omega}_1\right)\cap f_{HS}\left(\underline{\omega}_2\right)\right)\cap\left(g_{HS}\left(\underline{\omega}_1\right)\cap g_{HS}\left(\underline{\omega}_2\right)\right)$$

and thus

$$h_{HS}\left(\epsilon\underline{\omega}_1+(1-\epsilon)\underline{\omega}_2\right)\supseteq h_{HS}\left(\underline{\omega}_1\right)\cap h_{HS}\left(\underline{\omega}_2\right)$$

□

Remark 3.5. If $\left\{({h^i}_{HS},G_i):i\in\{1,2,3,...\}\right\}$ is any family of convex hypersoft sets, then the intersection $\bigcap_{i\in I}({h^i}_{HS},G_i)$ is a convex hypersoft set.

Remark 3.6. The union of any family $\left\{({h^i}_{HS},G_i):i\in\{1,2,3,...\}\right\}$ of convex hypersoft sets is not necessarily a convex hypersoft set.

Theorem 3.7. *(h_{HS},G) is convex hypersoft set iff for every $\epsilon\in J^\bullet$ and $\delta\in P(\text{II}),(h_{HS},G)^\delta$ is convex hypersoft set.*

Proof. Suppose (h_{HS},G) is convex hypersoft set. If $\underline{\omega}$, $\underline{\mu}\in G$ and $\delta\in P(\text{II})$, then $h_{HS}\left(\underline{\omega}\right)\supseteq\delta$ and $h_{HS}\left(\underline{\mu}\right)\supseteq\delta$, it implies that $h_{HS}\left(\underline{\omega}\right)\cap h_{HS}\left(\underline{\mu}\right)\supseteq\delta$.
So we have,

$$h_{HS}\left(\epsilon\underline{\omega}+(1-\epsilon)\,\underline{\mu}\right)\supseteq h_{HS}\left(\underline{\omega}\right)\cap h_{HS}\left(\underline{\mu}\right)\supseteq\delta$$

$\Rightarrow\quad h_{HS}\left(\epsilon\underline{\omega}+(1-\epsilon)\,\underline{\mu}\right)\supseteq\delta$
thus $(h_{HS},G)^\delta$ is convex hypersoft set.
Conversely suppose that $(h_{HS},G)^\delta$ is convex hypersoft set for every $\epsilon\in J^\bullet$. For $\underline{\omega}$, $\underline{\mu}\in G$, $(h_{HS},G)^\delta$ is convex hypersoft set with $\delta=h_{HS}\left(\underline{\omega}\right)\cap h_{HS}\left(\underline{\mu}\right)$. Since $h_{HS}\left(\underline{\omega}\right)\supseteq\delta$ and $h_{HS}\left(\underline{\mu}\right)\supseteq\delta$, we have $\underline{\omega}\in(h_{HS},G)^\delta$ and $\underline{\mu}\in(h_{HS},G)^\delta$,
$\Rightarrow\quad\epsilon\underline{\omega}+(1-\epsilon)\,\underline{\mu}\in(h_{HS},G)^\delta$.
Therefore,

$$h_{HS}\left(\epsilon\underline{\omega}+(1-\epsilon)\,\underline{\mu}\right)\supseteq\delta$$

So

$$h_{HS}\left(\epsilon\underline{\omega}+(1-\epsilon)\,\underline{\mu}\right)\supseteq h_{HS}\left(\underline{\omega}\right)\cap h_{HS}\left(\underline{\mu}\right),$$

Hence (h_{HS},G) is convex hypersoft set. □

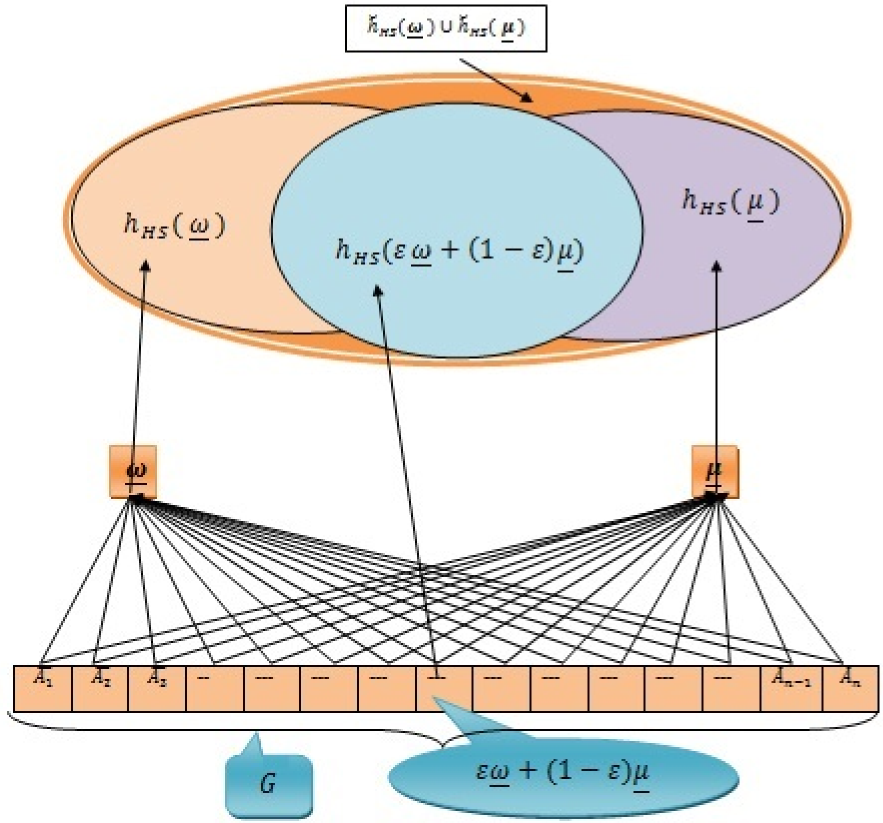

Figure 2. Concave hypersoft Set

Definition 3.8. Concave hypersoft Set

The hypersoft set (h_{HS}, G) on Λ is called a *concave hypersoft set* if

$$h_{HS}\left(\epsilon\underline{\omega} + (1-\epsilon)\,\underline{\mu}\right) \subseteq h_{HS}\left(\underline{\omega}\right) \cup h_{HS}\left(\underline{\mu}\right)$$

for every $\underline{\omega} = (A_1, A_2, A_3, ..., A_n)\,,\ \underline{\mu} = (B_1, B_2, B_3, ..., B_n) \in G$ where, $G = A_1 \times A_2 \times A_3 \times ..... \times A_n$ with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, 3, ..., n\}$; $h_{HS} : G \to P(\text{II})$ and $\epsilon \in J^{\bullet}$.

Example 3.9. Considering given data in Example 3.3, we have

$$h_{HS}(\underline{\omega}) \cup h_{HS}(\underline{\mu}) = h_{HS}(\{2,1,3\}) \cup h_{HS}(\{3,2,2\}) = \{t_1, t_5\} \cup \{t_1, t_3, t_4\} = \{t_1, t_3, t_4, t_5\} \quad (3)$$

it is vivid from equations (2) and (3), we have

$$h_{HS}\left(\epsilon\underline{\omega} + (1-\epsilon)\,\underline{\mu}\right) \subseteq h_{HS}\left(\underline{\omega}\right) \cup h_{HS}\left(\underline{\mu}\right)$$

Theorem 3.10. *$(f_{HS}, S) \cup (g_{HS}, T)$ is a concave hypersoft set when both (f_{HS}, S) and (g_{HS}, T) are concave hypersoft sets.*

Proof. Suppose that $(f_{HS}, S) \cup (g_{HS}, T) = (h_{HS}, G)$ with $G = S \cup T$, for $\underline{\omega}_1, \underline{\omega}_2 \in G; \epsilon \in J^{\bullet}$, we have then

$$h_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) = f_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \cup g_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right)$$

As (f_{HS}, S) and (g_{HS}, T) are concave hypersoft sets,

$$f_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \subseteq f_{HS}\left(\underline{\omega}_1\right) \cup f_{HS}\left(\underline{\omega}_2\right)$$

$$g_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \subseteq g_{HS}\left(\underline{\omega}_1\right) \cup g_{HS}\left(\underline{\omega}_2\right)$$

which implies

$$h_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \subseteq \left(f_{HS}\left(\underline{\omega}_1\right) \cup f_{HS}\left(\underline{\omega}_2\right)\right) \cup \left(g_{HS}\left(\underline{\omega}_1\right) \cup g_{HS}\left(\underline{\omega}_2\right)\right)$$

and thus

$$h_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \subseteq h_{HS}\left(\underline{\omega}_1\right) \cup h_{HS}\left(\underline{\omega}_2\right)$$

□

Remark 3.11. If $\left\{(\breve{h}^i{}_{HS}, G_i) : i \in \{1, 2, 3, ...\}\right\}$ is any family of concave hypersoft sets, then the union $\bigcup_{i \in I} (\breve{h}^i{}_{HS}, G_i)$ is a concave hypersoft set.

Theorem 3.12. *$(f_{HS}, S) \cap (g_{HS}, T)$ is a concave hypersoft set when both (f_{HS}, S) and (g_{HS}, T) are concave hypersoft sets.*

Proof. Suppose that $(f_{HS}, S) \cap (g_{HS}, T) = (h_{HS}, G)$ with $G = S \cap T$, for $\underline{\omega}_1, \underline{\omega}_2 \in G; \epsilon \in J^{\bullet}$, we have then

$$h_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) = f_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \cap g_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right)$$

As (f_{HS}, S) and (g_{HS}, T) are concave hypersoft sets,

$$f_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \subseteq f_{HS}\left(\underline{\omega}_1\right) \cup f_{HS}\left(\underline{\omega}_2\right)$$

$$g_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \subseteq g_{HS}\left(\underline{\omega}_1\right) \cup g_{HS}\left(\underline{\omega}_2\right)$$

which implies

$$h_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \subseteq \left(f_{HS}\left(\underline{\omega}_1\right) \cup f_{HS}\left(\underline{\omega}_2\right)\right) \cap \left(g_{HS}\left(\underline{\omega}_1\right) \cup g_{HS}\left(\underline{\omega}_2\right)\right)$$

and thus

$$h_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \subseteq h_{HS}\left(\underline{\omega}_1\right) \cup h_{HS}\left(\underline{\omega}_2\right)$$

□

Remark 3.13. The intersection of any family $\left\{(\breve{h}^i{}_{HS}, G_i) : i \in \{1, 2, 3, ...\}\right\}$ of concave hypersoft sets is a concave hypersoft set.

Theorem 3.14. *$(h_{HS}, G)^c$ is a convex hypersoft set when (h_{HS}, G) is a concave hypersoft set.*

Proof. Suppose that for $\underline{\omega}_1, \underline{\omega}_2 \in G,\ \epsilon \in J^\bullet$ and (h_{HS}, G) be concave hypersoft set.
Since (h_{HS}, G) is concave hypersoft set,

$$h_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \subseteq h_{HS}\left(\underline{\omega}_1\right) \cup h_{HS}\left(\underline{\omega}_2\right)$$

or

$$\amalg \backslash\, h_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \supseteq \amalg \backslash \left\{h_{HS}\left(\underline{\omega}_1\right) \cup h_{HS}\left(\underline{\omega}_2\right)\right\}$$

If $h_{HS}\left(\underline{\omega}_1\right) \supset h_{HS}\left(\underline{\omega}_2\right)$ then $h_{HS}\left(\underline{\omega}_1\right) \cup h_{HS}\left(\underline{\omega}_2\right) = h_{HS}\left(\underline{\omega}_1\right)$ therefore,

$$\amalg \backslash\, h_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \supseteq \amalg \backslash\ h_{HS}\left(\underline{\omega}_1\right). \tag{4}$$

If $h_{HS}\left(\underline{\omega}_1\right) \subset h_{HS}\left(\underline{\omega}_2\right)$ then $h_{HS}\left(\underline{\omega}_1\right) \cup h_{HS}\left(\underline{\omega}_2\right) = h_{HS}\left(\underline{\omega}_2\right)$ therefore,

$$\amalg \backslash\, h_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \supseteq \amalg \backslash\ h_{HS}\left(\underline{\omega}_2\right). \tag{5}$$

From (4) and (5) , we have

$$\amalg \backslash\, h_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \supseteq \left(\amalg \backslash h_{HS}\left(\underline{\omega}_1\right)\right) \cap \left(\amalg \backslash h_M\left(\underline{\omega}_2\right)\right).$$

So, $(h_{HS}, G)^c$ is a convex hypersoft set. □

Theorem 3.15. *$(h_{HS}, G)^c$ is a concave hypersoft set when (h_{HS}, G) is a convex hypersoft set.*

Proof. Suppose that for $\underline{\omega}_1, \underline{\omega}_2 \in G,\ \epsilon \in J^\bullet$ and (h_{HS}, G) be convex hypersoft set.
since (h_{HS}, G) is convex hypersoft set,

$$h_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \supseteq h_{HS}\left(\underline{\omega}_1\right) \cap h_{HS}\left(\underline{\omega}_2\right)$$

or

$$\amalg \backslash\, h_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \subseteq \amalg \backslash \left\{h_{HS}\left(\underline{\omega}_1\right) \cap h_{HS}\left(\underline{\omega}_2\right)\right\}$$

If $h_{HS}\left(\underline{\omega}_1\right) \supset h_{HS}\left(\underline{\omega}_2\right)$ then $h_{HS}\left(\underline{\omega}_1\right) \cap h_{HS}\left(\underline{\omega}_2\right) = h_{HS}\left(\underline{\omega}_2\right)$ therefore,

$$\amalg \backslash\, h_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \subseteq \amalg \backslash\ h_{HS}\left(\underline{\omega}_2\right). \tag{6}$$

If $h_{HS}\left(\underline{\omega}_1\right) \subset h_{HS}\left(\underline{\omega}_2\right)$ then $h_{HS}\left(\underline{\omega}_1\right) \cap h_{HS}\left(\underline{\omega}_2\right) = h_{HS}\left(\underline{\omega}_1\right)$ therefore,

$$\amalg \backslash\, h_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \subseteq \amalg \backslash\ h_{HS}\left(\underline{\omega}_1\right). \tag{7}$$

From (6) and (7) , we have

$$\amalg \backslash\, h_{HS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \supseteq \left(\amalg \backslash h_{HS}\left(\underline{\omega}_1\right)\right) \cup \left(\amalg \backslash h_M\left(\underline{\omega}_2\right)\right).$$

So $(h_{HS}, G)^c$ is a concave hypersoft set. □

Theorem 3.16. *(h_{HS}, G) is concave hypersoft set iff for every $\epsilon \in J^\bullet$ and $\delta \in P(\amalg), (h_{HS}, G)^\delta$ is concave hypersoft set.*

Proof. Suppose (h_{HS}, G) is concave hypersoft set. If $\underline{\omega}$, $\underline{\mu} \in G$ and $\delta \in P(\Pi)$, then $h_{HS}(\underline{\omega}) \supseteq \delta$ and $h_{HS}(\underline{\mu}) \supseteq \delta$, it implies that $h_{HS}(\underline{\omega}) \cup h_{HS}(\underline{\mu}) \supseteq \delta$.
So we have,

$$\delta \subseteq h_{HS}(\underline{\omega}) \cap h_{HS}(\underline{\mu}) \subseteq h_{HS}(\epsilon\underline{\omega} + (1-\epsilon)\underline{\mu}) \subseteq h_{HS}(\underline{\omega}) \cup h_{HS}(\underline{\mu})$$

$\Rightarrow \quad \delta \subseteq h_{HS}(\epsilon\underline{\omega} + (1-\epsilon)\underline{\mu})$
thus $(h_{HS}, G)^{\delta}$ is concave hypersoft set.
Conversely suppose that $(h_{HS}, G)^{\delta}$ is concave hypersoft set for every $\epsilon \in J^{\bullet}$. For $\underline{\omega}$, $\underline{\mu} \in G$, $(h_{HS}, G)^{\delta}$ is concave hypersoft set with $\delta = h_{HS}(\underline{\omega}) \cup h_{HS}(\underline{\mu})$. Since $h_{HS}(\underline{\omega}) \subseteq \delta$ and $h_{HS}(\underline{\mu}) \subseteq \delta$, we have $\underline{\omega} \in (h_{HS}, G)^{\delta}$ and $\underline{\mu} \in (h_{HS}, G)^{\delta}$,
$\Rightarrow \quad \epsilon\underline{\omega} + (1-\epsilon)\underline{\mu} \in (h_{HS}, G)^{\delta}$.
Therefore,

$$h_{HS}(\epsilon\underline{\omega} + (1-\epsilon)\underline{\mu}) \subseteq \delta$$

So

$$h_{HS}(\epsilon\underline{\omega} + (1-\epsilon)\underline{\mu}) \subseteq h_{HS}(\underline{\omega}) \cup h_{HS}(\underline{\mu}),$$

Hence (h_{HS}, G) is concave hypersoft set. □

4. **Conclusion**

In this study, convexity cum concavity on hypersoft sets, is conceptualized by adopting an abstract and analytical technique. This is novel addition in the literature and may enable the researchers to deal important applications of convexity under hypersoft environment with precise results. Moreover, some important results are established. Future work may include the introduction of strictly and strongly conexity cum concavity, convex hull, convex cone and many other types of convexity like (m, n)-convexity, ϕ-convexity, graded convexity, triangular convexity, concavoconvexity etc. on hypersoft set. It may also include the extension of this work by considering the modified versions of complement, intersection and union as discussed in [2]- [4].

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