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Neutrosophic $\underline{\mu}$-Topological spaces

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Abstract. In this paper, the concept of neutrosophic $\underline{\mu}-$topological spaces is introduced. We define and study the properties of neutrosophic $\underline{\mu}-$open sets, $\underline{\mu}-$closed sets, $\underline{\mu}-$interior and $\underline{\mu}-$closure. The set of all generalize neutrosophic pre-closed sets $GNPC(\tau)$ and the set of all neutrosophic α-open sets in a neutrosophic topological space (X,τ) can be considered as examples of generalized neutrosophic $\underline{\mu}-$topological spaces. The concept of neutrosophic $\underline{\mu}-continuity$ is defined and we studied their properties. We define and study the properties of neutrosophic $\underline{\mu}-compact$, $\underline{\mu}$-Lindelöf and $\underline{\mu}$-*countably compact* spaces. We prove that for a countable neutrosophic $\underline{\mu}$-space X: $\underline{\mu}$-*countably compactness* and $\underline{\mu}$-*compactness* are equivalent. We give an example of a neutrosophic $\underline{\mu}$-space X which has a neutrosophic countable $\underline{\mu}$-base but it is not neutrosophic $\underline{\mu}$-countably compact .

neutrosophic $\underline{\mu}-$topological spaces;$\underline{\mu}-$open; $\underline{\mu}-$closed; $\underline{\mu}-$interior; $\underline{\mu}-$closure; generalize neutrosophic pre-closed sets; neutrosophic α-open sets; neutrosophic $\underline{\mu}-continuity$; neutrosophic $\mu-compact$; neutrosophic $\underline{\mu}$-Lindelöf; neutrosophic $\underline{\mu}$-*countably compact space.*

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1. Introduction

The fuzzy set was introduced by Zadeh [24] in 1965, where each element had a degree of membership. The intuitionstic fuzzy set (Ifs for short) on a universe X was introduced by K. Atanassov [10–12] in 1983 as a generalization of fuzzy set, where besides the degree of membership we have the degree of non- membership of each element. The concept of neutrosophic sets first introduced by Smarandache [19, 22] as a generalization of intuitionistic fuzzy sets, where we have the degree of membership, the degree of indeterminacy and the degree of non-membership of each element in X. After the introduction of the neutrosophic sets, neutrosophic set operations have been investigated. Many researchers have studied topology on neutrosophic sets, such as Smarandache [22], Lupianez [15, 16] and Salama [17]. The neutrosophic interior, neutrosophic closure, neutrosophic exterior, neutrosophic boundary and neutrosophic subspace can be found in [20]. Neutrosophy has many applications specially in decision making, for more details about new trends of neutrosophic applications one can consult [1], [2], [3] and [4].

Definition 1.1. [19]: A *neutrosophic set* A on the universe of discourse X is defined as $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle; x \in X\}$ where $\mu, \sigma, \nu : X \to]^-0, 1^+[$ and $^-0 \leq \mu(x) + \sigma(x) + \nu(x) \leq 3^+$.

The class of all neutrosophic set on X will be denoted by $\mathcal{N}(X)$. We will exhibit the basic neutrosophic operations definitions (union, intersection and complement. Since there are different definitions of neutrosophic operations, we will organize the existing definitions into two types, in each type these operation will be consistent and functional.

Definition 1.2. [18][*Neutrosophic sets operations of Type.I*] Let $A, A_\alpha, B \in \mathcal{N}(X)$ such that $\alpha \in \Delta$. Then we define the neutrsophic:

(1) (*Inclusion*): $A \sqsubseteq B$ If $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \geq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.
(2) (*Equality*): $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.
(3) (*Intersection*) $\underset{\alpha\in\Delta}{\sqcap} A_\alpha(x) = \{\langle x, \underset{\alpha\in\Delta}{\wedge} \mu_{A_\alpha}(x), \underset{\alpha\in\Delta}{\vee} \sigma_A(x), \underset{\alpha\in\Delta}{\vee} \nu_A(x)\rangle; x \in X\}$.
(4) (*Union*) $\underset{\alpha\in\Delta}{\sqcup} A_\alpha(x) = \{\langle x, \underset{\alpha\in\Delta}{\vee} \mu_{A_\alpha}(x), \underset{\alpha\in\Delta}{\wedge} \sigma_A(x), \underset{\alpha\in\Delta}{\wedge} \nu_A(x)\rangle; x \in X\}$.
(5) (*Complement*) $A^c = \{\langle x, \nu_A(x), 1 - \sigma_A(x), \mu_A(x)\rangle; x \in X\}$
(6) (*Universal set*) $1_X = \{\langle x, 1, 0, 0\rangle; x \in X\}$; will be called the *neutrosophic universal set.*
(7) (*Empty set*) $0_X = \{\langle x, 0, 1, 1\rangle; x \in X\}$; will be called the *neutrosophic empty set.*

Definition 1.3. [18][*Neutrosophic sets operations of Type.II*] Let $A, A_\alpha, B \in \mathcal{N}(X)$ for every $\alpha \in \Delta$. Then we define the neutrsophic:

(1) (*Inclusion*): $A \sqsubseteq B$ If $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \leq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.
(2) (*Equality*): $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.
(3) (*Intersection*) $\underset{\alpha\in\Delta}{\sqcap} A_\alpha(x) = \{\langle x, \underset{\alpha\in\Delta}{\wedge} \mu_{A_\alpha}(x), \underset{\alpha\in\Delta}{\wedge} \sigma_A(x), \underset{\alpha\in\Delta}{\vee} \nu_A(x)\rangle; x \in X\}$.
(4) (*Union*) $\underset{\alpha\in\Delta}{\sqcup} A_\alpha(x) = \{\langle x, \underset{\alpha\in\Delta}{\vee} \mu_{A_\alpha}(x), \underset{\alpha\in\Delta}{\vee} \sigma_A(x), \underset{\alpha\in\Delta}{\wedge} \nu_A(x)\rangle; x \in X\}$.
(5) (*Complement*) $A^c = \{\langle x, \nu_A(x), 1 - \sigma_A(x), \mu_A(x)\rangle; x \in X\}$
(6) (*Universal set*) $1_X = \{\langle x, 1, 1, 0\rangle; x \in X\}$; will be called the *neutrosophic universal set.*
(7) (*Empty set*) $0_X = \{\langle x, 0, 0, 1\rangle; x \in X\}$; will be called the *neutrosophic empty set.*

Proposition 1.4. *[18] For any $A, B, C \in \mathcal{N}(X)$ we have:*

(1) $A \sqcap A = A$, $A \sqcup A = A$, $A \sqcap 0_X = 0_X$, $A \sqcup 0_X = A$, $A \sqcap 1_X = A$, $A \sqcup 1_X = 1_X$.
(2) $A \sqcap (B \sqcap C) = (A \sqcap B) \sqcap C$ *and* $A \sqcup (B \sqcup C) = (A \sqcup B) \sqcup C$.
(3) $A \sqcap (\underset{\alpha\in\Delta}{\sqcup} A_\alpha) = \underset{\alpha\in\Delta}{\sqcup} (A \sqcap A_\alpha)$.
(4) $A \sqcup (\underset{\alpha\in\Delta}{\sqcap} A_\alpha) = \underset{\alpha\in\Delta}{\sqcap} (A \sqcup A_\alpha)$.
(5) $(A^c)^c = A$.

(6) De Morgan's law*:*

(a) $(\underset{\alpha\in\Delta}{\sqcap} A_\alpha)^c = \underset{\alpha\in\Delta}{\sqcup} A_\alpha^c$.

(b) $(\underset{\alpha\in\Delta}{\sqcup} A_\alpha)^c = \underset{\alpha\in\Delta}{\sqcap} A_\alpha^c$.

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Definition 1.5. [18] [*Neutrosophic Topology*] Let $\tau \subseteq \mathcal{N}(X)$. Then τ is called a *neutrosophic topology* on X if

(1) $0_X, 1_X \in \tau$.

(2) The union of any number of neutrosophic sets in τ belongs to τ,

(3) The intersection of two neutrosophic sets in τ belongs to τ.

The pair (X, τ) is called a *neutrosophic topological space* over X. Moreover, the members of τ are said to be *neutrosophic open sets* in X. For any $A \in \mathcal{N}(X)$, If $A^c \in \tau$, then A is said to be *neutrosophic closed set* in X.

Definition 1.6. [20][*Neutrosophic interior*] Let (X, τ) be a neutrosophic topological space over X and $A \in \mathcal{N}(X)$. Then, the *neutrosophic interior* of A, denoted by $int(A)$ is the union of all neutrosophic open subsets of A.

Clearly that $int(A)$ is the biggest neutrosophic open set over X which containing A.

Theorem 1.7. *[20] Let (X, τ) be a neutrosophic topological space over X and $A, B \in \mathcal{N}(X)$. Then*

(1) $int(1_X) = 1_X$, $int(0_X) = 0_X$ *and* $int(A) \sqsubseteq A$.

(2) $int(int(A)) = int(A)$.

(3) $A \sqsubseteq B$ *implies* $int(A) \sqsubseteq int(B)$.

(4) $int(A \sqcap B) = int(A) \sqcap int(B)$.

Definition 1.8. [20][*Neutrosophic closure*] Let (X, τ) be a neutrosophic topological space over X and $A \in \mathcal{N}(X)$. Then, the *neutrosophic closure* of A, denoted by $cl(A)$ is the intersection of all neutrosophic closed super sets of A.

Clearly, $cl(A)$ is the smallest neutrosophic closed set over X which contains A.

Theorem 1.9. *[20] Let (X, τ) be a neutrosophic topological space over X and $A, B \in \mathcal{N}(X)$. Then,*

(1) $cl(1_X) = 1_X$, $cl(0_X) = 0_X$ *and* $A \sqsubseteq cl(A)$.

(2) $cl(cl(A)) = cl(A)$.

(3) $A \sqsubseteq B$ *implies* $cl(A) \sqsubseteq cl(B)$.

(4) $cl(A \sqcup B) = cl(A) \sqcup cl(B)$.

Definition 1.10. [5][*Neutrosophic pre-open and pre-closed*] Let (X, τ) be a neutrosophic topological space over X and $A \in \mathcal{N}(X)$. Then A is said to be *neutrosophic pre-open set* (NPOS), if $A \subseteq Int(Cl(A))$. The complement of a neutrosophic pre-open set is called *neutrosophi pre-closed set (NPCS).*

Definition 1.11. [6][*Neutrosophic α-open*] Let (X, τ) be a neutrosophic topological space over X and $A \in \mathcal{N}(X)$. A is said to be an *α-open set*, if $A \subseteq Int(Cl(Int(A))$. The set of all neutrosophic α-open sets in (X, τ) will be denoted by $N\alpha - O(\tau)$.

Definition 1.12. [5][*Neutrosophic pre-closure*] Let (X, τ) be a neutrosophic topological space over X and $A \in \mathcal{N}(X)$. The *neutrosophic pre-closure* of A, denoted by $pNCL(A)$ is the intersection of all neutrosophic pre-closed super sets of A.

Definition 1.13. [5][*Generalized Neutrosophic pre-closed sets*] Let (X, τ) be a neutrosophic topological space over X and $A \in \mathcal{N}(X)$. A is said to be a *neutrosophic generalized pre-closed set* (GNPCS) in (X, τ) if $pNCL(A) \subseteq B$ whenever $A \subseteq B$ and B is neutrosophic open. The set of all generalized neutrosophic pre-closed sets in (X, τ) will be denoted by $GNPC(\tau)$.

Theorem 1.14. *[5, 6] Let (X, τ) be a neutrosophic topological space over X. Then*

(1) *The union of any collection of α-open sets is an α-open set.*
(2) *The union of any collection of $GNPCs$ is $GNPC$.*

The following is an improvement of a definition in [14] makes it suitable for type.I and type.II neutrosophic sets.

Definition 1.15. Let X and Y be two nonempty sets and $\Omega : X \to Y$ be any function. Then for any netrosophic sets $A \in \mathcal{N}(X)$ and $B \in \mathcal{N}(Y)$ we have:

(1) The *Type.I(Type.II) pre-image* of B under Ω, denoted by $\Omega^{-1}(B)$, is the Neutrosophic set in X defined by

$$\Omega^{-1}(B) = \{\langle x, \mu_B(\Omega(x)), \sigma_B(\Omega(x)), \nu_B(\Omega(x))\rangle; x \in X\}$$

(2) *The Type.I (Type.II) image* of A under Ω, denoted by $\Omega(A)$, is the Neutrosophic set in Y defined by

$$\Omega(A) = \{\langle y, \Omega(\mu_A)(y), \Omega(\sigma_A)(y), (1 - \Omega(1 - \nu_A))(y)\rangle; y \in Y\} \text{ where}$$

$$(\mu_A)(y) = \begin{cases} \sup\limits_{x \in \Omega^{-1}(y)} \mu_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\ 0 & \text{if } \Omega^{-1}(y) = \emptyset \end{cases}$$

$$(\sigma_A)(y) = \begin{cases} \inf\limits_{x \in \Omega^{-1}(y)} \sigma_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\ 1 & \text{if } \Omega^{-1}(y) = \emptyset \end{cases} \qquad (Type.I)$$

$$(\sigma_A)(y) = \begin{cases} \sup\limits_{x\in\Omega^{-1}(y)} \sigma_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\ 0 & \text{if } \Omega^{-1}(y) = \emptyset \end{cases} \qquad (Type.II)$$

$$(1-\Omega(1-\nu_A))(y) = \begin{cases} \inf\limits_{x\in\Omega^{-1}(y)} \nu_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\ 1 & \text{if } \Omega^{-1}(y) = \emptyset \end{cases}$$

For the sake of simplicity we write $\Omega - (\nu_A$ instead of $(1 - \Omega(1 - \nu_A))$.

Note that the only difference between Type.I and Type.II images lies in the definition of the image of σ and this is important to make sure both Type.I and Type.II neutrosophic functions satisfy the following proposition.

Proposition 1.16. *[14] Let X and Y be two nonempty sets and $\Omega : X \to Y$ be any function. Let $A, A_\alpha \in \mathcal{N}(X)$ and $B, B_\alpha \in \mathcal{N}(Y)$. Then we have:*

(1) $A_1 \sqsubseteq A_2 \Rightarrow \Omega(A_1) \sqsubseteq \Omega(A_2)$.
(2) $B_1 \sqsubseteq B_2 \Rightarrow \Omega^{-1}(B_1) \sqsubseteq \Omega^{-1}(B_2)$.
(3) $A \sqsubseteq \Omega^{-1}(\Omega(A))$ *and equality holds if* Ω *is injective.*
(4) $\Omega(\Omega^{-1}(A)) \sqsubseteq A$ *and equality holds if* Ω *is surjective.*
(5) $\Omega(\bigsqcup\limits_{\alpha\in\Delta} A_\alpha) = \bigsqcup\limits_{\alpha\in\Delta} \Omega(A_\alpha)$.
(6) $\Omega(\bigsqcap\limits_{\alpha\in\Delta} A_\alpha) \sqsubseteq \bigsqcap\limits_{\alpha\in\Delta} \Omega(A_\alpha)$ *and equality holds when* Ω *is injective.*
(7) $\Omega^{-1}(\bigsqcup\limits_{\alpha\in\Delta} B_\alpha) = \bigsqcup\limits_{\alpha\in\Delta} \Omega^{-1}(B_\alpha)$.
(8) $\Omega^{-1}(\bigsqcap\limits_{\alpha\in\Delta} B_\alpha) = \bigsqcap\limits_{\alpha\in\Delta} \Omega^{-1}(B_\alpha)$.
(9) $\Omega^{-1}(1_N) = 1_N$, $\Omega^{-1}(0_N) = 0_N$.
(10) $\Omega(1_N) = 1_N$ *and* $\Omega(0_N) = 0_N$, *whenever* Ω *is surjective.*

Definition 1.17. Let X be a nonempty set and $0 < \alpha, \beta, \gamma < 1$. Then a neutrosophic set $A \in \mathcal{N}(X)$ is called:

(1) A *neutrosophic point of Type.I* if and only if there exists $x \in X$ such that $A = \{\langle x, \alpha, \beta, \gamma\rangle\} \cup \{\langle \acute{x}, 0, 1, 1\rangle; \acute{x} \neq x\}$.
(2) A *neutrosophic point of Type.II* if $A = \{\langle x, \alpha, \beta, \gamma\rangle\} \cup \{\langle \acute{x}, 0, 0, 1\rangle; \acute{x} \neq x\}$. Neutrosophic points will be denoted by $x_{\alpha,\beta,\gamma}$.

Now, we will exhibit some definitions and properties of μ-topological spaces. Á. Császár [13] introduced the notion of Generalized Topological Space (GTS). He also introduced the notion of

(μ_1;μ_2)-continuous function on GTS's. μ-compactness introduced in [23] and [21]. Countbaly $\mu-$paracompact introduced and studied in [8]. Strongly Generalized neighborhood systems introduced and studies in [9].

Let X be a nonempty set. A collection μ of subsets of X is called a generalized topology on X and the pair (X,μ) is called a generalized topological space, if μ satisfies the following two conditions:

(1) $\emptyset \in \mu$.

(2) Any union of elements of μ belongs to μ.

Let $\beta \subseteq \exp(X)$ and $\emptyset \in \beta$. Then β is called a $\mu - base$ for μ if $\mu = \{\bigcup \beta'; \beta' \subset \beta\}$,We also say μ is generated by β. If β is countable, then it said a countable $\mu - base$. A generalized topological space (X,μ) is said to be strong if $X \in \mu$. A subset B of X is called μ-open (resp. μ-closed) if $B \in \mu$ (resp. if $X - B \in \mu$). The set of all μ-open sets containing a point $x \in X$ will be denoted by μ_x (i.e. $\mu_x = \{U \in \mu; x \in U\}$).

Definition 1.18. Let (X,μ_1) and (X,μ_2) be two μ-topological space. A function $f:(X,\mu_1) \to (X,\mu_2)$ is said to be $(\mu_1,\mu_2) - continuous$ if and only if $f^{-1}(V) \in \mu_1$ whenever $V \in \mu_2$.

Definition 1.19. Let X be a generalized topological space and let $\mathfrak{F}$ be a collection of subsets of X. Then $\mathfrak{F}$ is said to be:

(1) A μ-cover of X if $X = \bigcup\{U; U \in \mathfrak{F}\}$.

(2) A μ-open cover of X if $\mathfrak{F}$ is a μ-cover of X and $U \in \mu$ for every $U \in \mathfrak{F}$.

Definition 1.20. Let X be a generalized topological space and let $\mathfrak{F}$ and $\mathfrak{C}$ be μ-covers of X. Then $\mathfrak{C}$ is said to be a μ-subcover of $\mathfrak{F}$, if $\mathfrak{C} \subseteq \mathfrak{F}$.

Definition 1.21. A generalized topological space X is said to be μ-compact (resp. μ-Lindelöf) if and only if every μ-open cover of X has a finite (resp. countable) μ-subcover.

The following theorem shows some differences between topological spaces and $\mu-$topological spaces.

Theorem 1.22.

(1) *In $\mu-$topological spaces $Int_\mu(A \cap B) = Int_\mu(A) \cap Int_\mu(B)$ is not satisfied where $Int_\mu(A)$ stands for interior of A.*

(2) *In $\mu-$topological spaces $Cl_\mu(A \cup B) = Cl_\mu(A) \cup Cl_\mu(B)$ is not satisfied where $Cl_\mu(A)$ stands for the closure of A in μ.*

(3) *[7] There exists a μ-normal space with a countable μ-base which has a μ-open cover with no μ-open point-finite refinement.*

2. Neutrosophic $\underline{\mu}$-Topological Spaces

In the literature of generlaized topological spaces the symbol μ is used to refer the $\mu-$topology and in neutrosophic sets it is used to refer the membership function μ, so, to avoid ambiguity, we will use the underlined $\underline{\mu}$ to refer the $\underline{\mu}$-topology and keep μ for the membership function in neutrosophic sets.

Definition 2.1 (*Neutrosophic $\underline{\mu}$-Topology*)**.** Let $\underline{\mu} \subseteq \mathcal{N}(X)$. Then $\underline{\mu}$ is called a *neutrosophic $\underline{\mu}$-topology* on X if

(1) $0_X \in \underline{\mu}$.
(2) The union of any number of neutrosophic sets in $\underline{\mu}$ belongs to $\underline{\mu}$.

The pair $(X, \underline{\mu})$ is called a *neutrosophic $\underline{\mu}$-topological space* over X. The members of $\underline{\mu}$ are said to be *neutrosophic $\underline{\mu}$-open sets* in X. If $1_X \in \underline{\mu}$, then $(X, \underline{\mu})$ is called a *strong neutrosophic $\underline{\mu}$-topological space*. For any $A \in \mathcal{N}(X)$, if $A^c \in \underline{\mu}$, then A is said to be *neutrosophic $\underline{\mu}$-closed set* in X. Since their are two types of neutrosophic sets, a neutrosophic $\underline{\mu}$-topology is said to be Type.I(Type.II) neutrosophic topology if its elements are treated as Type.I(Type.II) neutrosophic sets.

Example 2.2. Let $X = \{a, b, c\}$ and $A, B, C, \acute{C} \in \mathcal{N}(X)$ with:
$A = \{\langle a, 0.3, 0.5, 0.7\rangle, \langle b, 0.3, 0.4, 1\rangle\}$, $B = \{\langle a, 0.4, 0.7, 0.1\rangle, \langle b, 0.2, 0.6, 0.9\rangle\}$, $C = \{\langle a, 0.4, 0.5, 0.1\rangle, \langle b, 0.3, 0.4, 0.9\rangle\}$, $\acute{C} = \{\langle a, 0.4, 0.7, 0.1\rangle, \langle b, 0.3, 0.6, 0.9\rangle\}$. Then $\underline{\mu} = \{0_X, A, B, C\}$ is a Type.I neutrosophic $\underline{\mu} - topology$ and $\underline{\acute{\mu}} = \{1_X, 0_X, A, B, \acute{C}\}$ is a Type.II strong neutrosophic $mu - topology$. Neither $\underline{\mu}$ nor $\underline{\acute{\mu}}$ is neutrosophic topology. Note that ,in $(X, \underline{\mu})$, $A \sqcap B = \{\langle a, 0.3, 0.7, 0.7\rangle, \langle b, 0.2, 0.6, 1\rangle\}$ is not neutrosophic mu-open (here we apply type.I intersection). And in $(X, \underline{\acute{\mu}})$ we have $A \sqcap B = \{\langle a, 0.3, 0.5, 0.7\rangle, \langle b, 0.2, 0.4, 1\rangle\}$ is not neutrosophic mu-open (here we apply type.II intersection).

Most examples and theorems will be considered for Type.I neutrosophic sets, since the two types of neutrosophic sets have the same properties.

Definition 2.3 (*Neutrosophic $\underline{\mu}$-interior*)**.** Let $(X, \underline{\mu})$ be a neutrosophic topological space over X and $A \in \mathcal{N}(X)$. Then, the *neutrosophic $\underline{\mu}$-interior* of A, denoted by $int_{\underline{\mu}}(A)$ is the union of all neutrosophic $\underline{\mu}$-open subsets of A. Clearly $int_{\mu}(A)$ is the biggest neutrosophic $\underline{\mu}$-open set over X contained in A.

Theorem 2.4. *Let $(X, \underline{\mu})$ be a neutrosophic $\underline{\mu}$-topological space over X and $A, B \in \mathcal{N}(X)$. Then,*

(1) $int_{\underline{\mu}}(0_X) = 0_X$ *and* $int_{\mu}(A) \sqsubseteq A$.

(2) $int_{\underline{\mu}}(1_X) = 1_X$ *whenever* $\underline{\mu}$ *is a strong* $\underline{\mu}-$*topology.*

(3) $int_{\underline{\mu}}(int_{\underline{\mu}}(A)) = int_{\underline{\mu}}(A)$.

(4) $A \sqsubseteq B$ *implies* $int_{\underline{\mu}}(A) \sqsubseteq int_{\underline{\mu}}(B)$.

(5) $int_{\underline{\mu}}(A) = A$ *if and only if* $A \in \underline{\mu}$.

(6) *If* $A \sqsubseteq B$, *then* $int_{\underline{\mu}}(A) \sqsubseteq int_{\underline{\mu}}(B)$.

(7) $int_{\underline{\mu}}(A \sqcap B) \sqsubseteq int_{\underline{\mu}}(A) \sqcap int_{\underline{\mu}}(B)$. *Equality does not hold, see Example2.5.*

Proof. We will establish a proof for (4) and (7).

(4) Since $A \sqsubseteq B$, $\{U \in \underline{\mu}; U \sqsubseteq A\} \subseteq \{U \in \underline{\mu}; U \sqsubseteq B\}$. So that $\mu_{int_{\underline{\mu}}(A)}(x) = \sup\{\mu_U(x); U \in \underline{\mu}, U \sqsubseteq A\} \leq \sup\{\mu_U(x); U \in \underline{\mu}, U \sqsubseteq B\} = \mu_{int_{\underline{\mu}}(B)}(x)$, $\sigma_{int_{\underline{\mu}}(A)}(x) = \inf\{\sigma_U(x); U \in \underline{\mu}, U \sqsubseteq A\} \geq \inf\{\sigma_U(x); U \in \underline{\mu}, U \sqsubseteq B\} = \sigma_{int_{\underline{\mu}}(B)}(x)$, and $\nu_{int_{\underline{\mu}}(A)}(x) = \inf\{\nu_U(x); U \in \underline{\mu}, U \sqsubseteq A\} \geq \inf\{\nu_U(x); U \in \underline{\mu}, U \sqsubseteq B\} = \nu_{int_{\underline{\mu}}(B)}(x)$. Which means $int_{\underline{\mu}}(A) \sqsubseteq int_{\underline{\mu}}(B)$.

(7) Since $A \sqcap B \sqsubseteq A$ and B, $int_{\underline{\mu}}(A \sqcap B) \sqsubseteq int_{\underline{\mu}}(A)$ and $int_{\underline{\mu}}(A \sqcap B) \sqsubseteq int_{\underline{\mu}}(B)$ (by (4)), so we have $int_{\underline{\mu}}(A \sqcap B) \sqsubseteq int_{\underline{\mu}}(A) \sqcap int_{\underline{\mu}}(B)$. □

Example 2.5. Consider $(X, \underline{\mu})$ as in Example2.2. Note that:

(1) $int_{\underline{\mu}}(1_X) = 0_X \sqcup A \sqcup B \sqcup C = C \neq 1_X$.

(2) Since $A \sqcap B = \{\langle a, 0.3, 0.7, 0.7\rangle, \langle b, 0.2, 0.6, 1\rangle\}$ and there is no neutrosophic $\underline{\mu} - open$ set in $\underline{\mu}$ contained in $A \sqcap B$ except 0_X, we have $int_{\underline{\mu}}(A \sqcap B) = 0_X$, and since $A, B \in \underline{\mu}$, $int_{\underline{\mu}}(A) \sqcap int_{\underline{\mu}}(B) = A \sqcap B \neq int_{\underline{\mu}}(A \sqcap B) = 0_X$.

Definition 2.6 (*Neutrosophic* $\underline{\mu}$*-closure*)**.** Let $(X, \underline{\mu})$ be a neutrosophic $\underline{\mu}$-topological space over X and $A \in \mathcal{N}(X)$. Then, the *neutrosophic* $\underline{\mu}$*-closure* of A, denoted by $cl_{\underline{\mu}}(A)$, is the intersection of all neutrosophic $\underline{\mu}$-closed super sets of A.
Clearly $cl_{\underline{\mu}}(A)$ is the smallest neutrosophic $\underline{\mu}$-closed set over X which containing A.

Theorem 2.7. *Let* $(X, \underline{\mu})$ *be a neutrosophic* $\underline{\mu}$*-topological space over* X *and* $A, B \in \mathcal{N}(X)$. *Then,*

(1) $cl_{\underline{\mu}}(1_X) = 1_X$ *and* $A \sqsubseteq cl_{\underline{\mu}}(A)$.

(2) $cl_{\underline{\mu}}(0_X) = 0_X$ *whenever* $\underline{\mu}$ *is a strong* $\underline{\mu}-$*topology.*

(3) $cl_{\underline{\mu}}(cl_{\underline{\mu}}(A)) = cl_{\underline{\mu}}(A)$.

(4) $A \sqsubseteq B$ *implies* $cl_{\underline{\mu}}(A) \sqsubseteq cl_{\underline{\mu}}(B)$.

(5) A *is* $\underline{\mu}$*-closed if and only if* $cl_{\underline{\mu}}(A) = A$.

(6) $cl(A) \sqcup cl(B) \sqsubseteq cl(A \sqcup B)$. *The equality does not hold.*

Example 2.8. Consider $(X, \underline{\mu})$ as in Example2.2. The only $\underline{\mu}$-closed sets in $(X, \underline{\mu})$ are:

(1) $0_X^c = 1_X$.

(2) $A^c = \{\langle a, 0.7, 0.5, 0.3\rangle, \langle b, 1, 0.6, 0.3\rangle\}$.
(3) $B^c = \{\langle a, 0.1, 0.3, 0.4\rangle, \langle b, 0.9, 0.4, 0.2\rangle\}$.
(4) $C^c = \{\langle a, 0.1, 0.5, 0.4\rangle, \langle b, 0.9, 0.6, 0.3\rangle\}$

It is clear that $cl_{\underline{\mu}}(0_X) = 0_X^c \sqcap A^c \sqcap B^c \sqcap C^c = \{\langle a, 0.1, 0.5, 0.4\rangle, \langle b, 0.9, 0.6, 0.3\rangle\} \neq 0_X$. Let $H = A^c$ and $K = B^c$. Then $cl_{\underline{\mu}}(H) \sqcup cl_{\underline{\mu}}(K) = \{\langle a, 0.7, 0.3, 0.3\rangle, \langle b, 1, 0.4, 0.2\rangle\}$ and $cl_{\underline{\mu}}(H \sqcup K) = 1_X$, since the only neutrosophic $\underline{\mu}$-closed set containing $H \sqcup K = cl_{\underline{\mu}}(H) \sqcup cl_{\underline{\mu}}(K)$ is 1_X.

The following theorem shows the importance of generalized neutrosophic $\underline{\mu}$-topological spaces.

Theorem 2.9. *Let (X, τ) be a neutrosophic topological space over X. Then:*

(1) *The set $N\alpha - O(\tau)$ of all neutrosophic α-open sets over (X, τ) is a strong neutrosophic $\underline{\mu}$-topology over X.*
(2) *The set $GNPC(\tau)$ of all neutrosophic pre-closed sets in (X, τ) is a strong neutrosophic $\underline{\mu}$-topology over X.*

Proof. Easy! we just call Theorem1.14. □

Definition 2.10. Let $(X, \underline{\mu})$ and $(Y, \underline{\acute{\mu}})$ be two neutrosophic $\underline{\mu}$-topological spaces and let $\Omega : X \to Y$ be any function. Then Ω is said to be neutrosophic $(\underline{\mu}, \underline{\acute{\mu}})$-continuous if for any neutrosofpic point $x_{\alpha,\beta,\gamma}$ and for any neutrosophic $\underline{\acute{\mu}}$-open set $V \in \acute{\tau}$ such that $f(x_{\alpha,\beta,\gamma}) \in V$ there exists $U \in \tau$ such that $x_{\alpha,\beta,\gamma} \in U$ and $\Omega(U) \sqsubseteq V$.

Theorem 2.11. *Let X and Y be two nonempty sets and $\Omega : X \to Y$ be any function. Let $x_{\alpha,\beta,\gamma}$ be a neutrosophic point in X. Then $\Omega(x_{\alpha,\beta,\gamma}) = \Omega(x)_{\alpha,\beta,\gamma}$; that is the image of a neutrosophic point is a neutrosophic point.*

Proof. We will prove it for Type.I and Type.II neutrosophic sets. Let $A = x_{\alpha,\beta,\gamma}$ and $\Omega(x) = \acute{y}$. Then the Type.I (Type.II) image of A under Ω, denoted by $\Omega(A)$, is the Neutrosophic set:

$$\Omega(A) = \{\langle y, \Omega(\mu_A)(y), \Omega(\sigma_A)(y), (1 - \Omega(1 - \nu_A))(y)\rangle; y \in Y\}, \text{ where}$$

$$(\mu_A)(y) = \begin{cases} \sup\limits_{x \in \Omega^{-1}(y)} \mu_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\ 0 & \text{if } \Omega^{-1}(y) \neq \emptyset \end{cases} = \begin{cases} \alpha & \text{if } y = \acute{y} \\ 0 & \text{if } y \neq \acute{y} \end{cases}$$

$$(\sigma_A)(y) = \begin{cases} \inf\limits_{x \in \Omega^{-1}(y)} \sigma_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\ 1 & \text{if } \Omega^{-1}(y) \neq \emptyset \end{cases} = \begin{cases} \beta & \text{if } y = \acute{y} \\ 1 & \text{if } y \neq \acute{y} \end{cases} \qquad (Type.I)$$

$$(\sigma_A)(y) = \begin{cases} \sup\limits_{x\in\Omega^{-1}(y)} \sigma_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\ 0 & \text{if } \Omega^{-1}(y) \neq \emptyset \end{cases} = \begin{cases} \beta & \text{if } y = \acute{y} \\ 0 & \text{if } y \neq \acute{y} \end{cases} \qquad (Type.II)$$

$$(1-\Omega(1-\nu_A))(y) = \begin{cases} \inf\limits_{x\in\Omega^{-1}(y)} \nu_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\ 1 & \text{if } \Omega^{-1}(y) \neq \emptyset \end{cases} = \begin{cases} \gamma & \text{if } y = \acute{y} \\ 1 & \text{if } y \neq \acute{y} \end{cases}$$

That is -in Type.I and Type.II neutrosophic sets- $\Omega(x_{\alpha,\beta,\gamma}) = \acute{y}_{\alpha,\beta,\gamma}$ where $\acute{y} = \Omega(x)$. □

Definition 2.12. A neutrosophic point of type.I (type.II) $x_{\alpha,\beta,\gamma}$ is said to be in the neutrosophic set A -in symbols $x_{\alpha,\beta,\gamma} \in A$)- if and only if $\alpha < \mu_A(x), \beta > \sigma_A(x)$ and $\gamma > \nu_A(x)$ ($\alpha < \mu_A(x), \beta < \sigma_A(x)$ and $\gamma > \nu_A(x)$).

Lemma 2.13. *Let $A \in \mathcal{N}(X)$ and suppose that for every $x_{\alpha,\beta,\gamma} \in A$ there exists a neutrosophic set $B(x_{\alpha,\beta,\gamma}) \in \mathcal{N}(X)$ such that $x_{\alpha,\beta,\gamma} \in B(x_{\alpha,\beta,\gamma}) \sqsubseteq A$. Then $A = \sqcup\{B(x_{\alpha,\beta,\gamma}); x_{\alpha,\beta,\gamma} \in A\}$.*

Proof. The proof will be established for Type.I. Set $H = \sqcup\{B(x_{\alpha,\beta,\gamma}); x_{\alpha,\beta,\gamma} \in A\}$. It suffices to show that $A \sqsubseteq H$ and $H \sqsubseteq A$. First note that for every $B(x_{\alpha,\beta,\gamma}) \sqsubseteq A$ we have $\mu_{B(x_{\alpha,\beta,\gamma})}(x) \leq \mu_A(x)$, $\sigma_{B(x_{\alpha,\beta,\gamma})}(x) \geq \sigma_A(x)$ and $\nu_{B(x_{\alpha,\beta,\gamma})}(x) \geq \nu_A(x)$ for every $x \in X$. Let $x \in X$. Then $\mu_H(x) = \sup\{\mu_{B(x_{\alpha,\beta,\gamma})}; x_{\alpha,\beta,\gamma} \in A\} \leq \mu_A(x)$, $\sigma_H(x) = \inf\{\sigma_{B(x_{\alpha,\beta,\gamma})}; x_{\alpha,\beta,\gamma} \in A\} \geq \sigma_A(x)$, and $\nu_H(x) = \inf\{\nu_{B(x_{\alpha,\beta,\gamma})}; x_{\alpha,\beta,\gamma} \in A\} \geq \sigma_A(x)$, this means $H \sqsubseteq A$. To prove the converse, let $x \in X$ and let $\alpha_1 = \mu_A(x)$, $\beta_1 = \sigma_A(x)$, and $\gamma_1 = \nu_A(x)$. Consider the neutrosophic points $x_{\alpha,\beta,\gamma}$ such that $\alpha < \alpha_1$, $\beta > \beta_1$ and $\gamma > \gamma_1$. Then $x_{\alpha,\beta,\gamma} \in A$. Let $A_x = \cup\{B(x_{\alpha,\beta,\gamma}); \alpha < \alpha_1, \beta > \beta_1 and \gamma > \gamma_1\}$. It is clear that $A_x \sqsubseteq H$ so that $\mu_{A_x}(x) \leq \mu_H(x)$, $\sigma_{A_x}(x) \geq \sigma_H(x)$ and $\nu_{A_x}(x) \geq \nu_H(x)$. But $\mu_{A_x}(x) = \sup\{\mu_{A_{x_{\alpha,\beta,\gamma}}}(x); \alpha < \alpha_1, \beta > \beta_1, \gamma > \gamma_1\} = \alpha_1 = \mu_A(x)$, $\sigma_{A_x}(x) = \inf\{\sigma_{A_{x_{\alpha,\beta,\gamma}}}(x); \alpha < \alpha_1, \beta > \beta_1, \gamma > \gamma_1\} = \beta_1 = \sigma_A(x)$ and $\nu_{A_x}(x) = \sup\{\nu_{A_{x_{\alpha,\beta,\gamma}}}(x); \alpha < \alpha_1, \beta > \beta_1, \gamma > \gamma_1\} = \gamma_1 = \nu_A(x)$, which implies $\mu_A(x) \leq \mu_H(x)$, $\sigma_A \geq \sigma_H(x)$ and $\nu_A \geq \nu_H(x)$ or, equivalently, $A \sqsubseteq H$. □

Corollary 2.14. *Let $(X, \underline{\mu})$ be a neutrosophic topological space over X and let $A \in \mathcal{N}(X)$. Then A is neutrosophic $\underline{\mu}$-open in $(X, \underline{\mu})$ if and only if for every $x_{\alpha,\beta,\gamma} \in A$ there exists a neutrosophic $\underline{\mu}$-open set $B(x_{\alpha,\beta,\gamma}) \in \underline{\mu}$ such that $x_{\alpha,\beta,\gamma} \in B(x_{\alpha,\beta,\gamma}) \sqsubseteq A$.*

Definition 2.15. Let $(X, \underline{\mu})$ be a neutrosophic topological space over X. A sub-collection $\mathcal{B} \subseteq \underline{\mu}$ is called a neutrosophic $\underline{\mu} - base$ for $\underline{\mu}$ if and only if for any $U \in \underline{\mu}$ there exists $\acute{\mathcal{B}} \subseteq \mathcal{B}$ such that $U = \sqcup\{B; B \in \acute{\mathcal{B}}\}$.

Corollary 2.16. *Let $(X, \underline{\mu})$ be a neutrosophic topological space over X. Then a subcollection $\mathcal{B}$ of $\underline{\mu}$ is a neutrosophic $\underline{\mu}-$base for $\underline{\mu}$ if and only if for every $U \in \underline{\mu}$ and every $x_{\alpha,\beta,\gamma} \in U$ there exists $B \in \mathcal{B}$ such that $x_{\alpha,\beta,\gamma} \in B \sqsubseteq U$.*

Theorem 2.17. *Let $(X, \underline{\mu})$ and $(Y, \underline{\acute{\mu}})$ be two neutrosophic $\underline{\mu}$-topological spaces and let $\Omega : X \to Y$ be any function. Then Ω is neutrosophic $(\underline{\mu}, \underline{\acute{\mu}})$-continuous if and only if $\Omega^{-1}(V)$ is a neutrosophic $\underline{\mu}$-open set whenever V is a neutrosophc $\underline{\acute{\mu}}$-open set.*

Proof. Suppose that Ω is neutrosophic $(\underline{\mu}, \underline{\acute{\mu}})$-continuous, V be a neutrosophc $\underline{\acute{\mu}}$-open set, and $x_{\alpha,\beta,\gamma} \in \Omega^{-1}(V)$. Then $\Omega(x_{\alpha,\beta,\gamma}) = \Omega(x)_{\alpha,\beta,\gamma} \in \Omega(\Omega^{-1}(V)) \sqsubseteq V$ (we used theorem1.16(4)). Since Ω is $(\underline{\mu}, \underline{\acute{\mu}})$-continuous, there exists a neutrosophic $\underline{\mu}-$open set $V(x_{\alpha,\beta,\gamma})$ such that $x_{\alpha,\beta,\gamma} \in V(x_{\alpha,\beta,\gamma})$ and $\Omega(V(x_{\alpha,\beta,\gamma})) \sqsubseteq V$, which implies, by theorem1.16(3), $V(x_{\alpha,\beta,\gamma}) \sqsubseteq \Omega^{-1}(\Omega(V(x_{\alpha,\beta,\gamma}))) \sqsubseteq \Omega^{-1}(V)$, that is, by corollary2.14, $\Omega^{-1}(V)$ is $\underline{\mu} - open$. Conversely, suppose the condition of the theorem is true. To show that Ω is $(\underline{\mu}, \underline{\acute{\mu}})$-continuous let $x_{\alpha,\beta,\gamma}$ be a neutrosophic point in X and V is a neutrosophic $\underline{\acute{\mu}} - open$ set such that $\Omega(x_{\alpha,\beta,\gamma}) \in V$. By the condition of the theorem, $\Omega^{-1}(V)$ is neutrosophic $\underline{\mu}-open$ set, and from theorem1.16 (3) and (4) we have $x_{\alpha,\beta,\gamma} \in \Omega^{-1}(\Omega(x_{\alpha,\beta,\gamma})) \sqsubseteq \Omega^{-1}(V)$, and $\Omega(\Omega^{-1}(V)) \sqsubseteq V$, respectively. So we have $\Omega^{-1}(V)$ is neutrosophic $\underline{\mu}-open$, $x_{\alpha,\beta,\gamma} \in \Omega^{-1}(V)$ and $\Omega(\Omega^{-1}(V)) \sqsubseteq V$ which mean Ω is a neutrosophic $(\underline{\mu}, \underline{\acute{\mu}})$-continuous function. □

Theorem 2.18. *Let $(X, \underline{\mu})$ and $(Y, \underline{\acute{\mu}})$ be two neutrosophic $\underline{\mu}$-topological spaces, $\Omega : X \to Y$ be any function, and $\acute{\mathcal{B}}$ is a neutrosophic $\underline{\mu}-$base for $\underline{\acute{\mu}}$. Then Ω is neutrosophic $(\underline{\mu}, \underline{\acute{\mu}})$-continuous if and only if $\Omega^{-1}(V)$ is a neutrosophic $\underline{\mu}$-open set for every $V \in \acute{\mathcal{B}}$.*

Proof. $\Rightarrow$) Obvious!
$\Leftarrow$) Suppose that Ω satisfies the condition of the theorem, and let V be any neutrosophic $\underline{\acute{\mu}} - open$ set. Since $\acute{\mathcal{B}}$ is a neutrosophic $\underline{\mu} - base$ for $\underline{\acute{\mu}}$, there exists a sub-collection $\mathcal{B}^*$ from $\acute{\mathcal{B}}$ such that $V = \sqcup\{B; B \in \mathcal{B}^*\}$. But $\Omega^{-1}(V) = \Omega^{-1}(\sqcup\{B; B \in \mathcal{B}^*\}) = \sqcup\{\Omega^{-1}(B); B \in \mathcal{B}^*\}$. Since $\Omega^{-1}(B)$ is neutrosophic $\underline{\mu} - open$ for every $B \in \mathcal{B}^*$, $\Omega^{-1}(V)$ is neutrosophic $\underline{\mu} - open$, and so Ω is a neutrosophic $(\underline{\mu}, \underline{\acute{\mu}})$-continuous function. □

Definition 2.19. Let $(X, \underline{\mu})$ be a neutrosophic $\underline{\mu}$-topological space. A sub-collection $\mathcal{U} \subseteq \underline{\mu}$ is called a type.I (type.II) *neutrosophic $\underline{\mu}-$open cover* of X, if $1_X = \sqcup\{U; U \in \mathcal{U}\}$.

Definition 2.20. Let $(X, \underline{\mu})$ be a neutrosophic $\underline{\mu}$-topological space, and let $\mathcal{U}$ be a neutrosophic $\underline{\mu}-$open cover of X. A sub-collection $\acute{\mathcal{U}} \subseteq \mathcal{N}(X)$ is called a *neutrosophic $\underline{\mu}-$subcover* of X from $\mathcal{U}$, if $\acute{\mathcal{U}}$ is a neutrosophic $\underline{\mu}-$open cover of X and $\acute{\mathcal{U}} \subseteq \mathcal{U}$.

Corollary 2.21. *Let $(X, \underline{\mu})$ be a neutrosophic $\underline{\mu}$-topological space. A sub-collection $\mathcal{U} \subseteq \underline{\mu}$ is a $\underline{\mu}-$open cover of X if and only if for every $x_{\alpha,\beta,\gamma}$ in X there exists $U \in \mathcal{U}$ such that $x_{\alpha,\beta,\gamma} \in U$.*

Definition 2.22. A neutrosophic $\underline{\mu}$-topological space $(X, \underline{\mu})$ is called *neutrosophic $\underline{\mu}$-compact space* if every neutrosophic $\underline{\mu}-$open cover of X from $\underline{\mu}$ has a finite neutrosophic $\underline{\mu}$-subcover of X.

Theorem 2.23. *Let $\Omega : (X, \underline{\mu}) \to (Y, \underline{\acute{\mu}})$ be a neutrosophic $(\underline{\mu}, \underline{\acute{\mu}})$-continuous function. If $(X, \underline{\mu})$ is neutrosophic $\underline{\mu}-$compact, then $(Y, \underline{\acute{\mu}})$ is neutrosophic $\underline{\mu}-$compact.*

Proof. Let $\mathcal{V}$ be a neutrosophic $\underline{\mu}-$open cover of Y. Consider the collection $\mathcal{V}^{-1} = \{\Omega^{-1}(V); V \in \mathcal{V}\}$. Since Ω is neutrosophic $(\underline{\mu}, \underline{\acute{\mu}})$-continuous, $\mathcal{V}^{-1} \subseteq \underline{\mu}$. Set $A = \sqcup\{\Omega^{-1}(V); V \in \mathcal{V}\}$. To show that $A = 1_X$. But $A = \sqcup\{\Omega^{-1}(V); V \in \mathcal{V}\} = \Omega^{-1}(\sqcup\{V; V \in \mathcal{V}\}) = \Omega^{-1}(1_Y) = 1_X$ (we used Proposition 1.16,(9)); i.e. $\mathcal{V}^{-1}$ is a neutrosophic $\underline{\mu}-$*open* cover of X. Since X is neutrosophic $\underline{\mu} -$ *compact* space, $\mathcal{V}^{-1}$ has a finite neutrosophic $\underline{\mu} -$ *open* sub-cover $\mathcal{V}^{*-1}$. Suppose that $\mathcal{V}^{*-1} = \{\Omega^{-1}(V_i); i = 1, 2, ..., n\}$. Set $\mathcal{V}^* = \{V_i; i = 1, 2, ..., n\}$. It is clear that $\mathcal{V}^* \subseteq \mathcal{V}$. Since Ω is surjective, $\Omega(\Omega^{-1}(V_i)) = V_i$ for every $i = 1, 2, ..., n$, so we have $\sqcup\{V_i; i = 1, 2, ..., n\} = \sqcup\{\Omega(\Omega^{-1}(V_i)); i = 1, 2, ..., n\} = \Omega(\sqcup\{\Omega^{-1}(V_i); i = 1, 2, ..., n\}) = \Omega(1_X) = 1_Y$, that is $\mathcal{V}^*$ is a neutrosophic $\underline{\mu}-$subcover of X from $\mathcal{V}$. □

Theorem 2.24. *Let $(X, \underline{\mu})$ be a neutrosophic $\underline{\mu}$-topological space, and $\mathcal{B}$ be a neutrosophic $\underline{\mu} -$ base for $\underline{\mu}$. Then $(X, \underline{\mu})$ is neutrosophic $\underline{\mu}-$compact if and only if every neutrosophic $\underline{\mu} -$ open cover of X from $\mathcal{B}$ has a finite neutrosophic $\underline{\mu} -$ subcover.*

Proof. $\Rightarrow$) Obvious!

$\Leftarrow$) Suppose that X satisfies the condition of the theorem. Let $\mathcal{U}$ be a neutrosophic $\underline{\mu} -$ *open* cover of X. For every $U \in \mathcal{U}$ there exists $\mathcal{B}_U \subseteq \mathcal{B}$ such that $U = \sqcup\mathcal{B}_U$. Set $\mathcal{B}_1 = \{B; B \in \mathcal{B}_U, U \in \mathcal{U}\}$. It is clear that $\mathcal{B}_1$ is a neutrosophic $\underline{\mu} -$ *open* cover of X from $\mathcal{B}$, so it has a finite neutrosophic $\underline{\mu} -$ *subcover* $\mathcal{B}_1^*$. For every $B \in \mathcal{B}_1^*$ there exists $U_B \in \mathcal{U}$ such that $B \sqsubseteq U_B$. Let $\mathcal{U}^* = \{U_B; B \in \mathcal{B}_1^*\}$. Since $\mathcal{B}_1^*$ is a finite neutrosophic $\underline{\mu} -$ *open* cover of X, $\mathcal{U}^*$ is a finite $\underline{\mu} -$ *subcover* of X from $\mathcal{U}$, and X is neutrosophic $\underline{\mu} -$ *compact*. □

Definition 2.25. A neutrosophic $\underline{\mu}$-topological space $(X, \underline{\mu})$ is called:

(1) *neutrosophic* $\underline{\mu}$-Lindelöf space if every neutrosophic $\underline{\mu} -$ *open* cover of X from $\underline{\mu}$ has a countable neutrosophic $\underline{\mu}$-subcover of X.

(2) *neutrosophic* $\underline{\mu}$-countably compact space if every neutrosophic $\underline{\mu}-$*open* countable cover of X from $\underline{\mu}$ has a finite neutrosophic $\underline{\mu}$-subcover of X.

Theorem 2.26. *Every neutrosophic $\underline{\mu}$-topological space with a countable neutrosophic $\underline{\mu}-base$ is neutrosophic $\underline{\mu}$-Lindelöf.*

Proof. Let $(X, \underline{\mu})$ be a neutrosophic $\underline{\mu}$-topological space with a countable neutrosophic $\underline{\mu}-base$ $\mathcal{B}$. Let $\mathcal{U}$ be a neutrosophic $\underline{\mu}-open$ cover of X. For every $U \in \mathcal{U}$, there exists $\mathcal{B}_{\mathcal{U}} \subseteq \mathcal{B}$ such that $U = \sqcup \mathcal{B}_{\mathcal{U}}$. Let $\mathcal{B}^* = \cup\{\mathcal{B}_{\mathcal{U}}; U \in \mathcal{U}\}$. Since $\mathcal{U}$ is a neutrosophic $\underline{\mu}-open$ cover of X, $\mathcal{B}^*$ is a neutrosophic $\underline{\mu}-open$ cover of X. And since $\mathcal{B}^* \subseteq \mathcal{B}$, $\mathcal{B}^*$ is countable. We can write $\mathcal{B}^* = \{B_i; i = 1, 2, 3, ...\}$. For every $i = 1, 2, 3, ...$ pick a unique $U_i \in \mathcal{U}$ such that $B_i \sqsubseteq U_i$. Let $\mathcal{U}^* = \{U_i; i = 1, 2, ...\}$. Since $\mathcal{B}^*$ is a neutrosophic $\underline{\mu}-open$ cover of X, $\mathcal{U}^*$ is a neutrosophic $\underline{\mu}-open$ subcover of X from $\mathcal{U}$, and hence X is a neutrosophic $\underline{\mu}$-Lindelöf space. □

Theorem 2.27. *Every neutrosophic $\underline{\mu}$-Lindelöf and $\underline{\mu}$-*countably compact space *is $\underline{\mu}$-compact.*

Proof. Let $(X, \underline{\mu})$ be a neutrosophic $\underline{\mu}$-Lindelöf and $\underline{\mu}-countably$ compact space, and let $\mathcal{U}$ be a neutrosophic $\underline{\mu}-open$ cover of X. Since X is neutrosophic $\underline{\mu}$-Lindelöf, $\mathcal{U}$ has a countable neutrosophic $\underline{\mu}-subcover$ (say $\mathcal{U}_1$) of X from $\mathcal{U}$. And since X is neutrosophic $\underline{\mu}-countably$ compact, $\mathcal{U}_1$ has a neutrosophic $\underline{\mu}-finite$ subcover, say $\mathcal{U}_2$, from $\mathcal{U}_1$. It is clear that $\mathcal{U}_2$ is a neutrosophic $\underline{\mu}-finite$ subcover of X from $\mathcal{U}$, that means $(X, \underline{\mu})$ is a neutrosophic $\underline{\mu}-comapact$. □

Corollary 2.28. *Every neutrosophic $\underline{\mu}$-*countably compact space *with a neutrosophic countable $\underline{\mu}-base$ is $\underline{\mu}$-compact.*

Example 2.29. Let $X = \{a, b\}$ and $\beta = \{A_n; n = 1, 2, 3, ...\}$ where $A_n = \{\langle x, 1 - \frac{1}{2n}, \frac{1}{2n}, \frac{1}{2n}\rangle; x \in X\}$. Consider the neutrosophic $\underline{\mu}$-topology $\tau(\beta)$ generated by the neutrosophic μ-base β. Since $\tau(\beta)$ has a countable base, $\tau(\beta)$ is neutrosophic $\underline{\mu}$-Lindelöf. Note that $\tau(\beta)$ is strong neutrosophic μ-topological space, since β covers X, actually:
$\sqcup\beta = \sqcup\{A_n; n = 1, 2, 3, ...\} = \{\langle x, \vee_1^{\infty} 1 - \frac{1}{2n}, \wedge_1^{\infty} \frac{1}{2n}, \wedge_1^{\infty} \frac{1}{2n}\rangle; x \in X\} = \{\langle x, 1, 0, 0\rangle; x \in X\} = 1_X$. Now, we will show that $\tau(\beta)$ is not neutrosophic $\underline{\mu}$-countably paracompact (which implies it is not neutrosophic $\underline{\mu}$-compact). By contrapositive, suppose X is neutrosophic $\underline{\mu}$-countably paracompact. Then $\mathcal{U} = \beta$ is a countable neutrosophic $\underline{\mu}$-open cover of X. Since we suppose X is neutrosophic μ-countably paracompact, $\mathcal{U}$ has a finite $\underline{\mu}$-subcover , say $\mathcal{U}^* = \{A_{n1}, A_{n2}, ..., A_{nk}\}$. But $A_{n1} \sqcup A_{n2} \sqcup ... \sqcup A_{nk} = A_t$ where $t = \max\{n_1, n_2, ..., n_k\}$, and $A_t = \{\langle x, 1 - \frac{1}{2t}, \frac{1}{2t}, \frac{1}{2t}\rangle; x \in X\} \neq 1_X$, a contradiction. So X is not neutrosophic $\underline{\mu}$-countably paracompact and hence is not neutrosophic $\underline{\mu}$-compact.

The following theorem shows that neutrosophic $\underline{\mu}$-compact space and neutrusophic $\underline{\mu}$-countably compact space are equivalent if X is countable, which is not true in topological spaces.

Theorem 2.30. *For every countable neutrosophic $\underline{\mu}$-topological space X, the following two statements are equivalent:*

(1) *X is neutrosophic $\underline{\mu}$-compact.*
(2) *X is neutrosophic $\underline{\mu}$-countably compact.*

Proof. $\Rightarrow$) Obvious!
$\Leftarrow$) Suppose that X is a countable neutrosophic $\underline{\mu}$-countably compact space, and let $\mathcal{U}$ be a neutrosophic $\underline{\mu}$-open cover of X. For every $x \in X$ we define the following three subsets of $[0,1]$.

(1) $D^x_\mu = \{\mu_A(x); A \in \mathcal{U}\}$.
(2) $D^x_\sigma = \{\sigma_A(x); A \in \mathcal{U}\}$.
(3) $D^x_\nu = \{\nu_A(x); A \in \mathcal{U}\}$.

Let D^x_1, D^x_2 and D^x_3 be three countable dense subsets of D^x_μ, D^x_σ and D^x_ν respectively in the usual sense (the usual topology on the unit interval). Since $\mathcal{U}$ is a neutrosophic μ-open cover of X, we have $\sup D^x_1 = \sup D^x_\mu = 1$,$\inf D^x_2 = \inf D^x_\sigma = 0$ and $\inf D^x_3 = \inf D^x_\nu = 0$. Let $\mathcal{U}(x) = \{A \in \mathcal{U}; \mu_A(x) \in D^x_1, \sigma_A(x) \in D^x_2$ or $\nu_A(x) \in D^x_3\}$. It is clear that $\mathcal{U}(x)$ is countable. Let $\mathcal{U}^* = \cup\{\mathcal{U}(x); x \in X\}$. Since X is countable, $\mathcal{U}^*$ is a countable sub-collection from $\mathcal{U}$. We will show that $\mathcal{U}^*$ is a neutrosophic $\underline{\mu}$-cover of X. Set $B = \sqcup \mathcal{U}^*$. For every $x \in X$ we have:

(1) $\mu_B(x) = \vee\{\mu_A(x); A \in B\} \geq \vee\{\mu_A(x); A \in D^x_1\} = \sup D^x_1 = 1$.
(2) $\sigma_B(x) = \wedge\{\sigma_A(x); A \in B\} \geq \wedge\{\sigma_A(x); A \in D^x_1\} = \inf D^x_2 = 0$.
(3) $\nu_B(x) = \wedge\{\nu_A(x); A \in B\} \geq \wedge\{\vee_A(x); A \in D^x_1\} = \inf D^x_3 = 0$.

Which implies that $B = 1_X$ and $\mathcal{U}^*$ is a neutrosophic countable $\underline{\mu}$-open cover. Since X is a neutrosophic μ-countably compact space, $\mathcal{U}^*$ has a finite subcover , that is X is compact. $\square$

Question 2.31. *Are neutrosophic $\underline{\mu}$-compactness and neutrosophic $\underline{\mu}$-countably compactness equivalent.*

3. **Applications and further studies**

All existing studies are about neutrosophic topological spaces and since Neutrosophic $\underline{\mu}$-topological space is a generalization of neutrosophic topological spaces we can get more generalized results in Neutrosophic $\underline{\mu}$-topological space that are true for neutrosophic toplogical spaces, see for example Theorem 2.30, and some previous notations about neutrosophic sets can be considered as examples of neurosophic $\underline{\mu}$-topological spaces, see Theorem 2.9 which shows the relationship between $\underline{\mu}$-topological space and previous studies. In the future work we need to answer the question posted in this paper: Are neutrosophic $\underline{\mu}$-compactness and neutrosophic $\underline{\mu}$-countably compactness equivalent. Furthermore; many notations about neutrosophic $\underline{\mu}$-topological spaces need to be studied for example, first and second countable spaces, neighborhood systems, the relation between the usual topology defined on the interval [0,1] (which is the range of μ, σ and ν functions) and the neutrosophic $\underline{\mu}$-topology defined on X.

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