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Neutrosophic $\mu$-Topological spaces

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Abstract. In this paper, the concept of neutrosophic $\mu$-topological spaces is introduced. We define and study the properties of neutrosophic $\mu$-open sets, $\mu$-closed sets, $\mu$-interior and $\mu$-closure. The set of all generalize neutrosophic pre-closed sets $\text{GNPC}(\tau)$ and the set of all neutrosophic $\alpha$-open sets in a neutrosophic topological space $(X, \tau)$ can be considered as examples of generalized neutrosophic $\mu$-topological spaces. The concept of neutrosophic $\mu$-continuity is defined and we studied their properties. We define and study the properties of neutrosophic $\mu$-compact, $\mu$-Lindelöf and $\mu$-countably compact spaces. We prove that for a countable neutrosophic $\mu$-space $X$: $\mu$-countably compactness and $\mu$-compactness are equivalent. We give an example of a neutrosophic $\mu$-space $X$ which has a neutrosophic countable $\mu$-base but it is not neutrosophic $\mu$-countably compact.

neutrosophic $\mu$-topological spaces; $\mu$-open; $\mu$-closed; $\mu$-interior; $\mu$-closure; generalize neutrosophic pre-closed sets; neutrosophic $\alpha$-open sets; neutrosophic $\mu$-continuity; neutrosophic $\mu$-compact; neutrosophic $\mu$-Lindelöf; neutrosophic $\mu$-countably compact space.

1. Introduction

The fuzzy set was introduced by Zadeh [24] in 1965, where each element had a degree of membership. The intuitionistic fuzzy set (Ifs for short) on a universe $X$ was introduced by K. Atanassov [10–12] in 1983 as a generalization of fuzzy set, where besides the degree of membership we have the degree of non-membership of each element. The concept of neutrosophic sets first introduced by Smarandache [19,22] as a generalization of intuitionistic fuzzy sets, where we have the degree of membership, the degree of indeterminacy and the degree of non-membership of each element in $X$. After the introduction of the neutrosophic sets, neutrosophic set operations have been investigated. Many researchers have studied topology on neutrosophic sets, such as Smarandache [22], Lupianez [15,16] and Salama [17]. The neutrosophic interior, neutrosophic closure, neutrosophic exterior, neutrosophic boundary and neutrosophic subspace can be found in [20]. Neutrosophy has many applications specially
in decision making, for more details about new trends of neutrosophic applications one can consult [1], [2], [3] and [4].

Definition 1.1. [19]: A neutrosophic set $A$ on the universe of discourse $X$ is defined as
$$A = \{(x, \mu_A(x), \sigma_A(x), \nu_A(x)); x \in X\}$$
where $\mu, \sigma, \nu : X \to \mathbb{R}$ and $-1 \leq \mu(x) + \sigma(x) + \nu(x) \leq 1$.

The class of all neutrosophic set on $X$ will be denoted by $\mathcal{N}(X)$. We will exhibit the basic neutrosophic operations definitions (union, intersection and complement). Since there are different definitions of neutrosophic operations, we will organize the existing definitions into two types, in each type these operation will be consistent and functional.

Definition 1.2. [18] [Neutrosophic sets operations of Type.I] Let $A, A_A, B \in \mathcal{N}(X)$ such that $\alpha \in \Delta$. Then we define the neutrosophic:

1. (Inclusion): $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \geq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.
2. (Equality): $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
3. (Intersection) $\cap_{\alpha \in \Delta} A_\alpha(x) = \{ (x, \sigma_{A_\alpha}(x), \mu_{A_\alpha}(x), \nu_{A_\alpha}(x)); x \in X \}$.
4. (Union) $\cup_{\alpha \in \Delta} A_\alpha(x) = \{ (x, \sigma_{A_\alpha}(x), \mu_{A_\alpha}(x), \nu_{A_\alpha}(x)); x \in X \}$.
5. (Complement) $A^c = \{ (x, \nu_A(x), 1 - \sigma_A(x), \mu_A(x)); x \in X \}$.
6. (Universal set) $1_X = \{ (x, 1, 0, 0); x \in X \}$ will be called the neutrosophic universal set.
7. (Empty set) $0_X = \{ (x, 0, 1, 1); x \in X \}$ will be called the neutrosophic empty set.

Definition 1.3. [18] [Neutrosophic sets operations of Type.II] Let $A, A_A, B \in \mathcal{N}(X)$ for every $\alpha \in \Delta$. Then we define the neutrosophic:

1. (Inclusion): $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \leq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.
2. (Equality): $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
3. (Intersection) $\cap_{\alpha \in \Delta} A_\alpha(x) = \{ (x, \sigma_{A_\alpha}(x), \mu_{A_\alpha}(x), \nu_{A_\alpha}(x)); x \in X \}$.
4. (Union) $\cup_{\alpha \in \Delta} A_\alpha(x) = \{ (x, \sigma_{A_\alpha}(x), \mu_{A_\alpha}(x), \nu_{A_\alpha}(x)); x \in X \}$.
5. (Complement) $A^c = \{ (x, \nu_A(x), 1 - \sigma_A(x), \mu_A(x)); x \in X \}$.
6. (Universal set) $1_X = \{ (x, 1, 1, 0); x \in X \}$ will be called the neutrosophic universal set.
7. (Empty set) $0_X = \{ (x, 0, 0, 1); x \in X \}$ will be called the neutrosophic empty set.

Proposition 1.4. [18] For any $A, B, C \in \mathcal{N}(X)$ we have:

1. $A \cap A = A$, $A \cup A = A$, $A \cap 0_X = 0_X$, $A \cup 0_X = A$, $A \cap 1_X = A$, $A \cup 1_X = 1_X$.
2. $A \cap (B \cap C) = (A \cap B) \cap C$ and $A \cup (B \cup C) = (A \cup B) \cup C$.
3. $A \cap (\cup_{\alpha \in \Delta} A_\alpha) = \cup_{\alpha \in \Delta} (A \cap A_\alpha)$.
4. $A \cup (\cap_{\alpha \in \Delta} A_\alpha) = \cap_{\alpha \in \Delta} (A \cup A_\alpha)$.
5. $(A^c)^c = A$.
(6) De Morgan’s law:
(a) \((\bigcap_{\alpha \in \Delta} A_\alpha)^c = \bigcup_{\alpha \in \Delta} A_\alpha^c\).
(b) \((\bigcup_{\alpha \in \Delta} A_\alpha)^c = \bigcap_{\alpha \in \Delta} A_\alpha^c\).

**Definition 1.5.** [Neutrosophic Topology] Let \(\tau \subseteq \mathcal{N}(X)\). Then \(\tau\) is called a **neutrosophic topology** on \(X\) if
(1) \(0_X, 1_X \in \tau\).
(2) The union of any number of neutrosophic sets in \(\tau\) belongs to \(\tau\),
(3) The intersection of two neutrosophic sets in \(\tau\) belongs to \(\tau\).

The pair \((X, \tau)\) is called a **neutrosophic topological space** over \(X\). Moreover, the members of \(\tau\) are said to be **neutrosophic open sets** in \(X\). For any \(A \in \mathcal{N}(X)\), If \(A^c \in \tau\), then \(A\) is said to be **neutrosophic closed set** in \(X\).

**Definition 1.6.** [Neutrosophic interior] Let \((X, \tau)\) be a neutrosophic topological space over \(X\) and \(A \in \mathcal{N}(X)\). Then, the **neutrosophic interior** of \(A\), denoted by \(\text{int}(A)\) is the union of all neutrosophic open subsets of \(A\).

Clearly that \(\text{int}(A)\) is the biggest neutrosophic open set over \(X\) which containing \(A\).

**Theorem 1.7.** [20] Let \((X, \tau)\) be a neutrosophic topological space over \(X\) and \(A, B \in \mathcal{N}(X)\). Then
(1) \(\text{int}(1_X) = 1_X, \text{int}(0_X) = 0_X\) and \(\text{int}(A) \subseteq A\).
(2) \(\text{int}(\text{int}(A)) = \text{int}(A)\).
(3) \(A \subseteq B\) implies \(\text{int}(A) \subseteq \text{int}(B)\).
(4) \(\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)\).

**Definition 1.8.** [Neutrosophic closure] Let \((X, \tau)\) be a neutrosophic topological space over \(X\) and \(A \in \mathcal{N}(X)\). Then, the **neutrosophic closure** of \(A\), denoted by \(\text{cl}(A)\) is the intersection of all neutrosophic closed super sets of \(A\).

Clearly, \(\text{cl}(A)\) is the smallest neutrosophic closed set over \(X\) which contains \(A\).

**Theorem 1.9.** [20] Let \((X, \tau)\) be a neutrosophic topological space over \(X\) and \(A, B \in \mathcal{N}(X)\). Then,
(1) \(\text{cl}(1_X) = 1_X, \text{cl}(0_X) = 0_X\) and \(A \subseteq \text{cl}(A)\).
(2) \(\text{cl}(\text{cl}(A)) = \text{cl}(A)\).
(3) \(A \subseteq B\) implies \(\text{cl}(A) \subseteq \text{cl}(B)\).
(4) \(\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)\).
Definition 1.10. [Neutrosophic pre-open and pre-closed] Let \((X, \tau)\) be a neutrosophic topological space over \(X\) and \(A \in \mathcal{N}(X)\). Then \(A\) is said to be neutrosophic pre-open set (NPOS), if \(A \subseteq \text{Int} (\text{Cl}(A))\). The complement of a neutrosophic pre-open set is called neutrosophic pre-closed set (NPCS).

Definition 1.11. [Neutrosophic \(\alpha\)-open] Let \((X, \tau)\) be a neutrosophic topological space over \(X\) and \(A \in \mathcal{N}(X)\). \(A\) is said to be an \(\alpha\)-open set, if \(A \subseteq \text{Int} (\text{Cl}(\text{Int}(A)))\). The set of all neutrosophic \(\alpha\)-open sets in \((X, \tau)\) will be denoted by \(\mathcal{N}\alpha-O(\tau)\).

Definition 1.12. [Neutrosophic pre-closure] Let \((X, \tau)\) be a neutrosophic topological space over \(X\) and \(A \in \mathcal{N}(X)\). The neutrosophic pre-closure of \(A\), denoted by \(p\text{NCL}(A)\) is the intersection of all neutrosophic pre-closed super sets of \(A\).

Definition 1.13. [Generalized Neutrosophic pre-closed sets] Let \((X, \tau)\) be a neutrosophic topological space over \(X\) and \(A \in \mathcal{N}(X)\). \(A\) is said to be a neutrosophic generalized pre-closed set (GNPCS) in \((X, \tau)\) if \(p\text{NCL}(A) \subseteq B\) whenever \(A \subseteq B\) and \(B\) is neutrosophic open. The set of all generalized neutrosophic pre-closed sets in \((X, \tau)\) will be denoted by \(\text{GNPC}(\tau)\).

Theorem 1.14. [5, 6] Let \((X, \tau)\) be a neutrosophic topological space over \(X\). Then

1. The union of any collection of \(\alpha\)-open sets is an \(\alpha\)-open set.
2. The union of any collection of GNPCs is GNPC.

The following is an improvement of a definition in [14] makes it suitable for type.I and type.II neutrosophic sets.

Definition 1.15. Let \(X\) and \(Y\) be two nonempty sets and \(\Omega : X \to Y\) be any function. Then for any neutrosophic sets \(A \in \mathcal{N}(X)\) and \(B \in \mathcal{N}(Y)\) we have:

1. The Type.I (Type.II) pre-image of \(B\) under \(\Omega\), denoted by \(\Omega^{-1}(B)\), is the Neutrosophic set in \(X\) defined by
   \[
   \Omega^{-1}(B) = \{ (x, \mu_B(\Omega(x)), \sigma_B(\Omega(x)), \nu_B(\Omega(x))) ; x \in X \}
   \]

2. The Type.I (Type.II) image of \(A\) under \(\Omega\), denoted by \(\Omega(A)\), is the Neutrosophic set in \(Y\) defined by
   \[
   \Omega(A) = \{ (y, \Omega(\mu_A)(y), \Omega(\sigma_A)(y), (1 - \Omega(1 - \nu_A))(y)) ; y \in Y \}
   \]
   \[
   (\mu_A)(y) = \begin{cases} \sup_{x \in \Omega^{-1}(y)} \mu_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\ 0 & \text{if } \Omega^{-1}(y) = \emptyset \end{cases} \quad \text{(Type.I)}
   \]
   \[
   (\sigma_A)(y) = \begin{cases} \inf_{x \in \Omega^{-1}(y)} \sigma_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\ 1 & \text{if } \Omega^{-1}(y) = \emptyset \end{cases} \quad \text{(Type.I)}
   \]

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\[(\sigma_A)(y) = \begin{cases} 
\sup_{x \in \Omega^{-1}(y)} \sigma_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\
0 & \text{if } \Omega^{-1}(y) = \emptyset 
\end{cases} \quad (Type.II)\]

\[ (1 - \Omega(1 - \nu_A))(y) = \begin{cases} 
\inf_{x \in \Omega^{-1}(y)} \nu_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\
1 & \text{if } \Omega^{-1}(y) = \emptyset 
\end{cases} \]

For the sake of simplicity we write \(\Omega - (\nu_A\) instead of \((1 - \Omega(1 - \nu_A))\).

Note that the only difference between Type.I and Type.II images lies in the definition of the image of \(\sigma\) and this is important to make sure both Type.I and Type.II neutrosophic functions satisfy the following proposition.

**Proposition 1.16.** \([14]\) Let \(X\) and \(Y\) be two nonempty sets and \(\Omega : X \to Y\) be any function. Let \(A, A_\alpha \in \mathcal{N}(X)\) and \(B, B_\alpha \in \mathcal{N}(Y)\). Then we have:

1. \(A_1 \subseteq A_2 \Rightarrow \Omega(A_1) \subseteq \Omega(A_2)\).
2. \(B_1 \subseteq B_2 \Rightarrow \Omega^{-1}(B_1) \subseteq \Omega^{-1}(B_2)\).
3. \(A \subseteq \Omega^{-1}(\Omega(A))\) and equality holds if \(\Omega\) is injective.
4. \(\Omega(\Omega^{-1}(A)) \sqsubseteq A\) and equality holds if \(\Omega\) is surjective.
5. \(\Omega(\sqcup_{\alpha \in \Delta} A_\alpha) = \sqcup_{\alpha \in \Delta} \Omega(A_\alpha)\).
6. \(\Omega(\sqcap_{\alpha \in \Delta} A_\alpha) \sqsubseteq \sqcap_{\alpha \in \Delta} \Omega(A_\alpha)\) and equality holds when \(\Omega\) is injective.
7. \(\Omega^{-1}(\sqcup_{\alpha \in \Delta} B_\alpha) = \sqcup_{\alpha \in \Delta} \Omega^{-1}(B_\alpha)\).
8. \(\Omega^{-1}(\sqcap_{\alpha \in \Delta} B_\alpha) = \sqcap_{\alpha \in \Delta} \Omega^{-1}(B_\alpha)\).
9. \(\Omega^{-1}(1_N) = 1_N, \Omega^{-1}(0_N) = 0_N\).
10. \(\Omega(1_N) = 1_N\) and \(\Omega(0_N) = 0_N\), whenever \(\Omega\) is surjective.

**Definition 1.17.** Let \(X\) be a nonempty set and \(0 < \alpha, \beta, \gamma < 1\). Then a neutrosophic set \(A \in \mathcal{N}(X)\) is called:

1. A **neutrosophic point of Type.I** if and only if there exists \(x \in X\) such that \(A = \{\langle x, \alpha, \beta, \gamma \rangle\} \cup \{\langle \hat{x}, 0, 1, 1 \rangle; \hat{x} \neq x\}\).
2. A **neutrosophic point of Type.II** if \(A = \{\langle x, \alpha, \beta, \gamma \rangle\} \cup \{\langle \hat{x}, 0, 0, 1 \rangle; \hat{x} \neq x\}\). Neutrosophic points will be denoted by \(x_{\alpha, \beta, \gamma}\).

Now, we will exhibit some definitions and properties of \(\mu\)-topological spaces. Á. Császár \([13]\) introduced the notion of Generalized Topological Space (GTS). He also introduced the notion of Murad Arar and Saeid Jafari, Neutrosophic \(\mu\)-Topological spaces.
$(\mu_1;\mu_2)$-continuous function on GTS’s. $\mu$-compactness introduced in [23] and [21]. Countably $\mu$-paracompact introduced and studied in [8]. Strongly Generalized neighborhood systems introduced and studied in [9].

Let $X$ be a nonempty set. A collection $\mu$ of subsets of $X$ is called a generalized topology on $X$ and the pair $(X,\mu)$ is called a generalized topological space, if $\mu$ satisfies the following two conditions:

1. $\emptyset \in \mu$.
2. Any union of elements of $\mu$ belongs to $\mu$.

Let $\beta \subseteq \exp(X)$ and $\emptyset \in \beta$. Then $\beta$ is called a $\mu$-base for $\mu$ if $\mu = \{\bigcup \beta' ; \beta' \subset \beta\}$. We also say $\mu$ is generated by $\beta$. If $\beta$ is countable, then it is called a countable $\mu$-base. A generalized topological space $(X,\mu)$ is said to be strong if $X \in \mu$. A subset $B$ of $X$ is called $\mu$-open (resp. $\mu$-closed) if $B \in \mu$ (resp. if $X - B \in \mu$). The set of all $\mu$-open sets containing a point $x \in X$ will be denoted by $\mu_x$ (i.e. $\mu_x = \{U \in \mu ; x \in U\}$).

**Definition 1.18.** Let $(X,\mu_1)$ and $(X,\mu_2)$ be two $\mu$-topological space. A function $f : (X,\mu_1) \to (X,\mu_2)$ is said to be $(\mu_1,\mu_2)$-continuous if and only if $f^{-1}(V) \in \mu_1$ whenever $V \in \mu_2$.

**Definition 1.19.** Let $X$ be a generalized topological space and let $\mathcal{F}$ be a collection of subsets of $X$. Then $\mathcal{F}$ is said to be:

1. A $\mu$-cover of $X$ if $X = \bigcup\{U ; U \in \mathcal{F}\}$.
2. A $\mu$-open cover of $X$ if $\mathcal{F}$ is a $\mu$-cover of $X$ and $U \in \mu$ for every $U \in \mathcal{F}$.

**Definition 1.20.** Let $X$ be a generalized topological space and let $\mathcal{F}$ and $\mathcal{C}$ be $\mu$-covers of $X$. Then $\mathcal{C}$ is said to be a $\mu$-subcover of $\mathcal{F}$, if $\mathcal{C} \subseteq \mathcal{F}$.

**Definition 1.21.** A generalized topological space $X$ is said to be $\mu$-compact (resp. $\mu$-Lindelöf) if and only if every $\mu$-open cover of $X$ has a finite (resp. countable) $\mu$-subcover.

The following theorem shows some differences between topological spaces and $\mu$-topological spaces.

**Theorem 1.22.**

1. In $\mu$-topological spaces $\text{Int}_\mu(A \cap B) = \text{Int}_\mu(A) \cap \text{Int}_\mu(B)$ is not satisfied where $\text{Int}_\mu(A)$ stands for interior of $A$.
2. In $\mu$-topological spaces $\text{Cl}_\mu(A \cup B) = \text{Cl}_\mu(A) \cup \text{Cl}_\mu(B)$ is not satisfied where $\text{Cl}_\mu(A)$ stands for the closure of $A$ in $\mu$.

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(3) There exists a \( \mu \)-normal space with a countable \( \mu \)-base which has a \( \mu \)-open cover with no \( \mu \)-open point-finite refinement.

2. Neutrosophic \( \mu \)-Topological Spaces

In the literature of generalized topological spaces the symbol \( \mu \) is used to refer the \( \mu \)-topology and in neutrosophic sets it is used to refer the membership function \( \mu \), so, to avoid ambiguity, we will use the underlined \( \mu \) to refer the \( \mu \)-topology and keep \( \mu \) for the membership function in neutrosophic sets.

**Definition 2.1** (Neutrosophic \( \mu \)-Topology). Let \( \mu \subseteq \mathcal{N}(X) \). Then \( \mu \) is called a neutrosophic \( \mu \)-topology on \( X \) if

1. \( 0_X \in \mu \).
2. The union of any number of neutrosophic sets in \( \mu \) belongs to \( \mu \).

The pair \((X, \mu)\) is called a neutrosophic \( \mu \)-topological space over \( X \). The members of \( \mu \) are said to be neutrosophic \( \mu \)-open sets in \( X \). If \( 1_X \in \mu \), then \((X, \mu)\) is called a strong neutrosophic \( \mu \)-topological space. For any \( A \in \mathcal{N}(X) \), if \( A^c \in \mu \), then \( A \) is said to be neutrosophic \( \mu \)-closed set in \( X \). Since their are two types of neutrosophic sets, a neutrosophic \( \mu \)-topology is said to be Type.I/II neutrosophic topology if its elements are treated as Type.I/II neutrosophic sets.

**Example 2.2.** Let \( X = \{a, b, c\} \) and \( A, B, C, \hat{C} \in \mathcal{N}(X) \) with:

\[
A = \{(a, 0.3, 0.5, 0.7), (b, 0.3, 0.4, 1)\}, \quad B = \{(a, 0.4, 0.7, 0.1), (b, 0.2, 0.6, 0.9)\}, \quad C = \{(a, 0.4, 0.5, 0.1), (b, 0.3, 0.4, 0.9)\}, \quad \hat{C} = \{(a, 0.4, 0.7, 0.1), (b, 0.3, 0.6, 0.9)\}.
\]

Then \( \mu = \{0_X, A, B, C\} \) is a Type.I neutrosophic \( \mu \)-topology and \( \hat{\mu} = \{1_X, 0_X, A, B, \hat{C}\} \) is a Type.II strong neutrosophic \( \mu \)-topology. Neither \( \mu \) nor \( \hat{\mu} \) is neutrosophic topology. Note that, in \((X, \mu)\), \( A \cap B = \{(a, 0.3, 0.7, 0.7), (b, 0.2, 0.6, 1)\} \) is not neutrosophic \( \mu \)-open (here we apply type.I intersection). And in \((X, \hat{\mu})\) we have \( A \cap B = \{(a, 0.3, 0.5, 0.7), (b, 0.2, 0.4, 1)\} \) is not neutrosophic \( \mu \)-open (here we apply type.II intersection).

Most examples and theorems will be considered for Type.I neutrosophic sets, since the two types of neutrosophic sets have the same properties.

**Definition 2.3** (Neutrosophic \( \mu \)-interior). Let \((X, \mu)\) be a neutrosophic topological space over \( X \) and \( A \in \mathcal{N}(X) \). Then, the neutrosophic \( \mu \)-interior of \( A \), denoted by \( \text{int}_\mu(A) \) is the union of all neutrosophic \( \mu \)-open subsets of \( A \). Clearly \( \text{int}_\mu(A) \) is the biggest neutrosophic \( \mu \)-open set over \( X \) contained in \( A \).

**Theorem 2.4.** Let \((X, \mu)\) be a neutrosophic \( \mu \)-topological space over \( X \) and \( A, B \in \mathcal{N}(X) \). Then,
(1) \( \mathrm{int}_\mu(0_X) = 0_X \) and \( \mathrm{int}_\mu(A) \subseteq A \).

(2) \( \mathrm{int}_\mu(1_X) = 1_X \) whenever \( \mu \) is a strong \( \mu \)-topology.

(3) \( \mathrm{int}_\mu(\mathrm{int}_\mu(A)) = \mathrm{int}_\mu(A) \).

(4) \( A \subseteq B \) implies \( \mathrm{int}_\mu(A) \subseteq \mathrm{int}_\mu(B) \).

(5) \( \mathrm{int}_\mu(A) = A \) if and only if \( A \in \mu \).

(6) If \( A \subseteq B \), then \( \mathrm{int}_\mu(A) \subseteq \mathrm{int}_\mu(B) \).

(7) \( \mathrm{int}_\mu(A \cap B) \subseteq \mathrm{int}_\mu(A) \cap \mathrm{int}_\mu(B) \). Equality does not hold, see Example 2.5.

**Proof.** We will establish a proof for (4) and (7).

(4) Since \( A \subseteq B \), \{ \( U \in \mu; U \subseteq A \) \} \subseteq \{ \( U \in \mu; U \subseteq B \) \}. So that \( \mu_{\mathrm{int}_\mu(A)}(x) = \sup \{ \mu_U(x); U \in \mu, U \subseteq A \} \leq \sup \{ \mu_U(x); U \in \mu, U \subseteq B \} = \mu_{\mathrm{int}_\mu(B)}(x) \), \( \sigma_{\mathrm{int}_\mu(A)}(x) = \inf \{ \sigma_U(x); U \in \mu, U \subseteq A \} \geq \inf \{ \sigma_U(x); U \in \mu, U \subseteq B \} = \sigma_{\mathrm{int}_\mu(B)}(x) \), and \( \nu_{\mathrm{int}_\mu(A)}(x) = \inf \{ \nu_U(x); U \in \mu, U \subseteq A \} \geq \inf \{ \nu_U(x); U \in \mu, U \subseteq B \} = \nu_{\mathrm{int}_\mu(B)}(x) \). Which means \( \mathrm{int}_\mu(A) \subseteq \mathrm{int}_\mu(B) \).

(7) Since \( A \cap B \subseteq A \) and \( B \subseteq A \cap B \), \( \mathrm{int}_\mu(A \cap B) \subseteq \mathrm{int}_\mu(A) \) and \( \mathrm{int}_\mu(A \cap B) \subseteq \mathrm{int}_\mu(B) \) (by (4)), so we have \( \mathrm{int}_\mu(A \cap B) \subseteq \mathrm{int}_\mu(A) \cap \mathrm{int}_\mu(B) \). \( \square \)

**Example 2.5.** Consider \((X, \mu)\) as in Example 2.2. Note that:

(1) \( \mathrm{int}_\mu(1_X) = 0_X \cup A \cup B \cup C = C \neq 1_X \).

(2) Since \( A \cap B = \{ (a, 0.3, 0.7, 0.7), (b, 0.2, 0.6, 1) \} \) and there is no neutrosophic \( \mu \)-open set in \( \mu \) contained in \( A \cap B \) except \( 0_X \), we have \( \mathrm{int}_\mu(A \cap B) = 0_X \), and since \( A, B \in \mu \), \( \mathrm{int}_\mu(A) \cap \mathrm{int}_\mu(B) = A \cap B \neq \mathrm{int}_\mu(A \cap B) = 0_X \).

**Definition 2.6 (Neutrosophic \( \mu \)-closure).** Let \((X, \mu)\) be a neutrosophic \( \mu \)-topological space over \( X \) and \( A \in \mathcal{N}(X) \). Then, the neutrosophic \( \mu \)-closure of \( A \), denoted by \( \mathrm{cl}_\mu(A) \), is the intersection of all neutrosophic \( \mu \)-closed super sets of \( A \).

Clearly \( \mathrm{cl}_\mu(A) \) is the smallest neutrosophic \( \mu \)-closed set over \( X \) which containing \( A \).

**Theorem 2.7.** Let \((X, \mu)\) be a neutrosophic \( \mu \)-topological space over \( X \) and \( A, B \in \mathcal{N}(X) \). Then,

(1) \( \mathrm{cl}_\mu(1_X) = 1_X \) and \( A \subseteq \mathrm{cl}_\mu(A) \).

(2) \( \mathrm{cl}_\mu(0_X) = 0_X \) whenever \( \mu \) is a strong \( \mu \)-topology.

(3) \( \mathrm{cl}_\mu(\mathrm{cl}_\mu(A)) = \mathrm{cl}_\mu(A) \).

(4) \( A \subseteq B \) implies \( \mathrm{cl}_\mu(A) \subseteq \mathrm{cl}_\mu(B) \).

(5) \( A \) is \( \mu \)-closed if and only if \( \mathrm{cl}_\mu(A) = A \).

(6) \( \mathrm{cl}(A) \cup \mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B) \). The equality does not hold.

**Example 2.8.** Consider \((X, \mu)\) as in Example 2.2. The only \( \mu \)-closed sets in \((X, \mu)\) are:

(1) \( 0_X = 1_X \).

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(2) \( A^c = \{(a, 0.7, 0.5, 0.3), (b, 1, 0.6, 0.3)\} \).

(3) \( B^c = \{(a, 0.1, 0.3, 0.4), (b, 0.9, 0.4, 0.2)\} \).

(4) \( C^c = \{(a, 0.1, 0.5, 0.4), (b, 0.9, 0.6, 0.3)\} \).

It is clear that \( cl_\mu(0_X) = \emptyset \cap A^c \cap B^c \cap C^c = \{(a, 0.1, 0.5, 0.4), (b, 0.9, 0.6, 0.3)\} \neq 0_X \). Let \( H = A^c \) and \( K = B^c \). Then \( cl_\mu(H) \cup cl_\mu(K) = \{(a, 0.7, 0.3, 0.3), (b, 1, 0.4, 0.2)\} \) and \( cl_\mu(H \sqcup K) = 1_X \), since the only neutrosophic \( \mu \)-closed set containing \( H \sqcup K \) is \( cl_\mu(H) \cup cl_\mu(K) \) is \( 1_X \).

The following theorem shows the importance of generalized neutrosophic \( \mu \)-topological spaces.

**Theorem 2.9.** Let \( (X, \tau) \) be a neutrosophic topological space over \( X \). Then:

1. The set \( NO\alpha - O(\tau) \) of all neutrosophic \( \alpha \)-open sets over \( (X, \tau) \) is a strong neutrosophic \( \mu \)-topology over \( X \).
2. The set \( GNPC(\tau) \) of all neutrosophic pre-closed sets in \( (X, \tau) \) is a strong neutrosophic \( \mu \)-topology over \( X \).

**Proof.** Easy! we just call Theorem 1.14. \( \square \)

**Definition 2.10.** Let \( (X, \mu) \) and \( (Y, \hat{\mu}) \) be two neutrosophic \( \mu \)-topological spaces and let \( \Omega : X \to Y \) be any function. Then \( \Omega \) is said to be neutrosophic \( (\mu, \hat{\mu}) \)-continuous if for any neutrosophic point \( x_{\alpha, \beta, \gamma} \) and for any neutrosophic \( \hat{\mu} \)-open set \( V \in \tau \) such that \( f(x_{\alpha, \beta, \gamma}) \in V \) there exists \( U \in \tau \) such that \( x_{\alpha, \beta, \gamma} \in U \) and \( \Omega(U) \subseteq V \).

**Theorem 2.11.** Let \( X \) and \( Y \) be two nonempty sets and \( \Omega : X \to Y \) be any function. Let \( x_{\alpha, \beta, \gamma} \) be a neutrosophic point in \( X \). Then \( \Omega(x_{\alpha, \beta, \gamma}) = \Omega(x)_{\alpha, \beta, \gamma} \); that is the image of a neutrosophic point is a neutrosophic point.

**Proof.** We will prove it for Type.I and Type.II neutrosophic sets. Let \( A = x_{\alpha, \beta, \gamma} \) and \( \Omega(x) = \hat{y} \). Then the Type.I (Type.II) image of \( A \) under \( \Omega \), denoted by \( \Omega(A) \), is the Neutrosophic set:

\[
\Omega(A) = \{\langle y, \Omega(\mu_A)(y), \Omega(\sigma_A)(y), (1 - \Omega(1 - \nu_A))(y)\rangle; y \in Y\},
\]

where

\[
(\mu_A)(y) = \begin{cases}
\sup_{x \in \Omega^{-1}(y)} \mu_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\
0 & \text{if } \Omega^{-1}(y) = \emptyset
\end{cases}
= \begin{cases}
\alpha & \text{if } y = \hat{y} \\
0 & \text{if } y \neq \hat{y}
\end{cases}
\]

\[
(\sigma_A)(y) = \begin{cases}
\inf_{x \in \Omega^{-1}(y)} \sigma_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\
1 & \text{if } \Omega^{-1}(y) = \emptyset
\end{cases}
= \begin{cases}
\beta & \text{if } y = \hat{y} \\
1 & \text{if } y \neq \hat{y}
\end{cases}
\]

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Let \((\sigma_A)(y) = \begin{cases} \sup_{x \in \Omega^{-1}(y)} \sigma_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\ 0 & \text{if } \Omega^{-1}(y) = \emptyset \end{cases} \) and \((1 - \Omega(1 - \nu_A))(y) = \begin{cases} \inf_{x \in \Omega^{-1}(y)} \nu_A(x) & \text{if } \Omega^{-1}(y) \neq \emptyset \\ 1 & \text{if } \Omega^{-1}(y) = \emptyset \end{cases} \) for \(y \in \Omega(x)\) where \(\Omega(x)\) is the neutrosophic base of \(x\) in the neutrosophic topological space \((X,\mu)\).}

That is - in Type.I and Type.II neutrosophic sets- \(\Omega(x_{\alpha,\beta,\gamma}) = \hat{y}_{\alpha,\beta,\gamma}\) where \(\hat{y} = \Omega(x)\).

**Definition 2.12.** A neutrosophic point of type I (type II) \(x_{\alpha,\beta,\gamma}\) is said to be in the neutrosophic set \(A\) - in symbols \(x_{\alpha,\beta,\gamma} \in A\) - if and only if \(\alpha < \mu_A(x), \beta > \sigma_A(x)\) and \(\gamma > \nu_A(x)\) \((\alpha < \mu_A(x), \beta < \sigma_A(x)\) and \(\gamma > \nu_A(x))\).

**Lemma 2.13.** Let \(A \in \mathcal{N}(X)\) and suppose that for every \(x_{\alpha,\beta,\gamma} \in A\) there exists a neutrosophic set \(B(x_{\alpha,\beta,\gamma}) \in \mathcal{N}(X)\) such that \(x_{\alpha,\beta,\gamma} \in B(x_{\alpha,\beta,\gamma}) \subseteq A\). Then \(A = \bigcup\{B(x_{\alpha,\beta,\gamma}); x_{\alpha,\beta,\gamma} \in A\}\).

**Proof.** The proof will be established for Type I. Set \(H = \bigcup\{B(x_{\alpha,\beta,\gamma}); x_{\alpha,\beta,\gamma} \in A\}\). It suffices to show that \(A \subseteq H\) and \(H \subseteq A\). First note that for every \(B(x_{\alpha,\beta,\gamma}) \subseteq A\) we have \(\mu_B(x_{\alpha,\beta,\gamma})(x) \leq \mu_A(x), \sigma_B(x_{\alpha,\beta,\gamma})(x) \geq \sigma_A(x)\) and \(\nu_B(x_{\alpha,\beta,\gamma})(x) \geq \nu_A(x)\) for every \(x \in X\). Let \(x \in X\). Then \(\mu_H(x) = \sup\{\mu_B(x_{\alpha,\beta,\gamma}); x_{\alpha,\beta,\gamma} \in A\} \leq \mu_A(x), \sigma_H(x) = \inf\{\sigma_B(x_{\alpha,\beta,\gamma}); x_{\alpha,\beta,\gamma} \in A\} \geq \sigma_A(x)\), and \(\nu_H(x) = \inf\{\nu_B(x_{\alpha,\beta,\gamma}); x_{\alpha,\beta,\gamma} \in A\} \geq \nu_A(x)\), this means \(H \subseteq A\). To prove the converse, let \(x \in X\) and let \(\alpha_1 = \mu_A(x), \beta_1 = \sigma_A(x), \) and \(\gamma_1 = \nu_A(x)\). Consider the neutrosophic points \(x_{\alpha_2,\beta_2,\gamma_2}\) such that \(\alpha < \alpha_1, \beta > \beta_1\) and \(\gamma > \gamma_1\). Then \(x_{\alpha_2,\beta_2,\gamma_2} \in A\). Let \(A_x = \bigcup\{B(x_{\alpha,\beta,\gamma}); x_{\alpha,\beta,\gamma} \in A\}\). It is clear that \(A_x \subseteq H\) so that \(\mu_{A_x}(x) \leq \mu_H(x), \sigma_{A_x}(x) \geq \sigma_H(x)\) and \(\nu_{A_x}(x) \geq \nu_H(x)\). But \(\mu_{A_x}(x) = \sup\{\mu_{A_{\alpha_2,\beta_2,\gamma_2}}(x); \alpha < \alpha_1, \beta > \beta_1, \gamma > \gamma_1\} = \alpha_1 = \mu_A(x), \sigma_{A_x}(x) = \inf\{\sigma_{A_{\alpha_2,\beta_2,\gamma_2}}(x); \alpha < \alpha_1, \beta > \beta_1, \gamma > \gamma_1\} = \beta_1 = \sigma_A(x)\) and \(\nu_{A_x}(x) = \sup\{\nu_{A_{\alpha_2,\beta_2,\gamma_2}}(x); \alpha < \alpha_1, \beta > \beta_1, \gamma > \gamma_1\} = \gamma_1 = \nu_A(x)\), which implies \(\mu_A(x) \leq \mu_H(x), \sigma_A \geq \sigma_H(x)\) and \(\nu_A \geq \nu_H(x)\) or, equivalently, \(A \subseteq H\).}

**Corollary 2.14.** Let \((X,\mu)\) be a neutrosophic topological space over \(X\) and let \(A \in \mathcal{N}(X)\). Then \(A\) is neutrosophic \(\mu\)-open in \((X,\mu)\) if and only if for every \(x_{\alpha,\beta,\gamma} \in A\) there exists a neutrosophic \(\mu\)-open set \(B(x_{\alpha,\beta,\gamma}) \in \mu\) such that \(x_{\alpha,\beta,\gamma} \in B(x_{\alpha,\beta,\gamma}) \subseteq A\).

**Definition 2.15.** Let \((X,\mu)\) be a neutrosophic topological space over \(X\). A sub-collection \(B \subseteq \mu\) is called a neutrosophic \(\mu\)-base for \(\mu\) if and only if for any \(U \in \mu\) there exists \(\hat{B} \subseteq B\) such that \(U = \bigcup\{B; B \in \hat{B}\}\).
Corollary 2.16. Let $(X, \mu)$ be a neutrosophic topological space over $X$. Then a subcollection $B$ of $\mu$ is a neutrosophic $\mu$-base for $\mu$ if and only if for every $U \in \mu$ and every $x_{\alpha,\beta,\gamma} \in U$ there exists $B \in B$ such that $x_{\alpha,\beta,\gamma} \in B \subseteq U$.

Theorem 2.17. Let $(X, \mu)$ and $(Y, \hat{\mu})$ be two neutrosophic $\mu$-topological spaces and let $\Omega : X \rightarrow Y$ be any function. Then $\Omega$ is neutrosophic $(\mu, \hat{\mu})$-continuous if and only if $\Omega^{-1}(V)$ is a neutrosophic $\mu$-open set whenever $V$ is a neutrosophic $\hat{\mu}$-open set.

Proof. Suppose that $\Omega$ is neutrosophic $(\mu, \hat{\mu})$-continuous, $V$ be a neutrosophic $\hat{\mu}$-open set, and $x_{\alpha,\beta,\gamma} \in \Omega^{-1}(V)$. Then $\Omega(x_{\alpha,\beta,\gamma}) = \Omega(x_{\alpha,\beta,\gamma}) \in \Omega(\Omega^{-1}(V)) \subseteq V$ (we used theorem 1.16). Since $\Omega$ is $(\mu, \hat{\mu})$-continuous, there exists a neutrosophic $\mu$-open set $V(x_{\alpha,\beta,\gamma})$ such that $x_{\alpha,\beta,\gamma} \in V(x_{\alpha,\beta,\gamma})$ and $\Omega(V(x_{\alpha,\beta,\gamma})) \subseteq V$, which implies, by theorem 1.16(3), $V(x_{\alpha,\beta,\gamma}) \subseteq \Omega^{-1}(\Omega(V(x_{\alpha,\beta,\gamma}))) \subseteq \Omega^{-1}(V)$, that is, by corollary 2.14, $\Omega^{-1}(V)$ is $\mu$-open. Conversely, suppose the condition of the theorem is true. To show that $\Omega$ is $(\mu, \hat{\mu})$-continuous let $x_{\alpha,\beta,\gamma}$ be a neutrosophic point in $X$ and $V$ is a neutrosophic $\hat{\mu}$-open set such that $\Omega(x_{\alpha,\beta,\gamma}) \in V$. By the condition of the theorem, $\Omega^{-1}(V)$ is neutrosophic $\mu$-open set, and from theorem 1.16(3) and (4) we have $x_{\alpha,\beta,\gamma} \in \Omega^{-1}(\Omega(x_{\alpha,\beta,\gamma})) \subseteq \Omega^{-1}(V)$, and $\Omega(\Omega^{-1}(V)) \subseteq V$, respectively. So we have $\Omega^{-1}(V)$ is neutrosophic $\mu$-open, $x_{\alpha,\beta,\gamma} \in \Omega^{-1}(V)$ and $\Omega(\Omega^{-1}(V)) \subseteq V$ which mean $\Omega$ is a neutrosophic $(\mu, \hat{\mu})$-continuous function. □

Theorem 2.18. Let $(X, \mu)$ and $(Y, \hat{\mu})$ be two neutrosophic $\mu$-topological spaces, $\Omega : X \rightarrow Y$ be any function, and $B$ is a neutrosophic $\mu$-base for $\mu$. Then $\Omega$ is neutrosophic $(\mu, \hat{\mu})$-continuous if and only if $\Omega^{-1}(V)$ is a neutrosophic $\mu$-open set for every $V \in B$.

Proof. $\Rightarrow$) Obvious!

$\Leftarrow$) Suppose that $\Omega$ satisfies the condition of the theorem, and let $V$ be any neutrosophic $\hat{\mu}$-open set. Since $B$ is a neutrosophic $\mu$-base for $\mu$, there exists a sub-collection $B^*$ from $B$ such that $V = \sqcup\{B; B \in B^*\}$. But $\Omega^{-1}(V) = \Omega^{-1}(\sqcup\{B; B \in B^*\}) = \sqcup\{\Omega^{-1}(B); B \in B^*\}$. Since $\Omega^{-1}(B)$ is neutrosophic $\mu$-open for every $B \in B^*$, $\Omega^{-1}(V)$ is neutrosophic $\mu$-open, and so $\Omega$ is a neutrosophic $(\mu, \hat{\mu})$-continuous function. □

Definition 2.19. Let $(X, \mu)$ be a neutrosophic $\mu$-topological space. A sub-collection $\mathcal{U} \subseteq \mu$ is called a type.I (type.II) neutrosophic $\mu$-open cover of $X$, if $1_X = \sqcup\{U; U \in \mathcal{U}\}$.

Definition 2.20. Let $(X, \mu)$ be a neutrosophic $\mu$-topological space, and let $\mathcal{U}$ be a neutrosophic $\mu$-open cover of $X$. A sub-collection $\hat{\mathcal{U}} \subseteq \mathcal{N}(X)$ is called a neutrosophic $\mu$-subcover of $X$ from $\mathcal{U}$, if $\hat{\mathcal{U}}$ is a neutrosophic $\mu$-open cover of $X$ and $\hat{\mathcal{U}} \subseteq \mathcal{U}$. 

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Corollary 2.21. Let $(X,\mu)$ be a neutrosophic $\mu$-topological space. A sub-collection $U \subseteq \mu$ is a $\mu$-open cover of $X$ if and only if for every $x_{\alpha,\beta,\gamma}$ in $X$ there exists $U \in \mathcal{U}$ such that $x_{\alpha,\beta,\gamma} \in U$.

Definition 2.22. A neutrosophic $\mu$-topological space $(X,\mu)$ is called neutrosophic $\mu$-compact space if every neutrosophic $\mu$-open cover of $X$ from $\mu$ has a finite neutrosophic $\mu$-subcover of $X$.

Theorem 2.23. Let $\Omega : (X,\mu) \rightarrow (Y,\hat{\mu})$ be a neutrosophic $(\mu,\hat{\mu})$-continuous function. If $(X,\mu)$ is neutrosophic $\mu$-compact, then $(Y,\hat{\mu})$ is neutrosophic $\mu$-compact.

Proof. Let $\mathcal{V}$ be a neutrosophic $\mu$-open cover of $Y$. Consider the collection $\mathcal{V}^{-1} = \{\Omega^{-1}(V); V \in \mathcal{V}\}$. Since $\Omega$ is neutrosophic $(\mu,\hat{\mu})$-continuous, $\mathcal{V}^{-1} \subseteq \mu$. Set $A = \sqcup\{\Omega^{-1}(V); V \in \mathcal{V}\}$. To show that $A = 1_X$. But $A = \sqcup\{\Omega^{-1}(V); V \in \mathcal{V}\} = \Omega^{-1}(\sqcup\{V; V \in \mathcal{V}\}) = \Omega^{-1}(1_Y) = 1_X$ (we used Proposition [1.16](9)); i.e. $\mathcal{V}^{-1}$ is a neutrosophic $\mu$-open cover of $X$. Since $X$ is neutrosophic $\mu$-compact space, $\mathcal{V}^{-1}$ has a finite neutrosophic $\mu$-open sub-cover $\mathcal{V}^*-1$. Suppose that $\mathcal{V}^*-1 = \{\Omega^{-1}(V_i); i = 1, 2, ..., n\}$. Set $\mathcal{V}^* = \{V_i; i = 1, 2, ..., n\}$. It is clear that $\mathcal{V}^* \subseteq \mathcal{V}$. Since $\Omega$ is surjective, $\Omega(\Omega^{-1}(V_i)) = V_i$ for every $i = 1, 2, ..., n$, so we have $\sqcup\{V_i; i = 1, 2, ..., n\} = \sqcup\{\Omega(\Omega^{-1}(V_i)); i = 1, 2, ..., n\} = \Omega(\sqcup\{\Omega^{-1}(V_i); i = 1, 2, ..., n\}) = \Omega(1_X) = 1_Y$, that is $\mathcal{V}^*$ is a neutrosophic $\mu$-subcover of $X$ from $\mathcal{V}$.

Theorem 2.24. Let $(X,\mu)$ be a neutrosophic $\mu$-topological space, and $\mathcal{B}$ be a neutrosophic $\mu$-base for $\mu$. Then $(X,\mu)$ is neutrosophic $\mu$-compact if and only if every neutrosophic $\mu$-open cover of $X$ from $\mathcal{B}$ has a finite neutrosophic $\mu$-subcover.

Proof. $\Rightarrow$) Obvious!

$\Leftarrow$) Suppose that $X$ satisfies the condition of the theorem. Let $\mathcal{U}$ be a neutrosophic $\mu$-open cover of $X$. For every $U \in \mathcal{U}$ there exists $\mathcal{B}_U \subseteq \mathcal{B}$ such that $U = \sqcup\mathcal{B}_U$. Set $\mathcal{B}_1 = \{B; B \in \mathcal{B}_U, U \in \mathcal{U}\}$. It is clear that $\mathcal{B}_1$ is a neutrosophic $\mu$-open cover of $X$ from $\mathcal{B}$, so it has a finite neutrosophic $\mu$-subcover $\mathcal{B}_1^\ast$. For every $B \in \mathcal{B}_1^\ast$ there exists $U_B \in \mathcal{U}$ such that $B \subseteq U_B$. Let $\mathcal{U}^* = \{U_B; B \in \mathcal{B}_1^\ast\}$. Since $\mathcal{B}_1^\ast$ is a finite neutrosophic $\mu$-open cover of $X$, $\mathcal{U}^*$ is a finite $\mu$-subcover of $X$ from $\mathcal{U}$, and $X$ is neutrosophic $\mu$-compact.

Definition 2.25. A neutrosophic $\mu$-topological space $(X,\mu)$ is called:

1. neutrosophic $\mu$-Lindelöf space if every neutrosophic $\mu$-open cover of $X$ from $\mu$ has a countable neutrosophic $\mu$-subcover of $X$.

2. neutrosophic $\mu$-countably compact space if every neutrosophic $\mu$-open countable cover of $X$ from $\mu$ has a finite neutrosophic $\mu$-subcover of $X$.

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Theorem 2.26. Every neutrosophic $\mu$-topological space with a countable neutrosophic $\mu$-base is neutrosophic $\mu$-Lindelöf.

Proof. Let $(X, \mu)$ be a neutrosophic $\mu$-topological space with a countable neutrosophic $\mu$-base $\mathcal{B}$. Let $\mathcal{U}$ be a neutrosophic $\mu$-open cover of $X$. For every $U \in \mathcal{U}$, there exists $\mathcal{B}_U \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{B}_U$. Let $\mathcal{B}^* = \bigcup \{ \mathcal{B}_U; U \in \mathcal{U} \}$. Since $\mathcal{U}$ is a neutrosophic $\mu$-open cover of $X$, $\mathcal{B}^*$ is a neutrosophic $\mu$-open cover of $X$. And since $\mathcal{B}^* \subseteq \mathcal{B}$, $\mathcal{B}^*$ is countable. We can write $\mathcal{B}^* = \{ B_i; i = 1, 2, 3, ... \}$. For every $i = 1, 2, 3, ...$ pick a unique $U_i \in \mathcal{U}$ such that $B_i \subseteq U_i$. Let $\mathcal{U}^* = \{ U_i; i = 1, 2, 3, ... \}$. Since $\mathcal{B}^*$ is a neutrosophic $\mu$-open cover of $X$, $\mathcal{U}^*$ is a neutrosophic $\mu$-open subcover of $X$ from $\mathcal{U}$, and hence X is a neutrosophic $\mu$-Lindelöf space. \(\square\)

Theorem 2.27. Every neutrosophic $\mu$-Lindelöf and $\mu$-countably compact space is $\mu$-compact.

Proof. Let $(X, \mu)$ be a neutrosophic $\mu$-Lindelöf and $\mu$-countably compact space, and let $\mathcal{U}$ be a neutrosophic $\mu$-open cover of $X$. Since $X$ is neutrosophic $\mu$-Lindelöf, $\mathcal{U}$ has a countable neutrosophic $\mu$-subcover (say $\mathcal{U}_1$) of $X$ from $\mathcal{U}$. And since $X$ is neutrosophic $\mu$-countably compact, $\mathcal{U}_1$ has a neutrosophic $\mu$-finite subcover, say $\mathcal{U}_2$, from $\mathcal{U}_1$. It is clear that $\mathcal{U}_2$ is a neutrosophic $\mu$-finite subcover of $X$ from $\mathcal{U}$, that means $(X, \mu)$ is a neutrosophic $\mu$-compact. \(\square\)

Corollary 2.28. Every neutrosophic $\mu$-countably compact space with a neutrosophic countable $\mu$-base is $\mu$-compact.

Example 2.29. Let $X = \{ a, b \}$ and $\beta = \{ A_n; n = 1, 2, 3, ... \}$ where $A_n = \{ (x, 1 - \frac{1}{2^n}, \frac{1}{2^n}) \}; x \in X \}$. Consider the neutrosophic $\mu$-topology $\tau(\beta)$ generated by the neutrosophic $\mu$-base $\beta$. Since $\tau(\beta)$ has a countable base, $\tau(\beta)$ is neutrosophic $\mu$-Lindelöf. Note that $\tau(\beta)$ is strong neutrosophic $\mu$-topological space, since $\beta$ covers $X$, actually:

$\sqcup \beta = \sqcup \{ A_n; n = 1, 2, 3, ... \} = \{ (x, \sqrt[n]{1 - \frac{1}{2^n}}, \frac{1}{2^n}, \sqrt[n]{1 - \frac{1}{2^n}}); x \in X \} = \{ (x, 1, 0, 0); x \in X \} = 1_X$. Now, we will show that $\tau(\beta)$ is not neutrosophic $\mu$-countably paracompact (which implies it is not neutrosophic $\mu$-compact). By contrapositive, suppose $X$ is neutrosophic $\mu$-countably paracompact. Then $\mathcal{U} = \beta$ is a countable neutrosophic $\mu$-open cover of $X$. Since we suppose $X$ is neutrosophic $\mu$-countably paracompact, $\mathcal{U}$ has a finite $\mu$-subcover , say $\mathcal{U}^* = \{ A_{n_1}, A_{n_2}, ..., A_{n_k} \}$. But $A_{n_1} \cup A_{n_2} \cup ... \cup A_{n_k} = A_t$ where $t = \max\{ n_1, n_2, ..., n_k \}$, and $A_t = \{ (x, 1 - \frac{1}{2^n}, \frac{1}{2^n}, 1^n); x \in X \} \neq 1_X$, a contradiction. So $X$ is not neutrosophic $\mu$-countably paracompact and hence is not neutrosophic $\mu$-compact.

The following theorem shows that neutrosophic $\mu$-compact space and neutrosophic $\mu$-countably compact space are equivalent if $X$ is countable, which is not true in topological spaces.

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Theorem 2.30. For every countable neutrosophic $\mu$-topological space $X$, the following two statements are equivalent:

1. $X$ is neutrosophic $\mu$-compact.
2. $X$ is neutrosophic $\mu$-countably compact.

Proof. $\Rightarrow$ Obvious!

$\Leftarrow$ Suppose that $X$ is a countable neutrosophic $\mu$-countably compact space, and let $\mathcal{U}$ be a neutrosophic $\mu$-open cover of $X$. For every $x \in X$ we define the following three subsets of $[0,1]$.

1. $D^\mu_x = \{\mu_A(x); A \in \mathcal{U}\}$.
2. $D^\sigma_x = \{\sigma_A(x); A \in \mathcal{U}\}$.
3. $D^\nu_x = \{\nu_A(x); A \in \mathcal{U}\}$.

Let $D^\mu_1, D^\sigma_2$ and $D^\nu_3$ be three countable dense subsets of $D^\mu_x, D^\sigma_x$ and $D^\nu_x$ respectively in the usual sense (the usual topology on the unit interval). Since $\mathcal{U}$ is a neutrosophic $\mu$-open cover of $X$, we have $\sup D^\mu_1 = \sup D^\mu_x = 1$, $\inf D^\mu_3 = \inf D^\mu_x = 0$ and $\inf D^\sigma_2 = \inf D^\sigma_x = 0$. Let $\mathcal{U}(x) = \{A \in \mathcal{U}; \mu_A(x) \in D^\mu_1, \sigma_A(x) \in D^\sigma_2 \text{ or } \nu_A(x) \in D^\nu_3\}$. It is clear that $\mathcal{U}(x)$ is countable. Let $\mathcal{U}^* = \bigcup \{\mathcal{U}(x); x \in X\}$. Since $X$ is countable, $\mathcal{U}^*$ is a countable sub-collection from $\mathcal{U}$. We will show that $\mathcal{U}^*$ is a neutrosophic $\mu$-cover of $X$. Set $B = \cup \mathcal{U}^*$. For every $x \in X$ we have:

1. $\mu_B(x) = \vee \{\mu_A(x); A \in B\} \geq \{\mu_A(x); A \in D^\mu_1\} = \sup D^\mu_1 = 1$.
2. $\sigma_B(x) = \wedge \{\sigma_A(x); A \in B\} \geq \{\sigma_A(x); A \in D^\sigma_2\} = \inf D^\sigma_2 = 0$.
3. $\nu_B(x) = \wedge \{\nu_A(x); A \in B\} \geq \{\nu_A(x); A \in D^\nu_3\} = \inf D^\nu_3 = 0$.

Which implies that $B = 1_X$ and $\mathcal{U}^*$ is a neutrosophic countable $\mu$-open cover. Since $X$ is a neutrosophic $\mu$-countably compact space, $\mathcal{U}^*$ has a finite subcover, that is $X$ is compact.

Question 2.31. Are neutrosophic $\mu$-compactness and neutrosophic $\mu$-countably compactness equivalent.

3. Applications and further studies

All existing studies are about neutrosophic topological spaces and since Neutrosophic $\mu$-topological space is a generalization of neutrosophic topological spaces we can get more generalized results in Neutrosophic $\mu$-topological space that are true for neutrosophic topological spaces, see for example Theorem 2.30 and some previous notations about neutrosophic sets can be considered as examples of neutrosophic $\mu$-topological spaces, see Theorem 2.9 which shows the relationship between $\mu$-topological space and previous studies. In the future work we need to answer the question posted in this paper: Are neutrosophic $\mu$-compactness and Murad Arar and Saeid Jafari, Neutrosophic $\mu$-Topological spaces
neutrosophic $\mu$-countably compactness equivalent. Furthermore; many notations about neutrosophic $\mu$-topological spaces need to be studied for example, first and second countable spaces, neighborhood systems, the relation between the usual topology defined on the interval $[0,1]$ (which is the range of $\mu$, $\sigma$ and $\nu$ functions) and the neutrosophic $\mu$-topology defined on $X$.

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