A New Type of Neutrosophic Set in Pythagorean Fuzzy Environment and Applications to Multi-criteria Decision Making

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Abstract

In this paper, we introduce the concepts of Pythagorean fuzzy valued neutrosophic set (PFVNS) and Pythagorean fuzzy valued neutrosophic (PFVNV) constructed by considering Pythagorean fuzzy values (PFVs) instead of numbers for the degrees of the truth, the indeterminacy and the falsity, which is a new extension of intuitionistic fuzzy valued neutrosophic set (IFVNS). By means of PFVNSs, the degrees of the truth, the indeterminacy and the falsity can be given in Pythagorean fuzzy environment and more sensitive evaluations are made by a decision maker in decision making problems compared to IFVNSs. In other words, such sets enable a decision maker to evaluate the degrees of the truth, the indeterminacy and the falsity as PFVs to model the uncertainty in the evaluations. First of all, we propose the concepts of Pythagorean fuzzy t-norm and t-conorm and show that some Pythagorean fuzzy t-norms and t-conorms are expressed via ordinary continuous Archimedean t-norms and t-conorms. Then we define the concepts of PFVNS and PFVNV and provide a tool to construct a PFVNV from an ordinary neutrosophic fuzzy value. We also define some set theoretic operations between PFVNSs and some algebraic operations between PFVNVs via t-norms and t-conorms. With the help of these algebraic operations we propose some weighted aggregation operators. To measure discrimination information of PFVNVs, we define a simplified neutrosophic valued modified fuzzy cross-entropy measure. Moreover, we introduce a multi-criteria decision making method in Pythagorean fuzzy valued neutrosophic environment and practice the proposed theory to a real life multi-criteria decision making problem. Finally, we study the comparison analysis and the time complexity of the proposed method.

Keywords: Pythagorean fuzzy valued neutrosophic set; aggregation operators; multi-criteria decision making

1 Introduction

Zadeh[42] introduced the concept of fuzzy set (FS) defined by a membership function μ_A from a universal set X to closed unit interval $[0, 1]$ in order to handle the uncertainty in various real-world problems. Then using a membership function $\mu_A : X \to [0, 1]$ and a non-membership function $\nu_A : X \to [0, 1]$ under the condition $\mu_A(x) + \nu_A(x) \leq 1$ for $x \in X$, Atanassov[2] proposed the concept of intuitionistic fuzzy set (IFS) which is an extension of the concept of FS. The pair $\langle \mu_A(x), \nu_A(x) \rangle$ is called an intuitionistic fuzzy value (IFV) for a fixed $x \in X$. The theory of IFS has been used to solve problems in many applications as multi-criteria decision making (MCDM), classification, pattern recognition and clustering and it has been studied in many areas by researchers. The concepts of entropy and cross-entropy are important measurement methods used in the information theory and these concepts were proposed by Shannon.[23,24] Then the concept of cross-entropy was improved by Kullback and Leibler[13,14] and it was modified by Lin.[15] In order to measure

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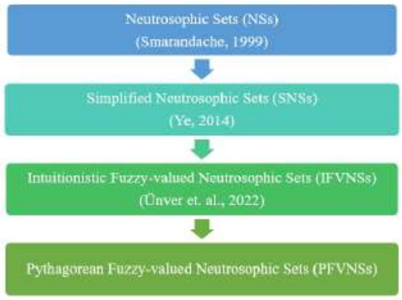

Figure 1: The development of theory of neutrosophic set

discrimination information between IFSs, Wei and Ye[31] introduced an improved intuitionistic fuzzy cross-entropy to overcome the drawback of intuitionistic fuzzy cross-entropy given by Vlachos and Sergiadis.[30] The concept of aggregation operator that transforms several input values into a single output value is an important tool in the decision making theory. A series of aggregation operators have been proposed by many researchers. Particularly, various generalizations of aggregation operators for IFSs (see e.g.[3,7]) were defined via several types of t-norms and t-conorms on $[0, 1]$. Further studies on MCDM with FSs and aggregation operators can be found in.[1,9,17,27,29,39–41]

As a generalization of IFS, the concept of Pythagorean fuzzy set (PFS) was developed by Yager,[33,34] which is defined by a membership function $\mu_A : X \to [0, 1]$ and a non-membership function $\nu_A : X \to [0, 1]$ under the condition $\mu_A^2(x) + \nu_A^2(x) \leq 1$ for $x \in X$. A PFS is more capable than an IFS to express uncertainty. In other words, PFSs enhance flexibility and practicability of IFSs and have wider area than IFSs while describing the uncertainty. Therefore theory of PFS has been studied by many researchers to model the uncertainty. Later Yager[34,35] proposed a range of aggregation operators for PFSs. After that, Peng et. al.[20] presented the axiomatic definitions of distance measure, similarity measure, entropy and inclusion measure for PFSs.

Smarandache[26] introduced the concept of neutrosophic set (NS) where each element has the degrees of truth, indeterminacy and falsify in the non-standard unit interval. There is no restriction on the membership functions for NSs. Due to the difficulty of applying neutrosophic sets to practical problems, the concept of simplified neutrosophic sets (SNSs) was proposed by Ye.[37] A SNS is constructed by a truth, an indeterminacy and a falsify membership function defined from X to $[0, 1]$. It has been used in various numerical applications. Moreover, the concept of single valued neutrosophic multi-set (SVNMS) that is characterized by sequences of truth, indeterminacy and falsify membership functions has been proposed by Ye and Ye.[36]

Schweizer and Sklar[21] introduced the concept of triangular norm (t-norm) and triangular conorm (t-conorm), following the concept of probabilistic metric spaces proposed by Menger,[16] which is a generalization of metric spaces. These notions are useful tools to define algebraic operations and aggregation operators for FSs. Deschrijver et. al.[6] proposed the concepts of intuitionistic fuzzy t-norm and intuitionistic fuzzy t-conorm, which are extensions of the concepts of t-norm and t-conorm, respectively, by turning into interval $[0, 1]$ to $\{(x_1, x_2) : x_1, x_2 \in [0, 1] \text{ and } x_1 + x_2 \leq 1\}$. In this paper, we introduce the concepts of Pythagorean fuzzy t-norm and Pythagorean fuzzy t-conorm which are extensions of notions of both ordinary and intuitionistic fuzzy t-norm and t-conorm, respectively. Ünver et. al.[28] proposed the concept of intuitionistic fuzzy valued neutrosophic multi-set (IFVNMS) by combining NS theory and IFS theory with the help of intuitionistic fuzzy values instead of numbers in membership sequences. Motivating from this idea, we propose the concepts of Pythagorean fuzzy valued neutrosophic set (PFVNS) and Pythagorean fuzzy valued neutrosophic value (PFVNV) by combining NS theory and PFS theory. A PFVNV consists of a triple of Pythagorean fuzzy values (PFVs) and a PFVNS consists of PFVNVs. Then we develop a method to transform fuzzy values and give some set theoretic and algebraic operations with the help of Pythagorean fuzzy t-norms and t-conorms. By using these algebraic operations we define some weighted arithmetic and geometric aggregation operators.

Entropy is a useful tool for measuring uncertain information. The concept of fuzzy entropy was introduced by Zadeh.[43] Later, a fuzzy cross-entropy measure and a symmetric discrimination information measure between FSs were proposed by Shang and Jiang.[22] Wei and Ye[31] defined a modified intuitionistic fuzzy cross-entropy measure between IFSs. With similar motivation, we introduce a simplified neutrosophic valued modified fuzzy

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cross-entropy for PFVNVs. Finally, using these concepts we provide a MCDM method in the Pythagorean fuzzy valued neutrosophic environment.

Some main contributions of the present study can be listed as follows:

- With the help of Pythagorean fuzzy t-norms and t-conorms we study in the fuzzy environment while conducting the aggregation process in decision making problems rather than defuzzified environment that provides us with more sensitive solutions.
- By means of PFVNSs, the degrees of the truth, the indeterminacy and the falsity can be given in the Pythagorean fuzzy environment and more sensitive evaluations are made by decision makers in real life problems. In other words, such sets enable a decision maker evaluate the degrees of the truth, the indeterminacy and the falsity as PFVs in decision making process. Thus, the uncertainty in the decision maker's evaluations is represented with the help of more capable FS notion.
- The proposed cross-entropy measure measures the discrimination of the information between PFVNVs in a fuzzy environment. Thus more sensitive evaluations can be made in decision making problems before defuzzifying the environment.

The rest of the paper is organized as follows. In Section 2, we introduce the concepts of Pythagorean fuzzy t-norm and t-conorm and show that some Pythagorean fuzzy t-norms and t-conorms are expressed via ordinary continuous Archimedean t-norms and t-conorms. In Section 3, we introduce the concept of PFVNS, which is an extension of the notion of intuitionistic fuzzy valued neutrosophic set (IFVNS) where an IFVNS is an IFVNMS with sequence length 1. Then we develop a method to transform simplified neutrosophic values (SNVs) to PFVNVs. We also give some set theoretical and algebraic operations via Pythagorean fuzzy t-norms and t-conorms for PFVNVs. In Section 4 we define some weighted aggregation operators using these algebraic operations. In Section 5, motivating by the modified intuitionistic fuzzy cross-entropy measure defined by Wei and Ye,[31] we give a simplified neutrosophic valued modified fuzzy cross-entropy. In Section 6, we propose a MCDM method and apply the proposed theory to a MCDM problem from the literature. We also compare the results of the proposed method with the existing results. Finally, we calculate the time complexity of the MCDM method. In Section 7, we conclude the paper.

2 Pythagorean Fuzzy t-norms and t-conorms

In this section, we recall some fundamental definitions about t-norms and t-conorms and define the concepts of Pythagorean fuzzy t-norm and t-conorm.

Definition 2.1. [10,21] A t-norm is a function $T : [0,1] \times [0,1] \to [0,1]$ that satisfies the following conditions:

(T1) $T(x,1) = x$ for all $x \in [0,1]$,

(T2) $T(x,y) = T(y,x)$ for all $x, y \in [0,1]$,

(T3) $T(x,T(y,z)) = T(T(x,y),z)$ for all $x,y,z \in [0,1]$,

(T4) $T(x,y) \leq T(x',y')$ whenever $x \leq x'$ and $y \leq y'$.

Definition 2.2. [10,21] A t-conorm is a function $S : [0,1] \times [0,1] \to [0,1]$ that satisfies the following conditions:

(S1) $S(x,0) = x$ for all $x \in [0,1]$,

(S2) $S(x,y) = S(y,x)$ for all $x, y \in [0,1]$,

(S3) $S(x,S(y,z)) = S(S(x,y),z)$ for all $x,y,z \in [0,1]$,

(S4) $S(x,y) \leq S(x',y')$ whenever $x \leq x'$ and $y \leq y'$.

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Definition 2.3. [33,34] A fuzzy complement is a function $N : [0,1] \to [0,1]$ satisfying the following conditions:

(N1) $N(0) = 1$ and $N(1) = 0$,

(N2) $N(a) \geq N(b)$ whenever $a \leq b$ for all $a, b \in [0,1]$,

(N3) Continuity,

(N4) $N(N(a)) = a$ for all $a \in [0,1]$.

The function $N : [0,1] \to [0,1]$ defined by $N(a) = (1-a^p)^{1/p}$ where $p \in (0,\infty)$ is a fuzzy complement introduced by Yager.[33,34] When $p = 2$, $N(a) = \sqrt{1-a^2}$ is called the Pythagorean fuzzy complement.

Definition 2.4. [12] Let T be a t-norm and let S be a t-conorm on $[0,1]$. If $T(x,y) = N(S(N(x), N(y)))$ and $S(x,y) = N(T(N(x), N(y)))$ for a fuzzy negator N, then T and S are said to be dual with respect to N.

Remark 2.5. Let T be a t-norm on $[0,1]$ and let $N(a) = \sqrt{1-a^2}$. Then the dual t-conorm S with respect to N is

$$S(x,y) = \sqrt{1 - T^2\big(\sqrt{1-x^2}, \sqrt{1-y^2}\big)}.$$

A strictly decreasing function $g : [0,1] \to [0,\infty]$ with $g(1) = 0$ is called the additive generator of a t-norm T if we have $T(x,y) = g^{-1}(g(x) + g(y))$ for all $(x,y) \in [0,1] \times [0,1]$.

Proposition 2.6. [12] *Let $g : [0,1] \to [0,\infty]$ be the additive generator of a t-norm T, let S be the dual t-conorm of T and let N be a fuzzy complement. The strictly increasing function $h : [0,1] \to [0,\infty]$ defined by $h(t) = g(N(t))$ is the additive generator of S and so $S(x,y) = h^{-1}(h(x) + h(y))$.*

Note that T is an Archimedean t-norm if and only if $T(x,x) < x$ for all $x \in (0,1)$ and S is an Archimedean t-conorm if and only if $S(x,x) > x$ for all $x \in (0,1)$.[10] Klement and Mesiar[11] proved that continuous Archimedean t-norms have useful representations via their additive generators as follows.

Theorem 2.7. [11] *Let T be a t-norm on $[0,1]$. The following are equivalent:*
(i) T is a continuous Archimedean t-norm.
(ii) T has a continuous additive generator.

Before introducing the concepts of Pythagorean fuzzy t-norm and t-conorm we recall the concept of PFS. Throughout this paper we assume $X = \{x_1, ..., x_n\}$ is a finite set.

Definition 2.8. [33,34] A PFS A on X is defined by

$$A = \{\langle x_j, \mu_A(x_j), \nu_A(x_j)\rangle : j = 1, ..., n\}$$

where $\mu_A, \nu_A : X \to [0,1]$ are the membership and non-membership functions respectively with condition that $\mu_A^2(x_j) + \nu_A^2(x_j) \leq 1$ for any $j = 1, ..., n$. For a fixed $j = 1, ..., n$ a PFV is denoted by

$$\alpha = \langle \mu_\alpha, \nu_\alpha \rangle = \langle \mu_A(x_j), \nu_A(x_j)\rangle.$$

Motivating by[6] we now introduce the notions of Pythagorean fuzzy t-norm and Pythagorean fuzzy t-conorm. Consider the set P^* defined by

$$P^* = \{x = (x_1, x_2) : x_1, x_2 \in [0,1] \text{ and } x_1^2 + x_2^2 \leq 1\}.$$

We use the partial order $\preceq$ on P^* that is defined by

$$x \preceq y \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2$$

for $x = (x_1, x_2), y = (y_1, y_2) \in P^*$.

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Definition 2.9. A Pythagorean fuzzy t-norm is a function $\mathcal{T} : P^* \times P^* \to P^*$ that satisfies the following conditions for any $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in P^*$.

(PT1) $\mathcal{T}(x, (1, 0)) = x$ for all $x \in P^*$,

(PT2) $\mathcal{T}(x, y) = \mathcal{T}(y, x)$ for all $x, y \in P^*$,

(PT3) $\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(y, \mathcal{T}(x, z))$ for all $x, y, z \in P^*$,

(PT4) $\mathcal{T}(x, y) \preceq \mathcal{T}(x', y')$ whenever $x \preceq x'$ and $y \preceq y'$.

Definition 2.10. A Pythagorean fuzzy t-conorm is a function $\mathcal{S} : P^* \times P^* \to P^*$ that satisfies the following conditions for any $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in P^*$.

(PS1) $\mathcal{S}(x, (0, 1)) = x$ for all $x \in P^*$,

(PS2) $\mathcal{S}(x, y) = \mathcal{S}(y, x)$ for all $x, y \in P^*$,

(PS3) $\mathcal{S}(x, \mathcal{S}(y, z)) = \mathcal{S}(y, \mathcal{S}(x, z))$ for all $x, y, z \in P^*$,

(PS4) $\mathcal{S}(x, y) \preceq \mathcal{S}(x', y')$ whenever $x \preceq x'$ and $y \preceq y'$.

We define the concept of Pythagorean fuzzy negator as an extension of the notion of fuzzy negator.

Definition 2.11. A Pythagorean fuzzy negator is a function $\mathcal{N} : P^* \to P^*$ satisfying the following conditions:

(PN1) $\mathcal{N}((1, 0)) = (0, 1)$ and $\mathcal{N}((0, 1)) = (1, 0)$,

(PN2) $\mathcal{N}(x) \succeq \mathcal{N}(y)$ whenever $x \preceq y$.

If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in P^*$, then we call $\mathcal{N}$ involutive. The mapping $\mathcal{N}_s : P^* \to P^*$ given by $\mathcal{N}_s((x_1, x_2)) = (x_2, x_1)$ is an involutive Pythagorean fuzzy negator which is called the standard negator.

Following theorem shows that a Pythagorean fuzzy t-conorm can be obtained from a Pythagorean fuzzy t-norm via Pythagorean fuzzy negators.

Theorem 2.12. *Let $\mathcal{T}$ be a Pythagorean fuzzy t-norm and let $\mathcal{N}$ be an involutive Pythagorean fuzzy negator. The function $\mathcal{S}$ defined by $\mathcal{S}(x, y) = \mathcal{N}(\mathcal{T}(\mathcal{N}(x), \mathcal{N}(y)))$ for any $x, y \in P^*$ is a t-conorm.*

Proof. (PS1) For all $x = (x_1, x_2) \in P^*$ we get

$$\begin{aligned}\mathcal{S}\big(x, (0, 1)\big) &= \mathcal{N}(\mathcal{T}(\mathcal{N}(x_1, x_2), \mathcal{N}(0, 1))) \\ &= \mathcal{N}(\mathcal{T}(\mathcal{N}(x_1, x_2), (1, 0))) \\ &= \mathcal{N}(\mathcal{N}(x_1, x_2)) \\ &= x.\end{aligned}$$

(PS2) For all $x, y \in P^*$ we obtain

$$\begin{aligned}\mathcal{S}(x, y) &= \mathcal{N}\big(\mathcal{T}(\mathcal{N}(x), \mathcal{N}(y))\big) \\ &= \mathcal{N}\big(\mathcal{T}(\mathcal{N}(y), \mathcal{N}(x))\big) \\ &= \mathcal{S}(y, x).\end{aligned}$$

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(PS3) For all $x, y, z \in P^*$ we have

$$\begin{aligned} \mathcal{S}(x, \mathcal{S}(y,z)) &= \mathcal{S}\big(x, \mathcal{N}(\mathcal{T}(\mathcal{N}(y), \mathcal{N}(z))\big) \\ &= \mathcal{N}\big(\mathcal{T}(\mathcal{N}(x), \mathcal{N}\big(\mathcal{N}(\mathcal{T}(\mathcal{N}(y), \mathcal{N}(z))\big)\big) \\ &= \mathcal{N}\big(\mathcal{T}(\mathcal{N}(x), (\mathcal{T}(\mathcal{N}(y), \mathcal{N}(z))\big) \\ &= \mathcal{N}\big(\mathcal{T}(\mathcal{N}(y), (\mathcal{T}(\mathcal{N}(x), \mathcal{N}(z))\big) \\ &= \mathcal{N}\big(\mathcal{T}(\mathcal{N}(y), \mathcal{N}\big(\mathcal{N}(\mathcal{T}(\mathcal{N}(x), \mathcal{N}(z))\big)\big) \\ &= \mathcal{S}\big(y, \mathcal{N}(\mathcal{T}(\mathcal{N}(x), \mathcal{N}(z))\big) \\ &= \mathcal{S}(y, \mathcal{S}(x,z)). \end{aligned}$$

(PS4) For all $x, x', y, y' \in P^*$ we have

$$\begin{aligned} x \preceq x' \text{ and } y \preceq y' &\Rightarrow \mathcal{N}(x) \succeq \mathcal{N}(x') \text{ and } \mathcal{N}(y) \succeq \mathcal{N}(y') \\ &\Rightarrow \mathcal{T}\big(\mathcal{N}(x), \mathcal{N}(y)\big) \succeq \mathcal{T}\big(\mathcal{N}(x'), \mathcal{N}(y')\big) \\ &\Rightarrow \mathcal{N}\big(\mathcal{T}\big(\mathcal{N}(x), \mathcal{N}(y)\big) \preceq \mathcal{N}\big(\mathcal{T}\big(\mathcal{N}(x'), \mathcal{N}(y')\big)\big) \\ &\Rightarrow \mathcal{S}(x,y) \preceq \mathcal{S}(x', y'). \end{aligned}$$

Therefore, $\mathcal{S}$ is a Pythagorean fuzzy t-conorm. □

The t-conorm $\mathcal{S}$ in Theorem 2.12 is called the dual Pythagorean fuzzy t-conorm of $\mathcal{T}$ with respect to negator $\mathcal{N}$. Following theorem states that Pythagorean fuzzy t-norms and t-conorms can be produced by ordinary t-norms and t-conorms.

Theorem 2.13. *Let T be a t-norm and let S be a t-conorm on $[0,1]$. If*

$$T(a,b) \le \sqrt{1 - S^2\big(\sqrt{1-a^2}, \sqrt{1-b^2}\big)} \text{ for all } a, b \in [0,1], \tag{1}$$

then the function $\mathcal{T} : P^* \times P^* \to P^*$ defined by

$$\mathcal{T}(x,y) = \big(T(x_1, y_1), S(x_2, y_2)\big)$$

is a Pythagorean fuzzy t-norm and the function $\mathcal{S} : P^* \times P^* \to P^*$ defined by

$$\mathcal{S}(x,y) = \big(S(x_1, y_1), T(x_2, y_2)\big)$$

is the dual Pythagorean fuzzy t-conorm of $\mathcal{T}$ with respect to $\mathcal{N}_s$.

Proof. As S is increasing, from (1) we have for any $x, y \in P^*$ that

$$\begin{aligned} T^2(x_1, x_2) + S^2(x_2, y_2) &\le 1 - S^2\Big(\sqrt{1-x_1^2}, \sqrt{1-x_2^2}\Big) + S^2(x_2, y_2) \\ &\le 1 - S^2\Big(\sqrt{1-x_1^2}, \sqrt{1-x_2^2}\Big) + S^2\Big(\sqrt{1-x_1^2}, \sqrt{1-x_2^2}\Big) \\ &= 1 \end{aligned}$$

which yields that $\mathcal{T}(x,y)$ is in P^*.
(PT1) For any $x = (x_1, x_2) \in P^*$ we have

$$\begin{aligned} \mathcal{T}(x, (1,0)) &= (T\,(x_1, 1)\,, S\,(x_2, 0)) \\ &= (x_1, x_2) \\ &= x. \end{aligned}$$

(PT2) For any $x, y \in P^*$ we get

$$\begin{aligned} \mathcal{T}(x,y) &= (T\,(x_1, y_1)\,, S\,(x_2, y_2)) \\ &= (T\,(y_1, x_1)\,, S\,(y_2, x_2)) \\ &= \mathcal{T}(y,x). \end{aligned}$$

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(PT3) For all $x, y, z \in P^*$ we obtain

$$\begin{aligned}\mathcal{T}(x,\mathcal{T}(y,z)) &= \mathcal{T}\big((x_1,x_2),\big(T(y_1,z_1),S(y_2,z_2)\big)\big)\\ &= \Big(T\big(x_1,T(y_1,z_1)\big),S\big(x_2,S(y_2,z_2)\big)\Big)\\ &= \Big(T\big(y_1,T(x_1,z_1)\big),S\big(y_2,S(x_2,z_2)\big)\Big)\\ &= \mathcal{T}\big((y_1,y_2),\big(T(x_1,z_1),S(x_2,z_2)\big)\big)\\ &= \mathcal{T}(y,\mathcal{T}(x,z)).\end{aligned}$$

(PT4) For all $x, x', y, y' \in P^*$ we have

$$\begin{aligned}x \preceq x' \text{ and } y \preceq y' &\Rightarrow x_1 \le x_1',\ x_2 \ge x_2' \text{ and } y_1 \le y_1',\ y_2 \ge y_2'\\ &\Rightarrow T(x_1,y_1) \le T(x_1',y_1') \text{ and } S(x_2,y_2) \ge S(x_2',y_2')\\ &\Rightarrow \mathcal{T}(x,y) \preceq \mathcal{T}(x',y').\end{aligned}$$

Therefore $\mathcal{T}$ is a Pythagorean fuzzy t-norm. Similarly, it can be shown that $\mathcal{S}$ is a Pythagorean fuzzy t-conorm. □

Continuous Archimedean t-norms and t-conorms on $[0,1]$ can be generated by their additive generators. So we can construct a Pythagorean fuzzy t-norm and a Pythagorean fuzzy t-conorm using these additive generators. Note that if a t-norm T and a t-conorm S are dual with respect to the Pythagorean fuzzy complement, then (1) is satisfied.

Corollary 2.14. *Let $g : [0,1] \to [0,\infty]$ be the additive generator of a continuous Archimedean t-norm T and let $h : [0,1] \to [0,\infty]$ be the additive generator of the dual continuous Archimedean t-conorm S where $h(t) = g(\sqrt{1-t^2})$. Then the function $\mathcal{T} : P^* \times P^* \to P^*$ defined by*

$$\mathcal{T}(x,y) = \big(g^{-1}(g(x_1)+g(y_1)), h^{-1}(h(x_2)+h(y_2))\big)$$

is a Pythagorean fuzzy t-norm and the function $\mathcal{S} : P^* \times P^* \to P^*$ defined by

$$\mathcal{S}(x,y) = \big(h^{-1}(h(x_1)+h(y_1)), g^{-1}(g(x_2)+g(y_2))\big)$$

is the dual Pythagorean fuzzy t-conorm of $\mathcal{S}$ with respect to $\mathcal{N}_{\mathcal{S}}$. In this case $\mathcal{T}$ and $\mathcal{S}$ are called Pythagorean fuzzy t-norm and Pythagorean fuzzy t-conorm generated by g and h, respectively.

Proof. It is trivial from Theorem 2.13 and Remark 2.5. □

Example 2.15. Consider the functions $g : [0,1] \to [0,\infty]$ defined by $g(t) = -\log t^2$ and $h : [0,1] \to [0,\infty]$ defined by $h(t) = -\log(1-t^2)$. Then the Algebraic Pythagorean fuzzy t-norm is

$$\mathcal{T}(x,y) = \Big(x_1y_1, \sqrt{x_2^2+y_2^2-x_2^2y_2^2}\Big)$$

and the Algebraic dual Pythagorean fuzzy t-conorm with respect to $\mathcal{N}_s$ is

$$\mathcal{S}(x,y) = \Big(\sqrt{x_1^2+y_1^2-x_1^2y_1^2}, x_2y_2\Big).$$

3 Pythagorean Fuzzy Valued Neutrosophic Sets

We start this section recalling the concept of IFVNS obtained by taking the sequence length equal to 1 in IFVNMSs. Then we define the concepts of PFVNS and PFVNV.

Definition 3.1. [28] An IFVNS A on X is defined by

$$A = \Big\{\langle x_j, \mathbb{T}_A^j, \mathbb{I}_A^j, \mathbb{F}_A^j\rangle : j = 1, \ldots, n\Big\}$$

where $\mathbb{T}_A^j$, $\mathbb{I}_A^j$ and $\mathbb{F}_A^j$ are the truth, the indeterminacy and the falsity membership pairs of IFVs, respectively, i.e., for $j = 1, \ldots, n$

$$\mathbb{T}_A^j = \big(\mu_{A,t}(x_j), \nu_{A,t}(x_j)\big) \text{ with } \mu_{A,t}(x_j), \nu_{A,t}(x_j) \in [0,1] \text{ such that } \mu_{A,t}(x_j) + \nu_{A,t}(x_j) \leq 1$$
$$\mathbb{I}_A^j = \big(\mu_{A,i}(x_i), \nu_{A,i}(x_j)\big) \text{ with } \mu_{A,i}(x_j), \nu_{A,i}(x_j) \in [0,1] \text{ such that } \mu_{A,i}(x_j) + \nu_{A,i}(x_j) \leq 1$$
$$\mathbb{F}_A^j = \big(\mu_{A,f}(x_j), \nu_{A,f}(x_j)\big) \text{ with } \mu_{A,f}(x_j), \nu_{A,f}(x_j) \in [0,1] \text{ such that } \mu_{A,f}(x_j) + \nu_{A,f}(x_j) \leq 1.$$

An IFVNV is denoted by

$$\alpha = \langle \mathbb{T}_\alpha, \mathbb{I}_\alpha, \mathbb{F}_\alpha\rangle := \langle \mathbb{T}_A^j, \mathbb{I}_A^j, \mathbb{F}_A^j\rangle$$

for a fixed $j = 1, ..., n$.

Example 3.2. Let $X = \{x_1, x_2\}$.

$$A = \{\langle x_1, (0.7, 0.3), (0.3, 0.5), (0.1, 0.8)\rangle, \langle x_2, (0.4, 0.3), (0.1, 0.6), (0.2, 0.75)\rangle\}$$

is an IFVNS.

Remark 3.3. The notion of IFVNS is a new extension of the notion of SNS, since each SNS has the form

$$\begin{aligned} A &= \Big\{\Big\langle x_j, T_A(x_j), I_A(x_j), F_A(x_j)\Big\rangle : j = 1, \ldots, n\Big\} \\ &= \Big\{\Big\langle x_j, \big(\mu_{A,t}(x_j), 0\big), \big(\mu_{A,i}(x_j), 0\big), \big(\mu_{A,f}(x_j), 0\big)\Big\rangle : j = 1, \ldots, n\Big\}. \end{aligned}$$

However, an IFVNS cannot be represented as a standard SNS (see, e.g., Example 3.2).

In some decision making problems, the sum of the membership and non-membership degrees that are determined by the decision makers can be larger than 1. Therefore, PFVs are more capable than IFVs for modelling vagueness in the degrees of the truth, the indeterminacy and the falsity in the practical problems as shown in Figure 2. Using PFVs for the degree of the truth, the indeterminacy and the falsity, we define the concept of PFVNS and propose some set theoretical and algebraic operations between PFVNSs. By utilizing PFVNSs, the uncertainty in the decision maker's evaluations can be modeled in a more capable environment.

Definition 3.4. A PFVNS A on X is defined by

$$A = \Big\{\langle x_j, \mathbb{T}_A^j, \mathbb{I}_A^j, \mathbb{F}_A^j\rangle : j = 1, \ldots, n\Big\}$$

where $\mathbb{T}_A^j$, $\mathbb{I}_A^j$ and $\mathbb{F}_A^j$ are the truth, the indeterminacy and the falsity membership pairs of PFVs, respectively, i.e., for $j = 1, \ldots, n$

$$\mathbb{T}_A^j = \big(\mu_{A,t}(x_j), \nu_{A,t}(x_j)\big) \text{ with } \mu_{A,t}(x_j), \nu_{A,t}(x_j) \in [0,1] \text{ such that } \mu_{A,t}^2(x_j) + \nu_{A,t}^2(x_j) \leq 1$$
$$\mathbb{I}_A^j = \big(\mu_{A,i}(x_i), \nu_{A,i}(x_j)\big) \text{ with } \mu_{A,i}(x_j), \nu_{A,i}(x_j) \in [0,1] \text{ such that } \mu_{A,i}^2(x_j) + \nu_{A,i}^2(x_j) \leq 1$$
$$\mathbb{F}_A^j = \big(\mu_{A,f}(x_j), \nu_{A,f}(x_j)\big) \text{ with } \mu_{A,f}(x_j), \nu_{A,f}(x_j) \in [0,1] \text{ such that } \mu_{A,f}^2(x_j) + \nu_{A,f}^2(x_j) \leq 1.$$

A PFVNV is denoted by

$$\alpha = \langle \mathbb{T}_\alpha, \mathbb{I}_\alpha, \mathbb{F}_\alpha\rangle := \langle \mathbb{T}_A^j, \mathbb{I}_A^j, \mathbb{F}_A^j\rangle$$

for a fixed $j = 1, ..., n$.

Example 3.5. Let $X = \{x_1, x_2\}$.

$$A = \{\langle x_1, (0.4, 0.3), (0.5, 0.5), (0.1, 0.9)\rangle, \langle x_2, (0.3, 0.7), (0.7, 0.6), (0.2, 0.2)\rangle\}$$

is a PFVNS.

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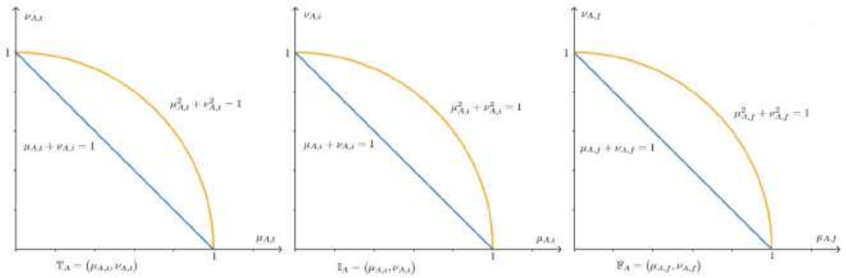

Figure 2: Comparison of concepts of PFV and IFV for the truth, the indeterminacy and the falsity membership degrees

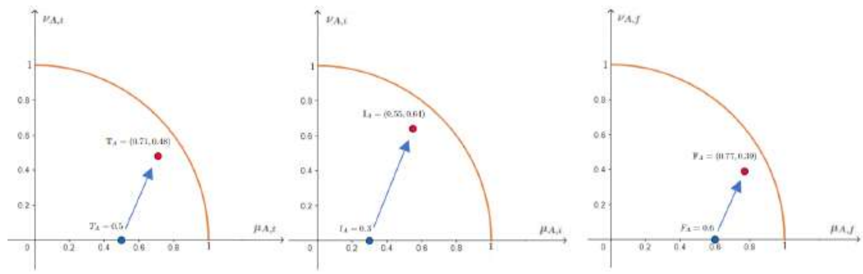

Figure 3: The SNV and PFVNV in Example 3.8

Remark 3.6. Each IFVNS is a PFVNS but the converse of this statement is not true in general. For example, A is a PFVNS, but A is not a $IFVNS$ in Example 3.5 since $0.7 + 0.6 = 1.3 > 1$ for $\mathbb{I}_A^2 = (0.7, 0.6)$.

SNVs can be converted to PFVNVs with the following method. A similar method was proposed for IFSs in.[25]

Proposition 3.7. *Let $\rho : [0,1] \to [0,1]$ be a function such that $\rho(t) \leq \sqrt{t}$ for any $t \in [0,1]$. Consider a number $\mu_\alpha \in [0,1]$. Then*

$$\alpha = \langle \sqrt{\mu_\alpha}, \rho(1-\mu_\alpha) \rangle$$

is a PFV.

Proof. Let $\mu_\alpha \in [0,1]$

$$\begin{aligned} \sqrt{\mu_\alpha} + \rho^2(1-\mu_\alpha) &\leq \mu_\alpha + \left(\sqrt{1-\mu_\alpha}\right)^2 \\ &= \mu_\alpha + 1 - \mu_\alpha \\ &= 1. \end{aligned}$$

So α is a PFV. □

We can transform a SNV into a $PFVNV$ by using Proposition 3.7 as in the following example.

Example 3.8. Consider the SNV $A = \langle 0.5, 0.3, 0.6 \rangle$ and the function $\rho : [0,1] \to [0,1]$ defined by $\rho(t) = sint$. From Proposition 3.7, we obtain a PFVNV from A as follows:

$$A = \langle (0.71, 0.48), (0.55, 0.64), (0.77, 0.39) \rangle.$$

We visualize this transformation in Figure 3.

Now we define the set operations between PFVNSs. Throughout this paper the set operations for PFSs are denoted by $\subset_{(pyt)}, \cup_{(pyt)}, \cap_{(pyt)}, (.)^{c_{(pyt)}}$" (see,[34]).

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Definition 3.9. Let

$$A=\left\{\langle x_j,\mathbb{T}_A^j,\mathbb{I}_A^j,\mathbb{F}_A^j\rangle : j=1,\ldots,n\right\}$$

and

$$B=\left\{\langle x_j,\mathbb{T}_B^j,\mathbb{I}_B^j,\mathbb{F}_B^j\rangle : j=1,\ldots,n\right\}$$

be two PFVNSs. Set operations among A and B are defined as follows.
a) $A\subset B$ if and only if for all j=1,...,n

$$\mathbb{T}_A^j \subset_{(pyt)} \mathbb{T}_B^j \text{ i.e. } \mu_{A,t}(x_j)\leq\mu_{B,t}(x_j) \text{ and } \nu_{A,t}(x_j)\geq\nu_{B,t}(x_j)$$
$$\mathbb{I}_A^j \supset_{(pyt)} \mathbb{I}_B^j \text{ i.e. } \mu_{A,i}(x_j)\geq\mu_{B,i}(x_j) \text{ and } \nu_{A,i}(x_j)\leq\nu_{B,i}(x_j)$$
$$\mathbb{F}_A^j \supset_{(pyt)} \mathbb{F}_B^j \text{ i.e. } \mu_{A,f}(x_j)\geq\mu_{B,f}(x_j) \text{ and } \nu_{A,f}(x_j)\leq\nu_{B,f}(x_j).$$

b) $A=B$ if and only if $A\subset B$ and $B\subset A$.
c) $A^c=\left\{\left\langle x_j,\mathbb{F}_A^j,\left(\mathbb{I}_A^j\right)^{c(pyt)},\mathbb{T}_A^j\right\rangle : j=1,\ldots,n\right\}$ where for all $j=1,\ldots,n$

$$\left(\mathbb{I}_A^j\right)^{c(pyt)}=\left(\nu_{A,i}(x_j),\mu_{A,i}(x_j)\right).$$

d) $A\cup B=\left\{\left\langle x_j,\mathbb{T}_A^j\cup_{(pyt)}\mathbb{T}_B^j,\mathbb{I}_A^j\cap_{(pyt)}\mathbb{I}_B^j,\mathbb{F}_A^j\cap_{(pyt)}\mathbb{F}_B^j\right\rangle : j=1,\ldots,n\right\}$ where

$$\begin{aligned}
\mathbb{T}_A^j\cup_{(pyt)}\mathbb{T}_B^j &= \Big(\max\left(\mu_{A,t}(x_j),\mu_{B,t}(x_j)\right),\min\left(\nu_{A,t}(x_j),\nu_{B,t}(x_j)\right)\Big)\\
\mathbb{I}_A^j\cap_{(pyt)}\mathbb{I}_B^j &= \Big(\min\left(\mu_{A,i}(x_j),\mu_{B,i}(x_j)\right),\max\left(\nu_{A,i}(x_j),\nu_{B,i}(x_j)\right)\Big)\\
\mathbb{F}_A^j\cap_{(pyt)}\mathbb{F}_B^j &= \Big(\min\left(\mu_{A,f}(x_j),\mu_{B,f}(x_j)\right),\max\left(\nu_{A,f}(x_j),\nu_{B,f}(x_j)\right)\Big).
\end{aligned}$$

e) $A\cap B=\left\{\left\langle x_j,\mathbb{T}_A^j\cap_{(pyt)}\mathbb{T}_B^j,\mathbb{I}_A^j\cup_{(pyt)}\mathbb{I}_B^j,\mathbb{F}_A^j\cup_{(pyt)}\mathbb{F}_B^j\right\rangle : j=1,\ldots,n\right\}$ where

$$\begin{aligned}
\mathbb{T}_A^j\cap_{(pyt)}\mathbb{T}_B^j &= \Big(\min\left(\mu_{A,t}(x_j),\mu_{B,t}(x_j)\right),\max\left(\nu_{A,t}(x_j),\nu_{B,t}(x_j)\right)\Big)\\
\mathbb{I}_A^j\cup_{(pyt)}\mathbb{I}_B^j &= \Big(\max\left(\mu_{A,i}(x_j),\mu_{B,i}(x_j)\right),\min\left(\nu_{A,i}(x_j),\nu_{B,i}(x_j)\right)\Big)\\
\mathbb{F}_A^j\cup_{(pyt)}\mathbb{F}_B^j &= \Big(\max\left(\mu_{A,f}(x_j),\mu_{B,f}(x_j)\right),\min\left(\nu_{A,f}(x_j),\nu_{B,f}(x_j)\right)\Big).
\end{aligned}$$

Example 3.10. Let $X=\{x_1,x_2\}$ and consider the PFVNSs

$$A=\left\{\left\langle x_1,(0.2,0.3),(0.7,0.4),(0.4,0.7)\right\rangle,\left\langle x_2,(0.4,0.8),(0.7,0.3),(0.7,0.1)\right\rangle\right\}$$

and

$$B=\left\{\left\langle x_1,(0.4,0.3),(0.4,0.5),(0.1,0.9)\right\rangle,\left\langle x_2,(0.6,0.5),(0.2,0.6),(0.2,0.2)\right\rangle\right\}.$$

Then $A\subset B$. On the other hand, it is easy to obtain that

$$A^c=\left\{\left\langle x_1,(0.4,0.7),(0.4,0.7),(0.2,0.3)\right\rangle,\left\langle x_2,(0.7,0.1),(0.3,0.7),(0.4,0.8)\right\rangle\right\}$$

$$A\cup B=\left\{\left\langle x_1,(0.4,0.3),(0.4,0.5),(0.1,0.9)\right\rangle,\left\langle x_2,(0.6,0.5),(0.2,0.6),(0.2,0.2)\right\rangle\right\}$$

and

$$A\cap B=\left\{\left\langle x_1,(0.2,0.3),(0.7,0.4),(0.4,0.7)\right\rangle,\left\langle x_2,(0.4,0.8),(0.7,0.3),(0.7,0.1)\right\rangle\right\}.$$

Next we show that set operations defined in Definition 3.9 satisy De Morgan's rules.

Theorem 3.11. *Let*

$$A=\left\{\langle x_j,\mathbb{T}_A^j,\mathbb{I}_A^j,\mathbb{F}_A^j\rangle : j=1,\ldots,n\right\}$$

and

$$B=\left\{\langle x_j,\mathbb{T}_B^j,\mathbb{I}_B^j,\mathbb{F}_B^j\rangle : j=1,\ldots,n\right\}$$

be two PFVNSs. The following are valid.
a) $(A\cup B)^c=A^c\cap B^c$
b) $(A\cap B)^c=A^c\cup B^c$.

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Proof. a) We get

$$
\begin{aligned}
(A\cup B)^c &= \Big(\Big\{\big\langle x_j, \mathbb{T}^j_A \cup_{(pyt)} \mathbb{T}^j_B, \mathbb{I}^j_A \cap_{(pyt)} \mathbb{I}^j_B, \mathbb{F}^j_A \cap_{(pyt)} \mathbb{F}^j_B\big\rangle : j=1,\ldots,n\Big\}\Big)^c \\
&= \Big\{\big\langle x_j, \mathbb{F}^j_A \cap_{(pyt)} \mathbb{F}^j_B, \Big(\mathbb{I}^j_A \cap_{(pyt)} \mathbb{I}^j_B\Big)^{c(pyt)}, \mathbb{T}^j_A \cup_{(pyt)} \mathbb{T}^j_B\big\rangle : j=1,\ldots,n\Big\} \\
&= \Big\{\big\langle x_j, \mathbb{F}^j_A \cap_{(pyt)} \mathbb{F}^j_B, \big(\mathbb{I}^j_A\big)^{c(pyt)} \cup_{(pyt)} \big(\mathbb{I}^j_B\big)^{c(pyt)}, \mathbb{T}^j_A \cup_{(pyt)} \mathbb{T}^j_B\big\rangle : j=1,\ldots,n\Big\} \\
&= A^c \cap B^c.
\end{aligned}
$$

b) We get

$$
\begin{aligned}
(A\cap B)^c &= \Big(\Big\{\big\langle x_j, \mathbb{T}^j_A \cap_{(pyt)} \mathbb{T}^j_B, \mathbb{I}^j_A \cup_{(pyt)} \mathbb{I}^j_B, \mathbb{F}^j_A \cup_{(pyt)} \mathbb{F}^j_B\big\rangle : j=1,\ldots,n\Big\}\Big)^c \\
&= \Big\{\big\langle x_j, \mathbb{F}^j_A \cup_{(pyt)} \mathbb{F}^j_B, \Big(\mathbb{I}^j_A \cup_{(pyt)} \mathbb{I}^j_B\Big)^{c(pyt)}, \mathbb{T}^j_A \cap_{(pyt)} \mathbb{T}^j_B\big\rangle : j=1,\ldots,n\Big\} \\
&= \Big\{\big\langle x_j, \mathbb{F}^j_A \cup_{(pyt)} \mathbb{F}^j_B, \big(\mathbb{I}^j_A\big)^{c(pyt)} \cap_{(pyt)} \big(\mathbb{I}^j_B\big)^{c(pyt)}, \mathbb{T}^j_A \cap_{(pyt)} \mathbb{T}^j_B\big\rangle : j=1,\ldots,n\Big\} \\
&= A^c \cup B^c.
\end{aligned}
$$

□

Now we introduce some algebraic operations for PFVNVs via Pythagorean fuzzy t-norms and t-conorms.

Definition 3.12. Let $\alpha = \langle \mathbb{T}_\alpha, \mathbb{I}_\alpha, \mathbb{F}_\alpha\rangle$ and $\beta = \langle \mathbb{T}_\beta, \mathbb{I}_\beta, \mathbb{F}_\beta\rangle$ be two PFVNVs, let $\mathcal{T}$ be a Pythagorean fuzzy t-norm and let $\mathcal{S}$ be the dual Pythagorean fuzzy t-conorm of $\mathcal{T}$ with respect to a Pythagorean fuzzy negator $\mathcal{N}$. Then

$$\alpha \oplus \beta := \Big\langle \Big(\mathcal{S}\big(\mathbb{T}_\alpha, \mathbb{T}_\beta\big), \mathcal{T}\big(\mathbb{I}_\alpha, \mathbb{I}_\beta\big), \mathcal{T}\big(\mathbb{F}_\alpha, \mathbb{F}_\beta\big)\Big)\Big\rangle$$

and

$$\alpha \otimes \beta := \Big\langle \Big(\mathcal{T}\big(\mathbb{T}_\alpha, \mathbb{T}_\beta\big), \mathcal{S}\big(\mathbb{I}_\alpha, \mathbb{I}_\beta\big), \mathcal{S}\big(\mathbb{F}_\alpha, \mathbb{F}_\beta\big)\Big)\Big\rangle$$

Remark 3.13. If g is the additive generator of a continuous Archimedean t-norm and $h(t) = g(\sqrt{1-t^2})$, then from Corollary 2.14 we get

$$
\begin{aligned}
\alpha \oplus \beta = &\Big\langle \Big(h^{-1}\big(h(\mu_{\alpha,t}) + h(\mu_{\beta,t})\big), g^{-1}\big(g(\nu_{\alpha,t}) + g(\nu_{\beta,t})\big)\Big), \\
&\Big(g^{-1}\big(g(\mu_{\alpha,i}) + g(\mu_{\beta,i})\big), h^{-1}\big(h(\nu_{\alpha,i}) + h(\nu_{\beta,i})\big)\Big), \\
&\Big(g^{-1}\big(g(\mu_{\alpha,f}) + g(\mu_{\beta,f})\big), h^{-1}\big(h(\nu_{\alpha,f}) + h(\nu_{\beta,f})\big)\Big)\Big\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha \otimes \beta = &\Big\langle \Big(g^{-1}\big(g(\mu_{\alpha,t}) + g(\mu_{\beta,t})\big), h^{-1}\big(h(\nu_{\alpha,t}) + h(\nu_{\beta,t})\big)\Big), \\
&\Big(h^{-1}\big(h(\mu_{\alpha,i}) + h(\mu_{\beta,i})\big), g^{-1}\big(g(\nu_{\alpha,i}) + g(\nu_{\beta,i})\big)\Big), \\
&\Big(h^{-1}\big(h(\mu_{\alpha,f}) + h(\mu_{\beta,f})\big), g^{-1}\big(g(\nu_{\alpha,f}) + g(\nu_{\beta,f})\big)\Big)\Big\rangle.
\end{aligned}
$$

Following proposition is the validation of that the sum and the product of two PFVNVs are also PFVNVs.

Proposition 3.14. *Let α and β be two PFVNVs, let $\mathcal{T}$ be A Pythagorean fuzzy t-norm and let $\mathcal{S}$ be the dual Pythagorean fuzzy t-conorm with respect to a Pythagorean fuzzy negator $\mathcal{N}$. Then, $\alpha \oplus \beta$ and $\alpha \otimes \beta$ are also PFVNVs.*

Proof. Since $\mathcal{T}$ and $\mathcal{S}$ have range P^*, $\alpha \oplus \beta$ and $\alpha \otimes \beta$ are PFVNVs. □

Next we define multiplication by non-negative constant and non-negative power of PFVNVs using additive generators of continuous Archimedean t-norm and t-conorm on $[0,1]$.

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Definition 3.15. Let $\alpha = \langle \mathbb{T}_\alpha, \mathbb{I}_\alpha, \mathbb{F}_\alpha \rangle$ be a PFVNV, let $\lambda \geq 0$, let $g : [0,1] \to [0,\infty]$ be the additive generator of a continuous Archimedean t-norm and let $h(t) = g(\sqrt{1-t^2})$. Then

$$\begin{aligned}\lambda\alpha &= \Big\langle \Big(h^{-1}\big(\lambda h(\mu_{\alpha,t})\big), g^{-1}\big(\lambda g(\nu_{\alpha,t})\big)\Big),\\ &\quad \Big(g^{-1}\big(\lambda g(\mu_{\alpha,i})\big), h^{-1}\big(\lambda h(\nu_{\alpha,i})\big)\Big),\\ &\quad \Big(g^{-1}\big(\lambda g(\mu_{\alpha,f})\big), h^{-1}\big(\lambda h(\nu_{\alpha,f})\big)\Big)\Big\rangle\end{aligned}$$

and

$$\begin{aligned}\alpha^\lambda &= \Big\langle \Big(g^{-1}\big(\lambda g(\mu_{\alpha,t})\big), h^{-1}\big(\lambda h(\nu_{\alpha,t})\big)\Big),\\ &\quad \Big(h^{-1}\big(\lambda h(\mu_{\alpha,i})\big), g^{-1}\big(\lambda g(\nu_{\alpha,i})\big)\Big),\\ &\quad \Big(h^{-1}\big(\lambda h(\mu_{\alpha,f})\big), g^{-1}\big(\lambda g(\nu_{\alpha,f})\big)\Big)\Big\rangle.\end{aligned}$$

The following proposition verifies that multiplication by constant and power of PFVNVs are also PFVNVs.

Proposition 3.16. *Let α be a PFVNV, let $\lambda \geq 0$, let $g : [0,1] \to [0,\infty]$ be the additive generator of a continuous Archimedean t-norm and let $h(t) = g(\sqrt{1-t^2})$. Then $\lambda\alpha$ and α^λ are PFVNVs.*

Proof. We know that $h^{-1}(t) = \sqrt{1-[g^{-1}(t)]^2}$ and $g(t) = h(\sqrt{1-t^2})$. Since $\mu_{\alpha,t} \leq \sqrt{1-\nu_{\alpha,t}^2}$ and h, h^{-1} are increasing, we obtain

$$\begin{aligned} 0 &\leq \big[h^{-1}(\lambda h(\mu_{\alpha,t}))\big]^2 + \big[g^{-1}(\lambda g(\nu_{\alpha,t}))\big]^2\\ &\leq \Big[h^{-1}\Big(\lambda h\Big(\sqrt{1-\nu_{\alpha,t}^2}\Big)\Big)\Big]^2 + \big[g^{-1}(\lambda g(\nu_{\alpha,t}))\big]^2\\ &= 1 - \Big[g^{-1}\Big(\lambda h\Big(\sqrt{1-\nu_{\alpha,t}^2}\Big)\Big]^2 + \Big[g^{-1}(\lambda g(\nu_{\alpha,t}))\Big]^2\\ &= 1 - \big[g^{-1}(\lambda g(\nu_{\alpha,t}))\big]^2 + \big[g^{-1}(\lambda g(\nu_{\alpha,t}))\big]^2\\ &= 1\end{aligned}$$

which yields that $\mathbb{T}_{\lambda\alpha}$ is a PFV. Similarly, it can be easily shown that $\mathbb{I}_{\lambda\alpha}, \mathbb{F}_{\lambda\alpha}.\mathbb{T}_{\alpha^\lambda}, \mathbb{I}_{\alpha^\lambda}$ and $\mathbb{F}_{\alpha^\lambda}$ are PFVs. Therefore $\lambda\alpha$ is a PFVNV. □

Following theorem gives some basic properties of algebraic operations.

Theorem 3.17. *Let $\alpha = \langle \mathbb{T}_\alpha, \mathbb{I}_\alpha, \mathbb{F}_\alpha \rangle$, $\beta = \langle \mathbb{T}_\beta, \mathbb{I}_\beta, \mathbb{F}_\beta \rangle$ and $\theta = \langle \mathbb{T}_\theta, \mathbb{I}_\theta, \mathbb{F}_\theta \rangle$ be PFVNVs, let $\lambda, \gamma \geq 0$, let g be the additive generator of a continuous Archimedean t-norm and let $h(t) = g(\sqrt{1-t^2})$. Then,*
i) $\alpha \oplus \beta = \beta \oplus \alpha$
ii) $\alpha \otimes \beta = \beta \otimes \alpha$
iii) $(\alpha \oplus \beta) \oplus \theta = \alpha \oplus (\beta \oplus \theta)$
iv) $(\alpha \otimes \beta) \otimes \theta = \alpha \otimes (\beta \otimes \theta)$
v) $\lambda(\alpha \oplus \beta) = \lambda\alpha \oplus \lambda\beta$
vi) $(\lambda + \gamma)\alpha = \alpha\lambda \oplus \gamma\alpha$
vii) $(\alpha \otimes \beta)^\lambda = \alpha^\lambda \otimes \beta^\lambda$
viii) $\alpha^\lambda \otimes \alpha^\gamma = \alpha^{\lambda+\gamma}$.

Proof. (i) and (ii) are trivial.
iii) It is clear from Definition 3.12 that

$$\mathbb{T}_{\alpha\oplus\beta} = \Big(h^{-1}\big(h(\mu_{\alpha,t}) + h(\mu_{\beta,t})\big), g^{-1}\big(g(\nu_{\alpha,t}) + g(\nu_{\beta,t})\big)\Big).$$

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Therefore, we obtain

$$\begin{aligned}
\mathbb{T}_{(\alpha\oplus\beta)\oplus\theta} &= \mathcal{S}(\mathbb{T}_{\alpha\oplus\beta},\mathbb{T}_{\theta}) \\
&= \Big(h^{-1}\Big(h\big(h^{-1}\big(h(\mu_{\alpha,t})+h(\mu_{\beta,t})\big)+h\big(\mu_{\theta,t}\big)\Big)\Big), \\
&\quad g^{-1}\Big(g\big(g^{-1}\big(g(\nu_{\alpha,t})+g(\nu_{\beta,t})\big)+g\big(\nu_{\theta,t}\big)\Big)\Big) \\
&= \Big(h^{-1}\Big(h(\mu_{\alpha,t})+h(\mu_{\beta,t})+h(\mu_{\theta,t})\Big), \\
&\quad g^{-1}\Big(g(\nu_{\alpha,t})+g(\nu_{\beta,t})+g\big(\nu_{\theta,t}\big)\Big)\Big) \\
&= \Big(h^{-1}\Big(h(\mu_{\alpha,t})+h\Big(h^{-1}\big(h(\mu_{\beta,t})+h(\mu_{\theta,t})\big)\Big)\Big), \\
&\quad g^{-1}\Big(g(\nu_{\alpha,t})+g\Big(g^{-1}\big(g(\nu_{\beta,t})+g\big(\nu_{\theta,t}\big)\big)\Big)\Big) \\
&= \mathcal{S}(\mathbb{T}_{\alpha},\mathbb{T}_{\beta\oplus\theta}) \\
&= \mathbb{T}_{\alpha\oplus(\beta\oplus\theta)}.
\end{aligned}$$

Similarly it can be easily obtained that $\mathbb{I}_{(\alpha\oplus\beta)\oplus\theta}=\mathbb{I}_{\alpha\oplus(\beta\oplus\theta)}$ and $\mathbb{F}_{(\alpha\oplus\beta)\oplus\theta}=\mathbb{F}_{\alpha\oplus(\beta\oplus\theta)}$.
iv) It is clear from Definition 3.12 that

$$\mathbb{T}_{\alpha\otimes\beta}=\Big(g^{-1}\big(g(\mu_{\alpha,t})+g(\mu_{\beta,t})\big),h^{-1}\big(h(\nu_{\alpha,t})+h(\nu_{\beta,t})\big)\Big).$$

Therefore, we have

$$\begin{aligned}
\mathbb{T}_{(\alpha\otimes\beta)\otimes\theta} &= \mathcal{T}(\mathbb{T}_{\alpha\otimes\beta},\mathbb{T}_{\theta}) \\
&= \Big(g^{-1}\Big(g\big(g^{-1}\big(g(\mu_{\alpha,t})+g(\mu_{\beta,t})\big)+g\big(\mu_{\theta,t}\big)\Big)\Big), \\
&\quad h^{-1}\Big(h\big(h^{-1}\big(h(\nu_{\alpha,t})+h(\nu_{\beta,t})\big)+h\big(\nu_{\theta,t}\big)\Big)\Big) \\
&= \Big(g^{-1}\Big(g(\mu_{\alpha,t})+g(\mu_{\beta,t})+g(\mu_{\theta,t})\Big), \\
&\quad h^{-1}\Big(h(\nu_{\alpha,t})+h(\nu_{\beta,t})+h\big(\nu_{\theta,t}\big)\Big)\Big) \\
&= \Big(g^{-1}\Big(g(\mu_{\alpha,t})+g\Big(g^{-1}\big(g(\mu_{\beta,t})+g(\mu_{\theta,t})\big)\Big)\Big), \\
&\quad h^{-1}\Big(h(\nu_{\alpha,t})+h\Big(h^{-1}\big(h(\nu_{\beta,t})+h\big(\nu_{\theta,t}\big)\big)\Big)\Big) \\
&= \mathcal{T}(\mathbb{T}_{\alpha},\mathbb{T}_{\beta\otimes\theta}) \\
&= \mathbb{T}_{\alpha\otimes(\beta\otimes\theta)}.
\end{aligned}$$

Similarly it can be seen that $\mathbb{I}_{(\alpha\otimes\beta)\otimes\theta}=\mathbb{I}_{\alpha\otimes(\beta\otimes\theta)}$ and $\mathbb{F}_{(\alpha\otimes\beta)\otimes\theta}=\mathbb{F}_{\alpha\otimes(\beta\otimes\theta)}$.
v) It is clear from Definition 3.12 and Definition 3.15 that

$$\mathbb{T}_{\alpha\oplus\beta}=(\mu_{\alpha\oplus\beta,t},\nu_{\alpha\oplus\beta,t})=\Big(h^{-1}\big(h(\mu_{\alpha,t})+h(\mu_{\beta,t})\big),g^{-1}\big(g(\nu_{\alpha,t})+g(\nu_{\beta,t})\big)\Big)$$

and

$$\mathbb{T}_{\lambda\alpha}=(\mu_{\lambda\alpha,t},\nu_{\lambda\alpha,t})=\Big(h^{-1}\big(\lambda h\big(\mu_{\alpha,t}\big)\big),g^{-1}\big(\lambda g\big(\nu_{\alpha,t}\big)\big)\Big).$$

Therefore, we get

$$\begin{aligned}
\mathbb{T}_{\lambda(\alpha\oplus\beta)} &= \Big(h^{-1}\big(\lambda h(\mu_{\alpha\oplus\beta,t})\big),g^{-1}\big(\lambda g(\nu_{\alpha\oplus\beta,t})\big)\Big) \\
&= \Big(h^{-1}\big(\lambda h(h^{-1}\big(h(\mu_{\alpha,t})+h(\mu_{\beta,t})\big))\big),g^{-1}\big(\lambda g(g^{-1}\big(g(\nu_{\alpha,t})+g(\nu_{\beta,t})\big))\big)\Big) \\
&= \Big(h^{-1}\big(\lambda h(\mu_{\alpha,t})+\lambda h(\mu_{\beta,t})\big)\big),g^{-1}\big(\lambda g(\nu_{\alpha,t})+\lambda g(\nu_{\beta,t})\big)\big)\Big) \\
&= \Big(h^{-1}\big(h\big(h^{-1}\big(\lambda h(\mu_{\alpha,t})\big)\big)+h\big(h^{-1}\big(\lambda h(\mu_{\beta,t})\big)\big)\big), \\
&\quad g^{-1}\big(g\big(g^{-1}\big(\lambda g(\nu_{\alpha,t})\big)\big)+g\big(g^{-1}\big(\lambda g(\nu_{\beta,t})\big)\big)\big)\Big) \\
&= \Big(h^{-1}\big(h\big(\mu_{\lambda\alpha,t}\big)+h\big(\mu_{\lambda\beta,t}\big)\big),g^{-1}\big(g\big(\nu_{\lambda\alpha,t}\big)+g\big(\nu_{\lambda\beta,t}\big)\big)\Big) \\
&= \mathbb{T}_{\lambda\alpha\oplus\lambda\beta}.
\end{aligned}$$

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Similarly it can be shown that $\mathbb{I}_{\lambda(\alpha\oplus\beta)} = \mathbb{I}_{\lambda\alpha\oplus\lambda\beta}$ and $\mathbb{F}_{\lambda(\alpha\oplus\beta)} = \mathbb{F}_{\lambda\alpha\oplus\lambda\beta}$.
vi) It is clear that

$$\begin{aligned}
\mathbb{T}_{(\lambda+\gamma)\alpha} &= \Big(h^{-1}\big((\lambda+\gamma)h\big(\mu_{\alpha,t}\big)\big), g^{-1}\big((\lambda+\gamma)g\big(\nu_{\alpha,t}\big)\big)\Big) \\
&= \Big(h^{-1}\big(\lambda h\big(\mu_{\alpha,t}\big)+\gamma h\big(\mu_{\alpha,t}\big)\big), g^{-1}\big(\lambda g\big(\nu_{\alpha,t}\big)+\gamma g\big(\nu_{\alpha,t}\big)\big)\Big) \\
&= \Big(h^{-1}\big(h\big(h^{-1}\big(\lambda h\big(\mu_{\alpha,t}\big)\big)\big)+h\big(h^{-1}\big(\gamma h\big(\mu_{\alpha,t}\big)\big)\big)\big), \\
&\quad g^{-1}\big(g\big(g^{-1}\big(\lambda g\big(\nu_{\alpha,t}\big)\big)\big)+g\big(g^{-1}\big(\gamma g\big(\nu_{\alpha,t}\big)\big)\big)\big)\Big) \\
&= \Big(h^{-1}\big(h\big(\mu_{\lambda\alpha,t}\big)+h\big(\mu_{\gamma\alpha,t}\big)\big), g^{-1}\big(g\big(\nu_{\lambda\alpha,t}\big)+g\big(\nu_{\gamma\alpha,t}\big)\big)\Big) \\
&= \mathbb{T}_{\lambda\alpha\oplus\gamma\alpha}.
\end{aligned}$$

Similarly it can be obtained that $\mathbb{I}_{(\lambda+\gamma)\alpha} = \mathbb{I}_{\lambda\alpha\oplus\gamma\alpha}$ and $\mathbb{F}_{(\lambda+\gamma)\alpha} = \mathbb{F}_{\lambda\alpha\oplus\gamma\alpha}$.
vii) We know that

$$\mathbb{T}_{\alpha\otimes\beta} = (\mu_{\alpha\otimes\beta,t}, \nu_{\alpha\otimes\beta,t}) = \Big(g^{-1}\big(g(\mu_{\alpha,t})+g(\mu_{\beta,t})\big), h^{-1}\big(h(\nu_{\alpha,t})+h(\nu_{\beta,t})\big)\Big)$$

and

$$\mathbb{T}_{\alpha^\lambda} = (\mu_{\alpha^\lambda,t}, \nu_{\alpha^\lambda,t}) = \Big(g^{-1}\big(\lambda g\big(\mu_{\alpha,t}\big)\big), h^{-1}\big(\lambda h\big(\nu_{\alpha,t}\big)\big)\Big).$$

Therefore, we obtain

$$\begin{aligned}
\mathbb{T}_{(\alpha\otimes\beta)^\lambda} &= \Big(g^{-1}\big(\lambda g(\mu_{\alpha\otimes\beta,t})\big), h^{-1}\big(\lambda h(\nu_{\alpha\otimes\beta,t})\big)\Big) \\
&= \Big(g^{-1}\big(\lambda g(g^{-1}\big(g(\mu_{\alpha,t})+g(\mu_{\beta,t})\big))\big), h^{-1}\big(\lambda h(h^{-1}\big(h(\nu_{\alpha,t})+h(\nu_{\beta,t})\big))\big)\Big) \\
&= \Big(g^{-1}\big(\lambda g(\mu_{\alpha,t})+\lambda g(\mu_{\beta,t})\big)\big), h^{-1}\big(\lambda h(\nu_{\alpha,t})+\lambda h(\nu_{\beta,t})\big)\big)\Big) \\
&= \Big(g^{-1}\big(g\big(g^{-1}\big(\lambda g(\mu_{\alpha,t})\big)+g\big(g^{-1}\big(\lambda g(\mu_{\beta,t})\big)\big)\big), \\
&\quad h^{-1}\big(h\big(h^{-1}\big(\lambda h(\nu_{\alpha,t})\big)+h\big(h^{-1}\big(\lambda h(\nu_{\beta,t})\big)\big)\big)\Big) \\
&= \Big(g^{-1}\big(g\big(\mu_{\alpha^\lambda,t}\big)+g\big(\mu_{\beta^\lambda,t}\big)\big), h^{-1}\big(h\big(\nu_{\alpha^\lambda,t}\big)+h\big(\nu_{\beta^\lambda,t}\big)\big)\Big) \\
&= \mathbb{T}_{\alpha^\lambda\oplus\beta^\lambda}.
\end{aligned}$$

Similarly it can be seen that $\mathbb{I}_{(\alpha\otimes\beta)^\lambda} = \mathbb{I}_{\alpha^\lambda\oplus\beta^\lambda}$ and $\mathbb{F}_{(\alpha\otimes\beta)^\lambda} = \mathbb{F}_{\alpha^\lambda\oplus\beta^\lambda}$.
viii) We have

$$\begin{aligned}
\mathbb{T}_{\alpha^{\lambda+\gamma}} &= \Big(g^{-1}\big((\lambda+\gamma)g\big(\mu_{\alpha,t}\big)\big), h^{-1}\big((\lambda+\gamma)h\big(\nu_{\alpha,t}\big)\big)\Big) \\
&= \Big(g^{-1}\big(\lambda g\big(\mu_{\alpha,t}\big)+\gamma g\big(\mu_{\alpha,t}\big)\big), h^{-1}\big(\lambda h\big(\nu_{\alpha,t}\big)+\gamma h\big(\nu_{\alpha,t}\big)\big)\Big) \\
&= \Big(g^{-1}\big(g\big(g^{-1}\big(\lambda g\big(\mu_{\alpha,t}\big)\big)\big)+g\big(h^{-1}\big(\gamma g\big(\mu_{\alpha,t}\big)\big)\big)\big), \\
&\quad h^{-1}\big(h\big(h^{-1}\big(\lambda h\big(\nu_{\alpha,t}\big)\big)\big)+h\big(h^{-1}\big(\gamma h\big(\nu_{\alpha,t}\big)\big)\big)\big)\Big) \\
&= \Big(g^{-1}\big(g\big(\mu_{\alpha^\lambda,t}\big)+g\big(\mu_{\alpha^\gamma,t}\big)\big), h^{-1}\big(h\big(\nu_{\alpha^\lambda,t}\big)+h\big(\nu_{\alpha^\gamma,t}\big)\big)\Big) \\
&= \mathbb{T}_{\alpha^\lambda\otimes\alpha^\gamma}.
\end{aligned}$$

Similarly it can be obtained that $\mathbb{I}_{\alpha^{\lambda+\gamma}} = \mathbb{I}_{\alpha^\lambda\otimes\alpha^\gamma}$ and $\mathbb{F}_{\alpha^{\lambda+\gamma}} = \mathbb{F}_{\alpha^\lambda\otimes\alpha^\gamma}$. □

4 Weighted Aggregation Operators for PFVNVs

Aggregation operators have a vital role while transforming input values represented by fuzzy values to a single output value. In this section, we introduce a weighted arithmetic aggregation operator and a weighted geometric aggregation operator for collections of PFVNVs by using algebraic operations given in Section 3.

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4.1 A Weighted Arithmetic Aggregation Operator

Definition 4.1. Let $\{\alpha_j = \langle \mathbb{T}_{\alpha_j}, \mathbb{I}_{\alpha_j}, \mathbb{F}_{\alpha_j} \rangle : j = 1, \ldots, n\}$ be a collection of PFVNVs. A weighted arithmetic aggregation operator $WA - PFVNV$ is defined by

$$WA - PFVNV(\alpha_1, \ldots, \alpha_n) := \bigoplus_{j=1}^{n} w_j \alpha_j$$

where $w = (w_1, \ldots, w_n)^T$ is a weight vector such that $0 \leq w_j \leq 1$ for any $j = 1, \ldots, n$ and $\sum_{j=1}^{n} w_j = 1$.

As seen in the next theorem, weighted arithmetic aggregation operators can be expressed via additive generators whenever t-norm and t-conorm are continuous Archimedean.

Theorem 4.2. *Let $\{\alpha_j = \langle \mathbb{T}_{\alpha_j}, \mathbb{I}_{\alpha_j}, \mathbb{F}_{\alpha_j} \rangle : j = 1, \ldots, n\}$ be a collection of PFVNVs, let $w = (w_1, \ldots, w_n)^T$ be a weight vector such that $0 \leq w_j \leq 1$ for any $j = 1, \ldots, n$ and $\sum_{j=1}^{n} w_j = 1$ and let g be the additive generator of a continuous Archimedean t-norm and let $h(t) = g(\sqrt{1-t^2})$. Then $WA - PFVVNV(\alpha_1, \ldots, \alpha_n)$ is a $PFVNV$ and we have*

$$\begin{aligned} WA - PFVNV(\alpha_1, \ldots, \alpha_n) \;=\; & \Big\langle \Big(h^{-1}\Big(\sum_{j=1}^{n} w_j h(\mu_{\alpha_j,t})\Big), g^{-1}\Big(\sum_{j=1}^{n} w_j g(\nu_{\alpha_j,t})\Big)\Big), \\ & \Big(g^{-1}\Big(\sum_{j=1}^{n} w_j g(\mu_{\alpha_j,i})\Big), h^{-1}\Big(\sum_{j=1}^{n} w_j h(\nu_{\alpha_j,i})\Big)\Big), \\ & \Big(g^{-1}\Big(\sum_{j=1}^{n} w_j g(\mu_{\alpha_j,f})\Big), h^{-1}\Big(\sum_{j=1}^{n} w_j h(\nu_{\alpha_j,f})\Big)\Big)\Big\rangle. \end{aligned}$$

Proof. It is clear from Propositions 3.14 and 3.16 that $WA - PFVVNV(\alpha_1, \ldots, \alpha_n)$ is a PFVNV. Using mathematical induction we show that the second part is valid. If $n = 2$, we obtain

$$\begin{aligned} \mathbb{T}_{w_1\alpha_1 \oplus w_2\alpha_2} &= \mathcal{S}(\mathbb{T}_{w_1\alpha_1}, \mathbb{T}_{w_2\alpha_2}) \\ &= \Big(h^{-1}\big(h(\mu_{w_1\alpha_1}) + h(\mu_{w_2\alpha_2})\big), g^{-1}\big(g(\nu_{w_1\alpha_1}) + g(\nu_{w_2\alpha_2})\big)\Big) \\ &= \Big(h^{-1}\big(h(h^{-1}(w_1 h(\mu_{\alpha_1}))) + h(h^{-1}(w_2 h(\mu_{\alpha_2})))\big), \\ &\quad g^{-1}\big(g(g^{-1}(w_1 g(\nu_{\alpha_1})) + g(g^{-1}(w_2 g(\nu_{\alpha_2})))\big)\Big) \\ &= \Big(h^{-1}\big(w_1 h(\mu_{\alpha_1}) + w_2 h(\mu_{\alpha_2})\big), g^{-1}\big(w_1 g(\nu_{\alpha_1}) + w_2 g(\nu_{\alpha_2})\big)\Big) \\ &= \Big(h^{-1}\Big(\sum_{j=1}^{2} w_j h(\mu_{\alpha_j})\Big), g^{-1}\Big(\sum_{j=1}^{2} w_j g(\nu_{\alpha_j})\Big)\Big). \end{aligned}$$

Now assume that

$$\mathbb{T}_{A_{n-1}} = \left(h^{-1}\left(\sum_{j=1}^{n-1} w_j h(\mu_{\alpha_j})\right), g^{-1}\left(\sum_{j=1}^{n-1} w_j g(\nu_{\alpha_j})\right)\right),$$

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where $A_n := \bigoplus_{j=1}^{n} w_j\alpha_j$. Then we obtain

$$\begin{aligned}
\mathbb{T}_{A_n} &= \mathbb{T}_{A_{n-1}} \oplus w_n\alpha_n \\
&= \mathcal{S}\big(\mathbb{T}_{A_{n-1}}, \mathbb{T}_{w_n\alpha_n}\big) \\
&= \mathcal{S}\Big(\Big(h^{-1}\Big(\sum_{j=1}^{n-1} w_j h(\mu_{\alpha_j})\Big), g^{-1}\Big(\sum_{j=1}^{n-1} w_j g(\nu_{\alpha_j})\Big)\Big), \\
&\Big(h^{-1}(w_n h(\mu_{\alpha_n,t})), g^{-1}(w_n g(\mu_{\alpha_n,t}))\Big)\Big) \\
&= \Big(h^{-1}\Big(h\Big(h^{-1}\Big(\sum_{j=1}^{n-1} w_j h(\mu_{\alpha_j})\Big)\Big) + h\Big(h^{-1}(w_n h(\mu_{\alpha_n,t}))\Big)\Big), \\
&g^{-1}\Big(g\Big(g^{-1}\Big(\sum_{j=1}^{n-1} w_j g(\nu_{\alpha_j})\Big)\Big) + g\Big(g^{-1}(w_n g(\mu_{\alpha_n,t}))\Big)\Big) \\
&= \Big(h^{-1}\Big(\sum_{j=1}^{n-1} w_j h(\mu_{\alpha_j}) + w_n h(\mu_{\alpha_n,t})\Big), g^{-1}\Big(\sum_{j=1}^{n-1} w_j g(\nu_{\alpha_j}) + w_n g(\mu_{\alpha_n,t})\Big)\Big) \\
&= \Big(h^{-1}\Big(\sum_{j=1}^{n} w_j h(\mu_{\alpha_j})\Big), g^{-1}\Big(\sum_{j=1}^{n} w_j g(\nu_{\alpha_j})\Big)\Big).
\end{aligned}$$

Similar proof is also valid for indeterminacy and falsity. □

Remark 4.3. Using particular additive generators we obtain some particular cases of $WA - PFVNV$ as follows:

a) Let g be the additive generator defined by $g(t) = -\log t^2$. Then we get Algebraic weighted arithmetic aggregation operator given by

$$\begin{aligned}
WA_A - PFVNV(\alpha_1, \ldots, \alpha_n) \quad = \quad & \Big\langle\Big(\sqrt{1 - \prod_{j=1}^{n}(1 - \mu^2_{\alpha_j,t})^{w_j}}, \prod_{j=1}^{n} \nu^{w_j}_{\alpha_j,t}\Big), \\
& \Big(\prod_{j=1}^{n} \mu^{w_j}_{\alpha_j,i}, \sqrt{1 - \prod_{j=1}^{n}(1 - \nu^2_{\alpha_j,i})^{w_j}}\Big), \\
& \Big(\prod_{j=1}^{n} \mu^{w_j}_{\alpha_j,f}, \sqrt{1 - \prod_{j=1}^{n}(1 - \nu^2_{\alpha_j,f})^{w_j}}\Big)\Big\rangle.
\end{aligned}$$

b) Let g be the additive generator defined by $g(t) = \log(\frac{2-t^2}{t^2})$. Then we get Einstein weighted arithmetic aggregation operator given by

$$\begin{aligned}
WA_E - PFVNV(\alpha_1, \ldots, \alpha_n) \quad = \quad & \Big\langle\Big(\sqrt{\frac{\prod_{j=1}^{n}(1+\mu^2_{\alpha_j,t})^{w_j} - \prod_{j=1}^{n}(1-\mu^2_{\alpha_j,t})^{w_j}}{\prod_{j=1}^{n}(1+\mu^2_{\alpha_j,t})^{w_j} + \prod_{j=1}^{n}(1-\mu^2_{\alpha_j,t})^{w_j}}}, \frac{\sqrt{2}\prod_{j=1}^{n}\nu^{w_i}_{\alpha_j,t}}{\sqrt{\prod_{j=1}^{n}(2-\nu^2_{\alpha_j,t})^{w_j} + \prod_{j=1}^{n}(\nu^2_{\alpha_i,t})^{w_j}}}\Big), \\
& \Big(\frac{\sqrt{2}\prod_{j=1}^{n}\mu^{w_j}_{\alpha_j,i}}{\sqrt{\prod_{j=1}^{n}(2-\mu^2_{\alpha_j,i})^{w_j} + \prod_{j=1}^{n}(\mu^2_{\alpha_j,i})^{w_j}}}, \sqrt{\frac{\prod_{j=1}^{n}(1+\nu^2_{\alpha_j,i})^{w_j} - \prod_{j=1}^{n}(1-\nu^2_{\alpha_j,i})^{w_j}}{\prod_{j=1}^{n}(1+\nu^2_{\alpha_j,i})^{w_j} + \prod_{j=1}^{n}(1-\nu^2_{\alpha_j,i})^{w_j}}}\Big), \\
& \Big(\frac{\sqrt{2}\prod_{j=1}^{n}\mu^{w_j}_{\alpha_j,f}}{\sqrt{\prod_{j=1}^{n}(2-\mu^2_{\alpha_j,f})^{w_j} + \prod_{j=1}^{n}(\mu^2_{\alpha_j,f})^{w_j}}}, \sqrt{\frac{\prod_{j=1}^{n}(1+\nu^2_{\alpha_j,f})^{w_j} - \prod_{j=1}^{n}(1-\nu^2_{\alpha_j,f})^{w_j}}{\prod_{j=1}^{n}(1+\nu^2_{\alpha_j,f})^{w_j} + \prod_{j=1}^{n}(1-\nu^2_{\alpha_j,f})^{w_j}}}\Big)\Big\rangle.
\end{aligned}$$

4.2 Weighted Geometric Aggregation Operator

Definition 4.4. Let $\{\alpha_j = \langle \mathbb{T}_{\alpha_j}, \mathbb{I}_{\alpha_j}, \mathbb{F}_{\alpha_j}\rangle : j = 1, \ldots, n\}$ be a collection of PFVNVs. A weighted geometric aggregation operator $WG - PFVVNV$ is defined by

$$WG - PFVVNV(\alpha_1, \ldots, \alpha_n) := \bigotimes_{j=1}^{n} \alpha_j^{w_j}$$

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where $w = (w_1, \ldots, w_n)^T$ is a weight vector such that $0 \leq w_j \leq 1$ for any $j = 1, \ldots, n$ and $\sum_{j=1}^{n} w_j = 1$.

As seen in the next theorem, weighted geometric aggregation operators can be expressed via additive generators whenever t-norm and t-conorm are continuous Archimedean.

Theorem 4.5. *Let $\{\alpha_j = \langle \mathbb{T}_{\alpha_j}, \mathbb{I}_{\alpha_j}, \mathbb{F}_{\alpha_j} \rangle : j = 1, \ldots, n\}$ be a collection of PFVNVs, let $w = (w_1, \ldots, w_n)^T$ be a weight vector such that $0 \leq w_j \leq 1$ for any $j = 1, \ldots, n$ and $\sum_{j=1}^{n} w_j = 1$, let g be the additive generator of a continuous Archimedean t-norm and let $h(t) = g(\sqrt{1-t^2})$. Then $WG - PFVVNV(\alpha_1, \ldots, \alpha_n)$ is a PFVNV and*

$$\begin{aligned} WG - PFVVNV(\alpha_1, \ldots, \alpha_n) &= \Big\langle \Big(g^{-1}\Big(\sum_{j=1}^{n} w_j g(\mu_{\alpha_j,t}) \Big), h^{-1}\Big(\sum_{j=1}^{n} w_j h(\nu_{\alpha_j,t}) \Big) \Big), \\ &\quad \Big(h^{-1}\Big(\sum_{j=1}^{n} w_j h(\mu_{\alpha_j,i}) \Big), g^{-1}\Big(\sum_{j=1}^{n} w_j g(\nu_{\alpha_j,i}) \Big) \Big), \\ &\quad \Big(h^{-1}\Big(\sum_{j=1}^{n} w_j h(\mu_{\alpha_j,f}) \Big), g^{-1}\Big(\sum_{j=1}^{n} w_j g(\nu_{\alpha_j,f}) \Big) \Big) \Big\rangle. \end{aligned}$$

Proof. It can be proved similar to Theorem 4.2. □

Remark 4.6. By using particular additive generators we also obtain some particular cases of $WG - PFVNV$ as follows.

a) Let g be the additive generator defined by $g(t) = -\log t^2$. Then we get Algebraic weighted geometric aggregation operator given by

$$\begin{aligned} WG_A - PFVVNV(\alpha_1, \ldots, \alpha_n) &= \Bigg\langle \Bigg(\prod_{j=1}^{n} \mu_{\alpha_j,t}^{w_j}, \sqrt{1 - \prod_{j=1}^{n} (1 - \nu_{\alpha_j,t}^2)^{w_j}} \Bigg), \\ &\quad \Bigg(\sqrt{1 - \prod_{j=1}^{n} (1 - \mu_{\alpha_j,i}^2)^{w_j}}, \prod_{j=1}^{n} \nu_{\alpha_j,i}^{w_j} \Bigg), \\ &\quad \Bigg(\sqrt{1 - \prod_{j=1}^{n} (1 - \mu_{\alpha_j,f}^2)^{w_j}}, \prod_{j=1}^{n} \nu_{\alpha_j,f}^{w_j} \Bigg) \Bigg\rangle. \end{aligned}$$

b) Let g be the additive generator defined by $g(t) = \log(\frac{2-t^2}{t^2})$. Then we get Einstein weighted geometric aggregation operator given by

$$\begin{aligned} WG_E - PFVVNV(\alpha_1, \ldots, \alpha_n) &= \Bigg\langle \Bigg(\frac{\sqrt{2} \prod_{j=1}^{n} \mu_{\alpha_j,t}^{w_j}}{\sqrt{\prod_{j=1}^{n} (2 - \mu_{\alpha_j,t}^2)^{w_j} + \prod_{j=1}^{n} (\mu_{\alpha_j,t}^2)^{w_j}}}, \sqrt{\frac{\prod_{j=1}^{n} (1 + \nu_{\alpha_j,t}^2)^{w_j} - \prod_{j=1}^{n} (1 - \nu_{\alpha_j,t}^2)^{w_j}}{\prod_{j=1}^{n} (1 + \nu_{\alpha_j,t}^2)^{w_j} + \prod_{j=1}^{n} (1 - \nu_{\alpha_j,t}^2)^{w_j}}} \Bigg), \\ &\quad \Bigg(\sqrt{\frac{\prod_{j=1}^{n} (1 + \mu_{\alpha_j,i}^2)^{w_j} - \prod_{j=1}^{n} (1 - \mu_{\alpha_j,i}^2)^{w_j}}{\prod_{j=1}^{n} (1 + \mu_{\alpha_j,i}^2)^{w_j} + \prod_{j=1}^{n} (1 - \mu_{\alpha_j,i}^2)^{w_j}}}, \frac{\sqrt{2} \prod_{j=1}^{n} \nu_{\alpha_j,i}^{w_i}}{\sqrt{\prod_{j=1}^{n} (2 - \nu_{\alpha_j,i}^2)^{w_j} + \prod_{j=1}^{n} (\nu_{\alpha_i,i}^2)^{w_j}}} \Bigg), \\ &\quad \Bigg(\sqrt{\frac{\prod_{j=1}^{n} (1 + \mu_{\alpha_j,f}^2)^{w_j} - \prod_{j=1}^{n} (1 - \mu_{\alpha_j,f}^2)^{w_j}}{\prod_{j=1}^{n} (1 + \mu_{\alpha_j,f}^2)^{w_j} + \prod_{j=1}^{n} (1 - \mu_{\alpha_j,f}^2)^{w_j}}}, \frac{\sqrt{2} \prod_{j=1}^{n} \nu_{\alpha_j,f}^{w_i}}{\sqrt{\prod_{j=1}^{n} (2 - \nu_{\alpha_j,f}^2)^{w_j} + \prod_{j=1}^{n} (\nu_{\alpha_i,f}^2)^{w_j}}} \Bigg) \Bigg\rangle. \end{aligned}$$

5 A Simplified Neutrosophic Valued Modified Fuzzy Cross-Entropy Measure for PFVNVs

The notion of SNS was proposed by Ye[37] to relieve the difficulty of applying NSs to practical problems.

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Definition 5.1. [37] A SNS A on X is given by

$$A = \{\langle x_j, T_A(x_j), I_A(x_j), F_A(x_j)\rangle : j = 1, ..., n\}$$

where $T_A, I_A, F_A : X \to [0,1]$ are the truth, indeterminacy and falsify membership functions. For convenience, a SNV is denoted by

$$\alpha = \langle T_\alpha, I_\alpha, F_\alpha\rangle = \langle T_A(x_j), I_A(x_j), F_A(x_j)\rangle$$

for a fixed $x_j \in X$.

Some algebraic operations of SNV is recalled in the next definition.

Definition 5.2. [18] Let $A = \langle T_A, I_A, F_A\rangle$ and $B = \langle T_B, I_B, F_B\rangle$ be SNVs. Assume that $g : [0,1] \to [0,\infty]$ is the additive generator of a continuous Archimedean t-norm and $h : [0,1] \to [0,\infty]$ is the additive generator of a continuous Archimedean t-conorm. Some algebraic operations between SNVs are defined as follow:
i) $A \oplus_{SNV} B = \left\langle h^{-1}(h(T_A) + h(T_B)), g^{-1}(g(I_A) + g(I_B)), g^{-1}(g(F_A) + g(F_B))\right\rangle$
ii) $\lambda A = \left\langle h^{-1}(\lambda h(T_A)), g^{-1}(\lambda g(I_A)), g^{-1}(\lambda g(F_A))\right\rangle$.

Definition 5.3. Let $\alpha = \langle \mathbb{T}_\alpha, \mathbb{I}_\alpha, \mathbb{F}_\alpha\rangle$ and $\beta = \langle \mathbb{T}_\beta, \mathbb{I}_\beta, \mathbb{F}_\beta\rangle$ be two PFVNVs. A simplified neutrosophic valued modified fuzzy cross-entropy measure between α and β is defined by

$$\begin{aligned} E(\alpha,\beta) \quad := \quad & \Bigg\langle \frac{1}{\ln 2}\left[\mu^2_{\alpha,t}\ln\frac{2\mu^2_{\alpha,t}}{\mu^2_{\alpha,t}+\mu^2_{\beta,t}} + \nu^2_{\alpha,t}\ln\frac{2\nu^2_{\alpha,t}}{\nu^2_{\alpha,t}+\nu^2_{\beta,t}} + \pi^2_{\alpha,t}\ln\frac{2\pi^2_{\alpha,t}}{\pi^2_{\alpha,t}+\pi^2_{\beta,t}}\right], \\ & 1-\frac{1}{\ln 2}\left[\mu^2_{\alpha,i}\ln\frac{2\mu^2_{\alpha,i}}{\mu^2_{\alpha,i}+\mu^2_{\beta,i}} + \nu^2_{\alpha,i}\ln\frac{2\nu^2_{\alpha,i}}{\nu^2_{\alpha,i}+\nu^2_{\beta,i}} + \pi^2_{\alpha,i}\ln\frac{2\pi^2_{\alpha,i}}{\pi^2_{\alpha,i}+\pi^2_{\beta,i}}\right], \\ & 1-\frac{1}{\ln 2}\left[\mu^2_{\alpha,f}\ln\frac{2\mu^2_{\alpha,f}}{\mu^2_{\alpha,f}+\mu^2_{\beta,f}} + \nu^2_{\alpha,f}\ln\frac{2\nu^2_{\alpha,f}}{\nu^2_{\alpha,f}+\nu^2_{\beta,f}} + \pi^2_{\alpha,f}\ln\frac{2\pi^2_{\alpha,f}}{\pi^2_{\alpha,f}+\pi^2_{\beta,f}}\right]\Bigg\rangle \end{aligned}$$

where $\pi^2_{\alpha,\cdot} = 1 - \mu^2_{\alpha,\cdot} - \nu^2_{\alpha,\cdot}$, and $\pi^2_{\beta,\cdot} = 1 - \mu^2_{\beta,\cdot} - \nu^2_{\beta,\cdot}$. As in,[5] we use the convention (based on continuity arguments) that $0\ln\frac{0}{p} = 0$ for $p \geq 0$.

The following proposition verifies that a simplified neutrosophic valued modified fuzzy cross-entropy measure between α and β is a SNV.

Proposition 5.4. *$E(\alpha,\beta)$ is a SNV.*

Proof. According to Shannon's inequality,[15] we know that each component of $E(\alpha,\beta)$ is equal or greater than 0. On the other hand, since the relation

$$\begin{aligned} \frac{1}{\ln 2}\left[\mu^2_{\alpha,\cdot}\ln\frac{2\mu^2_{\alpha,\cdot}}{\mu^2_{\alpha,\cdot}+\mu^2_{\beta,\cdot}} + \nu^2_{\alpha,\cdot}\ln\frac{2\nu^2_{\alpha,\cdot}}{\nu^2_{\alpha,\cdot}+\nu^2_{\beta,\cdot}} + \pi^2_{\alpha,\cdot}\ln\frac{2\pi^2_{\alpha,\cdot}}{\pi^2_{\alpha,\cdot}+\pi^2_{\beta,\cdot}}\right] & \leq \frac{1}{\ln 2}\Big(\mu^2_{\alpha,\cdot}\ln 2 + \nu^2_{\alpha,\cdot}\ln 2 + \pi^2_{\alpha,\cdot}\ln 2\Big) \\ & = 1 \end{aligned}$$

is true, each component of $E(\alpha,\beta)$ is equal or less than 1. Therefore $E(\alpha,\beta)$ is a SNV. □

It is clear that $E(\alpha,\beta) \neq E(\beta,\alpha)$ in general. To make the concept symmetric we can define the following.

Definition 5.5. Let α and β be two PFVNVs. A simplified neutrosophic valued symmetric discrimination information measure based on E is given by

$$E^*(\alpha,\beta) = \frac{1}{2}E(\alpha,\beta) \oplus_{SNV} \frac{1}{2}E(\beta,\alpha)$$

where multiplication by scalar $\frac{1}{2}$ is in the sense of (iii) of Definition 5.2.

E^* can be expressed via additive generators of continuous Archimedean t-norms and t-conorms.

Theorem 5.6. *Let α and β be two PFVNVs, let g be the additive generator of a continuous Archimedean t-norm and $h(t) = g(1-t)$. Then*

$$\begin{aligned} E^*(\alpha,\beta) &= \Bigg\langle h^{-1}\Bigg(\frac{1}{2}\Big[h\Big(T_{E(\alpha,\beta)}\Big)+h\Big(T_{E(\beta,\alpha)}\Big)\Big]\Bigg), g^{-1}\Bigg(\frac{1}{2}\Big[g\Big(I_{E(\alpha,\beta)}\Big)+g\Big(I_{E(\beta,\alpha)}\Big)\Big]\Bigg), \\ & g^{-1}\Bigg(\frac{1}{2}\Big[g(F_{E(\alpha,\beta)}\Big)+g\Big(F_{E(\beta,\alpha)}\Big)\Big]\Bigg)\Bigg\rangle. \end{aligned}$$

Proof. We have

$$\begin{aligned} E^*(\alpha,\beta) &= \frac{1}{2}E(\alpha,\beta)\oplus_{SNV}\frac{1}{2}E(\beta,\alpha) \\ &= \Bigg\langle h^{-1}\Bigg(h\Big(T_{\frac{1}{2}E(\alpha,\beta)}\Big)+h\Big(T_{\frac{1}{2}E(\beta,\alpha)}\Big)\Bigg), g^{-1}\Bigg(g\Big(I_{\frac{1}{2}E(\alpha,\beta)}\Big)+g\Big(I_{\frac{1}{2}E(\beta,\alpha)}\Big)\Bigg), \\ & g^{-1}\Bigg(g(F_{\frac{1}{2}E(\alpha,\beta)}\Big)+g\Big(F_{\frac{1}{2}E(\beta,\alpha)}\Big)\Bigg)\Bigg\rangle \\ &= \Bigg\langle h^{-1}\Bigg(h\Big(h^{-1}\Big\langle\frac{1}{2}h(T_{E(\alpha,\beta)})\Big)+h\Big(h^{-1}\Big(\frac{1}{2}h\big(T_{E(\beta,\alpha)}\big)\Big)\Bigg), \\ & g^{-1}\Bigg(g\Big(g^{-1}\Big(\frac{1}{2}g(I_{E(\alpha,\beta)})\Big)+g\Big(g^{-1}\Big(\frac{1}{2}g\big(I_{E(\beta,\alpha)}\big)\Big)\Bigg), \\ & g^{-1}\Bigg(g\Big(g^{-1}\Big(\frac{1}{2}g(I_{E(\alpha,\beta)})\Big)+g\Big(g^{-1}\Big(\frac{1}{2}g\big(I_{E(\beta,\alpha)}\big)\Big)\Bigg)\Bigg\rangle \end{aligned}$$

which yields that

$$\begin{aligned} E^*(\alpha,\beta) &= \Bigg\langle h^{-1}\Bigg(\frac{1}{2}\Big[h\Big(T_{E(\alpha,\beta)}\Big)+h\Big(T_{E(\beta,\alpha)}\Big)\Big]\Bigg), g^{-1}\Bigg(\frac{1}{2}\Big[g\Big(I_{E(\alpha,\beta)}\Big)+g\Big(I_{E(\beta,\alpha)}\Big)\Big]\Bigg), \\ & g^{-1}\Bigg(\frac{1}{2}\Big[g(F_{E(\alpha,\beta)}\Big)+g\Big(F_{E(\beta,\alpha)}\Big)\Big]\Bigg)\Bigg\rangle. \end{aligned}$$

□

By using particular additive generators we can obtain some particular cases of E^*.

Remark 5.7. Let g be the additive generator defined by $g(t) = -\log t$. Then we get Algebraic simplified neutrosophic valued symmetric discrimination information measure based on E given by

$$E^*_A(\alpha,\beta) = \Big\langle 1-\sqrt{(1-T_{E(\alpha,\beta)})(1-T_{E(\beta,\alpha)})}, \sqrt{I_{E(\alpha,\beta)}I_{E(\beta,\alpha)}}, \sqrt{F_{E(\alpha,\beta)}F_{E(\beta,\alpha)}}\Big\rangle.$$

Let g be the additive generator defined by $g(t) = \log(\frac{2-t}{t})$. Then we get Einstein simplified neutrosophic valued symmetric discrimination information measure based on E given by

$$\begin{aligned} E^*_E(\alpha,\beta) &= \Bigg\langle \frac{\sqrt{(1+T_{E(\alpha,\beta)})(1+T_{E(\beta,\alpha)})}-\sqrt{(1-T_{E(\alpha,\beta)})(1-T_{E(\beta,\alpha)})}}{\sqrt{(1+T_{E(\alpha,\beta)})(1+T_{E(\beta,\alpha)})}+\sqrt{(1-T_{E(\alpha,\beta)})(1-T_{E(\beta,\alpha)})}}, \\ & \frac{2\sqrt{I_{E(\alpha,\beta)}I_{E(\beta,\alpha)}}}{\sqrt{I_{E(\alpha,\beta)}I_{E(\beta,\alpha)}}+\sqrt{(2-I_{E(\alpha,\beta)})(2-I_{E(\beta,\alpha)})}}, \\ & \frac{2\sqrt{F_{E(\alpha,\beta)}F_{E(\beta,\alpha)}}}{\sqrt{F_{E(\alpha,\beta)}F_{E(\beta,\alpha)}}+\sqrt{(2-F_{E(\alpha,\beta)})(2-F_{E(\beta,\alpha)})}}\Bigg\rangle. \end{aligned}$$

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Since E^* is a SNV, a score function is needed to rank the values of E^*. In this paper, we use the following score function.

Definition 5.8. [32] Let $A = (T_A, I_A, F_A)$ be a SNV. A score function is defined by

$$N(A) = \frac{T_A + 1 - I_A + 1 - F_A}{3}.$$

Theorem 5.9. *Let α and β be two PFVNVs. The modified cross-entropy measure E^* satisfies the following properties.*

i) $E^*(\alpha,\alpha) = \langle 0,1,1\rangle,$

ii) $0 \leq N(E^*(\alpha,\beta)) \leq 1,$

iii) $E^*(\alpha,\beta) = E^*(\beta,\alpha),$

iv) $N(E^*(\alpha,\alpha)) = 0.$

Proof. **i)** We know that

$$\begin{aligned} E(\alpha,\alpha) &= \Bigg\langle \frac{1}{\ln 2}\left[\mu^2_{\alpha,t}\ln\frac{2\mu^2_{\alpha,t}}{\mu^2_{\alpha,t}+\mu^2_{\alpha,t}} + \nu^2_{\alpha,t}\ln\frac{2\nu^2_{\alpha,t}}{\nu^2_{\alpha,t}+\nu^2_{\alpha,t}} + \pi^2_{\alpha,t}\ln\frac{2\pi^2_{\alpha,t}}{\pi^2_{\alpha,t}+\pi^2_{\alpha,t}}\right], \\ &\quad 1-\frac{1}{\ln 2}\left[\mu^2_{\alpha,i}\ln\frac{2\mu^2_{\alpha,i}}{\mu^2_{\alpha,i}+\mu^2_{\alpha,i}} + \nu^2_{\alpha,i}\ln\frac{2\nu^2_{\alpha,i}}{\nu^2_{\alpha,i}+\nu^2_{\alpha,i}} + \pi^2_{\alpha,i}\ln\frac{2\pi^2_{\alpha,i}}{\pi^2_{\alpha,i}+\pi^2_{\alpha,i}}\right], \\ &\quad 1-\frac{1}{\ln 2}\left[\mu^2_{\alpha,f}\ln\frac{2\mu^2_{\alpha,f}}{\mu^2_{\alpha,f}+\mu^2_{\alpha,f}} + \nu^2_{\alpha,f}\ln\frac{2\nu^2_{\alpha,f}}{\nu^2_{\alpha,f}+\nu^2_{\alpha,f}} + \pi^2_{\alpha,f}\ln\frac{2\pi^2_{\alpha,f}}{\pi^2_{\alpha,f}+\pi^2_{\alpha,f}}\right]\Bigg\rangle \\ &= \langle 0,1,1\rangle. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} E^*(\alpha,\alpha) &= \Bigg\langle h^{-1}\left(\frac{1}{2}\left[h\Big(T_{E(\alpha,\alpha)}\Big) + h\Big(T_{E(\alpha,\alpha)}\Big)\right]\right), g^{-1}\left(\frac{1}{2}\left[g\Big(I_{E(\alpha,\alpha)}\Big) + g\Big(I_{E(\alpha,\alpha)}\Big)\right]\right), \\ &\quad g^{-1}\left(\frac{1}{2}\left[g(F_{E(\alpha,\alpha)}\Big) + g\Big(F_{E(\alpha,\alpha)}\Big)\right]\right)\Bigg\rangle \\ &= \Big\langle T_{E(\alpha,\alpha)}, I_{E(\alpha,\alpha)}, F_{E(\alpha,\alpha)}\Big\rangle \\ &= \langle 0,1,1\rangle. \end{aligned}$$

ii) Since $E^*(\alpha,\beta)$ is a SNV we obtain that $0 \leq N(E^*(\alpha,\beta)) \leq 1$.

iii) It is trivial from definition of E^*.

iv) It is trivial from (i) and definition of N.

□

6 An Application of $PFVNVs$ To A MCDM Problem

In this section, we provide a MCDM method in Pythagorean fuzzy valued neutrosophic environment. Then we solve a MCDM problem from the literature by using the proposed method.

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	C_1	C_2	C_3
A_1	$\langle 0.4, 0.2, 0.3\rangle$	$\langle 0.4, 0.2, 0.3\rangle$	$\langle 0.2, 0.2, 0.5\rangle$
A_2	$\langle 0.6, 0.1, 0.2\rangle$	$\langle 0.6, 0.1, 0.2\rangle$	$\langle 0.5, 0.2, 0.2\rangle$
A_3	$\langle 0.3, 0.2, 0.3\rangle$	$\langle 0.5, 0.2, 0.3\rangle$	$\langle 0.5, 0.3, 0.2\rangle$
A_4	$\langle 0.7, 0.0, 0.1\rangle$	$\langle 0.6, 0.1, 0.2\rangle$	$\langle 0.4, 0.3, 0.2\rangle$

Table 1: SNVs of alternatives according to criteria

6.1 A MCDM method

In this sub-section, we give a MCDM method in the Pythagorean fuzzy valued neutrosophic environment. The proposed method is applied to a MCDM problem adapted from the literature[37,38] to see the effectiveness of the proposed method. The steps can be summarized as follows.

Step 1: Form a MCDM problem with alternatives $A = \{A_1, \ldots, A_n\}$ and criteria $C = \{C_1, \ldots, C_k\}$.

Step 2: The weights of criteria are determined.

Step 3: The evaluation results of the alternatives are expressed by the expert as $SNVs$ for each criterion.

Step 4: Using Proposition 3.7, evaluation results of alternatives in Step 3 are converted from SNVs to PFVNVs.

Step 5: Evaluation values expressed as PFVNVs for each alternative according to criteria are transformed to a value expressed as a PFVNV by utilizing proposed weighted aggregation operators.

Step 6: Positive ideal alternative for each sample is defined by

$$A^+ = \langle(\mu_t^+, \nu_t^+), (\mu_i^+, \nu_i^+), (\mu_f^+, \nu_f^+)\rangle$$

with alternatives $A = \{A_1, \ldots, A_n\} = \{\langle(\mu_t^1, \nu_t^1), (\mu_i^1, \nu_i^1), (\mu_f^1, \nu_f^1)\rangle, \ldots, \langle(m_t^n, \nu_t^n), (\mu_i^n, \nu_i^n), (\mu_f^n, \nu_f^n)\rangle\}$ and

$\mu_t^+ = \max_{j\in\{1,\ldots,n\}} \mu_t^j$, $\nu_t^+ = \min_{j\in\{1,\ldots,n\}} \nu_t^j$, $\mu_i^+ = \min_{j\in\{1,\ldots,n\}} \mu_i^j$, $\nu_i^+ = \max_{j\in\{1,\ldots,n\}} \nu_i^j$, $\mu_f^+ = \min_{i\in\{1,\ldots,n\}} \mu_f^j$, $\nu_f^+ = \max_{i\in\{1,\ldots,n\}} \nu_f^j$.

Here, all of the criteria are assumed to be benefit criteria. If there exists a cost criterion, then we can take compliment.

Step 7: By using E^* defined in Definition 5.5 the symmetric discrimination information measure between aggregated value of each alternative and positive ideal alternative are calculated.

Step 8: The scores of these $SNVs$ are obtained by using a score function.

Step 9: Alternatives are ranked so that the minimum score value is the best alternative.

6.2 An application

As an application, we consider an investment company which wants to invest a sum of money in the best opinion. This problem is adapted from.[37,38] There are four alternatives for investing the money: (1) A_1 (a car company); (2) A_2 (a food company); (3) A_3 (a computer company); and (4) A_4 (an arms company). The investment company wants to make a decision using the following three criteria: (1) C_1 (the risk); (2) C_2 (the growth); and (3) C_3 (the the environmental impact). The weight vector of the criteria is considered by $w = (0.35, 0.25, 0.4)$ and the decision matrix given in Table 1 is used as in.[37,38]

Let $f_\delta(t) = sin\delta t$ with $\delta \in [0, 1]$. It is clear that $f_\delta(t) = sin\delta t \leq \sqrt{t}$ for $t \in [0, 1]$ (see also Figure 4). Now we convert $SNVs$ of alternatives in Table 1 to $PFVNVs$ by using Proposition 3.7. For $\delta = 1$, we

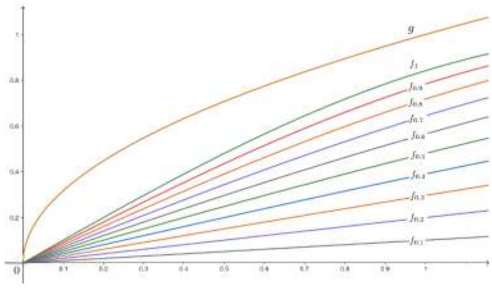

Figure 4: The graphs of $y = \sqrt{t}$ and f_δ with $\delta \in [0, 1]$

	C_1	C_2
A_1	$\langle(0.63, 0.56), (0.45, 0.72), (0.55, 0.64)\rangle$	$\langle(0.63, 0.56), (0.45, 0.72), (0.55, 0.64)\rangle$
A_2	$\langle(0.77, 0.39), (0.32, 0.78), (0.45, 0.72)\rangle$	$\langle(0.77, 0.39), (0.32, 0.78), (0.45, 0.72)\rangle$
A_3	$\langle(0.55, 0.64), (0.45, 0.72), (0.55, 0.64)\rangle$	$\langle(0.71, 0.48), (0.45, 0.72), (0.55, 0.64)\rangle$
A_4	$\langle(0.84, 0.3), (0.0, 0.84), (0.32, 0.78)\rangle$	$\langle(0.77, 0.39), (0.32, 0.78), (0.45, 0.72)\rangle$
	C_3	
A_1	$\langle(0.45, 0.72), (0.45, 0.72), (0.71, 0.48)\rangle$	
A_2	$\langle(0.71, 0.48), (0.45, 0.72), (0.45, 0.72)\rangle$	
A_3	$\langle(0.71, 0.48), (0.55, 0.64), (0.45, 0.72)\rangle$	
A_4	$\langle(0.63, 0, 56), (0.55, 0.64), (0.45, 0.72)\rangle$	

Table 2: PFVNVs of alternatives obtained for $\delta = 1$

	PFVNV
A_1	$\langle(0.57, 0.62), (0.43, 0.76), (0.6, 0.62)\rangle$
A_2	$\langle(0.76, 0.41), (0.37, 0.76), (0.43, 0.74)\rangle$
A_3	$\langle(0.6, 0.59), (0.55, 0.74), (0.45, 0.68)\rangle$
A_4	$\langle(0.74, 0.44), (0.0, 0.68), (0.48, 0.69)\rangle$
A^+	$\langle(0.76, 0.41), (0.0, 0.76), (0.43, 0.74)\rangle$

Table 3: Aggregated PFVNV according to $WA_A - PFVNV$ and positive ideal alternative A^+

Algebraic symmetric discrimination information measures	SNV
$E_A^*(A^+, A_1)$	$\langle 0.06, 0.97, 0.97\rangle$
$E_A^*(A^+, A_2)$	$\langle 0.0, 0.99, 1\rangle$
$E_A^*(A^+, A_3)$	$\langle 0.04, 0.93, 0.99\rangle$
$E_A^*(A^+, A_4)$	$\langle 0.003, 0.99, 0.99\rangle$

Table 4: Algebraic symmetric discrimination information measures between positive ideal alternative and alternatives

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	Score
$N(E_A^*(A^+, A_1))$	0.039
$N(E_A^*(A^+, A_2))$	**0.0048**
$N(E_A^*(A^+, A_3))$	0.041
$N(E_A^*(A^+, A_4))$	0.0051

Table 5: Scores of Algebraic symmetric discrimination information measures between A^+ and $A_j(j = 1, 2, 3, 4)$

Method	Ranking order	Best Alternative
Ye[37]	$A_1 \prec A_3 \prec A_2 \prec A_4$	A_4
Ye[38]	$A_1 \prec A_3 \prec A_4 \prec A_2$	A_2
Peng et al.[18]	$A_1 \prec A_3 \prec A_2 \prec A_4$	A_4
Proposed method	$A_3 \prec A_1 \prec A_4 \prec A_2$	A_2

Table 6: The comparison of some previous methods with proposed method

obtain Table 2. When $PFVNVs$ of each alternative with respect to criteria are aggregated by utilizing $WA_A - PFVNV$, Table 3 containing the positive ideal alternative is obtained.

Algebraic symmetric discrimination information measure E_A^* between positive ideal alternative A^+ and alternatives are calculated. The results are shown in Table 4.

Score function N recalled in Definition 5.8 is used to rank the value of E_A^*. The smaller the value of $N(E_A^*(A_j, A^+))$ is, the better the alternative A_j is. In the other words, the alternative A_j is closer to positive ideal alternative A^+ as $N(E_A^*(A_j, A^+))$ gets smaller. The results of score function are shown in Table 5. As a result, A_2 is the best alternative.

6.3 Comparative analysis

6.3.1 Comparison with the literature

We compare the results of some existing methods with the results of the proposed method in Sub-section 6.1. So as to solve the MCDM problem given in Sub-section 6.2, some methods were proposed under different fuzzy environments. Ye[38] gave a weighted correlation coefficient for SVNSs, utilized it to solve the same investment problem and obtained the ranking $A_1 \prec A_3 \prec A_4 \prec A_2$. Using different aggregation operators and a similarity measure for SNSs, Ye[37] solved the same problem and obtained the ranking $A_1 \prec A_3 \prec A_2 \prec A_4$. Similarly, Peng[18] proposed some aggregation operators via t-norms and t-conorms, applied them to investigate same problem and got the ranking $A_1 \prec A_3 \prec A_2 \prec A_4$. The best alternative obtained with the present method remains same with the one of[37] and[38] for some δ. The comparison results of proposed method with the existing ones are shown in Table 6 and illustrated in Figure 5.

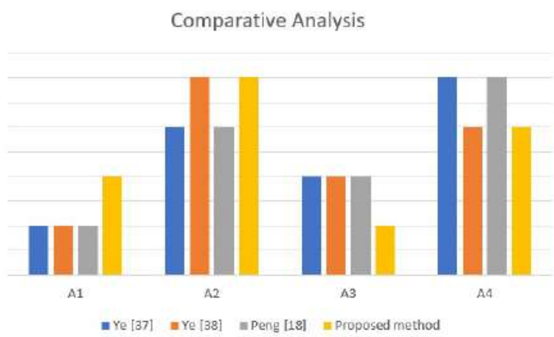

Figure 5: The column chart comparison of the other methods and proposed method

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δ	Ranking order	Best alternative
$\delta = 1$	$A_3 \prec A_1 \prec A_4 \prec A_2$	A_2
$\delta = 0.9$	$A_1 \prec A_3 \prec A_2 \prec A_4$	A_4
$\delta = 0.8$	$A_3 \prec A_1 \prec A_2 \prec A_4$	A_4
$\delta = 0.7$	$A_1 \prec A_3 \prec A_4 \prec A_2$	A_2
$\delta = 0.6$	$A_1 \prec A_3 \prec A_2 \prec A_4$	A_4
$\delta = 0.5$	$A_3 \prec A_1 \prec A_2 \prec A_4$	A_4
$\delta = 0.4$	$A_1 \prec A_3 \prec A_2 \prec A_4$	A_4
$\delta = 0.3$	$A_1 \prec A_3 \prec A_4 \prec A_2$	A_2
$\delta = 0.2$	$A_1 \prec A_3 \prec A_2 \prec A_4$	A_4
$\delta = 0.1$	$A_3 \prec A_1 \prec A_4 \prec A_2$	A_2

Table 7: The results with respect to $WA_A - PFVNV$ when the parameter δ varies from 0.1 and 1

δ	Ranking order	Best alternative
$\delta = 1$	$A_1 \prec A_3 \prec A_4 \prec A_2$	A_2
$\delta = 0.9$	$A_3 \prec A_1 \prec A_4 \prec A_2$	A_2
$\delta = 0.8$	$A_1 \prec A_3 \prec A_2 \prec A_4$	A_4
$\delta = 0.7$	$A_3 \prec A_1 \prec A_4 \prec A_2$	A_2
$\delta = 0.6$	$A_1 \prec A_3 \prec A_2 \prec A_4$	A_4
$\delta = 0.5$	$A_1 \prec A_3 \prec A_4 \prec A_2$	A_2
$\delta = 0.4$	$A_3 \prec A_1 \prec A_4 \prec A_2$	A_2
$\delta = 0.3$	$A_1 \prec A_3 \prec A_4 \prec A_2$	A_2
$\delta = 0.2$	$A_1 \prec A_3 \prec A_4 \prec A_2$	A_2
$\delta = 0.1$	$A_3 \prec A_1 \prec A_2 \prec A_4$	A_4

Table 8: The results with respect to $WG_A - PFVNV$ when the parameter δ varies from 0.1 and 1

6.3.2 Comparison with respect to parameter δ

We investigate the effect of the parameter δ on the ranking of alternatives with respect to $WA_A - PFVNV$ and $WG_A - PFVNV$. The results are summarized in Table 7 and Table 8. From these results, it is concluded that the ranking of the alternative is $A_3 \prec A_1 \prec A_4 \prec A_2$ when δ equals to 0.1 and 1 with respect to $WA_A - PFVNV$ while $A_1 \prec A_3 \prec A_2 \prec A_4$ when δ equals to 0.6 and 0.8 for $WG_A - PFVNV$. When δ equals to 0.1 and 1 with respect to $WA_A - PFVNV$, we conclude that A_2 is the best alternative, while by using $WG_A - PFVNV$, the best alternative is A_4 when δ equals to 0.6 and 0.8. The graphical representation of the behaviour of the alternatives with the δ variation is shown in Figure 6.

6.4 Time complexity of the proposed MCDM method

In this sub-section we analysis the complexity of the MCDM method proposed in Sub-section 6.1. Actually we evaluate the time complexity that depends on the number of times of multiplication, exponential, summation

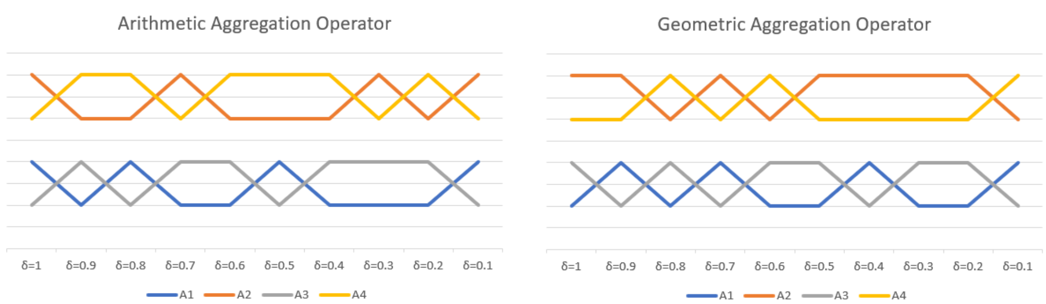

Figure 6: Effect of the parameter δ to the solution

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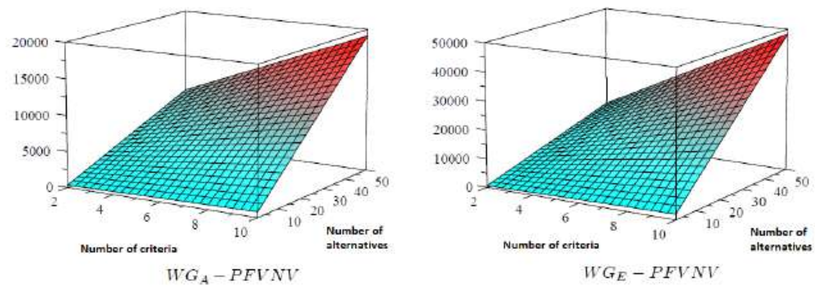

Figure 7: Time complexity of the proposed MCDM method

as in[4] and.[8] Consider a MCDM problem with n alternatives and k criteria. In Step 2 we need k operations, in Step 3 we need $3kn$ operations, in Step 4 we need $6kn$ operations, in Step 5 we need $3n\,(6k+2)$ operations if we use the aggregation operator $WG_A - PFVNV$ and we need $n(75k+21)$ operations if we use the aggregation operator $WG_E - PFVNV$. In Step 6 we need $3n$ operations, in Step 7 we need $98n$ operations and in Step 8 we need $5n$ operations. So the time complexity T_{nk} of the MCDM method is

$$T^A_{nk} = k + 27kn + 112n$$

for the aggregation operator $WG_A - PFVNV$ and

$$T^E_{nk} = k + 84kn + 127n$$

for the aggregation operator $WG_E - PFVNV$. Clearly the bi-variate functions $g_A, g_E : [2,\infty)^2 \to \mathbb{R}$ defined by $g_A(x,y) = x + 27xy + 112y$ and $g_E(x,y) = x + 84xy + 127y$ both take absolute minimum at point $(2,2)$. Figure 7 illustrates the change of the time complexity with respect to the change in the numbers of the alternatives and the criteria for $WG_A - PFVNV$ and $WG_E - PFVNV$.

7 Conclusion

The main aim of this study is to introduce the concept of PFVNS constructed by considering PFVs rather than numbers, inspired by IFVNSs. Thus, a PFVNS is an extension of IFVNSs. PFVNSs are used to express uncertainty in a more extended fuzzy environment. Therefore, larger information can be kept while the data is converted to a FS. In this way, information loss is prevented. In this study, some set operations between PFVNSs are proposed. We also introduce Pythagorean fuzzy t-norms and t-conorms with motivation from intuitionistic fuzzy t-norms and t-conorms. We show that some Pythagorean fuzzy t-norms and t-conorms are expressed via continuous Archimedean t-norms and continuous Archimedean t-conorms on $[0,1]$. Then we define some algebraic operations between PFVNVs by utilizing continuous Archimedean t-norm and continuous Archimedean t-conorms on $[0,1]$. By way of these algebraic operations, some weighted aggregation operators are proposed. Input values represented by PFVNVs are transformed to a single output value by using weighted aggregation operators. Also, we introduce a method that converts neutrosophic fuzzy values to PFVNVs. By defining a simplified neutrosophic valued modified fuzzy cross-entropy measure we manage to rank output values represented by PFVNVs with the help of a score function in simplified neutrosophic environment. Next a MCDM method is given to see practicability of the proposed theory. The proposed method is made use of to solve a MCDM problem adapted from the literature in Pythagorean fuzzy valued neutrosophic environment. A comparison analysis and a complexity analysis are also provided. In the future, different kind of aggregation operators and cross-entropy measures can be considered. The applications of the proposed theory can be extended to some decision making problems such as face recognition systems, classification and medical diagnosis.

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