Improvement of Proportional Conflict Redistribution Rules of Combination of Basic Belief Assignments

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This paper discusses and analyzes the behaviors of the Proportional Conflict Redistribution rules no. 5 (PCR5) and no. 6 (PCR6) to combine several distinct sources of evidence characterized by their basic belief assignments defined over the same frame of discernment. After a brief review of these rules, the paper shows through simple examples why their behaviors can sometimes increase the uncertainty more than necessary, which is detrimental to decision-making support drawn from the result of the combination. We present a theoretical improvement of these rules, and establish new PCR5$^+$ and PCR6$^+$ rules of combination. These new rules overcome the weakness of PCR5 and PCR6 rules by computing binary-keeping indexes that allow to keep only focal elements that play an effective role in the partial conflict redistribution. PCR5$^+$ and PCR6$^+$ rules are not associative but they preserve the neutrality of the vacuous belief assignment contrary to the PCR5 and PCR6 rules, and they make a more precise redistribution which does not increase improperly the mass of partial uncertainties.

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I. INTRODUCTION

There exist different theories based on distinct representations and modelings of uncertainty to deal with uncertain information to conduct information fusion [1]. The theory of probability [2], [3], the theory of fuzzy sets [4], [5], the possibility theory [6], [7], and the theory of belief functions [8]–[10] are the most well-known ones. This paper addresses the problem of information fusion in the mathematical framework of the belief functions introduced by Shafer from Dempster's works [11], [12]. The belief functions are often used in decision-making support applications because the experts are generally able to express only a belief in a hypothesis (or a set of hypotheses) from their partial knowledge, experience, and from their own perception of the reality. To conduct information fusion, we need some efficient rules of combination that are able to manage the conflicting sources of evidence (if any), or expert opinions expressed in terms of belief functions. Readers interested in belief functions can found classical related papers in [13] and in the special issue [14], which includes also a list of good selected papers. It is worth to mention that the recent book of Cuzzolin [15] includes 2137 references, with many of them related to belief functions.

In this paper, we adopt the notion of conflict introduced by Shafer in [8] (p. 65). This notion of conflict is often adopted by researchers working with belief functions, as in [16] (p. 17) for instance, because this notion is quite simple to understand. Different definitions and interpretations of conflict can be also found in [17]–[27] for readers interested in this topic. In this paper, two (or more) sources are said conflicting if they support incompatible (disjoint, or contradictory) hypotheses. We also work with distinct sources of evidences that are considered as (cognitively) independent and reliable. We neither consider, nor apply discounting techniques of belief assessments listed in [14] before combining them to keep the presentation and notations as simple as possible.[1]

While the conjunctive rule makes it possible to combine information between different sources of information by estimating the level of existing conflict, Dempster–Shafer (DS) rule [8], [16] proposes a distribution of this conflict on the hypotheses characterized by the sources of information. The normalization carried out by the DS rule may, however, be considered counter-intuitive especially when the level of conflict between the sources of information is high [28], [29], but also in some situations where the level of conflict between sources is low as shown in [30] showing a dictatorial behavior of DS rule. The Proportional Conflict Redistribution rules no. 5 (PCR5) [31] and no. 6 (PCR6) [32], [33] have been proposed to circumvent the problem of the DS rule to make a more judicious management of the conflict.

[1]Of course discounted belief assignments can also be combined by the rules presented in this paper.

In this paper, we put forward a flawed behavior of these combination rules in some cases attributed to the non-neutrality of the vacuous Basic Belief Assignment (BBA), and we propose an improvement of these two combination rules (denoted by PCR5$^+$ and PCR6$^+$) in order to ensure the neutrality property of the vacuous BBA (VBBA). This is achieved by discarding specific elements implied in the partial conflict and which are not useful for making the conflict redistribution.

In the Proportional Conflict Redistribution (PCR) rules [32]–[34], one redistributes the product of masses of belief of incompatible (i.e., conflicting) elements whose intersection is empty only to elements involved in this product and proportionally to their mass of belief. For instance, let's consider two elements A and B of a frame of discernment (FoD) with $A \cap B = \emptyset$, and three BBAs $m_1(\cdot)$, $m_2(\cdot)$, and $m_3(\cdot)$ defined on this FoD with $m_1(A) > 0$, $m_2(B) > 0$, and $m_3(A \cup B) > 0$. The product $m_1(A)m_2(B)m_3(A \cup B) > 0$ is called a conflicting product hereafter because $A \cap B \cap (A \cup B) = \emptyset$. Based on PCR5 (and PCR6) rule, we will redistribute the value of this product back to the focal[2] elements A, B, and $A \cup B$, and proportionally to $m_1(A)$, $m_2(B)$, and $m_3(A \cup B)$. In the improved PCR rules developed in this paper, we will redistribute this conflicting product only to the focal elements A and B since the focal element $A \cup B$ is neither in conflict with A, nor with B. Such an improvement in the PCR is made possible by defining a binary-keeping index for each focal element involved in the conflicting product. This index will allow the identification of elements of the conflicting product that will have an effective role in the proportional redistribution of conflicting product. All elements (if any) having a binary-keeping index equal to zero are discarded of the conflict redistribution process. This main idea is developed in this paper and illustrated with several examples. It allows to preserve the neutrality of the total ignorant source of evidence in the improved versions of PCR5 and PCR6 rules, which is often considered as a desirable property for a rule of combination of distinct and reliable sources of evidence.

For the reader not immersed in the belief mathematics notion, the comparative numerical examples of Example 1 of Section III-B as compared with Example 1 revisited of Section VII, provide a quick verification of the improvements.

This paper is organized as follows. We give the basics of belief functions in Section II. We present the PCR5 and PCR6 rules of combination in Section III with new general formulas in Subsection III-C, and associated examples in Section IV. The flawed behavior of PCR5 and PCR6 rules are highlighted in Section V through specific examples. Then, Section VI proposes the mathematical expression of the new improved PCR5$^+$ and PCR6$^+$ rules of combination, as well as the very detailed procedure to select the focal elements for these new proportional redistributions. Finally, comparative results for relevant examples are shown in Section VII in order to compare the PCR5 and PCR6 results with the PCR5$^+$ and PCR6$^+$ results. Concluding remarks are given in Section VIII. For convenience, two Matlab™ routines are also given in Appendix 3 of this paper for PCR5$^+$ and PCR6$^+$ rules of combination.

II. BASICS OF BELIEF FUNCTIONS

We consider a given finite set Θ of $n > 1$ distinct elements $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ corresponding to the FoD of the fusion problem, or the decision-making problem, under concern. All elements of Θ are mutually exclusive[3] and each element is an elementary choice of the potential decision to take. The power set of Θ is the set of all subsets of Θ (including empty set $\emptyset$ and Θ) and it is usually denoted 2^Θ because its cardinality equals $2^{|\Theta|}$. We adopt Shafer's formalism whereby propositions are represented by subsets [8] (Chap. 2, pp. 35–37). Hence, the propositions under concern are in one-to-one correspondance with subsets of Θ. We also use classical notations of set theory [35], i.e. $\emptyset$ for the empty set, $A \cup B$ for the union[4] of sets A and B (which is the set of all objects that are a member of the set A, or the set B, or both), $A \cap B$ for their intersection (which is the set of all objects that are members of both A and B), etc. A BBA given by a source of evidence is defined by Shafer [8] in his Mathematical Theory of Evidence (known also as Dempster–Shafer Theory (DST)) as $m(\cdot) : 2^\Theta \to [0, 1]$ satisfying

$$\begin{cases} m(\emptyset) = 0 \\ \sum_{A \in 2^\Theta} m(A) = 1, \end{cases} \tag{1}$$

where $m(A)$ is the mass of belief exactly committed to A, what we usually call the mass of A. A BBA is said proper (or normal) if it satisfies Shafer's definition (1). The subset $A \subseteq \Theta$ is called a focal element of the BBA $m(\cdot)$ if and only if $m(A) > 0$. The empty set is not a focal element of a BBA because $m(\emptyset) = 0$ according to definition (1). The set of all focal elements of a BBA $m(\cdot)$ is denoted $\mathcal{F}(m)$. Its mathematical definition is $\mathcal{F}(m) = \{X \in 2^\Theta | m(X) > 0\}$. The cardinality $|\mathcal{F}(m)|$ of the set $\mathcal{F}(m)$ is denoted $\mathcal{F}_m$. The order of focal elements of $\mathcal{F}(m)$ does not matter and all the focal elements are different. The set $\mathcal{F}(m)$ of focal elements of $m(\cdot)$ has at least one focal element, and at most $2^{|\Theta|} - 1$ focal elements.

[2]A focal element is an element (i.e., a subset) having a strictly positive mass of belief committed to it—see Section II elements.

[3]This standard assumption is called *Shafer's model of FoD* in Dezert–Smarandache theory (DSmT) framework [34].

[4]We prefer the notation $A \cup B$ for denoting the union of sets A and B, which is a formal mathematical notation for the union of two sets, instead of the notations AB or $\{A, B\}$ used by some authors.

Belief and plausibility functions are, respectively, defined from $m(\cdot)$ by [8]

$$Bel(A) = \sum_{X\in 2^\Theta | X\subseteq A} m(X) \tag{2}$$

and

$$Pl(A) = \sum_{X\in 2^\Theta | A\cap X\neq\emptyset} m(X) = 1 - \text{Bel}(\bar{A}). \tag{3}$$

where $\bar{A}$ represents the complement of A in Θ.

$Bel(A)$ and $Pl(A)$ are usually interpreted, respectively, as lower and upper bounds of an unknown (subjective) probability measure $P(A)$ [11], [12]. The functions $m(.)$, $Bel(.)$ and $Pl(.)$ are one-to-one. A belief function $Bel(.)$ is *Bayesian* if all Bel's focal elements are singletons [8] (Theorem 2.8, p. 45). In this case, $m(X) = Bel(X)$ for any (singleton) focal element X, and $m(.)$ is called a *Bayesian BBA*. Corresponding $Bel(\cdot)$ function is equal to $Pl(\cdot)$ and these functions can be interpreted as a same (possibly subjective) probability measure $P(\cdot)$. The VBBA representing a totally ignorant source is defined as $m_v(\Theta) = 1$.

III. COMBINATION OF BBAS

This section presents at first the conjunctive rule of combination which is one of the main rules to combine reliable sources of evidence and which allows to identify the conflicting information among the sources. Then we present the PCR5 [31] and PCR6 [32], [33] as alternatives of Dempster's rule of combination [8]. The development of these rules has been motivated by the counter-intuitive behavior of Dempster's rule [8] when combining high conflicting sources of evidences, but also when combining low conflicting sources of evidences as well[5]. The reader interested in this topic can refer to [13], [28]–[30] to see theoretical justifications and examples. In the following, and for simplicity, we restrain our presentation to the classical framework of belief functions, and we work with BBAs defined only on the power set 2^Θ of a FoD Θ. PCR rules have been defined originally for working with Dedekind's lattice as well, see Chapter 1 of [34] (Volume 2). In this paper, we present simple general expressions of PCR5 and PCR6 fusion rules because they are easier to understand than the original general formulas, and they afford expressions of the improved PCR5$^+$ and PCR6$^+$ rules in a direct and useful manner.

After a brief presentation of the main notations used in this paper, we will recall both PCR5 and PCR6 rules for historical and technical reasons. PCR5 has been developed at first, and then PCR6 has been proposed based on a modified redistribution principle inspired by PCR5. In this paper, we follow the logical and historical development of these PCR5 and PCR6 rules to make the presentation of their improved versions PCR5$^+$ and PCR6$^+$. It seems easier to understand PCR6$^+$ fusion formula once the PCR5$^+$ formula will have been established. By presenting both rules, we offer to the readers a global deeper view on how these new rules work and their fundamental and mathematical differences in their conflict redistribution principles. In the sequel, all the introduced examples assume the model of Shafer's FoD as in the classical DST framework.

A. Notations

When we make the combination of $S \geq 2$ BBAs by the conjunctive rule, or by the PCR5 and PCR6 fusion rules, we have to compute the product of the masses of the focal elements composing any possible S-tuple of focal elements. Each possible S-tuple is noted by[6]

$$\mathbf{X}_j \triangleq (X_{j_1}, X_{j_2}, \dots, X_{j_S}) \in \mathcal{F}(m_1)\times\mathcal{F}(m_2)\times\dots\times\mathcal{F}(m_S),$$

where $j_1 \in \{1, 2, \dots, \mathcal{F}_{m_1}\}$, $j_2 \in \{1, 2, \dots, \mathcal{F}_{m_2}\}$, ..., $j_S \in \{1, 2, \dots, \mathcal{F}_{m_S}\}$. The element X_{j_i} is the focal element of $m_i(\cdot)$ that makes the i-th component of the j-th S-tuple $\mathbf{X}_j$.

For notation convenience also, the cartesian product $\mathcal{F}(m_1) \times \mathcal{F}(m_2) \times \dots \times \mathcal{F}(m_S)$ is denoted by $\mathcal{F}(m_1, \dots, m_S)$ in the sequel.

We have $\mathcal{F} \triangleq |\mathcal{F}(m_1, \dots, m_S)| = \prod_{i=1}^S |\mathcal{F}(m_i)| = \prod_{i=1}^S \mathcal{F}_{m_i}$ products of masses of focal elements to consider and to calculate because we have $\mathcal{F}_{m_1}$ focal elements in $\mathcal{F}(m_1)$, $\mathcal{F}_{m_2}$ focal elements in $\mathcal{F}(m_2)$, ..., and $\mathcal{F}_{m_S}$ focal elements in $\mathcal{F}(m_S)$. Each product for $j = 1$ to $\mathcal{F}$ is of the form

$$\pi_j(X_{j_1} \cap X_{j_2} \cap \dots \cap X_{j_S}) \triangleq \prod_{i=1}^{S} m_i(X_{j_i}). \tag{4}$$

There are two types of products:

- $\pi_j(X_{j_1} \cap X_{j_2} \cap \dots \cap X_{j_S})$ is called a *non-conflicting (mass) product* if
$$X_{j_1} \cap X_{j_2} \cap \dots \cap X_{j_S} = X \neq \emptyset.$$
In this case, $\pi_j(X_{j_1} \cap X_{j_2} \cap \dots \cap X_{j_S})$ is also noted by $\pi_j(X)$ for short.
- $\pi_j(X_{j_1} \cap X_{j_2} \cap \dots \cap X_{j_S})$ is called a *conflicting (mass) product* if
$$X_{j_1} \cap X_{j_2} \cap \dots \cap X_{j_S} = \emptyset.$$
In this case, $\pi_j(X_{j_1} \cap X_{j_2} \cap \dots \cap X_{j_S})$ is also noted by $\pi_j(\emptyset)$ for short.

It is worth noting that an element $X \in 2^\Theta \setminus \{\emptyset\}$ may belong to sets of focal elements of the different BBAs to combine, and therefore a S-tuple $\mathbf{X}_j$ can have duplicate components. Because all the BBAs are normalized, we

[5]Which is known as the dictatorial behavior of Dempster's rule [30].

[6]The symbol $\triangleq$ means "equals by definition."

always have

$$\sum_{j=1}^{\mathcal{F}} \pi_j(X_{j_1} \cap X_{j_2} \cap \ldots \cap X_{j_S}) = 1. \quad (5)$$

As a simple example to illustrate our notations, let us consider two BBAs $m_1(\cdot)$ and $m_2(\cdot)$ defined over the FoD $\Theta = \{A, B, C\}$ with, respectively, two and three focal elements, say $\mathcal{F}(m_1) = \{A, B \cup C\}$ and $\mathcal{F}(m_2) = \{B, C, A \cup C\}$. Here $\mathcal{F}_{m_1} = |\mathcal{F}(m_1)| = 2$ and $\mathcal{F}_{m_2} = |\mathcal{F}(m_2)| = 3$. For $j_1 = 1$ (the first focal element of $m_1(\cdot)$) one has $X_{j_1} = A$, and for $j_1 = 2$ (the second focal element of $m_1(\cdot)$) one has $X_{j_1} = B \cup C$. Similarly, for $j_2 = 1$ (the first focal element of $m_2(\cdot)$) one has $X_{j_2} = B$, for $j_2 = 2$ (the 2nd focal element of $m_2(\cdot)$) one has $X_{j_2} = C$, and $j_2 = 3$ (the 3rd focal element of $m_2(\cdot)$) one has $X_{j_2} = A \cup C$. In this case we have $\mathcal{F} = \mathcal{F}_{m_1} \cdot \mathcal{F}_{m_2} = 6$ products of masses to consider in the conjunctive fusion rule (see next sub-section) which are

$$\pi_1(A \cap B) = m_1(A)m_2(B),$$
$$\pi_2(A \cap C) = m_1(A)m_2(C),$$
$$\pi_3(A \cap (A \cup C)) = m_1(A)m_2(A \cup C),$$
$$\pi_4((B \cup C) \cap B) = m_1(B \cup C)m_2(B),$$
$$\pi_5((B \cup C) \cap C) = m_1(B \cup C)m_2(C),$$
$$\pi_6((B \cup C) \cap (A \cup C)) = m_1(B \cup C)m_2(A \cup C).$$

The products π_1 and π_2 are called *conflicting products* because

- for π_1, the focal elements A and B involved in π_1 are incompatible (i.e., disjoint) because $A \cap B = \emptyset$. π_1 $(A \cap B)$ is of course equivalent to $\pi_j(X_{j_1} \cap X_{j_2})$ with $j = 1$ by taking $X_{j_1} = A$ and $X_{j_2} = B$; and
- for π_2, one has $A \cap C = \emptyset$. $\pi_2(A \cap C)$ is equivalent to $\pi_j(X_{j_1} \cap X_{j_2})$ with $j = 2$ by taking $X_{j_1} = A$ and $X_{j_2} = C$, etc.

The products π_3, ..., and π_6 are not conflicting products because the focal elements involved in each product have non-empty intersection. Because $m_1(A) + m_1(B \cup C) = 1$ and $m_2(B) + m_2(C) + m_2(A \cup C) = 1$, one has $(m_1(A) + m_1(B \cup C))(m_2(B) + m_2(C) + m_2(A \cup C)) = 1$, and therefore $\sum_{j=1}^{6} \pi_j = 1$. This illustrates the formula (5).

In this paper, $i \in \{1, \ldots, S\}$ represents the index of the i-th source of evidence characterized by the BBA $m_i(\cdot)$, and $j \in \{1, \ldots, \mathcal{F}\}$ represents the index of the j-th product $\pi_j(X_{j_1} \cap X_{j_2} \cap \ldots \cap X_{j_S})$.

B. The conjunctive rule of combination

Let's consider $S \geq 2$ distinct reliable sources of evidence characterized by their BBA $m_s(\cdot)$ $(s = 1, \ldots, S)$ defined on 2^Θ. Their conjunctive fusion[7] is defined for all $A \in 2^\Theta$ by

$$m_{1,2,\ldots,S}^{\text{Conj}}(A) = \sum_{\substack{\mathbf{X}_j \in \mathcal{F}(m_1,\ldots,m_S) \\ X_{j_1} \cap \ldots \cap X_{j_S} = A}} \pi_j(X_{j_1} \cap X_{j_2} \cap \ldots \cap X_{j_S})$$
$$= \sum_{\substack{\mathbf{X}_j \in \mathcal{F}(m_1,\ldots,m_S) \\ X_{j_1} \cap \ldots \cap X_{j_S} = A}} \prod_{i=1}^{S} m_i(X_{j_i}). \quad (6)$$

The symbol ⓝ is also used in the literature, for instance in [36], to note the conjunctive fusion operator, i.e., $m_{1,2,\ldots,S}^{\text{Conj}}(A) = [m_1 ⓝ m_2 ⓝ \ldots ⓝ m_S](A)$.

The total conflicting mass between the S sources of evidence, denoted $m_{1,2,\ldots,S}^{\text{Conj}}(\emptyset)$, is nothing but the sum of all existing conflicting mass products, that is

$$m_{1,2,\ldots,S}^{\text{Conj}}(\emptyset) = \sum_{\substack{\mathbf{X}_j \in \mathcal{F}(m_1,\ldots,m_S) \\ X_{j_1} \cap \ldots \cap X_{j_S} = \emptyset}} \pi_j(X_{j_1} \cap X_{j_2} \cap \ldots \cap X_{j_S})$$
$$= 1 - \sum_{A \in 2^\Theta \setminus \{\emptyset\}} m_{1,2,\ldots,S}^{\text{Conj}}(A). \quad (7)$$

Note that the combined BBA $m_{1,2,\ldots,S}^{\text{Conj}}(.)$ given in (6) is not a proper BBA because it does not satisfy Shafer's definition (1). In general, the S sources of evidence to combine do not fully agree, and we have consequently $m_{1,2,\ldots,S}^{\text{Conj}}(\emptyset) > 0$.

Dempster's rule of combination (called also *orthogonal sum* by Shafer [8], p. 6) coincides with the normalized version of the conjunctive rule. It is defined by $m_{1,2,\ldots,S}^{\text{DS}}(A) = m_{1,2,\ldots,S}^{\text{Conj}}(A)/(1 - m_{1,2,\ldots,S}^{\text{Conj}}(\emptyset))$, assuming $m_{1,2,\ldots,S}^{\text{Conj}}(\emptyset) \neq 1$. The DS upper notation refers to initials of Dempster and Shafer names because Dempster's rule has gained its popularity through Shafer's works on belief functions. Shafer uses the symbol $\oplus$ to note Dempster's fusion operator, i.e., $m_{1,2,\ldots,S}^{\text{DS}}(A) = [m_1 \oplus m_2 \oplus \ldots \oplus m_S](A)$ for $A \neq \emptyset$, and $m_{1,2,\ldots,S}^{\text{DS}}(\emptyset) = 0$. A probabilistic analysis of Dempster's rule of combination can be found in [37], and the geometry of Dempster's rule is analyzed in [38].

Example 1: Consider $\Theta = \{A, B\}$ and two following BBAs

$$m_1(A) = 0.1 \quad m_1(B) = 0.2 \quad m_1(A \cup B) = 0.7$$

$$m_2(A) = 0.4 \quad m_2(B) = 0.3 \quad m_2(A \cup B) = 0.3$$

We have $m_{1,2}^{\text{Conj}}(\emptyset) = 0.11$, and

$$m_{1,2}^{\text{Conj}}(A) = 0.35, \quad m_{1,2}^{\text{Conj}}(B) = 0.33, \quad m_{1,2}^{\text{Conj}}(\Theta) = 0.21.$$

[7]The conjunctive fusion rule is also called Smets' rule of combination by some authors because it has been widely used by Philippe Smets in his works related to belief functions. But Smets himself call it conjunctive rule, see his last paper [20] (p. 388).

Symbolically we denote the conjunctive fusion of S sources as $m^{\text{Conj}}_{1,2,\ldots,S} = \text{Conj}(m_1, m_2, \ldots, m_S)$. This conjunctive rule is commutative and associative. This means that the sources can be combined altogether in one step, or sequentially in any order and it does not matter. Also, the total ignorant source represented by the vacuous (non-informative) BBA has no impact in the fusion result—see Lemma 1 below.

Lemma 1: The VBBA m_v has a neutral impact in the conjunctive rule of combination, that is

$$\text{Conj}(m_1, m_2, \ldots, m_S, m_v) = \text{Conj}(m_1, m_2, \ldots, m_S). \tag{8}$$

Proof: see Appendix 1.

The main drawback of this fusion rule is that it does not generate a proper BBA because $m^{\text{Conj}}_{1,2,\ldots,S}(\emptyset) > 0$ in general, and also it can provide a fusion result $m^{\text{Conj}}_{1,2,\ldots,S}(\emptyset)$ that quickly tends to one after only few steps of a sequential fusion processing of the sources which is not very useful for decision-making support. This is because the empty set $\emptyset$ is the absorbing element for the conjunctive operation since $\emptyset \cap A = \emptyset$ for all $A \in 2^\Theta$ so that the mass committed to the empty set always increases through the repeated conjunctive fusion rule. The main interest of this rule is its ability to identify the partial conflicts and to provide a measure of the total level of conflict $m^{\text{Conj}}_{1,2,\ldots,S}(\emptyset)$ between the sources which can be used to manage (select or discard) the sources in the fusion process if one prefers, see [39] for an application in geophysics for instance.

C. PCR5 and PCR6 rules of combination

The PCR rules have been developed originally in the framework of DSmT [31], [32], [34] but they can work also in the classical framework of Shafer's belief functions as well. Six rules have been proposed and they are referred as PCR1, ..., PCR6 rules of combination having different complexities, PCR1 being the most simplest (but less effective) one. All these rules share the same general principle which consists of three steps:

- apply the conjunctive rule (6);
- calculate the conflicting mass products $\pi_j(\emptyset)$; and
- redistribute the conflicting mass products $\pi_j(\emptyset)$ proportionally on all non-empty sets involved in the conflict.

The way the conflicting mass product $\pi_j(\emptyset)$ is redistributed yields to different versions of PCR combination rules that work for any degree of conflict. The sophistication/complexity and preciseness of PCR rules increases from the first PCR1 rule up to the last rule PCR6. The main disadvantage of these rules, aside their complexity, is their non-associativity properties which impose to combine all the BBAs altogether with PCR rules rather than sequentially to expect the best fusion result.

In this paper, we focus on the presentation of PCR5 and PCR6 only because they are the most well-known advanced fusion rules used so far in the belief functions community. A detailed presentation of other rules of combination encountered in the literature can be found in [40]. Symbolically, the PCR5 fusion and the PCR6 fusion of $S \geq 2$ BBAs are, respectively, denoted $m^{\text{PCR5}}_{1,2,\ldots,S} = \text{PCR5}(m_1, m_2, \ldots, m_S)$, and $m^{\text{PCR6}}_{1,2,\ldots,S} = \text{PCR6}(m_1, m_2, \ldots, m_S)$.

Readers familiar with PCR rules could quickly read the example 1 given in section III-B, and the results obtained with classical and improved PCR5 and PCR6 rules in section VII to appreciate the discussion throughout the paper.

The PCR5 rule of combination [31]: This rule transfers the conflicting mass $\pi_j(\emptyset)$ to all the elements involved in this conflict and proportionally to their individual masses, so that a more sophisticate and specific distribution is done with the PCR5 fusion process with respect to other existing rules (including Dempster's rule). The PCR5 rule is presented in details (with justification and examples) in [34] (Vol. 2 and Vol. 3).

• The PCR5 fusion of two BBAs is obtained by $m^{\text{PCR5}}_{1,2}(\emptyset) = 0$, and for all $A \in 2^\Theta \setminus \{\emptyset\}$ by

$$m^{\text{PCR5}}_{1,2}(A) = m^{\text{Conj}}_{1,2}(A) + \sum_{\substack{X\in 2^\Theta \\ X\cap A=\emptyset}} \left[\frac{m_1(A)^2 m_2(X)}{m_1(A)+m_2(X)} + \frac{m_2(A)^2 m_1(X)}{m_2(A)+m_1(X)}\right], \tag{9}$$

where $m^{\text{Conj}}_{1,2}(A)$ is the conjunctive rule formula (6) with $S = 2$, and where all denominators in (9) are different from zero. If a denominator is zero, that fraction is discarded. All propositions/sets are in a canonical form. We take the disjunctive normal form, which is a disjunction of conjunctions, and it is unique in Boolean algebra and simplest. For example, $X = A \cap B \cap (A \cup B \cup C)$ it is not in a canonical form, but we simplify the formula and $X = A \cap B$ is in a canonical form.

The PCR5 formula (9) for two BBAs can also be expressed by considering only the focal elements of $m_1(\cdot)$ and $m_2(\cdot)$ as follows

$$\begin{aligned} m^{\text{PCR5}}_{1,2}(A) &= m^{\text{Conj}}_{1,2}(A) \\ &+ \sum_{\substack{(X_{j_1},X_{j_2})\in\mathcal{F}(m_1)\times\mathcal{F}(m_2) \\ X_{j_1}\cap X_{j_2}=\emptyset \\ X_{j_1}=A}} m_1(X_{j_1}) \cdot \frac{m_1(X_{j_1})m_2(X_{j_2})}{m_1(X_{j_1})+m_2(X_{j_2})} \\ &+ \sum_{\substack{(X_{j_1},X_{j_2})\in\mathcal{F}(m_1)\times\mathcal{F}(m_2) \\ X_{j_1}\cap X_{j_2}=\emptyset \\ X_{j_2}=A}} m_2(X_{j_2}) \cdot \frac{m_1(X_{j_1})m_2(X_{j_2})}{m_1(X_{j_1})+m_2(X_{j_2})}, \end{aligned} \tag{10}$$

or equivalently, with shorthand π_j notations, as

$$m_{1,2}^{\text{PCR5}}(A) = m_{1,2}^{\text{Conj}}(A) + \sum_{\substack{j\in\{1,\ldots,\mathcal{F}\}|\mathbf{X}_j\in\mathcal{F}(m_1,m_2)\\ X_{j_1}\cap X_{j_2}=\emptyset\\ A\in\mathbf{X}_j}} \left[m_{i\in\{1,2\}|X_{j_i}=A}(X_{j_i}) \cdot \frac{\pi_j(X_{j_1}\cap X_{j_2})}{m_1(X_{j_1})+m_2(X_{j_2})}\right], \quad (11)$$

where $\mathcal{F} = |\mathcal{F}(m_1)|\cdot|\mathcal{F}(m_2)|$ is the total number of products $\pi_j(X_{j_1}\cap X_{j_2}) = m_1(X_{j_1})m_2(X_{j_2})$, and $A \in \mathbf{X}_j$ means that at least one component of $\mathbf{X}_j$ equals A.

• The explicit formula of the PCR5 fusion of three BBAs is given in [41].

• A simple formulation of the general expression of the PCR5 fusion of $S > 2$ BBAs is obtained by redistributing each conflicting product $\pi_j(\emptyset) = \pi_j(X_{j_1}\cap\ldots\cap X_{j_S}=\emptyset) = \prod_{i=1}^{S} m_i(X_{j_i})$ to some elements of the power set of the FoD that are involved in the conflict. Each $\pi_j(\emptyset)$ is redistributed proportionally to elements involved in this conflict based on the PCR5 redistribution principle. When an element $A \in 2^\Theta$ is not involved in a conflicting product $\pi_j(\emptyset)$, i.e. $A \notin \mathbf{X}_j$, the conflicting product $\pi_j(\emptyset)$ is not redistributed to A. If an element A is involved in the conflict $X_{j_1}\cap\ldots\cap X_{j_S}=\emptyset$, i.e. $A\in\mathbf{X}_j$ and $\pi_j(\emptyset)$ occur, then the proportional redistribution of $\pi_j(\emptyset)$ to A is given by

$$x_j(A) \triangleq \left(\prod_{i\in\{1,\ldots,S\}|X_{j_i}=A} m_i(X_{j_i})\right) \cdot \frac{\pi_j(\emptyset)}{\sum_{X\in\mathbf{X}_j}\left(\prod_{i\in\{1,\ldots,S\}|X_{j_i}=X} m_i(X_{j_i})\right)}, \quad (12)$$

where $A \in \mathbf{X}_j$ means that at least one component of the S-tuple $\mathbf{X}_j = (X_{j_1},\ldots,X_{j_S}) \in \mathcal{F}(m_1,\ldots,m_S)$ equals A.

Finally the mass value of A obtained by the PCR5 rule is calculated by

$$m_{1,2,\ldots,S}^{\text{PCR5}}(A) = m_{1,2,\ldots,S}^{\text{Conj}}(A) + \sum_{j\in\{1,\ldots,\mathcal{F}\}|A\in\mathbf{X}_j\wedge\pi_j(\emptyset)} x_j(A), \quad (13)$$

where $A\in\mathbf{X}_j\wedge\pi_j(\emptyset)$ is a shorthand notation meaning that at least one component of the S-tuple $\mathbf{X}_j$ equals A and the components of $\mathbf{X}_j$ are conflicting, i.e., $X_{j_1}\cap\ldots\cap X_{j_S}=\emptyset$.

Therefore the general PCR5 formula can be expressed as $m_{1,2,\ldots,S}^{\text{PCR5}}(\emptyset) = 0$, and for $A \in 2^\Theta \setminus \{\emptyset\}$ by

$$m_{1,2,\ldots,S}^{\text{PCR5}}(A) = m_{1,2,\ldots,S}^{\text{Conj}}(A) + \sum_{j\in\{1,\ldots,\mathcal{F}\}|A\in\mathbf{X}_j\wedge\pi_j(\emptyset)} \left[\left(\prod_{i\in\{1,\ldots,S\}|X_{j_i}=A} m_i(X_{j_i})\right) \cdot \frac{\pi_j(\emptyset)}{\sum_{X\in\mathbf{X}_j}\left(\prod_{i\in\{1,\ldots,S\}|X_{j_i}=X} m_i(X_{j_i})\right)}\right]. \quad (14)$$

It is worth noting that the formula (14) is a generalization of the formula (11), i.e., (14) coincides with (11) when $S = 2$.

This general PCR5 formula is equivalent to the original PCR5 formula given in [31] but it involves only the focal elements of the BBAs to combine which makes the derivation more efficient (less computationally demanding) than the original general PCR5 formula, specially when each BBA has only few focal elements. We use this new general PCR5 formula because it is relatively simple and easy to improve it into PCR5$^+$ formula—see section VI-B. The extension of PCR5 for combining qualitative[8] BBAs can be found in [34] (Vol. 2 and 3) and in [33]. PCR5 rule is not associative and the best fusion result is obtained by combining the sources altogether at the same time when possible. A suboptimal fast fusion method using PCR5-based canonical decomposition [42] can be found in [43].

The PCR6 rule of combination [32]: A variant of PCR5 rule, called PCR6 rule, has been proposed by Martin and Osswald in [32], [33] for combining $S > 2$ sources. Because PCR6 coincides with PCR5 when one combines two sources, we do not provide the PCR6 formula for two sources which is the same as (9). The difference between PCR5 and PCR6 lies in the way the PCR is done as soon as three (or more) sources are involved in the fusion as it will be shown in the example 2 introduced in the next section. The explicit formula of the PCR6 fusion of three BBAs is given in [41] for convenience.

The PCR6 fusion of $S > 2$ BBAs is obtained by $m_{1,2,\ldots,S}^{\text{PCR6}}(\emptyset) = 0$, and for all $A \in 2^\Theta \setminus \{\emptyset\}$ by[9]

$$m_{1,2,\ldots,S}^{\text{PCR6}}(A) = m_{1,2,\ldots,S}^{\text{Conj}}(A) + \sum_{j\in\{1,\ldots,\mathcal{F}\}|A\in\mathbf{X}_j\wedge\pi_j(\emptyset)} \left[\left(\sum_{i\in\{1,\ldots,S\}|X_{j_i}=A} m_i(X_{j_i})\right) \cdot \frac{\pi_j(\emptyset)}{\sum_{X\in\mathbf{X}_j}\left(\sum_{i\in\{1,\ldots,S\}|X_{j_i}=X} m_i(X_{j_i})\right)}\right]. \quad (15)$$

The difference between the general PCR5 and PCR6 formulas is that the PCR5 proportional redistribution involves the products $\prod_{i\in\{1,\ldots,S\}|X_{j_i}=A} m_i(X_{j_i})$ of multiple same focal elements A (if any) in the conflict, whereas the PCR6 conflict redistribution principle works with their sum $\sum_{i\in\{1,\ldots,S\}|X_{j_i}=A} m_i(X_{j_i})$ instead. The next section presents some examples for PCR5 and PCR6 rules of combinations.

We use this general PCR6 formula instead of the original Martin-Osswald's PCR6 formula [32] because

[8]A qualitative BBA is a BBA whose values are labels (e.g., low, medium, and high) instead of real numbers.

[9]We wrote this PCR6 general formula in the style of PCR5 formula (14).

it is easier to improve it into PCR6$^+$ formula—see Section VI-B. From the implementation point of view, PCR6 is simpler to implement than PCR5. From the decision-making standpoint, PCR6 is better than PCR5 when $S > 2$ as reported by Martin and Osswald in [32] (see also the Example 3 in the next section) in their applications. For convenience, some Matlab™ codes of PCR5 and PCR6 fusion rules can be found in the appendix of [44], also in Chap. 7 of [34] (Vol. 3), or from Arnaud Martin's web page [45]. PCR6 code (in R programming language) can be found also in iBelief package developed by Kuang Zhou and Arnaud Martin from the BFAS[10] repository [46], or directly from [47] as well. When we have only two BBAs to combine, PCR5 and PCR6 rules provide the same result because formulas (14) and (15) coincide for $S = 2$.

In this paper, we have voluntarily chosen to present the two rules, PCR5 and PCR6, and their improved versions mainly for historical reasons and because these two rules have strong theoretical links as we have shown. By doing this, we offer the possibility to readers (and potential users) to test each of these advanced fusion methods and evaluate their performances on their own applications. Even though PCR6 is posterior to PCR5, since some researchers have implemented and are using PCR5 fusion rule, it appears important to introduce the improved version of this rule. Furthermore, PCR5 goes back exactly on the tracks of the conjunctive rule, while PCR6 does not.

IV. EXAMPLES FOR PCR5 AND PCR6 FUSION RULES

Here we provide two simple examples showing the difference of the results between PCR5 and PCR6 rules. For convenience, all numerical values given in the examples of this paper have been rounded to six decimal places when necessary.

Example 2: We consider the simplest FoD $\Theta = \{A, B\}$, and the three following BBAs

$$m_1(A) = 0.6, m_1(B) = 0.1, m_1(A \cup B) = 0.3$$
$$m_2(A) = 0.5, m_2(B) = 0.3, m_2(A \cup B) = 0.2$$
$$m_3(A) = 0.4, m_3(B) = 0.1, m_3(A \cup B) = 0.5$$

Because $\mathcal{F}_{m_1} = |\mathcal{F}(m_1)| = 3$, $\mathcal{F}_{m_2} = |\mathcal{F}(m_2)| = 3$ and $\mathcal{F}_{m_3} = |\mathcal{F}(m_3)| = 3$, we have $\mathcal{F} = \mathcal{F}_{m_1} \cdot \mathcal{F}_{m_2} \cdot \mathcal{F}_{m_3} = 27$ products to consider. Fifteen products are non-conflicting and will enter in the calculation of $m^{\text{Conj}}_{1,2,3}(A)$, $m^{\text{Conj}}_{1,2,3}(B)$, and $m^{\text{Conj}}_{1,2,3}(A \cup B)$, and 12 products are conflicting products that will need to be proportionally redistributed. The conjunctive combination of these three BBAs is

$$\begin{aligned} m^{\text{Conj}}_{1,2,3}(A) &= m_1(A)m_2(A)m_3(A) \\ &\quad + m_1(A)m_2(A)m_3(A \cup B) \\ &\quad + m_1(A)m_2(A \cup B)m_3(A) \\ &\quad + m_1(A \cup B)m_2(A)m_3(A) \\ &\quad + m_1(A)m_2(A \cup B)m_3(A \cup B) \\ &\quad + m_1(A \cup B)m_2(A)m_3(A \cup B) \\ &\quad + m_1(A \cup B)m_2(A \cup B)m_3(A) \\ &= 0.5370, \end{aligned}$$

$$\begin{aligned} m^{\text{Conj}}_{1,2,3}(B) &= m_1(B)m_2(B)m_3(B) \\ &\quad + m_1(B)m_2(B)m_3(A \cup B) \\ &\quad + m_1(B)m_2(A \cup B)m_3(B) \\ &\quad + m_1(A \cup B)m_2(B)m_3(B) \\ &\quad + m_1(B)m_2(A \cup B)m_3(A \cup B) \\ &\quad + m_1(A \cup B)m_2(B)m_3(A \cup B) \\ &\quad + m_1(A \cup B)m_2(A \cup B)m_3(B) \\ &= 0.0900, \end{aligned}$$

$$\begin{aligned} m^{\text{Conj}}_{1,2,3}(A \cup B) &= m_1(A \cup B)m_2(A \cup B)m_3(A \cup B) \\ &= 0.3 \cdot 0.2 \cdot 0.5 = 0.0300, \end{aligned}$$

and

$$\begin{aligned} m^{\text{Conj}}_{1,2,3}(\emptyset) &= 1 - m^{\text{Conj}}_{1,2,3}(A) - m^{\text{Conj}}_{1,2,3}(B) - m^{\text{Conj}}_{1,2,3}(A \cup B) \\ &= 0.3430, \end{aligned}$$

In this example, we have 12 partial conflicts, noted $\pi_j(\emptyset)$ $(j = 1, \ldots, 12)$, which are given by the following products

$$\pi_1(\emptyset) = m_1(A)m_2(A)m_3(B) = 0.0300,$$
$$\pi_2(\emptyset) = m_1(A)m_2(B)m_3(A) = 0.0720,$$
$$\pi_3(\emptyset) = m_1(B)m_2(A)m_3(A) = 0.0200,$$
$$\pi_4(\emptyset) = m_1(B)m_2(B)m_3(A) = 0.0120,$$
$$\pi_5(\emptyset) = m_1(B)m_2(A)m_3(B) = 0.0050,$$
$$\pi_6(\emptyset) = m_1(A)m_2(B)m_3(B) = 0.0180,$$
$$\pi_7(\emptyset) = m_1(A \cup B)m_2(A)m_3(B) = 0.0150,$$
$$\pi_8(\emptyset) = m_1(A \cup B)m_2(B)m_3(A) = 0.0360,$$
$$\pi_9(\emptyset) = m_1(B)m_2(A)m_3(A \cup B) = 0.0250,$$
$$\pi_{10}(\emptyset) = m_1(A)m_2(B)m_3(A \cup B) = 0.0900,$$
$$\pi_{11}(\emptyset) = m_1(A)m_2(A \cup B)m_3(B) = 0.0120,$$
$$\pi_{12}(\emptyset) = m_1(B)m_2(A \cup B)m_3(A) = 0.0080.$$

In applying the PCR5 formula (14), and the PCR6 formula (15), we obtain finally $m^{\text{PCR5}}_{1,2,3}(\emptyset) = m^{\text{PCR6}}_{1,2,3}(\emptyset) = 0$,

[10]Belief Functions and Applications Society.

and[11]

$$m_{1,2,3}^{\text{PCR5}}(A) \approx 0.723281,$$
$$m_{1,2,3}^{\text{PCR5}}(B) \approx 0.182460,$$
$$m_{1,2,3}^{\text{PCR5}}(A \cup B) \approx 0.094259,$$

and

$$m_{1,2,3}^{\text{PCR6}}(A) \approx 0.743496,$$
$$m_{1,2,3}^{\text{PCR6}}(B) \approx 0.162245,$$
$$m_{1,2,3}^{\text{PCR6}}(A \cup B) \approx 0.094259.$$

We see a difference between the BBAs $m_{1,2,3}^{\text{PCR5}}$ and $m_{1,2,3}^{\text{PCR6}}$, which is normal because the PCR principles are quite different. Using the PCR5 fusion rule the first partial conflicting mass $\pi_1(\emptyset) = m_1(A)m_2(A)m_3(B) = 0.03$ will be redistributed back to A and B proportionally to $m_1(A)m_2(A)$ and to $m_3(B)$ as follows

$$\frac{x_1(A)}{m_1(A)m_2(A)} = \frac{x_1(B)}{m_3(B)} = \frac{\pi_1(\emptyset)}{m_1(A)m_2(A) + m_3(B)},$$

whence

$$x_1(A) = \frac{m_1(A)m_2(A)\pi_1(\emptyset)}{m_1(A)m_2(A) + m_3(B)} = 0.0225,$$
$$x_1(B) = \frac{m_3(B)\pi_1(\emptyset)}{m_1(A)m_2(A) + m_3(B)} = 0.0075.$$

We can verify $\pi_1(\emptyset) = x_1(A) + x_1(B) = 0.03$.

Using the PCR6 fusion rule the first partial conflicting mass $\pi_1(\emptyset) = 0.03$ will be redistributed back to A and B proportionally to $(m_1(A)+m_2(A))$ and to $m_3(B)$. So we will get the following redistributions $x_1(A) = 0.0275$ for A and $x_1(B) = 0.0025$ for B because

$$\frac{x_1(A)}{m_1(A) + m_2(A)} = \frac{x_1(B)}{m_3(B)} = \frac{\pi_1(\emptyset)}{m_1(A) + m_2(A) + m_3(B)},$$

whence

$$x_1(A) = \frac{(m_1(A) + m_2(A))\pi_1(\emptyset)}{m_1(A) + m_2(A) + m_3(B)} = 0.0275,$$
$$x_1(B) = \frac{m_3(B)\pi_1(\emptyset)}{m_1(A) + m_2(A) + m_3(B)} = 0.0025.$$

We can verify $\pi_1(\emptyset) = x_1(A) + x_1(B) = 0.03$.

Note that for all the partial conflicts having no duplicate element involved in the conflicting product $\pi_j(\emptyset)$ we make the same redistribution with PCR5 rule and with PCR6 rule. For instance, for $\pi_7(\emptyset) = m_1(A \cup B)m_2(A)m_3(B) = 0.0150$ we get

$$\frac{x_7(A \cup B)}{m_1(A \cup B)} = \frac{x_7(A)}{m_2(A)} = \frac{x_7(B)}{m_3(B)} = \frac{\pi_7(\emptyset)}{m_1(A \cup B) + m_2(A) + m_3(B)},$$

whence $\pi_7(\emptyset) = x_7(A \cup B) + x_7(A) + x_7(B) = 0.0150$ with

$$x_7(A \cup B) = \frac{m_1(A \cup B)\pi_7(\emptyset)}{m_1(A \cup B) + m_2(A) + m_3(B)} = 0.0050,$$
$$x_7(A) = \frac{m_2(A)\pi_7(\emptyset)}{m_1(A \cup B) + m_2(A) + m_3(B)} \approx 0.0083,$$
$$x_7(B) = \frac{m_3(B)\pi_7(\emptyset)}{m_1(A \cup B) + m_2(A) + m_3(B)} \approx 0.0017.$$

The next example shows also the difference between PCR5 and PCR6 rules, and it justifies why PCR6 rule is usually preferred to PCR5 rule in applications.

Example 3: we consider the FoD $\Theta = \{A, B, C\}$, and the four very simple BBAs defined by

$$m_1(A \cup B) = 1, m_2(B) = 1, m_3(A \cup B) = 1, \text{ and } m_4(C) = 1.$$

These BBAs are in conflict because the intersection of their focal elements is $(A \cup B) \cap A \cap (A \cup B) \cap C = \emptyset$. In this example, one has only one product of masses to calculate, which is $\pi_1((A \cup B) \cap A \cap (A \cup B) \cap C) = m_1(A \cup B)m_2(A)m_3(A \cup B)m_4(C) = 1$. In fact this product is a conflicting product denoted $\pi_1(\emptyset)$. We can also denote it $\pi(\emptyset)$ because the index $j = 1$ is useless in this case. Moreover, these BBAs are also in total conflict because $\pi(\emptyset) = m_1(A \cup B)m_2(A)m_3(A \cup B)m_4(C) = 1$.

If one applies the PCR5 rule principle we get

$$\frac{x(A \cup B)}{m_1(A \cup B)m_3(A \cup B)} = \frac{x(B)}{m_2(B)} = \frac{x(C)}{m_4(C)} = \frac{\pi(\emptyset)}{m_1(A \cup B)m_3(A \cup B) + m_2(B) + m_4(C)},$$

whence $x(A \cup B) = 1/3, x(B) = 1/3$ and $x(C) = 1/3$ so that

$$m_{1,2,3,4}^{\text{PCR5}}(A \cup B) = x(A \cup B) = 1/3,$$
$$m_{1,2,3,4}^{\text{PCR5}}(B) = x(B) = 1/3,$$
$$m_{1,2,3,4}^{\text{PCR5}}(C) = x(C) = 1/3.$$

This PCR5 result appears counter-intuitive because three sources among the four sources exclude definitely the hypothesis C because one has $Pl_1(C) = Pl_2(C) = Pl_3(C) = 0$, so it is intuitively expected that after the combination of all the four BBAs the mass committed to C should not be greater than $1/4 = 0.25$.

If one applies the PCR6 rule principle, we get

$$\frac{x(A \cup B)}{m_1(A \cup B) + m_3(A \cup B)} = \frac{x(B)}{m_2(B)} = \frac{x(C)}{m_4(C)} = \frac{\pi(\emptyset)}{m_1(A \cup B) + m_3(A \cup B) + m_2(B) + m_4(C)},$$

whence $x(A \cup B) = 2/4, x(B) = 1/4$ and $x(C) = 1/4$ so that

[11]The symbol $\approx$ means "approximately equal to."

$$m^{\mathrm{PCR6}}_{1,2,3,4}(A\cup B)=x(A\cup B)=0.5,$$
$$m^{\mathrm{PCR6}}_{1,2,3,4}(B)=x(B)=0.25,$$
$$m^{\mathrm{PCR6}}_{1,2,3,4}(C)=x(C)=0.25,$$

which is in better agreement with what we intuitively expect because $m^{\mathrm{PCR6}}_{1,2,3,4}(C)$ is not greater than 1/4. Of course in this example, Dempster's rule of combination cannot be simply applied because the conflict is total yielding a division by zero in Dempster's rule formula [8], but by using eventually some discounting methods to modify the BBAs to combine.

V. FLAWED BEHAVIOR OF PCR5 AND PCR6 RULES

The PCR5 and PCR6 rules of combination are not associative which means that the fusion of the BBAs must be done using general formulas (14) or (15) if one has more than two BBAs to combine, which is not very convenient. Therefore, the sequential PCR5 or PCR6 combination of $S>2$ BBAs are not in general equal to the global PCR5 or PCR6 fusion of the S BBAs altogether because the order of the combination of the sources does matter in the sequential combination. In general (i.e. when conflicts exist between the sources of evidence to combine) one has for $S>2$

$$\mathrm{PCR5}(m_1,m_2,\dots,m_S)\neq \mathrm{PCR5}(\mathrm{PCR5}(\mathrm{PCR5}(m_1,m_2),m_3),\dots,m_S) \quad (16)$$

and

$$\mathrm{PCR6}(m_1,m_2,\dots,m_S)\neq \mathrm{PCR6}(\mathrm{PCR6}(\mathrm{PCR6}(m_1,m_2),m_3),\dots,m_S), \quad (17)$$

and also for $S>2$ PCR5 fusion result is generally different of PCR6 fusion result that is

$$\mathrm{PCR5}(m_1,m_2,\dots,m_S)\neq \mathrm{PCR6}(m_1,m_2,\dots,m_S). \quad (18)$$

Formula (18) says that in general PCR5 is different from PCR6, of course except the case when we combine only two sources. PCR5 and PCR6 rules can become computationally intractable for combining a large number of sources and for working with large FoD. This is a well-known limitation of these rules, but this is the price to pay to get better results than with classical rules.

Aside the complexity of these rules, it is worth to mention that the neutral impact property of the VBBA m_v is lost in general when considering the PCR5 or PCR6 combination of $S>2$ BBAs altogether, that is

$$\mathrm{PCR5}(m_1,\dots,m_{S-1},m_v)\neq \mathrm{PCR5}(m_1,\dots,m_{S-1}) \quad (19)$$

and

$$\mathrm{PCR6}(m_1,\dots,m_{S-1},m_v)\neq \mathrm{PCR6}(m_1,\dots,m_{S-1}) \quad (20)$$

Formula (19) and (20) show that in general PCR5 and PCR6 do not have the ignorant source as a neutral element. This is due to the redistribution principles used in PCR5 and in PCR6 rules. Example 4 shows the non-neutral impact of the VBBA in PCR5 and PCR6 rules for convenience. Note that the VBBA has a neutral impact in the fusion result if and only if one has only two BBAs to combine with PCR5, or PCR6, and one of them is the VBBA because in this case there is no possible (partial) conflict to redistribute between any BBA $m(\cdot)$ defined over the FoD Θ and the VBBA $m_v(\cdot)$. That is, for any BBA $m_1(\cdot)$ one always has

$$\mathrm{PCR5}(m_1,m_v)=\mathrm{PCR6}(m_1,m_v)=m_1. \quad (21)$$

Example 4: we consider the FoD $\Theta=\{A,B\}$ having only two elements, and the following four BBAs as follows:

$$m_1(A)=0.6, m_1(B)=0.1, m_1(A\cup B)=0.3,$$
$$m_2(A)=0.5, m_2(B)=0.3, m_2(A\cup B)=0.2,$$
$$m_3(A)=0.4, m_3(B)=0.1, m_3(A\cup B)=0.5,$$
$$m_4(A\cup B)=1.$$

BBAs m_1, m_2, and m_3 are as in example 2, and the BBA m_4 is nothing but the VBBA m_v defined over this FoD Θ.

In example 2, we did obtain with $\mathrm{PCR5}(m_1,m_2,m_3)$ and with $\mathrm{PCR5}(m_1,m_2,m_3,m_4)$ the following resulting BBAs

$$m^{\mathrm{PCR5}}_{1,2,3}(A)\approx 0.723281,$$
$$m^{\mathrm{PCR5}}_{1,2,3}(B)\approx 0.182460,$$
$$m^{\mathrm{PCR5}}_{1,2,3}(A\cup B)\approx 0.094259,$$

and

$$m^{\mathrm{PCR5}}_{1,2,3,4}(A)\approx 0.654604,$$
$$m^{\mathrm{PCR5}}_{1,2,3,4}(B)\approx 0.144825,$$
$$m^{\mathrm{PCR5}}_{1,2,3,4}(A\cup B)\approx 0.200571.$$

Clearly, $\mathrm{PCR5}(m_1,m_2,m_3)\neq \mathrm{PCR5}(m_1,m_2,m_3,m_4)$ even if m_4 is the VBBA.

Analogously, we did obtain with $\mathrm{PCR6}(m_1,m_2,m_3)$ and with $\mathrm{PCR6}(m_1,m_2,m_3,m_4)$

$$m^{\mathrm{PCR6}}_{1,2,3}(A)\approx 0.743496,$$
$$m^{\mathrm{PCR6}}_{1,2,3}(B)\approx 0.162245,$$
$$m^{\mathrm{PCR6}}_{1,2,3}(A\cup B)\approx 0.094259,$$

and

$$m^{\mathrm{PCR6}}_{1,2,3,4}(A) \approx 0.647113,$$

$$m^{\mathrm{PCR6}}_{1,2,3,4}(B) \approx 0.128342,$$

$$m^{\mathrm{PCR6}}_{1,2,3,4}(A \cup B) \approx 0.224545.$$

Hence, $\mathrm{PCR6}(m_1, m_2, m_3) \neq \mathrm{PCR6}(m_1, m_2, m_3, m_4)$, even if m_4 is the VBBA.

This example 4 shows clearly that the VBBA does not have a neutral impact in the PCR5 and PCR6 rules of combination. In fact, adding more VBBAs m_v in the PCR5 or PCR6 fusion will increase more and more the mass of $A \cup B$ while decreasing more and more the masses of A and of B with PCR5, and PCR6. When the number of VBBAs m_v increases, we will have[12] $m^{\mathrm{PCR5/6}}_{1,2,3,m_v,\dots,m_v}(A \cup B) \to 1$, $m^{\mathrm{PCR5/6}}_{1,2,3,m_v,\dots,m_v}(A) \to 0$, and $m^{\mathrm{PCR5/6}}_{1,2,3,m_v,\dots,m_v}(B) \to 0$.

This is unsatisfactory because the VBBA brings no useful information to exploit, and it is naturally expected that it must not impact the fusion result in the combination of BBAs. This can be seen as a flaw of the behavior of PCR5 and PCR6 rules of combination.

To emphasize this flaw, we give in the example 5 a case where the mass committed to some partial uncertainties can increase more than necessary with PCR5 and with PCR6 rules of combination. This is detrimental for the quality of the fusion result and for decision-making because the result is more uncertain than it should be, and consequently the decision is more difficult to make.

Example 5: we consider the FoD $\Theta = \{A, B, C, D, E\}$, and the following three BBAs

$$\begin{cases} m_1(A \cup B) = 0.70 \\ m_1(C \cup D) = 0.06 \\ m_1(A \cup B \cup C \cup D) = 0.15 \\ m_1(E) = 0.09 \end{cases}$$

and

$$\begin{cases} m_2(A \cup B) = 0.06 \\ m_2(C \cup D) = 0.50 \\ m_2(A \cup B \cup C \cup D) = 0.04 \\ m_2(E) = 0.40 \end{cases}$$

and

$$\begin{cases} m_3(B) = 0.01 \\ m_3(A \cup B \cup C \cup D \cup E) = 0.99. \end{cases}$$

Note that the BBA m_3 is not equal to the VBBA but it is very close to the VBBA because $m_3(\Theta)$ is close to one.

If we make the $\mathrm{PCR6}(m_1, m_2)$ fusion of only the two BBAs m_1 and m_2 altogether, which is also equal to $\mathrm{PCR5}(m_1, m_2)$, we obtain

$$\begin{cases} m^{\mathrm{PCR6}}_{1,2}(A \cup B) \approx 0.465309 \\ m^{\mathrm{PCR6}}_{1,2}(C \cup D) \approx 0.296299 \\ m^{\mathrm{PCR6}}_{1,2}(A \cup B \cup C \cup D) \approx 0.023471 \\ m^{\mathrm{PCR6}}_{1,2}(E) \approx 0.214921 \end{cases}$$

[12]The notation $m^{\mathrm{PCR5/6}}$ indicates "m^{PCR5} or m^{PCR6}" for convenience.

If we make the $\mathrm{PCR6}(m_1, m_2, m_3)$ fusion of all these three BBAs altogether we obtain

$$\begin{cases} m^{\mathrm{PCR6}}_{1,2,3}(B) \approx 0.000962 \\ m^{\mathrm{PCR6}}_{1,2,3}(A \cup B) \approx 0.286107 \\ m^{\mathrm{PCR6}}_{1,2,3}(C \cup D) \approx 0.203454 \\ m^{\mathrm{PCR6}}_{1,2,3}(A \cup B \cup C \cup D) \approx 0.012203 \\ m^{\mathrm{PCR6}}_{1,2,3}(E) \approx 0.116038 \\ m^{\mathrm{PCR6}}_{1,2,3}(A \cup B \cup C \cup D \cup E) \approx 0.381236 \end{cases}$$

One sees that combining the BBAs m_1, m_2 with the BBA m_3 (where m_3 is close to VBBA, and therefore m_3 is almost non-informative) generates a big increase of the belief of the uncertainty in the resulting BBA. This behaviour is clearly counter-intuitive because if the source is almost vacuous, only a small degradation of the uncertainty is expected and in the limit case when m_3 is the VBBA no impact of m_3 on the fusion result should occur. Note that this behavior also occurs with $\mathrm{PCR5}(m_1, m_2, m_3)$ because one has for this example

$$\begin{cases} m^{\mathrm{PCR5}}_{1,2,3}(B) \approx 0.001103 \\ m^{\mathrm{PCR5}}_{1,2,3}(A \cup B) \approx 0.286107 \\ m^{\mathrm{PCR5}}_{1,2,3}(C \cup D) \approx 0.203384 \\ m^{\mathrm{PCR5}}_{1,2,3}(A \cup B \cup C \cup D) \approx 0.012203 \\ m^{\mathrm{PCR5}}_{1,2,3}(E) \approx 0.115967 \\ m^{\mathrm{PCR5}}_{1,2,3}(A \cup B \cup C \cup D \cup E) \approx 0.381236 \end{cases}$$

The deep analysis of the partial conflict redistributions done in this interesting example reveals clearly the flaw of the principles of PCR5 and PCR6 rules of combination. Indeed, for this example, one has $\mathcal{F}_{m_1} \cdot \mathcal{F}_{m_2} \cdot \mathcal{F}_{m_3} = 4 \cdot 4 \cdot 2 = 32$ products $\pi_j(X_{j_1} \cap X_{j_2} \cap X_{j_3}) = m_1(X_{j_1})m_2(X_{j_2})m_3(X_{j_3})$ to calculate, where $X_{j_1} \in \mathcal{F}(m_1) = \{A \cup B, C \cup D, A \cup B \cup C \cup D, E\}$, $X_{j_2} \in \mathcal{F}(m_2) = \{A \cup B, C \cup D, A \cup B \cup C \cup D, E\}$, and $X_{j_3} \in \mathcal{F}(m_3) = \{B, A \cup B \cup C \cup D \cup E\}$. Among these 32 possible conjunctions of focal elements, 20 products correspond to partial conflicts when $X_{j_1} \cap X_{j_2} \cap X_{j_3} = \emptyset$, which need to be redistributed properly to some elements of $2^\Theta \setminus \{\emptyset\}$ according to the PCR5, or the PCR6 redistribution principles.

More precisely, we have to consider all the following products π_j for calculating the result

$$\pi_1(B) = m_1(A \cup B)m_2(A \cup B)m_3(B) = 0.00042,$$

$$\pi_2(A \cup B) = m_1(A \cup B)m_2(A \cup B)m_3(\Theta) = 0.04158,$$

$$\pi_3(\emptyset) = m_1(A \cup B)m_2(C \cup D)m_3(B) = 0.0035,$$

$$\pi_4(\emptyset) = m_1(A \cup B)m_2(C \cup D)m_3(\Theta) = 0.3465,$$

$$\pi_5(B) = m_1(A \cup B)m_2(A \cup B \cup C \cup D)m_3(B) = 0.00028,$$

$$\pi_6(A \cup B) = m_1(A \cup B)m_2(A \cup B \cup C \cup D)m_3(\Theta) = 0.02772,$$

$$\pi_7(\emptyset) = m_1(A \cup B)m_2(E)m_3(B) = 0.0028,$$

$$\pi_8(\emptyset) = m_1(A \cup B)m_2(E)m_3(\Theta) = 0.2772,$$

$$\pi_9(\emptyset) = m_1(C \cup D)m_2(A \cup B)m_3(B) = 0.000036,$$

$$\pi_{10}(\emptyset) = m_1(C \cup D)m_2(A \cup B)m_3(\Theta) = 0.003564,$$

$$\pi_{11}(\emptyset) = m_1(C \cup D)m_2(C \cup D)m_3(B) = 0.0003,$$
$$\pi_{12}(C \cup D) = m_1(C \cup D)m_2(C \cup D)m_3(\Theta) = 0.0297,$$
$$\pi_{13}(\emptyset) = m_1(C \cup D)m_2(A \cup B \cup C \cup D)m_3(B) = 0.000024,$$
$$\pi_{14}(C \cup D) = m_1(C \cup D)m_2(A \cup B \cup C \cup D)m_3(\Theta) = 0.002376,$$
$$\pi_{15}(\emptyset) = m_1(C \cup D)m_2(E)m_3(B) = 0.00024,$$
$$\pi_{16}(\emptyset) = m_1(C \cup D)m_2(E)m_3(\Theta) = 0.02376,$$
$$\pi_{17}(B) = m_1(A \cup B \cup C \cup D)m_2(A \cup B)m_3(B) = 0.00009,$$
$$\pi_{18}(A \cup B) = m_1(A \cup B \cup C \cup D)m_2(A \cup B)m_3(\Theta) = 0.00891,$$
$$\pi_{19}(\emptyset) = m_1(A \cup B \cup C \cup D)m_2(C \cup D)m_3(B) = 0.00075,$$
$$\pi_{20}(C \cup D) = m_1(A \cup B \cup C \cup D)m_2(C \cup D)m_3(\Theta) = 0.07425,$$
$$\pi_{21}(B) = m_1(A \cup B \cup C \cup D)m_2(A \cup B \cup C \cup D)m_3(B) = 0.00006,$$
$$\pi_{22}(A \cup B \cup C \cup D) = m_1(A \cup B \cup C \cup D)m_2(A \cup B \cup C \cup D) \cdot m_3(\Theta) = 0.00594,$$
$$\pi_{23}(\emptyset) = m_1(A \cup B \cup C \cup D)m_2(E)m_3(B) = 0.0006,$$
$$\pi_{24}(\emptyset) = m_1(A \cup B \cup C \cup D)m_2(E)m_3(\Theta) = 0.0594,$$
$$\pi_{25}(\emptyset) = m_1(E)m_2(A \cup B)m_3(B) = 0.000054,$$
$$\pi_{26}(\emptyset) = m_1(E)m_2(A \cup B)m_3(\Theta) = 0.005346,$$
$$\pi_{27}(\emptyset) = m_1(E)m_2(C \cup D)m_3(B) = 0.00045,$$
$$\pi_{28}(\emptyset) = m_1(E)m_2(C \cup D)m_3(\Theta) = 0.04455,$$
$$\pi_{29}(\emptyset) = m_1(E)m_2(A \cup B \cup C \cup D)m_3(B) = 0.000036,$$
$$\pi_{30}(\emptyset) = m_1(E)m_2(A \cup B \cup C \cup D)m_3(\Theta) = 0.003564,$$
$$\pi_{31}(\emptyset) = m_1(E)m_2(E)m_3(B) = 0.00036,$$
$$\pi_{32}(E) = m_1(E)m_2(E)m_3(\Theta) = 0.03564.$$

The conjunctive rule gives

$$m^{\text{Conj}}_{1,2,3}(B) = \pi_1(B) + \pi_5(B) + \pi_{17}(B) + \pi_{21}(B) = 0.00085,$$
$$m^{\text{Conj}}_{1,2,3}(A \cup B) = \pi_2(A \cup B) + \pi_6(A \cup B) + \pi_{18}(A \cup B) = 0.07821,$$
$$m^{\text{Conj}}_{1,2,3}(C \cup D) = \pi_{12}(C \cup D) + \pi_{14}(C \cup D) + \pi_{20}(C \cup D) = 0.106326,$$
$$m^{\text{Conj}}_{1,2,3}(A \cup B \cup C \cup D) = \pi_{22}(A \cup B \cup C \cup D) = 0.00594,$$
$$m^{\text{Conj}}_{1,2,3}(E) = \pi_{32}(E) = 0.03564.$$

The total conflicting mass between these three BBAs is

$$m^{\text{Conj}}_{1,2,3}(\emptyset) = \sum_{j=3,4,7,\ldots,11,13,15,16,19,23,\ldots,31} \pi_j(\emptyset)$$
$$= 1 - m^{\text{Conj}}_{1,2,3}(B) - m^{\text{Conj}}_{1,2,3}(A \cup B) - m^{\text{Conj}}_{1,2,3}(C \cup D) - m^{\text{Conj}}_{1,2,3}(A \cup B \cup C \cup D) - m^{\text{Conj}}_{1,2,3}(E) = 0.773034.$$

Let us examine how the $m^{\text{PCR5}}_{1,2,3}(\Theta) \approx 0.381236$ value is obtained based on the PCR5 redistribution principle. Based on the structures of $\pi_j(\emptyset)$ products, we have to consider only products involving a proportional redistribution to Θ. So we get a proportional redistribution to Θ only from the following products

$$\pi_4(\emptyset) = m_1(A \cup B)m_2(C \cup D)m_3(\Theta) = 0.3465,$$
$$\pi_8(\emptyset) = m_1(A \cup B)m_2(E)m_3(\Theta) = 0.2772,$$
$$\pi_{10}(\emptyset) = m_1(C \cup D)m_2(A \cup B)m_3(\Theta) = 0.003564,$$
$$\pi_{16}(\emptyset) = m_1(C \cup D)m_2(E)m_3(\Theta) = 0.02376,$$
$$\pi_{24}(\emptyset) = m_1(A \cup B \cup C \cup D)m_2(E)m_3(\Theta) = 0.0594,$$
$$\pi_{26}(\emptyset) = m_1(E)m_2(A \cup B)m_3(\Theta) = 0.005346,$$
$$\pi_{28}(\emptyset) = m_1(E)m_2(C \cup D)m_3(\Theta) = 0.04455,$$
$$\pi_{30}(\emptyset) = m_1(E)m_2(A \cup B \cup C \cup D)m_3(\Theta) = 0.003564.$$

Because there is no duplicate focal elements in each of these products, the PCR5 and PCR6 redistributions to Θ will be the same in this example.

The proportional redistribution of $\pi_4(\emptyset)$ to Θ is

$$x_4(\Theta) = \frac{m_3(\Theta)\pi_4(\emptyset)}{m_1(A \cup B) + m_2(C \cup D) + m_3(\Theta)} \approx 0.156637.$$

The proportional redistribution of $\pi_8(\emptyset)$ to Θ is

$$x_8(\Theta) = \frac{m_3(\Theta)\pi_8(\emptyset)}{m_1(A \cup B) + m_2(E) + m_3(\Theta)} \approx 0.131305.$$

The proportional redistribution of $\pi_{10}(\emptyset)$ to Θ is

$$x_{10}(\Theta) = \frac{m_3(\Theta)\pi_{10}(\emptyset)}{m_1(C \cup D) + m_2(A \cup B) + m_3(\Theta)} \approx 0.003179.$$

The proportional redistribution of $\pi_{16}(\emptyset)$ to Θ is

$$x_{16}(\Theta) = \frac{m_3(\Theta)\pi_{16}(\emptyset)}{m_1(C \cup D) + m_2(E) + m_3(\Theta)} \approx 0.016222.$$

The proportional redistribution of $\pi_{24}(\emptyset)$ to Θ is

$$x_{24}(\Theta) = \frac{m_3(\Theta)\pi_{24}(\emptyset)}{m_1(A \cup B \cup C \cup D) + m_2(E) + m_3(\Theta)} \approx 0.038186.$$

The proportional redistribution of $\pi_{26}(\emptyset)$ to Θ is

$$x_{26}(\Theta) = \frac{m_3(\Theta)\pi_{26}(\emptyset)}{m_1(E) + m_2(A \cup B) + m_3(\Theta)} \approx 0.004643.$$

The proportional redistribution of $\pi_{28}(\emptyset)$ to Θ is

$$x_{28}(\Theta) = \frac{m_3(\Theta)\pi_{28}(\emptyset)}{m_1(E) + m_2(C \cup D) + m_3(\Theta)} \approx 0.027914.$$

The proportional redistribution of $\pi_{30}(\emptyset)$ to Θ is

$$x_{30}(\Theta) = \frac{m_3(\Theta)\pi_{30}(\emptyset)}{m_1(E) + m_2(A \cup B \cup C \cup D) + m_3(\Theta)} \approx 0.003150.$$

Therefore, we finally obtain the quite big value for the mass committed to Θ

$$m_{1,2,3}^{\text{PCR5}}(\Theta) = x_4(\Theta) + x_8(\Theta) + x_{10}(\Theta) + x_{16}(\Theta) + x_{24}(\Theta) + x_{26}(\Theta) + x_{28}(\Theta) + x_{30}(\Theta) \approx 0.381236.$$

We see clearly why PCR5 (and PCR6) redistributes some mass to uncertainty Θ although the focal element Θ is not in conflict with other focal elements involved in each product $\pi_4(\emptyset), \pi_8(\emptyset), \pi_{10}(\emptyset), \pi_{16}(\emptyset), \pi_{24}(\emptyset), \pi_{26}(\emptyset)$, $\pi_{28}(\emptyset)$, and $\pi_{30}(\emptyset)$, which is an undesirable behavior that we want to avoid. That is why we propose in the next section some improvement of PCR5 and PCR6 rules of combination.

VI. IMPROVEMENT OF PCR5 AND PCR6 RULES

To circumvent the weakness of the orignal PCR5 and PCR6 redistribution principles, we propose an improvement of these rules that will be denoted as PCR5$^+$ and PCR6$^+$ in the sequel. These new rules are not redundant with PCR5 nor with PCR6 when combining more than two BBAs altogether.

The very simple and basic idea to improve PCR5 and PCR6 redistribution principles is to discard the elements that contain all the other elements implied in the partial conflict $\pi_j(\emptyset)$ calculation. Indeed, the elements discarded are regarded as non-informative and not useful for making the conflict redistribution.

For instance, if we consider the previous example 5, the conflicting mass with PCR5$^+$ and PCR6$^+$ for the conflicting product $\pi_4(\emptyset) = m_1(A\cup B)m_2(C\cup D)m_3(\Theta)$ will be proportionally redistributed back only to $A\cup B$ and to $C \cup D$ but not to Θ because $A \cup B \subseteq \Theta$ and $C \cup D \subseteq \Theta$. Thus, with *PCR5*$^+$ and *PCR6*$^+$ rules, we will make the following redistribution:

$$\frac{x_4(A \cup B)}{m_1(A \cup B)} = \frac{x_4(C \cup D)}{m_2(C \cup D)} = \frac{\pi_4(\emptyset)}{m_1(A \cup B) + m_2(C \cup D)}$$

Here, $x_4(\Theta)$ is set to 0 with PCR5$^+$ and PCR6$^+$ principles because no proportion of $\pi_4(\emptyset)$ must be redistributed to Θ.

However, with PCR5 and PCR6 rule we make the redistributions according to

$$\frac{x_4(A \cup B)}{m_1(A \cup B)} = \frac{x_4(C \cup D)}{m_2(C \cup D)} = \frac{x_4(\Theta)}{m_3(\Theta)} = \frac{\pi_4(\emptyset)}{m_1(A \cup B) + m_2(C \cup D) + m_3(\Theta)}.$$

A. Selection of focal elements for proportional redistribution

The main issue to improve PCR5 and PCR6 rules of combination is how to identify in each conflicting product $\pi_j(\emptyset)$ the set of elements to keep for making the improved proportional redistribution.

In this section, we propose a solution of this problem that can be easily implemented. For convenience, we give also the basic Matlab$^{\text{TM}}$codes of PCR5$^+$ and PCR6$^+$ in Appendix 3.

Let us consider $\pi_j(\emptyset) = m_1(X_{j_1})m_2(X_{j_2})\ldots m_S(X_{j_S})$ a conflicting product[13] where $X_{j_1} \cap X_{j_2} \cap \ldots \cap X_{j_S} = \emptyset$. We denote by $\mathcal{X}_j = \{X_1, \ldots, X_{s_j}, s_j \leq S\}$ the set of all distinct components of the S-tuple $\mathbf{X}_j$ related with the conflicting product $\pi_j(\emptyset)$. The order of the elements in $\mathcal{X}_j$ does not matter. The number s_j of elements in $\mathcal{X}_j$ can be less than S because it is possible to have duplicate focal elements in $\pi_j(\emptyset)$. We consider in $\mathcal{X}_j$ only the distinct focal elements involved in $\pi_j(\emptyset)$ (see the next example) and we will define their binary *keeping-index indicator* which will allow to know if each element of $\mathcal{X}_j$ needs to be kept in the PCR, or not, in the improved PCR5 and PCR6 rules of combination.

For each element $X_l \in \mathcal{X}_j$ we first define its binary *containing indicator* $\delta_j(X_{l'}, X_l)$ with respect to $X_{l'} \in \mathcal{X}_j$ to characterize if X_l contains (includes) $X_{l'}$ in wide sense, or not. Therefore, we take $\delta_j(X_{l'}, X_l) = 1$ if $X_{l'} \cap X_l = X_{l'}$, or equivalently if $X_{l'} \subseteq X_l$, and $\delta_j(X_{l'}, X_l) = 0$ otherwise. The definition of this binary *containing indicator* is summarized by the formula

$$\delta_j(X_{l'}, X_l) \triangleq \begin{cases} 1 & \text{if } X_{l'} \subseteq X_l, \\ 0 & \text{if } X_{l'} \nsubseteq X_l. \end{cases} \tag{22}$$

Of course $\delta_j(X_l, X_l) = 1$ because $X_l \cap X_l = X_l$, and we have $\delta_j(X_{l'}, X_l) = 0$ as soon as $|X_{l'}| > |X_l|$, where $|X_{l'}|$ and $|X_l|$ are the cardinalities of $X_{l'}$ and X_l, respectively. We have also $\delta_j(X_{l'}, X_l) = 0$ when $X_{l'} \cap X_l \neq X_{l'}$. For $X_l = \Theta$, we have $\delta_j(X_{l'}, X_l) = \delta_j(X_{l'}, \Theta) = 1$ for any $X_{l'} \in \mathcal{X}_j$.

To know if a focal element $X_{j_i} \in \mathbf{X}_j$ must be kept, or not, in the proportional redistribution of the j-th conflicting mass $\pi_j(\emptyset)$ with PCR5$^+$ and PCR6$^+$ rules, we have to determinate its binary *keeping-index* $\kappa_j(X_{j_i})$. For this, we define $\kappa_j(X_{j_i}) \in \{0, 1\}$ as follows

$$\kappa_j(X_{j_i}) \triangleq 1 - \prod_{\substack{X_{l'}, X_l \in \mathcal{X}_j \\ X_{l'} \neq X_l \\ |X_{j_i}| \leq |X_l| \\ |X_{l'}| \leq |X_l|}} \delta_j(X_{l'}, X_l) \tag{23}$$

The value $\kappa_j(X_{j_i}) = 1$ stipulates that the focal element $X_{j_i} \in \mathbf{X}_j$ must receive some proportional redistribution from the conflicting mass $\pi_j(\emptyset)$. The value $\kappa_j(X_{j_i}) = 0$ indicates that the focal element X_{j_i} will not be involved in the proportional redistribution of the conflicting mass $\pi_j(\emptyset)$.

[13] We consider $S > 2$ BBAs because for $S = 2$ BBAs, no improper increasing of uncertainty occurs with PCR5 or PCR6.

The binary keeping-index can also be defined equivalently as

$$\kappa_j(X_{j_i}) = \begin{cases} 1 \text{ if } c(X_{j_i}) \text{ is true} \\ 1 - \prod_{\substack{X_{l'} \in \mathcal{X}_j \\ X_{l'} \neq X_{j_i} \\ |X_{l'}| \leq |X_{j_i}|}} \delta_j(X_{l'}, X_{j_i}) \text{ if } c(X_{j_i}) \text{ is false,} \end{cases} \tag{24}$$

where the condition $c(X_{j_i})$ is defined as

$$c(X_{j_i}) \triangleq \exists X_l \in \mathcal{X}_j \text{ such } |X_l| > |X_{j_i}| \text{ and } \kappa_j(X_l) = 1.$$

Because this second definition of $\kappa_j(X_{j_i})$ is self-referencing, we need to calculate the binary keeping indexes iteratively starting by the element of $\mathcal{X}_j$ of highest cardinality (say X), then for elements of $\mathcal{X}_j$ of cardinality $|X|-1$ (if any), then for elements of $\mathcal{X}_j$ of cardinality $|X|-2$ (if any), etc. From the implementation standpoint the definition (24) is more efficient than the direct definition (23).

Remark 1: We always have $\kappa_j(\Theta) = 0$ if $\Theta \in \mathcal{X}_j$ because Θ always includes all other focal elements of $\mathcal{X}_j$ and Θ has the highest cardinality, so $\delta_j(X_{l'}, \Theta) = 1$ for all $X_{l'} \in \mathcal{X}_j$. Therefore the binary keeping index formula (23) reduces to

$$\kappa_j(\Theta) = 1 - \prod_{X_{l'} \in \mathcal{X}_j} \delta_j(X_{l'}, \Theta) = 1 - \underbrace{1 \cdot 1 \cdot \ldots \cdot 1}_{|\mathcal{X}_j| \text{ terms}} = 0.$$

Remark 2: For a given FoD and a given number of BBAs to combine, it is always possible to calculate offline the values of the binary keeping indexes of focal elements of all possible combinations of focal elements involved in conflicting products $\pi_j(\emptyset) > 0$ because the binary keeping index depends only on the structure of the focal elements, and not on the numerical mass values of the focal elements. This remark is important, especially in applications where we have thousands or millions of fusion steps to make because we will not have to recalculate in each fusion step the binary keeping-indexes for each $\pi_j(\emptyset)$ even if the input BBAs values to combine change.

Remark 3: It is worth to recall that PCR5$^+$ and PCR6$^+$ have interest if and only if we have more than two ($S > 2$) BBAs to combine. If we have only two BBAs to combine ($S = 2$) we always get $m_{\text{PCR5}} = m_{\text{PCR5}^+} = m_{\text{PCR6}} = m_{\text{PCR6}^+}$ because in this case the PCR5, PCR5$^+$, PCR6, and PCR6$^+$ rules coincide.

For convenience, we illustrate the calculation of these binary keeping-indexes based on the direct calculation (23) for different examples.

Example 6: We consider the FoD $\Theta = \{A, B, C, D\}$, six BBAs, and the j-th conflicting (assumed strictly positive) product whose structure is as follows

$$\pi_j(\emptyset) = m_1(A)m_2(B \cup C)m_3(A \cup C)m_4(B \cup C) \cdot m_5(A \cup B \cup C)m_6(A \cup B \cup C \cup D).$$

In this product $\pi_j(\emptyset)$, we have the duplicate focal element $B \cup C$ because it appears both in $m_2(B \cup C)$ and in $m_4(B \cup C)$. The focal elements entering in each BBA of $\pi_j(\emptyset)$ are respectively $X_{j_1} = A$, $X_{j_2} = B \cup C$, $X_{j_3} = A \cup C$, $X_{j_4} = B \cup C$, $X_{j_5} = A \cup B \cup C$, and $X_{j_6} = A \cup B \cup C \cup D = \Theta$. So we have to consider only the following set of distinct focal elements for this $\pi_j(\emptyset)$ product

$$\mathcal{X}_j = \{X_1 = A, X_2 = B \cup C, X_3 = A \cup C, X_4 = A \cup B \cup C, X_5 = A \cup B \cup C \cup D\}.$$

Therefore, considering only $X_{l'} \neq X_l$ and $|X_{l'}| \leq |X_l|$ that are conditions entering in formula (23), we have the following binary containing indicator $\delta_j(X_{l'}, X_l)$ values:

$\delta_j(X_1, X_2) = 0$ because $(X_1 = A) \nsubseteq (X_2 = B \cup C)$,

$\delta_j(X_1, X_3) = 1$ because $(X_1 = A) \subseteq (X_3 = A \cup C)$,

$\delta_j(X_1, X_4) = 1$ because $(X_1 = A) \subseteq (X_4 = A \cup B \cup C)$,

$\delta_j(X_1, X_5) = 1$ because $(X_1 = A) \subseteq (X_5 = \Theta)$,

$\delta_j(X_2, X_3) = 0$ because $(X_2 = B \cup C) \nsubseteq (X_3 = A \cup C)$,

$\delta_j(X_2, X_4) = 1$ because $(X_2 = B \cup C) \subseteq (X_4 = A \cup B \cup C)$,

$\delta_j(X_2, X_5) = 1$ because $(X_2 = B \cup C) \subseteq (X_5 = \Theta)$,

$\delta_j(X_3, X_2) = 0$ because $(X_3 = A \cup C) \nsubseteq (X_2 = B \cup C)$,

$\delta_j(X_3, X_4) = 1$ because $(X_3 = A \cup C) \subseteq (X_4 = A \cup B \cup C)$,

$\delta_j(X_3, X_5) = 1$ because $(X_3 = A \cup C) \subseteq (X_5 = \Theta)$,

$\delta_j(X_4, X_5) = 1$ because $(X_4 = A \cup B \cup C) \subseteq (X_5 = \Theta)$.

The binary keeping indexes $\kappa_j(X_{j_i})$ for $i = 1, 2, \ldots, 6$ are calculated based on the formula (23) as follows:

- For the focal element $X_{j_1} = A = X_1$ of $\mathcal{X}_j$ having $|X_{j_1}| = 1$, we get

$$\begin{aligned}\kappa_j(A) &= 1 - \prod_{\substack{X_{l'}, X_l \in \mathcal{X}_j \\ X_{l'} \neq X_l \\ |X_{j_1}| \leq |X_l| \\ |X_{l'}| \leq |X_l|}} \delta_j(X_{l'}, X_l) \\ &= 1 - [\delta_j(X_1, X_2)\delta_j(X_1, X_3)\delta_j(X_1, X_4)\delta_j(X_1, X_5) \\ &\quad \cdot \delta_j(X_2, X_3)\delta_j(X_2, X_4)\delta_j(X_2, X_5)\delta_j(X_3, X_2) \\ &\quad \cdot \delta_j(X_3, X_4)\delta_j(X_3, X_5)\delta_j(X_4, X_5)] \\ &= 1 - 0 \cdot 1 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 = 1.\end{aligned}$$

  Hence, the focal element $X_{j_1} = A$ will be kept in the proportional redistribution of the conflicting mass $\pi_j(\emptyset)$.

- For the focal element $X_{j_2} = B \cup C = X_2$ of $\mathcal{X}_j$ having $|X_{j_2}| = 2$, we get

$$\begin{aligned}\kappa_j(B \cup C) &= 1 - \prod_{\substack{X_{l'}, X_l \in \mathcal{X}_j \\ X_{l'} \neq X_l \\ |X_{j_2}| \leq |X_l| \\ |X_{l'}| \leq |X_l|}} \delta_j(X_{l'}, X_l) \\ &= 1 - [\delta_j(X_1, X_2)\delta_j(X_1, X_3)\delta_j(X_1, X_4)\end{aligned}$$

$$\cdot \delta_j(X_1, X_5)\delta_j(X_2, X_3)\delta_j(X_2, X_4) \cdot \delta_j(X_2, X_5)\delta_j(X_3, X_2)\delta_j(X_3, X_4) \cdot \delta_j(X_3, X_5)\delta_j(X_4, X_5)]$$
$$= 1 - 0 \cdot 1 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 = 1.$$

Hence, the focal element $X_{j_2} = B \cup C$ will be kept in the proportional redistribution of the conflicting mass $\pi_j(\emptyset)$.

- For the focal element $X_{j_3} = A \cup C = X_3$ of $\mathcal{X}_j$ having $|X_{j_3}| = 2$, we get

$$\kappa_j(A \cup C) = 1 - \prod_{\substack{X_{l'}, X_l \in \mathcal{X}_j \\ X_{l'} \neq X_l \\ |X_{j_3}| \leq |X_l| \\ |X_{l'}| \leq |X_l|}} \delta_j(X_{l'}, X_l)$$
$$= 1 - [\delta_j(X_1, X_2)\delta_j(X_1, X_3)\delta_j(X_1, X_4) \cdot \delta_j(X_1, X_5)\delta_j(X_2, X_3)\delta_j(X_2, X_4) \cdot \delta_j(X_2, X_5)\delta_j(X_3, X_2)\delta_j(X_3, X_4) \cdot \delta_j(X_3, X_5)\delta_j(X_4, X_5)]$$
$$= 1 - 0 \cdot 1 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 = 1.$$

Hence, the focal element $X_{j_3} = A \cup C$ will be kept in the proportional redistribution of the conflicting mass $\pi_j(\emptyset)$.

- For the duplicate focal element $X_{j_4} = B \cup C$ of $\mathcal{X}_j$ having $|X_{j_4}| = 2$, we have $\kappa_j(X_{j_4}) = 1$ because $X_{j_4} = X_{j_2}$ and $\kappa_j(X_{j_2}) = 1$.
- For the focal element $X_{j_5} = A \cup B \cup C = X_4$ of $\mathcal{X}_j$ having $|X_{j_5}| = 3$, we get

$$\kappa_j(A \cup B \cup C) = 1 - \prod_{\substack{X_{l'}, X_l \in \mathcal{X}_j \\ X_{l'} \neq X_l \\ |X_{j_5}| \leq |X_l| \\ |X_{l'}| \leq |X_l|}} \delta_j(X_{l'}, X_l)$$
$$= 1 - [\delta_j(X_1, X_4)\delta_j(X_1, X_5) \cdot \delta_j(X_2, X_4)\delta_j(X_2, X_5)\delta_j(X_3, X_4) \cdot \delta_j(X_3, X_5)\delta_j(X_4, X_5)]$$
$$= 1 - 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 0.$$

Hence, the focal element $X_{j_5} = A \cup B \cup C$ will be discarded in the proportional redistribution of the conflicting mass $\pi_j(\emptyset)$.

- For the focal element $X_{j_6} = A \cup B \cup C \cup D = \Theta = X_5$ of $\mathcal{X}_j$ having $|X_{j_6}| = 4$, we get

$$\kappa_j(\Theta) = 1 - \prod_{\substack{X_{l'}, X_l \in \mathcal{X}_j \\ X_{l'} \neq X_l \\ |X_{j_6}| \leq |X_l| \\ |X_{l'}| \leq |X_l|}} \delta_j(X_{l'}, X_l)$$
$$= 1 - \delta_j(X_1, X_5)\delta_j(X_2, X_5)\delta_j(X_3, X_5)\delta_j(X_4, X_5)$$
$$= 1 - 1 \cdot 1 \cdot 1 \cdot 1 = 0.$$

This result illustrates the validity of the aforementioned remark 1. Hence, the focal element $X_{j_5} = A \cup B \cup C \cup D = \Theta$ will be discarded in the proportional redistribution of the conflicting mass $\pi_j(\emptyset)$.

In summary, the conflicting product $\pi_j(\emptyset) = m_1(A)m_2(B\cup C)m_3(A\cup C)m_4(B\cup C)m_5(A\cup B\cup C)m_6(\Theta)$ will be redistributed only to the three focal elements A, $B \cup C$, and $A \cup C$ with the improved rules PCR5$^+$ and PCR6$^+$, whereas it would have been redistributed to all five focal elements A, $B \cup C$, $A \cup C$, $A \cup B \cup C$, and Θ with the classical PCR5 and PCR6 rules. Thus, two focal elements were discarded.

Example 7: This example is somehow an extension of example 6 by including a new element E in the FoD. So, the FoD is $\Theta = \{A, B, C, D, E\}$, seven BBAs, and the j-th conflicting (assumed strictly positive) product whose structure is as follows

$$\pi_j(\emptyset) = m_1(A\cup E)m_2(B\cup C\cup E)m_3(A\cup C\cup E)m_4(B\cup C\cup E) \cdot m_5(A \cup B \cup C \cup E)m_6(A \cup B \cup C \cup D \cup E)m_7(A).$$

In this product $\pi_j(\emptyset)$, we have the duplicate focal element $B\cup C\cup E$ because it appears both in $m_2(B\cup C\cup E)$ and in $m_4(B \cup C \cup E)$. The focal elements entering in each BBA of $\pi_j(\emptyset)$ are, respectively, $X_{j_1} = A \cup E$, $X_{j_2} = B \cup C \cup E$, $X_{j_3} = A \cup C \cup E$, $X_{j_4} = B \cup C \cup E$, $X_{j_5} = A \cup B \cup C \cup E$, $X_{j_6} = A \cup B \cup C \cup D \cup E = \Theta$, and $X_{j_7} = A$. So we have to consider only the following set of distinct focal elements for this $\pi_j(\emptyset)$ product

$$\mathcal{X}_j = \{X_1 = A \cup E, X_2 = B \cup C \cup E, X_3 = A \cup C \cup E, X_4 = A\cup B\cup C\cup E, X_5 = A\cup B\cup C\cup D\cup E, X_6 = A\}.$$

Therefore, considering only $X_{l'} \neq X_l$ and $|X_{l'}| \leq |X_l|$ that are conditions entering in formula (23), we have the following binary containing indicator $\delta_j(X_{l'}, X_l)$ values:

$\delta_j(X_6, X_1) = 1$ because $(X_6 = A) \subseteq (X_1 = A \cup E)$,

$\delta_j(X_6, X_2) = 0$ because $(X_6 = A) \nsubseteq (X_2 = B \cup C \cup E)$,

$\delta_j(X_6, X_3) = 1$ because $(X_6 = A) \subseteq (X_3 = A \cup C \cup E)$,

$\delta_j(X_6, X_4) = 1$ because $(X_6 = A) \subseteq (X_4 = A \cup B \cup C \cup E)$,

$\delta_j(X_6, X_5) = 1$ because $(X_6 = A) \subseteq (X_5 = \Theta)$,

$\delta_j(X_1, X_2) = 0$ because $(X_1 = A \cup E) \nsubseteq (X_2 = B \cup C \cup E)$,

$\delta_j(X_1, X_3) = 1$ because $(X_1 = A \cup E) \subseteq (X_3 = A \cup C \cup E)$,

$\delta_j(X_1, X_4) = 1$ because $(X_1 = A \cup E) \subseteq (X_4 = A \cup B \cup C \cup E)$,

$\delta_j(X_1, X_5) = 1$ because $(X_1 = A \cup E) \subseteq (X_5 = \Theta)$,

$\delta_j(X_2, X_3) = 0$ because $(X_2 = B \cup C \cup E) \nsubseteq (X_3 = A \cup C \cup E)$,

$\delta_j(X_2, X_4) = 1$ because $(X_2 = B \cup C \cup E) \subseteq (X_4 = A \cup B \cup C \cup E)$,

$\delta_j(X_2, X_5) = 1$ because $(X_2 = B \cup C \cup E) \subseteq (X_5 = \Theta)$,

$\delta_j(X_3, X_2) = 0$ because $(X_3 = A \cup C \cup E) \nsubseteq (X_2 = B \cup C \cup E)$,

$\delta_j(X_3, X_4) = 1$ because $(X_3 = A \cup C \cup E) \subseteq (X_4 = A \cup B \cup C \cup E)$,

$\delta_j(X_3, X_5) = 1$ because $(X_3 = A \cup C \cup E) \subseteq (X_5 = \Theta)$,

$\delta_j(X_4, X_5) = 1$ because $(X_4 = A \cup B \cup C \cup E) \subseteq (X_5 = \Theta)$.

The binary keeping indexes $\kappa_j(X_{j_i})$ for $i = 1, 2, \ldots, 7$ are calculated based on the formula (23) as follows

- For the focal element $X_{j_1} = A \cup E = X_1$ of $\mathcal{X}_j$ having $|X_{j_1}| = 2$, we get

$$\kappa_j(X_{j_1}) = 1 - \prod_{\substack{X_{l'}, X_l \in \mathcal{X}_j \\ X_{l'} \neq X_l \\ |X_{j_1}| \leq |X_l| \\ |X_{l'}| \leq |X_l|}} \delta_i(X_{l'}, X_l)$$
$$= 1 - [\delta_j(X_1, X_2)\delta_j(X_1, X_3)\delta_j(X_1, X_4)\delta_j(X_1, X_5) \cdot \delta_j(X_2, X_3)\delta_j(X_2, X_4)\delta_j(X_2, X_5)\delta_j(X_3, X_2) \cdot \delta_j(X_3, X_4)\delta_j(X_3, X_5)\delta_j(X_4, X_5)\delta_j(X_6, X_1) \cdot \delta_j(X_6, X_2)\delta_j(X_6, X_3)\delta_j(X_6, X_4)\delta_j(X_6, X_5)]$$
$$= 1 - 0 \cdot 1 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 = 1.$$

  Hence, the focal element $X_{j_1} = A \cup E$ will be kept in the proportional redistribution of the conflicting mass $\pi_j(\emptyset)$.

- For the focal element $X_{j_2} = B \cup C \cup E = X_2$ of $\mathcal{X}_j$ having $|X_{j_2}| = 3$, we get

$$\kappa_j(X_{j_2}) = 1 - \prod_{\substack{X_{l'}, X_l \in \mathcal{X}_j \\ X_{l'} \neq X_l \\ |X_{j_2}| \leq |X_l| \\ |X_{l'}| \leq |X_l|}} \delta_j(X_{l'}, X_l)$$
$$= 1 - [\delta_j(X_1, X_2)\delta_j(X_1, X_3)\delta_j(X_1, X_4)\delta_j(X_1, X_5) \cdot \delta_j(X_2, X_3)\delta_j(X_2, X_4)\delta_j(X_2, X_5)\delta_j(X_3, X_2) \cdot \delta_j(X_3, X_4)\delta_j(X_3, X_5)\delta_j(X_4, X_5)\delta_j(X_6, X_2) \cdot \delta_j(X_6, X_3)\delta_j(X_6, X_4)\delta_j(X_6, X_5)]$$
$$= 1 - 0 \cdot 1 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 = 1.$$

  Hence, the focal element $X_{j_2} = B \cup C \cup E$ will also be kept in the proportional redistribution of the conflicting mass $\pi_j(\emptyset)$.

- For the focal element $X_{j_3} = A \cup C \cup E = X_3$ of $\mathcal{X}_j$ having $|X_{j_3}| = 3$, we get

$$\kappa_j(X_{j_3}) = 1 - \prod_{\substack{X_{l'}, X_l \in \mathcal{X}_j \\ X_{l'} \neq X_l \\ |X_{j_3}| \leq |X_l| \\ |X_{l'}| \leq |X_l|}} \delta_j(X_{l'}, X_l)$$
$$= 1 - [\delta_j(X_1, X_2)\delta_j(X_1, X_3)\delta_j(X_1, X_4)\delta_j(X_1, X_5) \cdot \delta_j(X_2, X_3)\delta_j(X_2, X_4)\delta_j(X_2, X_5)\delta_j(X_3, X_2) \cdot \delta_j(X_3, X_4)\delta_j(X_3, X_5)\delta_j(X_4, X_5)\delta_j(X_6, X_2) \cdot \delta_j(X_6, X_3)\delta_j(X_6, X_4)\delta_j(X_6, X_5)]$$
$$= 1 - 0 \cdot 1 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 = 1.$$

  Hence, the focal element $X_{j_3} = A \cup C \cup E$ is also kept in the redistribution.

- For the duplicate focal element $X_{j_4} = B \cup C \cup E$ having $|X_{j_4}| = 3$, we have $\kappa_j(X_{j_4}) = 1$ because $X_{j_4} = X_{j_2}$ and $\kappa_j(X_{j_2}) = 1$.

- For the focal element $X_{j_5} = A \cup B \cup C \cup E = X_4$ having $|X_{j_5}| = 4$, we get

$$\kappa_j(X_{j_5}) = 1 - \prod_{\substack{X_{l'}, X_l \in \mathcal{X}_j \\ X_{l'} \neq X_l \\ |X_{j_5}| \leq |X_l| \\ |X_{l'}| \leq |X_l|}} \delta_j(X_{l'}, X_l)$$
$$= 1 - [\delta_j(X_1, X_4)\delta_j(X_1, X_5)\delta_j(X_2, X_4)\delta_j(X_2, X_5) \cdot \delta_j(X_3, X_4)\delta_j(X_3, X_5)\delta_j(X_4, X_5)\delta_j(X_6, X_4) \cdot \delta_j(X_6, X_5)]$$
$$= 1 - 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 0.$$

  Hence, the focal element $X_{j_5} = A \cup B \cup C \cup E$ must be ignored in the proportional redistribution.

- For the focal element $X_{j_6} = A \cup B \cup C \cup D \cup E = \Theta = X_5$ having $|X_{j_6}| = 5$, we get

$$\kappa_j(X_{j_6}) = 1 - \prod_{\substack{X_{l'}, X_l \in \mathcal{X}_j \\ X_{l'} \neq X_l \\ |X_{j_6}| \leq |X_l| \\ |X_{l'}| \leq |X_l|}} \delta_j(X_{l'}, X_l)$$
$$= 1 - [\delta_j(X_1, X_5)\delta_j(X_2, X_5)\delta_j(X_3, X_5)\delta_j(X_4, X_5) \cdot \delta_j(X_6, X_5)]$$
$$= 1 - 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 0.$$

  This result illustrates the validity of the aforementioned remark 1. Hence, the focal element $X_{j_6} = A \cup B \cup C \cup D \cup E$ must be ignored in the proportional redistribution.

- For the focal element $X_{j_7} = A = X_6$ having $|X_{j_7}| = 1$, we get naturally (see our previous remark 1)

$$\kappa_j(X_{j_7}) = 1 - \prod_{\substack{X_{l'}, X_l \in \mathcal{X}_j \\ X_{l'} \neq X_l \\ |X_{j_7}| \leq |X_l| \\ |X_{l'}| \leq |X_l|}} \delta_j(X_{l'}, X_l)$$
$$= 1 - [\delta_j(X_1, X_2)\delta_j(X_1, X_3)\delta_j(X_1, X_4)\delta_j(X_1, X_5) \cdot \delta_j(X_2, X_3)\delta_j(X_2, X_4)\delta_j(X_2, X_5)\delta_j(X_3, X_2) \cdot \delta_j(X_3, X_4)\delta_j(X_3, X_5)\delta_j(X_4, X_5)\delta_j(X_6, X_2) \cdot \delta_j(X_6, X_3)\delta_j(X_6, X_4)\delta_j(X_6, X_5)]$$
$$= 1 - 0 \cdot 1 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 = 1.$$

  Hence, the focal element $X_{j_7} = A$ must be kept in the proportional redistribution.

In summary, the conflicting product $\pi_j(\emptyset) = m_1(A \cup E)m_2(B \cup C \cup E)m_3(A \cup C \cup E)m_4(B \cup C \cup E)m_5(A \cup B \cup C \cup E)m_6(\Theta)m_7(A)$ will be redistributed only to focal elements $A\cup E$, $B\cup C\cup E$, $A\cup C\cup E$, and A with the improved rules PCR5$^+$ and PCR6$^+$, whereas it would have been redistributed to all focal elements $A\cup E$, $B\cup C\cup E$, $A\cup C\cup E$, $A\cup B\cup C\cup E$, Θ, and A with the classical PCR5 and PCR6 rules.

Example 8: This is a somehow simplified version of example 6. We consider the FoD $\Theta=\{A,B,C,D\}$, only five BBAs, and suppose that the j-th conflicting (assumed strictly positive) product is as follows

$$\pi_j(\emptyset)=m_1(A)m_2(B\cup C)m_3(A\cup C)m_4(B\cup C)\cdot m_5(A\cup B\cup C\cup D).$$

Based on (23), it can be verified[14] that the binary keeping indexes of focal elements involved in conflicting products are

$$\kappa_j(A)=1,$$
$$\kappa_j(B\cup C)=1,$$
$$\kappa_j(A\cup C)=1,$$
$$\kappa_j(A\cup B\cup C\cup D)=0.$$

Example 9: We consider the FoD $\Theta=\{A,B,C,D\}$, seven BBAs, and suppose that the j-th conflicting (assumed strictly positive) product is as follows

$$\pi_j(\emptyset)=m_1(A)m_2(B\cup C)m_3(A\cup C)m_4(B\cup C)\cdot m_5(A\cup B\cup C\cup D)m_6(A\cup B\cup C)m_7(A\cup B\cup C).$$

Based on (23), it can be verified that the binary keeping indexes of focal elements involved in conflicting products are

$$\kappa_j(A)=1,$$
$$\kappa_j(B\cup C)=1,$$
$$\kappa_j(A\cup C)=1,$$
$$\kappa_j(A\cup B\cup C\cup D)=0,$$
$$\kappa_j(A\cup B\cup C)=0.$$

Example 10: We consider the FoD $\Theta=\{A,B,C\}$, three BBAs, and suppose that the j-th conflicting (assumed strictly positive) product is as follows

$$\pi_j(\emptyset)=m_1(A)m_2(B\cup C)m_3(A\cup C).$$

Based on (23), it can be verified that the binary keeping indexes of focal elements involved in conflicting products are

$$\kappa_j(A)=1,$$
$$\kappa_j(B\cup C)=1,$$
$$\kappa_j(A\cup C)=1.$$

[14]The verification is left to the reader.

Example 11: We consider the FoD $\Theta=\{A,B,C\}$, four BBAs, and suppose that the j-th conflicting (assumed strictly positive) product is as follows

$$\pi_j(\emptyset)=m_1(A)m_2(B\cup C)m_3(A\cup C)m_4(A\cup B).$$

Based on (23), it can be verified that the binary keeping-indexes of focal elements involved in conflicting products are

$$\kappa_j(A)=1,$$
$$\kappa_j(B\cup C)=1,$$
$$\kappa_j(A\cup C)=1,$$
$$\kappa_j(A\cup B)=1.$$

Example 12: We consider the FoD $\Theta=\{A,B,C\}$, three BBAs, and suppose that the j-th conflicting (assumed strictly positive) product is as follows

$$\pi_j(\emptyset)=m_1(A\cup B\cup C)m_2(A)m_3(B\cup C).$$

Based on (23), it can be verified that the binary keeping-indexes of focal elements involved in conflicting products are

$$\kappa_j(A\cup B\cup C)=0,$$
$$\kappa_j(A)=1,$$
$$\kappa_j(B\cup C)=1.$$

Example 13: We consider the FoD $\Theta=\{A,B,C,D\}$, and the three following BBAs

$$m_1(A\cup B)=0.8, m_1(C\cup D)=0.2,$$
$$m_2(A\cup B)=0.4, m_2(C\cup D)=0.6,$$
$$m_3(B)=0.1, m_3(A\cup B\cup C\cup D)=0.9.$$

We have $\mathcal{F}=|\mathcal{F}(m_1)|\cdot|\mathcal{F}(m_2)|\cdot|\mathcal{F}(m_3)|=2\cdot 2\cdot 2=8$ products π_j ($j=1,\ldots,\mathcal{F}$) entering in the fusion process as follows

$$\pi_1(B)=m_1(A\cup B)m_2(A\cup B)m_3(B)=0.032,$$
$$\pi_2(A\cup B)=m_1(A\cup B)m_2(A\cup B)m_3(\Theta)=0.288,$$
$$\pi_3(\emptyset)=m_1(A\cup B)m_2(C\cup D)m_3(B)=0.048,$$
$$\pi_4(\emptyset)=m_1(A\cup B)m_2(C\cup D)m_3(\Theta)=0.432,$$
$$\pi_5(\emptyset)=m_1(C\cup D)m_2(A\cup B)m_3(B)=0.008,$$
$$\pi_6(\emptyset)=m_1(C\cup D)m_2(A\cup B)m_3(\Theta)=0.072,$$
$$\pi_7(\emptyset)=m_1(C\cup D)m_2(C\cup D)m_3(B)=0.012,$$
$$\pi_8(C\cup D)=m_1(C\cup D)m_2(C\cup D)m_3(\Theta)=0.108.$$

Based on (23), it can be verified[15] that the binary keeping-indexes of focal elements involved in conflicting products $\pi_3(\emptyset)$ to $\pi_7(\emptyset)$ are

$$\kappa_3(A\cup B)=1, \kappa_3(C\cup D)=1, \kappa_3(B)=1,$$

[15]The verification is left to the reader.

$$\kappa_4(A\cup B)=1, \kappa_4(C\cup D)=1, \kappa_4(\Theta)=0,$$
$$\kappa_5(C\cup D)=1, \kappa_5(A\cup B)=1, \kappa_5(B)=1,$$
$$\kappa_6(C\cup D)=1, \kappa_6(A\cup B)=1, \kappa_6(\Theta)=0,$$
$$\kappa_7(C\cup D)=1, \kappa_7(B)=1.$$

In summary, once the binary keeping-index of $\kappa_j(X_{j_i})$ of all focal elements X_{j_i} involved in a conflicting product $\pi_j(\emptyset)$ are calculated, we can apply PCR5 or PCR6 redistribution principle only with the focal elements for which $\kappa_j(X_{j_i}) = 1$. With this new improved method of proportional redistribution, PCR5$^+$ and PCR6$^+$ rules will never increase the mass of non-conflicting elements involved in each $\pi_j(\emptyset)$ (if any), and in doing this way, we will preserve the neutrality of the vacuous belief assignment in the PCR5$^+$ and PCR6$^+$ fusion rules, which is a very desirable behavior.

B. Expressions of PCR5$^+$ and PCR6$^+$ fusion rules

The expressions of PCR5$^+$ and PCR6$^+$ fusion rules are proper modifications of PCR5 and PCR6 formulas (14) and (15) taking into account the selection of focal elements on which the proportional redistribution must apply thanks to the value of their binary keeping index.

The PCR5$^+$ fusion of $S > 2$ BBAs is obtained by $m^{\text{PCR5}^+}_{1,2,\dots,S}(\emptyset)=0$, and for all $A\in 2^\Theta\setminus\{\emptyset\}$ by

$$m^{\text{PCR5}^+}_{1,2,\dots,S}(A)=m^{\text{Conj}}_{1,2,\dots,S}(A) + \sum_{j\in\{1,\dots,\mathcal{F}\}|A\in\mathbf{X}_j\wedge\pi_j(\emptyset)} \Bigg[\Big(\kappa_j(A)\prod_{i\in\{1,\dots,S\}|X_{j_i}=A} m_i(X_{j_i})\Big) \cdot \frac{\pi_j(\emptyset)}{\sum_{X\in\mathbf{X}_j}\Big(\kappa_j(X)\prod_{i\in\{1,\dots,S\}|X_{j_i}=X} m_i(X_{j_i})\Big)}\Bigg]. \quad (25)$$

The PCR6$^+$ fusion of $S > 2$ BBAs is obtained by $m^{\text{PCR6}^+}_{1,2,\dots,S}(\emptyset)=0$, and for all $A\in 2^\Theta\setminus\{\emptyset\}$ by

$$m^{\text{PCR6}^+}_{1,2,\dots,S}(A)=m^{\text{Conj}}_{1,2,\dots,S}(A) + \sum_{j\in\{1,\dots,\mathcal{F}\}|A\in\mathbf{X}_j\wedge\pi_j(\emptyset)} \Bigg[\Big(\kappa_j(A)\sum_{i\in\{1,\dots,S\}|X_{j_i}=A} m_i(X_{j_i})\Big) \cdot \frac{\pi_j(\emptyset)}{\sum_{X\in\mathbf{X}_j}\Big(\kappa_j(X)\sum_{i\in\{1,\dots,S\}|X_{j_i}=X} m_i(X_{j_i})\Big)}\Bigg], \quad (26)$$

where $\kappa_j(A)$ and $\kappa_j(X)$ are, respectively, the binary keeping indexes of elements A and X involved in the conflicting product $\pi_j(\emptyset)$, that are calculated by the formula (23) or (24).

Remark 4: It is worth mentioning that PCR5$^+$ formula (25) is totally consistent with PCR5 formula (14) when all binary keeping-indexes are equal to one. Similarly, the PCR6$^+$ formula (26) reduces to PCR6 formula (15) if all binary keeping indexes equal one.

Theorem: The VBBA m_v has a neutral impact in PCR5$^+$ and PCR6$^+$ rules of combination.

Proof: see Appendix 2.

C. On the complexity of PCR5$^+$ and PCR6$^+$ fusion rules

The complexity of PCR5 and PCR6 rules is difficult to establish precisely because the number of computations highly depends on the structure of focal elements of the BBAs to combine, but definitely it is higher than Dempster's rule of combination. What about the complexity of PCR5$^+$ and PCR6$^+$ fusion rules? On one hand, PCR5$^+$ and PCR6$^+$ seem more complex than PCR5 and PCR6 rules because one needs extra computational burden with respect to PCR5 and PCR6 rules to calculate the binary keeping indexes. But in fact, the calculation of binary keeping indexes do not depend on the mass values of focal elements but only on their structure. Hence, the binary keeping indexes can be calculated off-line once for all for many possible structures of focal elements of BBAs to combine. On the other hand, if the binary keeping index calculation is done off-line, then PCR5$^+$ and PCR6$^+$ become less complex than PCR5 and PCR6 rule because some elements are discarded with PCR5$^+$ and PCR6$^+$ making the redistribution simpler and more effective than with PCR5 and PCR6 rules. It is not possible to say for sure if globally PCR5$^+$ and PCR6$^+$ are more (or less) complex than PCR5 and PCR6 because it really depends on the fusion problem under consideration and the structure of focal elements of BBAs to combine. If the sources of evidence to combine generate many partial conflicts to redistribute, including many elements to discard, then PCR5$^+$ and PCR6$^+$ are more advantageous than PCR5 and PCR6 in terms of reduction of complexity.

VII. EXAMPLES FOR PCR5$^+$ AND PCR6$^+$ FUSION RULES

Here we compare the results obtained with PCR5$^+$ and PCR6$^+$ with respect to those drawn from PCR5 and PCR6 rules on the examples from 1 to 13 in the previous sections. Since these following examples, for PCR5$^+$ and PCR6$^+$ fusion rules, respectively, consider the same FoD and BBAs as those presented, they will be denoted as "revisited examples."

Example 1 (revisited): Consider $\Theta=\{A,B\}$ and two following BBAs

$$m_1(A)=0.1 \quad m_1(B)=0.2 \quad m_1(A\cup B)=0.7$$

$$m_2(A)=0.4 \quad m_2(B)=0.3 \quad m_2(A\cup B)=0.3$$

Because there is only two BBAs to combine, we have

$$PCR5(m_1,m_2)=PCR6(m_1,m_2),$$
$$PCR5^+(m_1,m_2)=PCR6^+(m_1,m_2).$$

We have $m_{1,2}^{\text{Conj}}(A) = 0.35$, $m_{1,2}^{\text{Conj}}(B) = 0.33$, and $m_{1,2}^{\text{Conj}}(\Theta) = 0.21$, and we have the two conflicting products $\pi_1(\emptyset) = m_1(A)m_2(B) = 0.03$ and $\pi_2(\emptyset) = m_2(A)m_1(B) = 0.08$ to redistribute.

Applying PCR5 principle for $\pi_1(\emptyset) = 0.03$ we get

$$\frac{x_1(A)}{m_1(A)} = \frac{x_1(B)}{m_2(B)} = \frac{\pi_1(\emptyset)}{m_1(A) + m_2(B)},$$

whence $x_1(A) = 0.1 \cdot \frac{0.03}{0.1+0.3} = 0.0075$ and $x_1(B) = 0.3 \cdot \frac{0.03}{0.1+0.3} = 0.0225$.

Applying PCR5 principle for $\pi_2(\emptyset) = 0.08$ we get

$$\frac{x_2(A)}{m_2(A)} = \frac{x_2(B)}{m_1(B)} = \frac{\pi_2(\emptyset)}{m_2(A) + m_1(B)},$$

whence $x_2(A) = 0.4 \cdot \frac{0.08}{0.4+0.2} \approx 0.0533$ and $x_2(B) = 0.2 \cdot \frac{0.08}{0.4+0.2} \approx 0.0267$.

Therefore we get

$$m_{1,2}^{\text{PCR5}}(A) = m_{1,2}^{\text{PCR6}}(A) = m_{1,2}^{\text{Conj}}(A) + x_1(A) + x_2(A)$$
$$= 0.35 + 0.0075 + 0.0533 = 0.4108,$$
$$m_{1,2}^{\text{PCR5}}(B) = m_{1,2}^{\text{PCR6}}(B) = m_{1,2}^{\text{Conj}}(B) + x_1(B) + x_2(B)$$
$$= 0.33 + 0.0225 + 0.0267 = 0.3792,$$
$$m_{1,2}^{\text{PCR5}}(A \cup B) = m_{1,2}^{\text{PCR6}}(A \cup B) = m_{1,2}^{\text{Conj}}(A \cup B) = 0.21.$$

If we want to apply PCR5$^+$, or PCR6$^+$, rule we need to compute the binary keeping indexes of each focal element entering in the conflicting products $\pi_1(\emptyset)$ and $\pi_2(\emptyset)$. In this example, for $\pi_1(\emptyset) = m_1(A)m_2(B)$, we have $\mathcal{X}_1 = \{A, B\}$, and for $\pi_2(\emptyset) = m_2(A)m_1(B)$, we have $\mathcal{X}_2 = \{A, B\}$. Applying formula (22), we get $\delta_1(A, B) = 0$ because $A \nsubseteq B$, and $\delta_1(B, A) = 0$ because $B \nsubseteq A$ (and also $\delta_2(A, B) = 0$ and $\delta_2(B, A) = 0$). Applying formula (23) we get the binary keeping indexes $\kappa_1(A) = 1$, $\kappa_1(B) = 1$, $\kappa_2(A) = 1$, and $\kappa_2(B) = 1$, indicating that the redistribution of $\pi_1(\emptyset)$ must operate on all elements of $\mathcal{X}_1 = \{A, B\}$, and the redistribution of $\pi_2(\emptyset)$ must also operate on all elements of $\mathcal{X}_2 = \{A, B\}$, so there is no element that must be discarded for making the improved redistribution in this example. Therefore PCR5$^+$, or PCR6$^+$ results coincide with $PCR5$ and $PCR6$ results, that is $m^{\text{PCR5}}(\cdot) = m^{\text{PCR6}}(\cdot) = m^{\text{PCR5}^+}(\cdot) = m^{\text{PCR6}^+}(\cdot)$ which is normal.

Example 2 (revisited): Consider $\Theta = \{A, B\}$ and the three following BBAs

$$m_1(A) = 0.6, m_1(B) = 0.1, m_1(A \cup B) = 0.3,$$
$$m_2(A) = 0.5, m_2(B) = 0.3, m_2(A \cup B) = 0.2,$$
$$m_3(A) = 0.4, m_3(B) = 0.1, m_3(A \cup B) = 0.5.$$

As shown in Section IV, for this example, one has the following 12 conflicting products to redistribute when applying PCR5 or PCR6 fusion formulas.

$$\pi_1(\emptyset) = m_1(A)m_2(A)m_3(B) = 0.0300,$$
$$\pi_2(\emptyset) = m_1(A)m_2(B)m_3(A) = 0.0720,$$
$$\pi_3(\emptyset) = m_1(B)m_2(A)m_3(A) = 0.0200,$$
$$\pi_4(\emptyset) = m_1(B)m_2(B)m_3(A) = 0.0120,$$
$$\pi_5(\emptyset) = m_1(B)m_2(A)m_3(B) = 0.0050,$$
$$\pi_6(\emptyset) = m_1(A)m_2(B)m_3(B) = 0.0180,$$
$$\pi_7(\emptyset) = m_1(A \cup B)m_2(A)m_3(B) = 0.0150,$$
$$\pi_8(\emptyset) = m_1(A \cup B)m_2(B)m_3(A) = 0.0360,$$
$$\pi_9(\emptyset) = m_1(B)m_2(A)m_3(A \cup B) = 0.0250,$$
$$\pi_{10}(\emptyset) = m_1(A)m_2(B)m_3(A \cup B) = 0.0900,$$
$$\pi_{11}(\emptyset) = m_1(A)m_2(A \cup B)m_3(B) = 0.0120,$$
$$\pi_{12}(\emptyset) = m_1(B)m_2(A \cup B)m_3(A) = 0.0080.$$

With PCR5 and PCR6, the products $\pi_1(\emptyset)$ to $\pi_6(\emptyset)$ are redistributed to A and B only, whereas the products $\pi_7(\emptyset)$ to $\pi_{12}(\emptyset)$ are redistributed to A, B, and $A \cup B$. Applying PCR5 formula (14) and PCR6 formula (15), we obtain $m_{1,2,3}^{\text{PCR5}}(\emptyset) = m_{1,2,3}^{\text{PCR6}}(\emptyset) = 0$ and

$$\begin{cases} m_{1,2,3}^{\text{PCR5}}(A) \approx 0.723281 \\ m_{1,2,3}^{\text{PCR5}}(B) \approx 0.182460 \\ m_{1,2,3}^{\text{PCR5}}(A \cup B) \approx 0.094259 \end{cases} \quad \text{and} \quad \begin{cases} m_{1,2,3}^{\text{PCR6}}(A) \approx 0.743496 \\ m_{1,2,3}^{\text{PCR6}}(B) \approx 0.162245 \\ m_{1,2,3}^{\text{PCR6}}(A \cup B) \approx 0.094259 \end{cases}$$

The calculation of the binary keeping indexes by the formula (23) gives in this example

$$\begin{cases} \kappa_j(A) = 1, \kappa_j(B) = 1, \quad \text{for } j = 1, \ldots, 6 \\ \kappa_j(A) = 1, \kappa_j(B) = 1, \kappa_j(A \cup B) = 0, \quad \text{for } j = 7, \ldots, 12. \end{cases}$$

Therefore, if we apply the PCR5$^+$ and PCR6$^+$ improved rules of combination, we redistribute the products $\pi_1(\emptyset)$ to $\pi_6(\emptyset)$ to A and B (as for PCR5 and PCR6 rule), but the products $\pi_7(\emptyset)$ to $\pi_{12}(\emptyset)$ will be redistributed to A, B only, and not to $A \cup B$ because $\kappa_j(A \cup B) = 0$ for $j = 7, \ldots, 12$. So finally, we obtain $m_{1,2,3}^{\text{PCR5}^+}(\emptyset) = m_{1,2,3}^{\text{PCR6}^+}(\emptyset) = 0$ and

$$\begin{cases} m_{1,2,3}^{\text{PCR5}^+}(A) \approx 0.768631 \\ m_{1,2,3}^{\text{PCR5}^+}(B) \approx 0.201369 \\ m_{1,2,3}^{\text{PCR5}^+}(A \cup B) = 0.03 \end{cases} \quad \text{and} \quad \begin{cases} m_{1,2,3}^{\text{PCR6}^+}(A) \approx 0.788847 \\ m_{1,2,3}^{\text{PCR6}^+}(B) \approx 0.181153 \\ m_{1,2,3}^{\text{PCR6}^+}(A \cup B) = 0.03 \end{cases}$$

We can verify that we obtain a more precise redistribution with PCR5$^+$ (respectively PCR6$^+$) rule with respect to PCR5 (respectively PCR6) rule because

TABLE I
Example 5: Results of PCR5$^+$ versus PCR5

Focal elements	$m_{1,2,3}^{\text{PCR5}}(\cdot)$	$m_{1,2,3}^{\text{PCR5}^+}(\cdot)$
B	0.001103	0.001107
$A \cup B$	0.286107	0.464483
$C \cup D$	0.203385	0.296186
$A \cup B \cup C \cup D$	0.012203	0.023408
E	0.115966	0.214816
$A \cup B \cup C \cup D \cup E$	0.381236	0

TABLE II
Example 5: Results of PCR6$^+$ versus PCR6

Focal elements	$m_{1,2,3}^{\mathrm{PCR6}}(\cdot)$	$m_{1,2,3}^{\mathrm{PCR6}^+}(\cdot)$
B	0.000962	0.000967
$A \cup B$	0.286107	0.464483
$C \cup D$	0.203454	0.296255
$A \cup B \cup C \cup D$	0.012203	0.023408
E	0.116038	0.214887
$A \cup B \cup C \cup D \cup E$	0.381236	0

$m_{1,2,3}^{\mathrm{PCR5}^+}(A \cup B) < m_{1,2,3}^{\mathrm{PCR5}}(A \cup B)$ and also $m_{1,2,3}^{\mathrm{PCR6}^+}(A \cup B) < m_{1,2,3}^{\mathrm{PCR6}}(A \cup B)$.

Example 3 (revisited): we consider $\Theta = \{A, B, C\}$, and the four very simple BBAs defined by

$$m_1(A\cup B) = 1, m_2(B) = 1, m_3(A\cup B) = 1, \text{ and } m_4(C) = 1.$$

These four BBAs are in total conflict because $(A \cup B) \cap A \cap (A \cup B) \cap C = \emptyset$, and one has only one product $\pi(\emptyset) = m_1(A \cup B)m_2(A)m_3(A \cup B)m_4(C) = 1$ to consider, so $j = 1$ in this case and it can be omitted in the notations of the binary keeping indexes.

As shown previously, one has

$$\begin{cases} m_{1,2,3,4}^{\mathrm{PCR5}}(A \cup B) = 1/3 \\ m_{1,2,3,4}^{\mathrm{PCR5}}(B) = 1/3 \\ m_{1,2,3,4}^{\mathrm{PCR5}}(C) = 1/3 \end{cases} \quad \text{and} \quad \begin{cases} m_{1,2,3,4}^{\mathrm{PCR6}}(A \cup B) = 0.5 \\ m_{1,2,3,4}^{\mathrm{PCR6}}(B) = 0.25 \\ m_{1,2,3,4}^{\mathrm{PCR6}}(C) = 0.25 \end{cases}$$

Because all focal elements $A \cup B$, A, and C entering in $\pi(\emptyset)$ are conflicting then one has the binary keeping-indexes $\kappa(A \cup B) = 1$, $\kappa(A) = 1$ and $\kappa(C) = 1$, i.e., all these elements will receive a redistribution of the conflicting mass $\pi(\emptyset)$. Therefore there is no restriction for making the redistribution. Consequently, $PCR5^+$ result coincides with PCR5 result, and $PCR6^+$ result coincides with PCR6 result.

Example 4 (revisited): we consider $\Theta = \{A, B\}$, and the following four BBAs

$$m_1(A) = 0.6, m_1(B) = 0.1, m_1(A \cup B) = 0.3,$$

$$m_2(A) = 0.5, m_2(B) = 0.3, m_2(A \cup B) = 0.2,$$

$$m_3(A) = 0.4, m_3(B) = 0.1, m_3(A \cup B) = 0.5,$$

$$m_4(A \cup B) = 1 \quad (m_4 \text{ is the VBBA}).$$

The BBAs m_1, m_2, and m_3 are the same as in example 2, and the BBA m_4 is the VBBA. We have already shown that $\mathrm{PCR5}(m_1, m_2, m_3) \neq \mathrm{PCR5}(m_1, m_2, m_3, m_4)$ even if m_4 is the VBBA, and

$$\begin{cases} m_{1,2,3,4}^{\mathrm{PCR5}}(A) \approx 0.654604 \\ m_{1,2,3,4}^{\mathrm{PCR5}}(B) \approx 0.144825 \\ m_{1,2,3,4}^{\mathrm{PCR5}}(A \cup B) \approx 0.200571 \end{cases}$$

Similarly, $\mathrm{PCR6}(m_1, m_2, m_3) \neq \mathrm{PCR6}(m_1, m_2, m_3, m_4)$, and

$$\begin{cases} m_{1,2,3,4}^{\mathrm{PCR6}}(A) \approx 0.647113 \\ m_{1,2,3,4}^{\mathrm{PCR6}}(B) \approx 0.128342 \\ m_{1,2,3,4}^{\mathrm{PCR6}}(A \cup B) \approx 0.224545 \end{cases}$$

Applying the PCR5$^+$ formula (25) and the PCR6$^+$ formula (26) we will obtain $m_{1,2,3}^{\mathrm{PCR5}^+}(\emptyset) = m_{1,2,3,4}^{\mathrm{PCR6}^+}(\emptyset) = 0$ and

$$\begin{cases} m_{1,2,3,4}^{\mathrm{PCR5}^+}(A) \approx 0.768631 \\ m_{1,2,3,4}^{\mathrm{PCR5}^+}(B) \approx 0.201369 \\ m_{1,2,3,4}^{\mathrm{PCR5}^+}(A \cup B) = 0.03 \end{cases} \quad \text{and} \quad \begin{cases} m_{1,2,3,4}^{\mathrm{PCR6}^+}(A) \approx 0.788847 \\ m_{1,2,3,4}^{\mathrm{PCR6}^+}(B) \approx 0.181153 \\ m_{1,2,3,4}^{\mathrm{PCR6}^+}(A \cup B) = 0.03 \end{cases}$$

One has $\mathrm{PCR5}^+(m_1, m_2, m_3, m_4) = \mathrm{PCR5}^+(m_1, m_2, m_3)$ and $\mathrm{PCR6}^+(m_1, m_2, m_3, m_4) = \mathrm{PCR6}^+(m_1, m_2, m_3)$ because with the improved proportional redistribution of PCR5$^+$ and PCR6$^+$ rules, the VBBA has always a neutral impact in the fusion result, which is what we intuitively expect.

Example 5 (revisited): we consider $\Theta = \{A, B, C, D, E\}$, and the following three BBAs

$$\begin{cases} m_1(A \cup B) = 0.70 \\ m_1(C \cup D) = 0.06 \\ m_1(A \cup B \cup C \cup D) = 0.15 \\ m_1(E) = 0.09 \end{cases}$$

TABLE III
Example 6: Results of PCR5$^+$ versus PCR5

Focal elements	$m_{1,2,3,4,5,6}^{\mathrm{PCR5}}(\cdot)$	$m_{1,2,3,4,5,6}^{\mathrm{PCR5}^+}(\cdot)$
A	1/5	1/3
$A \cup C$	1/5	1/3
$B \cup C$	1/5	1/3
$A \cup B \cup C$	1/5	0
$A \cup B \cup C \cup D$	1/5	0

TABLE IV
Example 6: Results of PCR6$^+$ versus PCR6

Focal elements	$m_{1,2,3,4,5,6}^{\mathrm{PCR6}}(\cdot)$	$m_{1,2,3,4,5,6}^{\mathrm{PCR6}^+}(\cdot)$
A	1/6	1/4
$A \cup C$	1/6	1/4
$B \cup C$	1/3	1/2
$A \cup B \cup C$	1/6	0
$A \cup B \cup C \cup D$	1/6	0

TABLE V
Example 7: Results of PCR5$^+$ versus PCR5

Focal elements	$m_{1,2,3,4,5,6,7}^{\mathrm{PCR5}}(\cdot)$	$m_{1,2,3,4,5,6,7}^{\mathrm{PCR5}^+}(\cdot)$
A	1/6	1/4
$A \cup E$	1/6	1/4
$A \cup C \cup E$	1/6	1/4
$B \cup C \cup E$	1/6	1/4
$A \cup B \cup C \cup E$	1/6	0
$A \cup B \cup C \cup D \cup E$	1/6	0

TABLE VI
Example 7: Results of PCR6$^+$ versus PCR6

Focal elements	$m^{\text{PCR6}}_{1,2,3,4,5,6,7}(\cdot)$	$m^{\text{PCR6}^+}_{1,2,3,4,5,6,7}(\cdot)$
A	1/7	1/5
$A\cup E$	1/7	1/5
$A\cup C\cup E$	1/7	1/5
$B\cup C\cup E$	2/7	2/5
$A\cup B\cup C\cup E$	1/7	0
$A\cup B\cup C\cup D\cup E$	1/7	0

and

$$\begin{cases} m_2(A\cup B)=0.06\\ m_2(C\cup D)=0.50\\ m_2(A\cup B\cup C\cup D)=0.04\\ m_2(E)=0.40\end{cases}$$

and

$$\begin{cases} m_3(B)=0.01\\ m_3(A\cup B\cup C\cup D\cup E)=0.99\end{cases}$$

Note that the BBA m_3 is not equal to the VBBA but it is very close to the VBBA because $m_3(\Theta)$ is close to one.

If we consider the fusion of only the two first BBAs m_1 and m_2, we have $\text{PCR6}(m_1,m_2) = \text{PCR6}^+(m_1,m_2) = \text{PCR5}(m_1,m_2) = \text{PCR5}^+(m_1,m_2)$ because all these rules coincide when combining two BBAs.

$$\begin{cases} m^{\text{PCR6}}_{1,2}(A\cup B)\approx 0.465309\\ m^{\text{PCR6}}_{1,2}(C\cup D)\approx 0.296299\\ m^{\text{PCR6}}_{1,2}(A\cup B\cup C\cup D)\approx 0.023471\\ m^{\text{PCR6}}_{1,2}(E)\approx 0.214921\end{cases}$$

If we make the PCR5, PCR5$^+$, PCR6, and PCR6$^+$ fusion of these three BBAs altogether we obtain now different results which is normal, because for $S>2$, one has $\text{PCR5}^+(m_1,\ldots,m_S)\neq \text{PCR5}(m_1,\ldots,m_S)$ and $\text{PCR6}^+(m_1,\ldots,m_S)\neq \text{PCR6}(m_1,\ldots,m_S)$ in general. So, in this example 5, we get results shown in Tables I and II.

These values highlight the great ignorance of the results proposed by PCR5 and PCR6 when the third (almost vacuous) source of information is taken into account. Indeed, $m^{\text{PCR5}}_{1,2,3}(\Theta) = m^{\text{PCR6}}_{1,2,3}(\Theta)$ is the greatest mass among the set of hypotheses, whereas the results proposed with PCR5$^+$ and PCR6$^+$ combination rules discard the ignorant information and propose results closer to those obtained by merging two sources. Indeed, the largest mass is allocated to $A\cup B$.

The next examples 6–12 are very simple examples involving only categorical BBAs so that only one conflicting product (equals to one) needs to be redistributed based on PCR5, PCR6, PCR5$^+$, and PCR6$^+$ rules. These examples offer the possibility to the reader to do the derivations manually for making a verification of our results.

Example 6 (revisited): we consider $\Theta=\{A,B,C,D\}$, and the following categorical BBAs $m_1(A)=1, m_2(B\cup C)=1, m_3(A\cup C)=1, m_4(B\cup C)=1, m_5(A\cup B\cup C)=1$, and $m_6(A\cup B\cup C\cup D)=1$. If we make the PCR5, PCR5$^+$, PCR6, and PCR6$^+$ fusion of these six BBAs altogether, we obtain results given in Tables III and IV.

In this example, we have only one conflicting product $\pi_1(\emptyset)$ to redistribute which is given by

$$\pi_1(\emptyset)=m_1(A)m_2(B\cup C)m_3(A\cup C)m_4(B\cup C)\cdot m_5(A\cup B\cup C)m_6(A\cup B\cup C\cup D).$$

Because $\kappa_1(A\cup B\cup C)=0$ and $\kappa_1(A\cup B\cup C\cup D)=0$, these two disjunctions are discarded and more mass is committed to A, $A\cup C$ and $B\cup C$ with PCR5$^+$ and PCR6$^+$ rules. There is more mass allocated to $B\cup C$ with PCR6$^+$ and PCR6 than with PCR5$^+$ and PCR5 because two sources of information support this hypothesis.

Example 7 (revisited): we consider $\Theta=\{A,B,C,D,E\}$, and the following seven categorical BBAs $m_1(A\cup E)=1$, $m_2(B\cup C\cup E)=1$, $m_3(A\cup C\cup E)=1$, $m_4(B\cup C\cup E)=1$, $m_5(A\cup B\cup C\cup E)=1, m_6(A\cup B\cup C\cup D\cup E)=1$, and $m_7(A)=1$. If we make the PCR5, PCR5$^+$, PCR6,

TABLE VII
Example 8: Results of PCR5$^+$ versus PCR5

Focal elements	$m^{\text{PCR5}}_{1,2,3,4,5}(\cdot)$	$m^{\text{PCR5}^+}_{1,2,3,4,5}(\cdot)$
A	1/4	1/3
$A\cup C$	1/4	1/3
$B\cup C$	1/4	1/3
$A\cup B\cup C\cup D$	1/4	0

TABLE VIII
Example 8: Results of PCR6$^+$ versus PCR6

Focal elements	$m^{\text{PCR6}}_{1,2,3,4,5}(\cdot)$	$m^{\text{PCR6}^+}_{1,2,3,4,5}(\cdot)$
A	1/5	1/4
$A\cup C$	1/5	1/4
$B\cup C$	2/5	1/2
$A\cup B\cup C\cup D$	1/5	0

TABLE IX
Example 9: Results of PCR5$^+$ versus PCR5

Focal elements	$m^{\text{PCR5}}_{1,2,3,4,5,6,7}(\cdot)$	$m^{\text{PCR5}^+}_{1,2,3,4,5,6,7}(\cdot)$
A	1/5	1/3
$A\cup C$	1/5	1/3
$B\cup C$	1/5	1/3
$A\cup B\cup C$	1/5	0
$A\cup B\cup C\cup D$	1/5	0

TABLE X
Example 9: Results of PCR6$^+$ versus PCR6

Focal elements	$m^{PCR6}_{1,2,3,4,5,6,7}(\cdot)$	$m^{PCR6^+}_{1,2,3,4,5,6,7}(\cdot)$
A	1/7	1/4
$A \cup C$	1/7	1/4
$B \cup C$	2/7	1/2
$A \cup B \cup C$	2/7	0
$A \cup B \cup C \cup D$	1/7	0

TABLE XII
Example 12: Results of PCR5, PCR5$^+$

Focal elements	$m^{PCR5}_{1,2,3}(\cdot)$	$m^{PCR5^+}_{1,2,3}(\cdot)$
A	1/3	1/2
$B \cup C$	1/3	1/2
$A \cup B \cup C$	1/3	0

and PCR6$^+$ fusion of these seven BBAs altogether, we obtain results given in Tables V and VI.

In this example 7, we have only one conflicting product $\pi_1(\emptyset)$ to redistribute which is given by

$$\pi_1(\emptyset) = m_1(A \cup E)m_2(B \cup C \cup E)m_3(A \cup C \cup E) \cdot m_4(B \cup C \cup E)m_5(A \cup B \cup C \cup E) \cdot m_6(A \cup B \cup C \cup D \cup E)m_7(A).$$

Because $\kappa_1(A \cup B \cup C \cup E) = 0$ and $\kappa_1(A \cup B \cup C \cup D \cup E) = 0$, these two disjunctions are discarded and more mass is committed to $A, A \cup E, A \cup C \cup E$, and $B \cup C \cup E$ with PCR5$^+$ and PCR6$^+$ rules. There is more mass allocated to $B \cup C \cup E$ with PCR6$^+$ and PCR6 than with PCR5$^+$ and PCR5 because two sources of information support this hypothesis.

Example 8 (revisited): we consider $\Theta = \{A, B, C, D\}$, and the following categorical BBAs $m_1(A) = 1, m_2(B \cup C) = 1, m_3(A \cup C) = 1, m_4(B \cup C) = 1$, and $m_5(A \cup B \cup C \cup D) = 1$. If we make the PCR5, PCR5$^+$, PCR6, and PCR6$^+$ fusion of these seven BBAs altogether we obtain results given in Tables VII and VIII.

Because $\kappa_1(A \cup B \cup C \cup D) = 0$, this disjunction is discarded and more mass is committed to $A, A \cup C$, and $B \cup C$ with PCR5$^+$ and PCR6$^+$ rules. There is more mass allocated to $B \cup C$ with PCR6$^+$ and PCR6 than with PCR5$^+$ and PCR5 because two sources of information support this hypothesis.

Example 9 (revisited): we consider $\Theta = \{A, B, C, D\}$, and the following seven categorical BBAs $m_1(A) = 1, m_2(B \cup C) = 1, m_3(A \cup C) = 1, m_4(B \cup C) = 1, m_5(A \cup B \cup C \cup D) = 1, m_6(A \cup B \cup C) = 1$, and $m_7(A \cup B \cup C) = 1$. If we make the PCR5, PCR5$^+$, PCR6, and PCR6$^+$ fusion of these seven BBAs altogether, we obtain results given in Tables IX and X.

Because $\kappa_1(A \cup B \cup C \cup D) = 0$ and $\kappa_1(A \cup B \cup C) = 0$, these disjunctions are discarded and more mass is committed to $A, A \cup C$ and $B \cup C$ with PCR5$^+$ and PCR6$^+$ rules. There is more mass allocated to $B \cup C$ with PCR6$^+$ and PCR6 than with PCR5$^+$ and PCR5 because two sources of information support this hypothesis. Similarly, more mass is allocated to $(A \cup B \cup C)$ with PCR6 than PCR5 since two sources of information support this hypothesis.

Example 10 (revisited): we consider $\Theta = \{A, B, C\}$, and the following three categorical BBAs $m_1(A) = 1, m_2(B \cup C) = 1$, and $m_3(A \cup C) = 1$. We have only one conflicting product $\pi_1(\emptyset) = m_1(A)m_2(B \cup C) m_3(A \cup C) = 1$ to redistribute, and for this example, we have $\kappa_1(A) = 1, \kappa_1(A \cup C) = 1$, and $\kappa_1(B \cup C) = 1$, which means that all focal elements $A, A \cup C$, and $B \cup C$ must be kept, and they must receive a mass through the proportional redistribution principle. Hence, in this example, we have $m^{PCR5}_{1,2,3} = m^{PCR6}_{1,2,3} = m^{PCR5^+}_{1,2,3} = m^{PCR6^+}_{1,2,3}$, and the combined masses are evenly distributed as shown in the Table XI.

Example 11 (revisited): we consider $\Theta = \{A, B, C\}$, and the following four categorical BBAs $m_1(A) = 1, m_2(B \cup C) = 1, m_3(A \cup C) = 1$, and $m_4(A \cup B) = 1$. Because we have only one conflicting product $\pi_1(\emptyset) = m_1(A)m_2(B \cup C)m_3(A \cup C)m_4(A \cup B) = 1$ and $\kappa_1(A) = 1, \kappa_1(A \cup B) = 1, \kappa_1(A \cup C) = 1$ and $\kappa_1(B \cup C) = 1$, no hypothesis is discarded in the PCR, and we get $m^{PCR5}_{1,2,3,4} = m^{PCR6}_{1,2,3,4} = m^{PCR5^+}_{1,2,3,4} = m^{PCR6^+}_{1,2,3,4}$ with the merged masses being evenly distributed, that is $m^{PCR5}_{1,2,3,4}(A) = 1/4, m^{PCR5}_{1,2,3,4}(A \cup B) = 1/4, m^{PCR5}_{1,2,3,4}(A \cup C) = 1/4$, and $m^{PCR5}_{1,2,3,4}(B \cup C) = 1/4$.

Example 12 (revisited): we consider $\Theta = \{A, B, C\}$, and the following three categorical BBAs, $m_1(A \cup B \cup C) = 1, m_2(A) = 1$, and $m_3(B \cup C) = 1$. If we make the PCR5 fusion and the PCR5$^+$ fusion of these three BBAs altogether, we obtain results given in Table XII. Because $\pi_1(\emptyset) = m_1(A \cup B \cup C)m_2(A)m_3(B \cup C)$, we

TABLE XI
Example 10: Results of PCR5, PCR5$^+$, PCR6, PCR6$^+$

Focal elements	$m^{PCR5}_{1,2,3}(\cdot)$	$m^{PCR5^+}_{1,2,3}(\cdot)$	$m^{PCR6}_{1,2,3}(\cdot)$	$m^{PCR6^+}_{1,2,3}(\cdot)$
A	1/3	1/3	1/3	1/3
$A \cup C$	1/3	1/3	1/3	1/3
$B \cup C$	1/3	1/3	1/3	1/3

TABLE XIII
Example 13: Results of PCR5$^+$ versus PCR5

Focal elements	$m^{PCR5}_{1,2,3}(\cdot)$	$m^{PCR5^+}_{1,2,3}(\cdot)$
B	0.041797	0.041797
$A \cup B$	0.487632	0.613029
$C \cup D$	0.258327	0.345174
$A \cup B \cup C \cup D$	0.212244	0

TABLE XIV
Example 13: Results of PCR6$^+$ versus PCR6

Focal elements	$m_{1,2,3}^{\mathrm{PCR6}}(\cdot)$	$m_{1,2,3}^{\mathrm{PCR6}^+}(\cdot)$
B	0.037676	0.037676
$A \cup B$	0.487632	0.613029
$C \cup D$	0.262448	0.349295
$A \cup B \cup C \cup D$	0.212244	0

get $\kappa_1(A \cup B \cup C) = 0, \kappa_1(A) = 1$, and $\kappa_1(B \cup C) = 1$ based on (23). Therefore, using the PCR5$^+$ combination rule, we get a redistribution of the conflicting mass $\pi_1(\emptyset) = 1$ only between A and $B \cup C$. In this example we have $m_{1,2,3}^{\mathrm{PCR5}} = m_{1,2,3}^{\mathrm{PCR6}}$, and $m_{1,2,3}^{\mathrm{PCR5}^+} = m_{1,2,3,4}^{\mathrm{PCR6}^+}$ because no mass is allocated on the same hypothesis by two different sources.

Example 13 (revisited): we consider $\Theta = \{A, B, C, D\}$, and the three following BBAs

$$m_1(A \cup B) = 0.8, m_1(C \cup D) = 0.2,$$

$$m_2(A \cup B) = 0.4, m_2(C \cup D) = 0.6,$$

$$m_3(B) = 0.1, m_3(A \cup B \cup C \cup D) = 0.9.$$

If we make the PCR5, PCR5$^+$, PCR6, and PCR6$^+$ fusion of these seven BBAs altogether, we obtain results given in Tables XIII and XIV.

Because $\kappa_j(\Theta) = 0$ for any conflicting product $\pi_j(\emptyset)$ involving Θ, this hypothesis is discarded in the redistribution of $\pi_4(\emptyset)$ and of $\pi_6(\emptyset)$ (see example 13 in Subsection VI-A for details), and therefore more mass is redistributed to $A \cup B$ and $C \cup D$ with PCR5$^+$ and PCR6$^+$ rules. No more mass is committed to B with PCR5$^+$ and PCR6$^+$, respectively, in comparison with PCR5 and PCR6. This is because B is not implied in any partial conflict with Θ (cf. Subsection VI-A for details).

VIII. CONCLUSION

In this paper, after having demonstrated the flawed behavior of PCR5 and PCR6 rules of combination for $S > 2$ BBAs (including possibly VBBAs), we proposed improvements to correct these behaviors. A computation of a binary keeping index has been detailed, which makes it possible to discard ignorant information sources for the calculation of each partial conflict. This binary keeping index has been integrated into the original formulations of PCR5 and PCR6 in order to ensure the neutrality property of the VBBA and to propose two new combination rules for a number of sources greater than 2: PCR5$^+$ and PCR6$^+$ rules. The interest of such combination rules could prove to be particularly important in an application case identifying many ignorant sources of information. In such a scenario, the preponderant ignorance of a certain number of sources will no longer obscure a more precise characterization provided by other sources.

These new rules of combination have been already applied to risk analysis issues for geophysical and geotechnical data fusion in order to reinforce the levee protection characterizations [48].

APPENDIX 1: PROOF OF THE LEMMA 1

We prove that: $m_{1,2,\ldots,S,S+1}^{\mathrm{Conj}}(A) = m_{1,2,\ldots,S}^{\mathrm{Conj}}(A)$, for any $A \in 2^\Theta \setminus \{\emptyset\}$, where $m_{S+1}(\Theta) = 1$ is the VBBA m_v. The set of focal elements of $m_{S+1}(\cdot)$ is $\mathcal{F}(m_{S+1}) = \{\Theta\}$, therefore $\mathcal{F}_{m_{S+1}} = 1$ and $X_{j_{S+1}} = \Theta$. Based on the formula (6) written for $S+1$ BBAs, we have

$$m_{1,2,\ldots,S,S+1}^{\mathrm{Conj}}(A) = \sum_{\substack{\mathbf{X}_j \in \mathcal{F}(m_1,\ldots,m_S,m_{S+1}) \\ X_{j_1}\cap\ldots\cap X_{j_S}\cap X_{j_{S+1}} = A}} \pi_j(X_{j_1} \cap \ldots \cap X_{j_S} \cap X_{j_{S+1}})$$

$$= \sum_{\substack{\mathbf{X}_j \in \mathcal{F}(m_1,\ldots,m_S,m_{S+1}) \\ X_{j_1}\cap\ldots\cap X_{j_S}\cap \Theta = A}} \prod_{i=1}^{S+1} m_i(X_{j_i}). \quad (27)$$

Because $X_{j_{S+1}} = \Theta$ is constant and $m_{S+1}(X_{j_{S+1}}) = m_{S+1}(\Theta) = 1$, one has

$$\prod_{i=1}^{S+1} m_i(X_{j_i}) = \left(\prod_{i=1}^{S} m_i(X_{j_i})\right) \cdot m_{S+1}(\Theta) = \prod_{i=1}^{S} m_i(X_{j_i}),$$

and $X_{j_1} \cap \ldots \cap X_{j_S} \cap X_{j_{S+1}} = X_{j_1} \cap \ldots \cap X_{j_S} \cap \Theta = X_{j_1} \cap \ldots \cap X_{j_S}$. Therefore the formula (27) becomes

$$m_{1,2,\ldots,S,S+1}^{\mathrm{Conj}}(A) = \sum_{\substack{\mathbf{X}_j \in \mathcal{F}(m_1,\ldots,m_S,m_{S+1}) \\ X_{j_1}\cap\ldots\cap X_{j_S}\cap \Theta = A}} \prod_{i=1}^{S+1} m_i(X_{j_i})$$

$$= \sum_{\substack{\mathbf{X}_j \in \mathcal{F}(m_1,\ldots,m_S) \\ X_{j_1}\cap\ldots\cap X_{j_S} = A}} \prod_{i=1}^{S} m_i(X_{j_i})$$

$$= m_{1,2,\ldots,S}^{\mathrm{Conj}}(A),$$

which completes the proof of the Lemma 1.

APPENDIX 2: PROOF OF THE THEOREM

We prove that $\mathrm{PCR5}^+(m_1, \ldots, m_S, m_{S+1}) = \mathrm{PCR5}^+(m_1, \ldots, m_S)$, or equivalently that $m_{1,2,\ldots,S+1}^{\mathrm{PCR5}^+}(A) = m_{1,2,\ldots,S}^{\mathrm{PCR5}^+}(A)$ for any $A \in 2^\Theta \setminus \{\emptyset\}$, where $m_{S+1}(X_{j_{S+1}}) = m_{S+1}(\Theta) = 1$ is the VBBA. It is worth noting that $m_{1,2,\ldots,S,S+1}^{\mathrm{Conj}}(A) = m_{1,2,\ldots,S}^{\mathrm{Conj}}(A)$ for any $A \in 2^\Theta \setminus \{\emptyset\}$ because the VBBA $m_{S+1}(.)$ is the neutral element of the conjunctive rule (see Lemma 1). It is important to note that when considering $A = \Theta$, we have always $m^{\text{PCR5}^+}_{1,2,\ldots,S+1}(\Theta) = m^{\text{Conj}}_{1,2,\ldots,S,S+1}(\Theta) = m^{\text{Conj}}_{1,2,\ldots,S}(\Theta) = m^{\text{PCR5}^+}_{1,2,\ldots,S}(\Theta)$ because the binary keeping index of Θ is always equal to zero (see remark 1), i.e., $\kappa_j(\Theta) = 0$. Therefore all the redistribution terms to Θ in PCR5$^+$ (and in PCR6$^+$) formula are equal to zero when $A = \Theta$. So, we just have to consider $A \neq \Theta$ to make the proof.

Because $m_{S+1}(\cdot)$ is the VBBA, its set of focal elements is $\mathcal{F}(m_{S+1}) = \{\Theta\}$ and it contains only one focal element, i.e. $|\mathcal{F}(m_{S+1})| = 1$. Therefore

$$\mathcal{F} = |\mathcal{F}(m_1)| \cdot |\mathcal{F}(m_2)| \cdot \ldots \cdot |\mathcal{F}(m_S)| \cdot |\mathcal{F}(m_{S+1})| \quad (28)$$

$$= |\mathcal{F}(m_1)| \cdot |\mathcal{F}(m_2)| \cdot \ldots \cdot |\mathcal{F}(m_S)|. \quad (29)$$

This means that the number of conflicting products $\pi_j(\emptyset)$ associated to the $S+1$-tuple $\mathbf{X_j} = (X_{j_1}, \ldots, X_{j_S}, \Theta) \in \mathcal{F}(m_1, \ldots, m_S, m_{S+1})$ is equal to the number of conflicting products $\pi_j(\emptyset)$ associated to S-tuple $\mathbf{X_j} = (X_{j_1}, \ldots, X_{j_S}) \in \mathcal{F}(m_1, \ldots, m_S)$. Moreover, we always have

$$\prod_{i=1}^{S+1} m_i(X_{j_i}) = \left(\prod_{i=1}^{S} m_i(X_{j_i})\right) \cdot m_{S+1}(\Theta) = \prod_{i=1}^{S} m_i(X_{j_i}).$$

Hence, we always have

$$\pi_j(X_{j_1} \cap \ldots \cap X_{j_S} \cap \Theta = \emptyset) = \pi_j(X_{j_1} \cap \ldots \cap X_{j_S} = \emptyset),$$

because $X_{j_1} \cap \ldots \cap X_{j_S} \cap \Theta = X_{j_1} \cap \ldots \cap X_{j_S}$.

Based on the formula (25) written for $S+1$ BBAs, we have

$$m^{\text{PCR5}^+}_{1,2,\ldots,S,S+1}(A) = m^{\text{Conj}}_{1,2,\ldots,S,S+1}(A) + \sum_{j\in\{1,\ldots,\mathcal{F}\}|A\in\mathbf{X}_j\wedge\pi_j(\emptyset)} \left[\left(\kappa_j(A) \prod_{i\in\{1,\ldots,S+1\}|X_{j_i}=A} m_i(X_{j_i})\right) \cdot \frac{\pi_j(X_{j_1} \cap \ldots \cap X_{j_S} \cap \Theta = \emptyset)}{\sum_{X\in\mathbf{X}_j}\left(\kappa_j(X) \prod_{i\in\{1,\ldots,S+1\}|X_{j_i}=X} m_i(X_{j_i})\right)}\right], \quad (30)$$

where $\mathcal{F}$ is given by (28).

Because $X_{j_{S+1}} = \Theta$ and because we consider $A \neq \Theta$, we have always

$$\prod_{i\in\{1,\ldots,S+1\}|X_{j_i}=A} m_i(X_{j_i}) = \prod_{i\in\{1,\ldots,S\}|X_{j_i}=A} m_i(X_{j_i}).$$

Whether $X \in \mathbf{X_j} = (X_{j_1}, \ldots, X_{j_S})$ or $X \in \mathbf{X_j} = (X_{j_1}, \ldots, X_{j_S}, \Theta)$ the value of $\kappa_j(X)$ is the same since the additional binary *containing indicator* $\delta_j(X, \Theta)$ entering in the product of the computation of the binary *keeping-index* is always equal to 1 and does not modify $\kappa_j(X)$ value, and of course when $X = A$. Because the binary keeping-index entering in the numerator and denominator of formula (30) removes the factor $m_{S+1}(\Theta)$ from all products it belongs to (since Θ includes all elements of the product it belongs to), the formula (30) reduces to the following formula

$$m^{\text{PCR5}^+}_{1,2,\ldots,S,S+1}(A) = m^{\text{Conj}}_{1,2,\ldots,S}(A) + \sum_{j\in\{1,\ldots,\mathcal{F}\}|A\in\mathbf{X}_j\wedge\pi_j(\emptyset)} \left[\left(\kappa_j(A) \prod_{i\in\{1,\ldots,S\}|X_{j_i}=A} m_i(X_{j_i})\right) \cdot \frac{\pi_j(X_{j_1} \cap \ldots \cap X_{j_S} = \emptyset)}{\sum_{X\in\mathbf{X}_j}\left(\kappa_j(X) \prod_{i\in\{1,\ldots,S\}|X_{j_i}=X} m_i(X_{j_i})\right)}\right] = m^{\text{PCR5}^+}_{1,2,\ldots,S}(A), \quad (31)$$

where $\mathbf{X}_j$ represents now the S-tuple $(X_{j_1}, \ldots, X_{j_S})$, and $\pi_j(\emptyset) = \pi_j(X_{j_1} \cap \ldots \cap X_{j_S} = \emptyset)$.

So, we have proved PCR5$^+(m_1, \ldots, m_S, m_{S+1})$ = PCR5$^+(m_1, \ldots, m_S)$ when m_{S+1} is the VBBA. Similarly, we can prove that PCR6$^+(m_1, \ldots, m_S, m_{S+1})$ = PCR6$^+(m_1, \ldots, m_S)$ when m_{S+1} is the VBBA. This completes the proof of the theorem.

APPENDIX 3: CODES OF PCR5$^+$ AND PCR6$^+$ RULES

For convenience, we provide two basic Matlab$^{\text{TM}}$codes for PCR5$^+$ and PCR6$^+$ for the fusion of $S \geq 2$ BBAs for working with 2^Θ, i.e. working with Shafer's model. No input verification of input is done in the routines. It is assumed that the input matrix BBA is correct, both in dimension and in content. The derivation of all possible combinations is done with combvec(Combinations, vec) instruction which is included in the Matlab$^{\text{TM}}$ neural networks toolbox. This combvec call can be a very time-consuming task when the size of the problem increases. A standalone version of these codes is also available upon request to the authors. The j-th column of the BBA input matrix corresponds to the (vertical) BBA vector $m_j(.)$ associated with the j-th source s_j. Each element of a BBA matrix is in [0,1] and the sum of each column must be one. If N is the cardinality of the frame Θ and if S is the number of sources, then the size of the BBA input matrix is $((2^N) - 1)) \times S$. Each column of the BBA matrix must use the classical binary encoding of elements. For example, if $\Theta = \{A, B, C\}$, then we encode the elements of $2^\Theta \setminus \{\emptyset\}$ by the binary sequence $001 \equiv A$, $010 \equiv B, 011 \equiv A \cup B, \ldots, 111 \equiv A \cup B \cup C$. The mass of empty set is not included in the BBA vector because its is always set to zero. These codes can be used and shared for free for research purposes only. Commercial uses of these codes, or adaptation of them in any programming language, is not allowed without written agreement of the authors. These codes are provided by the copyright holders "as is" and any express or implied warranties are disclaimed. The copyright holder will not be liable for any direct, or indirect damages of the use of these codes. The authors would appreciate any feedback in the use of these codes, and publication using these codes should cite this paper in agreement for their use.

```
%===========================================================================
function [mPCR5plus]=PCR5plusfusion(BBA)
%===========================================================================
% Authors and copyrights: Theo Dezert & Jean Dezert
% Input:  BBA=[m1 m2 ... mS]= Matrix of BBAs to combine with PCR5+
% Output: mPCR5plus is PCR5+(m1,m2,...,mS) fusion result
%===========================================================================
NbrSources=size(BBA,2);CardTheta=log2(size(BBA,1)+1);
if(NbrSources==1), mPCR5plus=BBA(:,1);return, end
mPCR5plus=zeros(size(BBA,1),1);FocalElem = cell(NbrSources,1);
for i=1:NbrSources, FocalElem{i}=find(BBA(:,i)> 0)';end
Combinations=combvec(FocalElem{1:NbrSources})';
for c=1:size(Combinations,1)
    PC=Combinations(c,:);masseConj=diag(BBA(PC,:))';
    massConj=prod(diag(BBA(PC,:))',2);Intersections=PC(1);
    for s=2:NbrSources, Intersections=bitand(Intersections,PC(s)); end
    if(Intersections~=0)
       mPCR5plus(Intersections)=mPCR5plus(Intersections)+massConj;
    else
       Binary=[];CardPC=[];KeepIndex=[];
       for i=1:NbrSources
            Binary(i,:)=bitget(PC(i),CardTheta:-1:1,'int8');
            CardPC(i,:)=sum(Binary(i,:)==1);
       end
       for j=1:NbrSources
           delta=[];
           for js=1:NbrSources
               if CardPC(js)>=CardPC(j)
                  for jp=1:NbrSources
                      if PC(jp)~=PC(js) && CardPC(jp)<=CardPC(js)
                         if sum(Binary(jp,:)<=Binary(js,:))==CardTheta
                            delta=[delta 1];
                         else
                            delta=[delta 0];
                         end
                      end
                  end
               end
           end
           if isempty(delta)==1
              KeepIndex(j,1)=1;
           else
              KeepIndex(j,1)=1-prod(delta);
           end
       end
       KeepIndex=KeepIndex';
       for i=1:NbrSources
           if KeepIndex(i)==1, KeepIndex(i)=masseConj(i); end
       end
       UQ=unique(PC);Proportions=0*UQ;DenPCR5=0;
       for u=1:size(UQ,2)
           SamePropositions=find(PC==UQ(u));
           MassProd=prod(KeepIndex(SamePropositions));
           Proportions(u)= MassProd*massConj;DenPCR5=DenPCR5+MassProd;
       end
       Proportions=Proportions/DenPCR5;
       for u=1:size(UQ,2),mPCR5plus(UQ(u))=mPCR5plus(UQ(u))+Proportions(u); end
    end
end
```

```
%===========================================================================
function [mPCR6plus]=PCR6plusfusion(BBA)
%===========================================================================
% Authors and copyrights: Theo Dezert & Jean Dezert
% Input:  BBA=[m1 m2 ... mS]= Matrix of BBAs to combine with PCR6+
% Output: mPCR6plus is PCR6+(m1,m2,...,mS) fusion result
%===========================================================================
NbrSources=size(BBA,2);CardTheta=log2(size(BBA,1)+1);
if(NbrSources==1), mPCR6plus=BBA(:,1);return, end
mPCR6plus=zeros(size(BBA,1),1);FocalElem = cell(NbrSources,1);
for i=1:NbrSources, FocalElem{i}=find(BBA(:,i)> 0)';end
Combinations=combvec(FocalElem{1:NbrSources})';
for c=1:size(Combinations,1)
    PC=Combinations(c,:);masseConj=diag(BBA(PC,:))';
    massConj=prod(diag(BBA(PC,:))',2);Intersections=PC(1);
    for s=2:NbrSources, Intersections=bitand(Intersections,PC(s));end
    if(Intersections~=0)
       mPCR6plus(Intersections)=mPCR6plus(Intersections)+massConj;
    else
       Binary=[];CardPC=[];KeepIndex=[];
       for i=1:NbrSources
            Binary(i,:)=bitget(PC(i),CardTheta:-1:1,'int8');
            CardPC(i,:)=sum(Binary(i,:)==1);
       end
       for j=1:NbrSources
           delta=[];
           for js=1:NbrSources
               if CardPC(js)>=CardPC(j)
                  for jp=1:NbrSources
                      if PC(jp)~=PC(js) && CardPC(jp)<=CardPC(js)
                         if sum(Binary(jp,:)<=Binary(js,:))==CardTheta
                            delta=[delta 1];
                         else
                            delta=[delta 0];
                         end
                      end
                  end
               end
           end
           if isempty(delta)==1
              KeepIndex(j,1)=1;
           else
              KeepIndex(j,1)=1-prod(delta);
           end
       end
       KeepIndex=KeepIndex';IgnoringSetOfFE=find(KeepIndex==0);
       masseConj(IgnoringSetOfFE)=[];PC(IgnoringSetOfFE)=[];
       for s=1:numel(masseConj)
           Proportion= masseConj(s)*(massConj/(sum(masseConj,2)));
           mPCR6plus(PC(s))=mPCR6plus(PC(s))+Proportion;
       end
    end
end
```

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