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Fixed Point Results for Contraction Theorems in Neutrosophic Metric Spaces

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Abstract. In this article, we present fixed and common fixed point results for Banach and Edelstein contraction theorems in neutrosophic metric spaces. Then some properties and examples are given for neutrosophic metric spaces. Thus, we added a new path in neutrosophic theory to obtain fixed point results. we investigate and prove some contraction theorems that are extended to neutrosophic metric space with the assistance of Grabiec.

Keywords: Fixed point; Neutrosophic Metric Space; Banach Contraction; Edelstein Contraction.

1. Introduction

Fuzzy Sets was presented by Zadeh [20] as a class of elements with a grade of membership. Kramosil and Michalek [9] defined new notion called Fuzzy Metric Space (FMS). Later, many authors have examined the concept of fuzzy metric in various aspects. In 1984 Kaleva and Seikkala [8] have characterized the FMS, where separation between any two points to be positive number. In particular, George and Veeramani [4, 5] redefined the concept of fuzzy metric space with the assistance of continuous t-norm, and continuous t-co norm. FMS has utilized in applied science fields such as fixed point theory, decision making, medical imaging and signal processing. Heilpern [7] defined fuzzy contraction for Fixed point theorem. Park [14] defined Intuitionistic Fuzzy Metric Space (IFMS) from the concept of FMS and given some fixed point results. Fixed point theorems related to FMS and IFMS given by Alaca et al [2] and nemerous researchers [13, 19].In 1998, Smarandache [16] characterized the new concept called neutrosophic logic and neutrosophic set. In the idea of neutrosophic sets, there is T degree of membership, I degree of indeterminacy and F degree of non-membership. A neutrosophic value is appeared by (T, I, F). Hence, neutrosophic logic and neutrosophic set assists us to brief many uncertainties in our lives. In addition, several researchers have made significant development on this theory [26–30]. Recently, Baset et al. [22–25] explored the neutrosophic applications in different fields such as model for sustainable supply chain risk management, resource levelling problem in construction projects, Decision Making and financial performance evaluation of manufacturing industries. In fact, the idea of fuzzy sets deals with only a degree of membership. In addition, the concept of intuitionistic fuzzy set established while adding degree of non - membership with degree of membership. But these degrees are characterized relatively one another. Therefore, neutrosophic set is a generalized state of fuzzy and intuitionistic fuzzy set by incorporating degree of indeterminacy. In 2019, Kirisci et al [10, 11] defined neutrosophic metric space as a generalization of IFMS and brings about fixed point theorems in complete neutrosophic metric space.

In this paper, we investigate and prove some contraction theorems that are extended to neutrosophic metric space with the assistance of Grabiec [6].

2. Preliminaries

Definition 2.1 [17] Let Σ be a non-empty fixed set. A Neutrosophic Set (NS for short) N in Σ is an object having the form $N = \{\langle a, \xi_N(a), \varrho_N(a), \nu_N(a)\rangle : a \in \Sigma\}$ where the functions $\xi_N(a), \varrho_N(a)$ and $\nu_N(a)$ represent the degree of membership, degree of indeterminacy and the degree of non-membership respectively of each element a $\in$ N to the set Σ.

A neutrosophic set $N = \{\langle a, \xi_N(a), \varrho_N(a), \nu_N(a)\rangle : a \in \Sigma\}$ is expressed as an ordered triple $N = \langle a, \xi_N(a), \varrho_N(a), \nu_N(a)\rangle$ in Σ.

In NS, there is no restriction on $(\xi_N(a), \varrho_N(a), \nu_N(a))$ other than they are subsets of $]^{-}0, 1^{+}[$

Remark 2.2 [10] Neutrosophic Set N is included in another Neutrosophic set Γ ($N \subseteq \Gamma$) if and only if

$$inf\ \xi_N(a) \leq inf \xi_\Gamma(a) \quad sup\ \xi_N(a) \leq sup\ \xi_\Gamma(a)$$
$$inf\ \varrho_N(a) \geq inf \varrho_\Gamma(a) \quad sup\ \varrho_N(a) \geq sup\ \varrho_\Gamma(a)$$
$$inf\ \nu_N(a) \geq inf \nu_\Gamma(a) \quad sup\ \nu_N(a) \geq sup\ \nu_\Gamma(a)$$

Triangular Norms (TNs) were initiated by menger. Triangular co norms(TCs) knowns as dual operations of triangular norms (TNs).

Definition 2.3 [4] A binary operation $\star : [0, 1] \times [0, 1] \to [0, 1]$ is called continuous t - norm (CTN) if it satisfies the following conditions;

For all $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0, 1]$

(i) $\varepsilon_1 \star 0 = \varepsilon_1$;

(ii) If $\varepsilon_1 \leq \varepsilon_3$ and $\varepsilon_2 \leq \varepsilon_4$ then $\varepsilon_1 \star \varepsilon_2 \leq \varepsilon_3 \star \varepsilon_4$;

(iii) $\star$ is continuous;

(iv) $\star$ is commutative and associative.

Definition 2.4 [4] A binary operation $\diamond : [0,1] \times [0,1] \rightarrow [0,1]$ is called continuous t - co norm (CTC) if it satisfies the following conditions;

For all $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0, 1]$

(i) $\varepsilon_1 \diamond 0 = \varepsilon_1$;

(ii) If $\varepsilon_1 \leq \varepsilon_3$ and $\varepsilon_2 \leq \varepsilon_4$ then $\varepsilon_1 \diamond \varepsilon_2 \leq \varepsilon_3 \diamond \varepsilon_4$;

(iii) $\diamond$ is continuous;

(iv) $\diamond$ is commutative and associative.

Remark 2.5 From the definitions of CTN and CTC, we note that if we take $0 < \varepsilon_1, \varepsilon_2 < 1$ for $\varepsilon_1 < \varepsilon_2$ then there exist $0 < \varepsilon_3, \varepsilon_4 < 1$ such that $\varepsilon_1 \star \varepsilon_3 \geq \varepsilon_2$ and $\varepsilon_1 \geq \varepsilon_2 \diamond \varepsilon_4$.

Further we choose $\varepsilon_5 \in (0,1)$ then there exists $\varepsilon_6, \varepsilon_7 \in (0,1)$ such that $\varepsilon_6 \star \varepsilon_6 \geq \varepsilon_5$ and $\varepsilon_7 \diamond \varepsilon_7 \leq \varepsilon_5$.

Definition 2.6 [13] A Sequence $\{t_n\}$ is called s - non-decreasing sequence if there exists $m_0 \in \mathbb{N}$ such that $t_m \leq t_{m+1}$ for all $m > m_0$.

3. **Neutrosophic Metric Space**

In this section, we apply neutrosophic theory to generalize the Intuitionistic fuzzy metric space. we also discuss some properties and examples in it.

Definition 3.1 A 6 - tuple $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$is called Neutrosophic Metric Space(NMS), if Σ is an arbitrary non empty set, $\star$ is a neutrosophic CTN and $\diamond$ is a neutrosophic CTC and Ξ, Θ, Υ are neutrosophic sets on $\Sigma^2 \times \mathbb{R}^+$ satisfying the following conditions:

For all $\zeta, \eta, \omega \in \Sigma, \lambda \in \mathbb{R}^+$

(i) $0 \leq \Xi(\zeta, \eta, \lambda) \leq 1;\ 0 \leq \Theta(\zeta, \eta, \lambda) \leq 1;\ 0 \leq \Upsilon(\zeta, \eta, \lambda) \leq 1$;

(ii) $\Xi(\zeta, \eta, \lambda) + \Theta(\zeta, \eta, \lambda) + \Upsilon(\zeta, \eta, \lambda) \leq 3$;

(iii) $\Xi(\zeta, \eta, \lambda) = 1$ if and only if $\zeta = \eta$;

(iv) $\Xi(\zeta, \eta, \lambda) = \Xi(\eta, \zeta, \lambda)$ for $\lambda > 0$;

(v) $\Xi(\zeta, \eta, \lambda) \star \Xi(\eta, \zeta, \mu) \leq \Xi(\zeta, \omega, \lambda + \mu)$, for all $\lambda, \mu > 0$;

(vi) $\Xi(\zeta, \eta, .) : [0, \infty) \rightarrow [0, 1]$ is neutrosophic continuous ;

(vii) $lim_{\lambda \rightarrow \infty} \Xi(\zeta, \eta, \lambda) = 1$ for all $\lambda > 0$;

(viii) $\Theta(\zeta, \eta, \lambda) = 0$ if and only if $\zeta = \eta$;

(ix) $\Theta(\zeta, \eta, \lambda) = \Theta(\eta, \zeta, \lambda)$ for $\lambda > 0$;

(x) $\Theta(\zeta, \eta, \lambda) \diamond \Theta(\zeta, \omega, \mu) \geq \Theta(\zeta, \omega, \lambda + \mu)$, for all $\lambda, \mu > 0$;

(xi) $\Theta(\zeta, \eta, .) : [0, \infty) \rightarrow [0, 1]$ is neutrosophic continuous ;

(xii) $lim_{\lambda \rightarrow \infty} \Theta(\zeta, \eta, \lambda) = 0$ for all $\lambda > 0$;

(xiii) $\Upsilon(\zeta, \eta, \lambda) = 0$ if and only if $\zeta = \eta$;

(xiv) $\Upsilon(\zeta,\eta,\lambda) = \Upsilon(\eta,\zeta,\lambda)$ for $\lambda > 0$;

(xv) $\Upsilon(\zeta,\eta,\lambda) \diamond \Upsilon(\zeta,\omega,\mu) \geq \Upsilon(\zeta,\omega,\lambda+\mu)$, for all $\lambda,\mu > 0$;

(xvi) $\Upsilon(\zeta,\eta,.) : [0,\infty) \to [0,1]$ is neutrosophic continuous ;

(xvii) $lim_{\lambda\to\infty}\Upsilon(\zeta,\eta,\lambda) = 0$ for all $\lambda > 0$;

(xviii) If $\lambda > 0$ then $\Xi(\zeta,\eta,\lambda) = 0, \Theta(\zeta,\eta,\lambda) = 1, \Upsilon(\zeta,\eta,\lambda) = 1$.

Then (Ξ,Θ,Υ) is called Neutrosophic Metric on Σ. The functons Ξ,Θ and Υ denote degree of closedness, neturalness and non - closedness between ζ and η with respect to λ respectively.

Example 3.2 Let (Σ, d) be a metric space. Define $\zeta\star\eta = min\{\zeta,\eta\}$ and $\zeta\diamond\eta = max\{\zeta,\eta\}$, and $\Xi,\Theta,\Upsilon : \Sigma^2\times\mathbb{R}^+ \to [0,1]$ defined by , we define

$$\Xi(\zeta,\eta,\lambda) = \frac{\lambda}{\lambda + d(\zeta,\eta)};\quad \Theta(\zeta,\eta,\lambda) = \frac{d(\zeta,\eta)}{\lambda + d(\zeta,\eta)};\quad \Upsilon(\zeta,\eta,\lambda) = \frac{d(\zeta,\eta)}{\lambda}$$

for all $\zeta,\eta \in \Sigma$ and $\lambda > 0$. Then $(\Sigma,\Xi,\Theta,\Upsilon,\star,\diamond)$ is called neutrosophic metric space induced by a metric d the standard neutrosophic metric.

Example 3.3 If we take $\Sigma = \mathbb{N}$, consider the CTN, CTC are $\zeta\star\eta = min\{\zeta,\eta\}$ and $\zeta\diamond\eta = max\{\zeta,\eta\}$, $\Xi,\Theta,\Upsilon : \Sigma^2\times\mathbb{R}^+ \to [0,1]$ defined by

$$\Xi(\zeta,\eta,\lambda) = \begin{cases} \frac{\zeta}{\eta} & if \quad \zeta \leq \eta \\ \frac{\eta}{\zeta} & if \quad \eta \leq \zeta \end{cases}$$

$$\Theta(\zeta,\eta,\lambda) = \begin{cases} \frac{\eta-\zeta}{\eta} & if \quad \zeta \leq \eta \\ \frac{\zeta-\eta}{\zeta} & if \quad \eta \leq \zeta \end{cases}$$

$$\Upsilon(\zeta,\eta,\lambda) = \begin{cases} \eta-\zeta & if \quad \zeta \leq \eta \\ \zeta-\eta & if \quad \eta \leq \zeta \end{cases}$$

for all $\zeta,\eta \in \Sigma$ *and* $\lambda > 0$. Then $\Xi,\Theta,\Upsilon : \Sigma^2\times\mathbb{R}^+ \to [0,1]$ is a NMS.

Remark 3.4 In Neutrosophic Metric space Ξ is non - decreasing , Θ is a non - increasing , Υ is decreasing for all $\zeta,\eta \in \Sigma$.

Definition 3.5 Let $(\Sigma,\Xi,\Theta,\Upsilon,\star,\diamond)$ be neutrosophic metric space . Then

(a) a sequence $\{\zeta_n\}$ in Σ is converging to a point $\zeta \in \Sigma$ if for each $\lambda > 0$

$$lim_{\lambda\to\infty}\Xi(\zeta,\eta,\lambda) = 1;\quad lim_{\lambda\to\infty}\Theta(\zeta,\eta,\lambda) = 0;\quad lim_{\lambda\to\infty}\Upsilon(\zeta,\eta,\lambda) = 0.$$

(b) a sequence ζ_n in Σ is said to be Cauchy if for each $\epsilon > 0$ and $\lambda > 0$ there exist $N \in \mathbb{N}$ such that $\Xi(\zeta_n,\zeta_m,\lambda) > 1-\epsilon$; $\Theta(\zeta_n,\zeta_m,\lambda) < \epsilon$; $\Upsilon(\zeta_n,\zeta_m,\lambda) < \epsilon$ for all n, m $\leq$ N.

(c) $(\Sigma,\Xi,\Theta,\Upsilon,\star,\diamond)$ is said to be complete neutrosophic metric space if every Cauchy sequence is convergent.

(d) $(\Sigma,\Xi,\Theta,\Upsilon,\star,\diamond)$ is called compact neutrosophic metric space if every sequence contains convergent sub sequence.

4. **Main Results**

Theorem 4.1 (Neutrosophic Banach Contraction Theorem) Let $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ be a complete neutrosophic metric space. Let $\digamma : \Sigma \to \Sigma$ be a function satisfying

$$\Xi(\digamma\zeta, \digamma\eta, \lambda) \geq \Xi(\zeta, \eta, \lambda); \quad \Theta(\digamma\zeta, \digamma\eta, \lambda) \leq \Theta(\zeta, \eta, \lambda); \quad \Upsilon(\digamma\zeta, \digamma\eta, \lambda) \leq \Upsilon(\zeta, \eta, \lambda) \qquad (4.1.1)$$

for all $\zeta, \eta \in \Sigma$. $0 < k < 1$. Then $\digamma$ has unique fixed point.
Proof: Let $\zeta \in \Sigma$ and $\{\zeta_n\} = \digamma^n(a)$ $(n \in \mathbb{N})$. By Mathematical induction, we obtain

$$\Xi(\zeta_n, \zeta_{n+1}, \lambda) \geq \Xi(\zeta, \zeta_1, \frac{\lambda}{k^n}); \Theta(\zeta_n, \zeta_{n+1}, \lambda) \leq \Theta(\zeta, \zeta_1, \frac{\lambda}{k^n}); \Upsilon(\zeta_n, \zeta_{n+1}, \lambda) \leq \Upsilon(\zeta, \zeta_1, \frac{\lambda}{k^n}) \;....(4.1.2)$$

for all $n > 0$ and $\lambda > 0$. Thus for any non-negative integer p, we have

$$\begin{aligned}
\Xi(\zeta_n, \zeta_{n+p}, \lambda) &\geq \Xi(\zeta, \zeta_{n+1}, \frac{\lambda}{p}) \star \cdots^{(p-times)} \cdots \star \Xi(\zeta_{n+p-1}, \zeta_{n+p}, \frac{\lambda}{p}) \\
&\geq \Xi(\zeta, \zeta_1, \frac{\lambda}{pk^n}) \star \cdots^{(p-times)} \cdots \star \Xi(\zeta, \zeta_1, \frac{\lambda}{pk^{n+p-1}}) \\
\Theta(\zeta_n, \zeta_{n+p}, \lambda) &\leq \Theta(\zeta, \zeta_{n+1}, \frac{\lambda}{p}) \diamond \cdots^{(p-times)} \cdots \diamond \Theta(\zeta_{n+p-1}, \zeta_{n+p}, \frac{\lambda}{p}) \\
&\leq \Theta(\zeta, \zeta_1, \frac{\lambda}{pk^n}) \diamond \cdots^{(p-times)} \cdots \diamond \Theta(\zeta, \zeta_1, \frac{\lambda}{pk^{n+p-1}}) \\
\Upsilon(\zeta_n, \zeta_{n+p}, \lambda) &\leq \Upsilon(\zeta, \zeta_{n+1}, \frac{\lambda}{p}) \diamond \cdots^{(p-times)} \cdots \diamond \Upsilon(\zeta_{n+p-1}, \zeta_{n+p}, \frac{\lambda}{p}) \\
&\leq \Upsilon(\zeta, \zeta_1, \frac{\lambda}{pk^n}) \diamond \cdots^{(p-times)} \cdots \diamond \Upsilon(\zeta, \zeta_1, \frac{\lambda}{pk^{n+p-1}})
\end{aligned}$$

by (4.1.2) and the definition of NMS conditions, we get

$$\begin{aligned}
lim_{n\to\infty}\Xi(\zeta_n, \zeta_{n+p}, \lambda) &\geq 1 \star \cdots^{(p-times)} \cdots \star 1 = 1 \\
lim_{n\to\infty}\Theta(\zeta_n, \zeta_{n+p}, \lambda) &\leq 0 \diamond \cdots^{(p-times)} \cdots \diamond 0 = 0 \\
lim_{n\to\infty}\Upsilon(\zeta_n, \zeta_{n+p}, \lambda) &\leq 0 \diamond \cdots^{(p-times)} \cdots \diamond 0 = 0.
\end{aligned}$$

Therefore, $\{\zeta_n\}$ is Cauchy sequence and it is convergent to a limit, let the limit point is η. Thus, we get

$$\begin{aligned}
\Xi(\digamma\eta, \eta, t) &\geq \Xi(\digamma\eta, \digamma\zeta_n, \frac{\lambda}{2}) \star \Xi(\zeta_{n+1}, \eta, \frac{\lambda}{2}) \\
&\geq \Xi(\eta, \zeta_n, \frac{\lambda}{2k}) \star \Xi(\zeta_{n+1}, \eta, \frac{\lambda}{2}) \to 1 \star 1 = 1. \\
\Theta(\digamma\eta, \eta, \lambda) &\leq \Theta(\digamma\eta, \digamma\zeta_n, \frac{\lambda}{2}) \diamond \Theta(\zeta_{n+1}, \eta, \frac{\lambda}{2}) \\
&\leq \Theta(\eta, \zeta_n, \frac{\lambda}{2k}) \diamond \Theta(\zeta_{n+1}, \eta, \frac{\lambda}{2}) \to 0 \diamond 0 = 0. \\
\Upsilon(\digamma\eta, \eta, \lambda) &\leq \Upsilon(\digamma\eta, \digamma\zeta_n, \frac{\lambda}{2}) \diamond \Upsilon(\zeta_{n+1}, \eta, \frac{\lambda}{2})) \\
&\leq \Upsilon(\eta, \zeta_n, \frac{\lambda}{2k}) \diamond \Upsilon(\zeta_{n+1}, \eta, \frac{\lambda}{2}) \to 0 \diamond 0 = 0.
\end{aligned}$$

Since we see that

$$\Xi(\zeta,\eta,\lambda)=1 \ \ iff \ \ \zeta=\eta; \quad \Theta(\zeta,\eta,\lambda)=0 \ \ iff \ \ \zeta=\eta; \quad \Upsilon(\zeta,\eta,\lambda)=0 \ \ iff \ \ \zeta=\eta$$

we get $\digamma\eta = \eta$, which is the fixed point of Neutrosophic metric space.
To show the uniqueness, let us assume that $\digamma\omega = \omega$ for some $\omega \in \Sigma$

$$\begin{aligned}
&1 \geq \Xi(\zeta,\omega,\lambda) = \Xi(\digamma\eta,\digamma\omega,\lambda) \geq \Xi(\zeta,\omega,\frac{\lambda}{k}) = \Xi(\digamma\eta,\digamma\eta,\frac{\lambda}{k}) \geq \Xi(\zeta,\omega,\frac{\lambda}{k^2}) \\
&\quad \geq \cdots \geq \Xi(\zeta,\omega,\frac{\lambda}{k^n}) \to 1 \ as \ n \to \infty \\
&0 \leq \Theta(\zeta,\omega,\lambda) = \Theta(\digamma\eta,\digamma\omega\digamma\omega,\lambda) \leq \Theta(\zeta,\omega,\frac{\lambda}{k}) = \Theta(\digamma\eta,\digamma\omega,\frac{\lambda}{k}) \leq \Theta(\zeta,\omega,\frac{\lambda}{k^2}) \\
&\quad \leq \cdots \leq \Theta(\zeta,\omega,\frac{\lambda}{k^n}) \to 0 \ as \ n \to \infty \\
&0 \leq \Upsilon(\zeta,\omega,\lambda) = \Upsilon(\digamma\eta,\digamma\omega,\lambda) \leq \Upsilon(\zeta,\omega,\frac{\lambda}{k}) = \Upsilon(\digamma\eta,\digamma\omega,\frac{\lambda}{k}) \leq \Upsilon(\zeta,\omega,\frac{\lambda}{k^2}) \\
&\quad \leq \cdots \leq \Upsilon(\zeta,\omega,\frac{\lambda}{k^n}) \to 0 \ as \ n \to \infty.
\end{aligned}$$

From the definition of NMS, We get $\eta = \omega$. Therefor, $\digamma$ has a unique fixed point.

Lemma 4.2 (a) If $lim_{n\to\infty}\zeta_n = \zeta$ and $lim_{n\to\infty}\eta_n = \eta$, then

$$\begin{aligned}
\Xi(\zeta,\eta,\lambda-\epsilon) &\leq lim_{n\to\infty}inf \ \Xi(\zeta_n,\eta_n,\lambda) \\
\Theta(\zeta,\eta,\lambda-\epsilon) &\geq lim_{n\to\infty}sup \ \Theta(\zeta_n,\eta_n,\lambda) \\
\Upsilon(\zeta,\eta,\lambda-\epsilon) &\geq lim_{n\to\infty}sup \ \Upsilon(\zeta_n,\eta_n,\lambda)
\end{aligned}$$

(b) If $lim_{n\to\infty}\zeta_n = \zeta$ and $lim_{n\to\infty}\eta_n = \eta$, then

$$\begin{aligned}
\Xi(\zeta,\eta,\lambda+\epsilon) &\geq lim_{n\to\infty}sup \ \Xi(\zeta_n,\eta_n,\lambda) \\
\Theta(\zeta,\eta,\lambda+\epsilon) &\leq lim_{n\to\infty}inf \ \Theta(\zeta_n,\eta_n,\lambda) \\
\Upsilon(\zeta,\eta,\lambda+\epsilon) &\leq lim_{n\to\infty}inf \ \Upsilon(\zeta_n,\eta_n,\lambda)
\end{aligned}$$

for all $\lambda > 0$ and $0 < \epsilon < \lambda$.

Proof for(a): By the definition of NMS, conditions (v),(x) and (xv)

$$\Xi(\zeta_n, \eta_n, \lambda) \geq \Xi(\zeta_n, \zeta, \frac{\epsilon}{2}) \star \Xi(\zeta, \eta, \lambda - \epsilon) \star \Xi(\eta, \eta_n, \frac{\epsilon}{2})$$

$$lim_{n\to\infty}inf\ \Xi(\zeta_n, \eta_n, \lambda) \geq 1 \star \Xi(\zeta, \eta, \lambda - \epsilon) \star 1$$

$$Hence, \quad lim_{n\to\infty}inf\ \Xi(\zeta_n, \eta_n, \lambda) \geq \Xi(\zeta, \eta, \lambda - \epsilon)$$

$$\Theta(\zeta_n, \eta_n, \lambda) \leq \Theta(\zeta_n, \zeta, \frac{\epsilon}{2}) \diamond \Theta(\zeta, \eta, \lambda - \epsilon) \diamond \Theta(\eta, \eta_n, \frac{\epsilon}{2})$$

$$lim_{n\to\infty}sup\ \Theta(\zeta_n, \eta_n, \lambda) \leq 0 \diamond \Theta(\zeta, \eta, \lambda - \epsilon) \diamond 0$$

$$Hence, \quad lim_{n\to\infty}sup\ \Theta(\zeta_n, \eta_n, \lambda) \leq \Theta(\zeta, \eta, \lambda - \epsilon)$$

$$\Upsilon(\zeta_n, \eta_n, \lambda) \leq \Upsilon(\zeta_n, \zeta, \frac{\epsilon}{2}) \diamond \Upsilon(\zeta, \eta, \lambda - \epsilon) \diamond \Upsilon(\eta, \eta_n, \frac{\epsilon}{2})$$

$$lim_{n\to\infty}sup\ \Upsilon(\zeta_n, \eta_n, \lambda) \leq 0 \diamond \Upsilon(\zeta, \eta, \lambda - \epsilon) \diamond 0$$

$$Hence, \quad lim_{n\to\infty}sup\ \Upsilon(\zeta_n, \eta_n, \lambda) \leq \Theta(\zeta, \eta, \lambda - \epsilon)$$

Proof for (b):By the definition of NMS, conditions (v),(x) and (xv)

$$\Xi(\zeta, \eta, \lambda + \epsilon) \geq \Xi(\zeta, \zeta_n, \frac{\epsilon}{2}) \star \Xi(\zeta_n, \eta_n, \epsilon) \star \Xi(\eta_n, \eta, \frac{\epsilon}{2})$$

$$Hence, \quad \Xi(\zeta, \eta, \lambda + \epsilon) \geq lim_{n\to\infty}sup\Xi(\zeta_n, \eta_n, \epsilon)$$

$$\Theta(\zeta, \eta, \lambda + \epsilon) \leq \Xi(\zeta, \zeta_n, \frac{\epsilon}{2}) \diamond \Theta(\zeta_n, \eta_n, \epsilon) \diamond \Theta(\eta_n, \eta, \frac{\epsilon}{2})$$

$$Hence, \quad \Theta(\zeta, \eta, \lambda + \epsilon) \leq lim_{n\to\infty}inf\Theta(\zeta_n, \eta_n, \epsilon)$$

$$\Upsilon(\zeta, \eta, \lambda + \epsilon) \leq \Upsilon(\zeta, \zeta_n, \frac{\epsilon}{2}) \diamond \Upsilon(\zeta_n, \eta_n, \epsilon) \diamond \Upsilon(\eta_n, \eta, \frac{\epsilon}{2})$$

$$Hence, \quad \Upsilon(\zeta, \eta, \lambda + \epsilon) \leq lim_{n\to\infty}inf\Upsilon(\zeta_n, \eta_n, \epsilon)$$

Corollary 4.3 If $lim_{n\to\infty}\zeta_n = a$ and $lim_{n\to\infty}\eta_n = \eta$, then

$$(a) \quad \Xi(\zeta, \eta, \lambda) \leq lim_{n\to\infty}inf\ \Xi(\zeta_n, \eta_n, \lambda);$$

$$\Theta(\zeta, \eta, \lambda) \geq lim_{n\to\infty}sup\ \Theta(\zeta_n, \eta_n, \lambda);$$

$$\Upsilon(\zeta, \eta, \lambda) \geq lim_{n\to\infty}sup\ \Upsilon(\zeta_n, \eta_n, \lambda)....(4.3.1)$$

$$(b) \quad \Xi(\zeta, \eta, \lambda) \geq lim_{n\to\infty}sup\ \Xi(\zeta_n, \eta_n, \lambda)$$

$$\Theta(\zeta, \eta, \lambda) \leq lim_{n\to\infty}inf\ \Theta(\zeta_n, \eta_n, \lambda)$$

$$\Upsilon(\zeta, \eta, \lambda) \leq lim_{n\to\infty}inf\ \Upsilon\zeta_n, \eta_n, \lambda)....(4.3.2)$$

for all $\lambda > 0$ and $0 < \epsilon < \lambda$.

Theorem 4.4 (Neutrosophic Edelstein Contraction Theorem) Let $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ be compact neutrosophic metric space. Let $\digamma : \Sigma \to \Sigma$ be a function satisfying

$$\Xi(\digamma\zeta, \digamma\eta, .) > \Xi(\zeta, \eta, .); \Theta(\digamma\zeta, \digamma\eta, .) < \Theta(\zeta, \eta, .); \Upsilon(\digamma\zeta, \digamma\eta, .) < \Upsilon(\zeta, \eta, .).....(4.4.1)$$

Then $\digamma$ has fixed point.

Proof: Let $a \in \Sigma$ and $a_n = \digamma^n \zeta \quad (n \in \mathbb{N})$. Assume $\zeta_n \neq \zeta_{n+1}$ for each n (If not $\digamma \zeta_n = \zeta_n$) consequently $a_n \neq a_{n+1} \quad (n \neq m)$, For otherwise we get

$$\Xi(\zeta_n, \zeta_{n+1}, .) = \Xi(\zeta_m, \zeta_{m+1}, .) > \Xi(\zeta_{m-1}, \zeta_m, .) > \cdots > \Xi(\zeta_n, \zeta_{n+1}, .)$$
$$\Theta(\zeta_n, \zeta_{n+1}, .) = \Theta(\zeta_m, \zeta_{m+1}, .) < \Theta(\zeta_{m-1}, \zeta_m, .) < \cdots < \Theta(\zeta_n, \zeta_{n+1}, .)$$
$$\Upsilon(\zeta_n, \zeta_{n+1}, .) = \Upsilon(\zeta_m, \zeta_{m+1}, .) < \Upsilon(\zeta_{m-1}, \zeta_m, .) < \cdots < \Upsilon(\zeta_n, \zeta_{n+1}, .)$$

where $m > n$, which is a contradiction. Since Σ is compact set, $\{\zeta_n\}$ has convergent sub sequence $\{\zeta_{n_i}\}$. Let $\eta = lim_{i\to\infty}\zeta_{n_i}$, Also we assume that η such that $\digamma\eta \in \{\zeta_{n_i}; i \in \mathbb{N}\}$. According to the above assumption, we may now write,

$$\Xi(\digamma\zeta_{n_i}, \digamma\eta, .) > \Xi(\zeta_{n_i}, \eta, .); \quad \Theta(\digamma\zeta_{n_i}, \digamma\eta, .) < \Theta(\zeta_{n_i}, \eta, .); \quad \Upsilon(\digamma\zeta_{n_i}, \digamma\eta, .) < \Upsilon(\zeta_{n_i}, \eta, .)$$

for all $i \in \mathbb{N}$. Then by equation (4.3.1) we obtain

$$lim\ inf \Xi(\digamma\zeta_{n_i}, \digamma\eta, \lambda) \geq lim\ \Xi(\zeta_{n_i}, \eta, \lambda) = \Xi(\eta, \eta, \lambda) = 1$$
$$lim\ sup \Theta(\digamma\zeta_{n_i}, \digamma\eta, \lambda) \leq lim\ \Theta(\zeta_{n_i}, \eta, \lambda) = \Theta(\eta, \eta, \lambda) = 0$$
$$lim\ sup \Upsilon(\digamma\zeta_{n_i}, \digamma\eta, \lambda) \leq lim\ \Upsilon(\zeta_{n_i}, \eta, \lambda) = \Upsilon(\eta, \eta, \lambda) = 0$$

for each $\lambda > 0$. Hence

$$lim\ \digamma\zeta_{n_i} = \digamma\eta....(4.4.2)$$

Simillarly

$$lim\ \digamma^2\zeta_{n_i} = lim \digamma^2\eta...(4.4.3)$$

(we recall that $lim\ \digamma\zeta_{n_i} = \digamma\eta$ for all $(i \in \mathbb{N})$), Now observe that,

$$\begin{aligned}
\Xi(\zeta_{n_i}, \digamma\zeta_{n_i}, \lambda) &\leq \Xi(\digamma\zeta_{n_i}, \digamma^2\zeta_{n_i}, \lambda) \leq \cdots \leq \Xi(\zeta_{n_i}, \digamma\zeta_{n_i}, \lambda) \\
&\leq \Xi(\digamma\zeta_{n_i}, \digamma^2\zeta_{n_i}, \lambda) \leq \cdots \leq \Xi(\digamma\zeta_{n_{i+1}}, \digamma^2\zeta_{n_{i+1}}, \lambda) \\
&\leq \Xi(\digamma\zeta_{n_{i+1}}, \digamma^2\zeta_{n_{i+1}}, \lambda) \leq \cdots \leq 1. \\
\Theta(\zeta_{n_i}, \digamma\zeta_{n_i}, \lambda) &\geq \Theta(\digamma\zeta_{n_i}, \digamma^2\zeta_{n_i}, \lambda) \geq \cdots \geq \Theta(\zeta_{n_i}, \digamma\zeta_{n_i}, \lambda) \\
&\geq \Theta(\digamma\zeta_{n_i}, \digamma^2\zeta_{n_i}, \lambda) \geq \cdots \geq \Theta(\digamma\zeta_{n_{i+1}}, \digamma^2\zeta_{n_{i+1}}, \lambda) \\
&\geq \Theta(\digamma\zeta_{n_{i+1}}, \digamma^2\zeta_{n_{i+1}}, \lambda) \geq \cdots \geq 0. \\
\Upsilon(\zeta_{n_i}, \digamma\zeta_{n_i}, \lambda) &\geq \Upsilon(\digamma\zeta_{n_i}, \digamma^2\zeta_{n_i}, \lambda) \geq \cdots \geq \Upsilon(\zeta_{n_i}, \digamma\zeta_{n_i}, \lambda) \\
&\geq \Upsilon(\digamma\zeta_{n_i}, \digamma^2\zeta_{n_i}, \lambda) \geq \cdots \geq \Upsilon(\digamma\zeta_{n_{i+1}}, \digamma^2\zeta_{n_{i+1}}, \lambda) \\
&\geq \Upsilon(\digamma\zeta_{n_{i+1}}, \digamma^2\zeta_{n_{i+1}}, \lambda) \geq \cdots \geq 0.
\end{aligned}$$

for all $\lambda > 0$. Thus $\{\Xi(\zeta_{n_i}, F\zeta_{n_i}, \lambda)\}, \{\Theta(\zeta_{n_i}, F\zeta_{n_i}, \lambda)\}, \{\Upsilon(\zeta_{n_i}, F\zeta_{n_i}, \lambda)\}$ and $\{(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda)\}$ $(\lambda > 0)$ are convergent to a common limit point . So by equations (4.3.1) , (4.3.2) and (4.4.1) and we get,

$$\begin{aligned}
\Xi(\eta, F\eta, \lambda) &\geq lim\ sup\ \Xi(\zeta_{n_i}, F\zeta_{n_i}, \lambda) = lim\ sup\ (F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda)\\
&\geq lim\ inf \Xi(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda)\\
&\geq \Xi(F\eta, F^2\eta, \lambda)\\
\Theta(\eta, F\eta, \lambda) &\leq lim\ inf\ \Theta(\zeta_{n_i}, F\zeta_{n_i}, \lambda) = lim\ inf\ \Theta(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda)\\
&\leq lim\ sup \Theta(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda)\\
&\leq \Theta(F\eta, F^2\eta, \lambda)\\
\Upsilon(\eta, F\eta, \lambda) &\leq liminf\ \Upsilon(\zeta_{n_i}, F\zeta_{n_i}, \lambda) = lim\ inf\ \Upsilon(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda)\\
&\leq lim\ sup \Upsilon(F\zeta F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda)\\
&\leq \Upsilon(F\eta, F^2\eta, \lambda)
\end{aligned}$$

for all $\lambda > 0$. Suppose $b \neq F\eta$, By equation (4.4.1)

$$\Xi(\eta, F\eta, .) < \Xi(F\eta, F^2\eta, .); \quad \Theta(\eta, F\eta, .) > \theta(F\eta, F^2\eta, .); \quad \Upsilon(\eta, F\eta, .) > \Upsilon(F\eta, F^2\eta, .).$$

which is a contradiction , because all the above functions are left continuous , non -decreasing and right continuous , non - increasing respectively. Hence $\eta = F\eta$ is a fixed point.
To prove the uniqueness of the fixed point, let us consider $F(\zeta) = \omega$ for some $\zeta \in \Sigma$.
Then

$$\begin{aligned}
1 &\geq \Xi(\zeta, \omega, \lambda) = \Xi(F\eta, F\omega, \lambda) \geq \Xi(\zeta, \omega, \frac{\lambda}{k}) = \Xi(F\eta, F\omega, \frac{\lambda}{k}) \geq \cdots \geq \Xi(\zeta, \omega, \frac{\lambda}{k^n})\\
0 &\leq \Theta(\zeta, \omega, \lambda) = \Theta(F\eta, F\omega, \lambda) \leq \Theta(\zeta, \omega, \frac{\lambda}{k}) = \Theta(F\eta, F\omega, \frac{\lambda}{k}) \leq \cdots \leq \Theta(\zeta, \omega, \frac{\lambda}{k^n})\\
0 &\leq \Upsilon(\zeta, \omega, \lambda) = \Upsilon(F\eta, F\omega, \lambda) \leq \Upsilon(\zeta, \omega, \frac{\lambda}{k}) = \Upsilon(F\omega, F\omega, \frac{\lambda}{k}) \leq \cdots \leq \Upsilon(\zeta, \omega, \frac{\lambda}{k^n})
\end{aligned}$$

Now , we easily verify that $\{\frac{\lambda}{k^n}\}$ is an s - increasing sequence, then by assumption for a given $\epsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that

$$\Xi(\zeta, \omega, \frac{\lambda}{k^n}) \geq 1 - \epsilon; \quad \Theta(\zeta, \omega, \frac{\lambda}{k^n}) \leq \epsilon; \quad \Upsilon(\zeta, \omega, \frac{\lambda}{k^n}) \leq \epsilon.$$

Clearly

$$lim_{n\to\infty}\Xi(\zeta, \omega, \frac{\lambda}{k^n}) = 1; \ lim_{n\to\infty}\Theta(\zeta, \omega, \frac{\lambda}{k^n}) = 0; \ lim_{n\to\infty}\Upsilon(\zeta, \omega, \frac{\lambda}{k^n}) = 0.$$

Hence $\Xi(\zeta, \omega, \lambda) = 1; \ \Theta(\zeta, \omega, \lambda) = 0; \ \Upsilon(\zeta, \omega, \lambda) = 0$. Thus $\eta = \omega$. Hence proved.

Conclusion: In this study, we have investigated the concept of Neutrosophic Metric Space and its properties. We have proved fixed point results for contraction theorems in the setting of neutrosophic metric Space. There is a scope to establish many fixed point results in the areas such as fuzzy metric, generalized fuzzy metric, bipolar and partial fuzzy metric spaces by using the concept of Neutrosophic Set.

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